HARMONIC MAPS INTO GRASSMANN MANIFOLDS

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ABSTRACT. A harmonic map from a Riemannian manifold into a Grassmann manifold is characterized by a vector bundle, a space of sections of this bundle and a Laplace operator. We apply our main theorem (a generalization of Theorem of Takahashi) to generalize the theory of do Carmo and Wallach and to describe the moduli space of harmonic maps satisfying the gauge and the Einstein–Hermitian conditions from a compact Riemannian manifold into a Grassmannian. The geometric meaning of the compactification of the moduli space is interpreted and it is shown that the compactified moduli space is connected and convex. As applications, several rigidity results are exhibited and we also construct moduli spaces of holomorphic isometric embeddings of the complex projective line into complex quadrics of low degree. The compactification of the moduli space leads to classification theorems for equivariant harmonic maps.

CONTENTS

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1. Introduction

What this paper attempts to do is to bring ideas from the gauge theory of vector bundles into the theory of harmonic maps.

One of our main purposes is to generalize Theorem of Takahashi [32]. Let $f : M \to S^{N-1}$ be a smooth map from a Riemannian manifold (M, g) into the standard sphere S^{N-1}, which can be considered as a unit sphere of the Euclidean space $\mathbf{R}^N$. If we fix an orthonormal basis $e_1, \cdots, e_N$ of $\mathbf{R}^N$ and the associated co-ordinates are denoted by $(x_1, \cdots, x_N)$, then each co-ordinate function x_A $(A = 1, \cdots, N)$ can be regarded as a function on S^{N-1} by restriction. We can pull-back each x_A by $f : M \to S^{N-1}$ to obtain a function on M, which is also denoted by the same symbol. Then (a version of) Theorem of Takahashi [32] states

Theorem 1.1. *A map $f : M \to S^{N-1}$ is a harmonic map if and only if there exists a function $h : M \to \mathbf{R}$ such that $\Delta x_A = h x_A$ for all $A = 1, \cdots, N$, where Δ is the Laplace operator of (M, g). Under these conditions, we have $h = |df|^2$.*

In the proof of Theorem 1.1, the position vector $f(x) \in \mathbf{R}^N$ $(x \in M)$, considered as $f : M \to \mathbf{R}^N$, plays a central role.

First of all, our concern is with a map from (M, g) into a real or complex Grassmann manifold $Gr_p(W)$ parametrizing p-dimensional subspaces of W with a standard metric of Fubini-Study type, where W is a real or complex vector space with an inner product or a Hermitian inner product. We abbreviate an inner product and a Hermitian inner product to a scalar product denoted by $(\cdot, \cdot)$. To emphasize the role of the scalar product $(\cdot, \cdot)$, we denote by $(Gr_p(W), (\cdot, \cdot))$ a Grassmannian with a metric of Fubini-Study type induced by $(\cdot, \cdot)$.

Let $S \to Gr_p(W)$ be the tautological vector bundle over $Gr_p(W)$. Since $S \to Gr_p(W)$ is a subbundle of a trivial bundle $\underline{W} = Gr_p(W) \times W \to Gr_p(W)$, we have a quotient bundle $Q \to Gr_p(W)$, which is called the *universal quotient bundle*. The scalar product on W gives an identification of $Q \to Gr_p(W)$ with the orthogonal complement of $S \to Gr_p(W)$ in $\underline{W} \to Gr_p(W)$. Consequently, vector bundles $S, Q \to Gr_p(W)$ are equipped with induced fiber metrics and connections.

When the standard sphere S^{N-1} is identified with the real Grassmannian of oriented $(N-1)$-planes in $\mathbf{R}^N$, the position vector $f(x) \in \mathbf{R}^N$ can be considered as a section of the universal quotient bundle $Q \to Gr_{N-1}(\mathbf{R}^N)$ which is also regarded as the normal bundle of S^{N-1} into $\mathbf{R}^N$. Then we use the inner product on $\mathbf{R}^N$ to recover functions $x_1, \cdots, x_N$ by relations $x_A(x) = (e_A, f(x))$ and thus $\mathbf{R}^N$ induces sections of the universal quotient bundle as $x_A(x) f(x)$. The differential of the position vector can be recognized as the second fundamental form of the subbundle $Q \to \underline{\mathbf{R}^N}$ in the sense of Griffiths [16] and Kobayashi [18] in a natural manner. Since the bundle has a preferred connection, the Laplace operator acting on sections of the bundle is well-defined, which is indeed a Laplace operator acting on functions due to the trivialization by $f(x)$ of the pull-back bundle $f^*Q \to M$. Hence we can reformulate Theorem 1.1 from the viewpoint of vector bundles when these geometric structures are pulled back. Our first result is

Main Theorem 1. (Theorem 3.5) *Let (M, g) be a Riemannian manifold and $f : M \to Gr_p(W)$ a smooth map. We fix a scalar product $(\cdot, \cdot)$ on W, which gives a Riemannian structure on $Gr_p(W)$. We regard W as a space of sections of the pull-back bundle $f^*Q \to M$.*

Then, the following three conditions are equivalent.

(1) *$f : (M, g) \to (Gr_p(W), (\cdot, \cdot))$ is a harmonic map.*
(2) *W has the zero property for the Laplacian acting on sections of the pull-back bundle $f^*Q \to M$ of the universal quotient bundle.*
(3) *There exists an endomorphism A of the bundle $f^*Q \to M$ such that $\Delta t + At = 0$ for an arbitrary $t \in W$.*

Under these conditions, a bundle endomorphism A turns out to be the mean curvature operator of $f : M \to Gr_p(W)$ and

$$|df|^2 = -\mathrm{trace}\, A \ (when\, W = \mathbf{R}^N), \quad |df|^2 = -2\mathrm{trace}\, A \ (when\, W = \mathbf{C}^N).$$

See Definition 3.4 for the zero property for the Laplacian. The bundle endomorphism A on $f^*Q \to M$, called the *mean curvature operator*, is also defined in §3 and plays a crucial role in the sequel.

In §4, we introduce three natural functionals and use the Gauss-Codazzi equations for vector bundles to compute their Euler-Lagrange equations in terms of the second fundamental forms. The first two functionals are well-known: they are a modification of the Yang-Mills functional for the pull-back connections of the universal quotient bundle and the energy functional of a map. (The modification means that the Yang-Mills type functional is defined on the space of mappings into Grassmannians, which is different from the ordinary Yang-Mills functional defined on the space of connections on the fixed bundle.) The third functional is obtained as the L^2 norm of the mean curvature operator of mappings. This functional has a lower bound which relates to the total energy of the map. A map which minimizes the third functional has the property that its mean curvature operator is proportional to the identity of the pull-back bundle. This property shall be referred to as the *Einstein-Hermitian* condition. A minimal immersion is a special case of a harmonic map with constant energy density. In cases of isometric minimal immersions into the sphere and holomorphic isometric immersions into the complex projective space regarded as $Gr_n(\mathbf{C}^{n+1}) = \mathbf{P}(\mathbf{C}^{n+1^*})$, the pull-back connections are product connections and Hermitian Yang-Mills connections on line bundles, respectively, and the mean curvature operators can be considered as constant functions. However, when the target is a Grassmannian of higher rank, we could have various connections as the pull-back connections and various bundle endomorphisms as the mean curvature operators, which we call admissible connections and admissible endomorphisms for harmonic maps, respectively, or an admissible pair for short. To determine admissible pairs could give us some difficulties in describing a set of harmonic maps. Even if we could specify an admissible pair of a connection ∇ and a bundle endomorphism A, we could have various harmonic maps with (∇, A) as admissible pair, which is the subject in §5.

A vector bundle together with a finite-dimensional vector space of sections of the bundle induces a map into a Grassmannian. Such a map is called the *induced map* or *classifying map* in [4] (Definition 5.1). A famous example of induced maps is the Kodaira embedding of an algebraic manifold into the complex projective space, which is induced by a holomorphic line bundle and the space of holomorphic sections of it. Though it is well-known that any holomorphic map between Kähler manifolds is a harmonic map, our Main Theorem 1 with Lemma 4.2 yields the fact in the case where the target is $Gr_p(\mathbf{C}^N)$ or a complex hyperquadric Q^N.

In the theory of do Carmo-Wallach [7], Theorem of Takahashi with representation theory of the special orthogonal groups is applied to classify isometric minimal immersions of spheres into spheres. Toth and D'ambra obtain the exact dimension of the moduli spaces of those immersions and extend the do Carmo-Wallach theory to the case that the domain manifold is a compact isotropy irreducible Riemannian homogeneous space [35] (see also Wallach [36]). Section 5 is devoted to developing a generalization of the theory of do Carmo and Wallach by Main Theorem 1. Instead of representation theory, we make use of geometry of vector bundles. We are now concerned with a harmonic map from a compact Riemannian manifold into a Grassmannian, satisfying the *gauge* and the *Einstein-Hermitian* conditions (see §§4 and 5 for the definition): by the gauge condition on a harmonic map, we specify a vector bundle with a fiber metric and a connection obtained as the pull-back of the universal quotient bundle. The Einstein-Hermitian condition relates a harmonic map to an eigenspace of the Laplace operator. After fixing a vector bundle with a metric and a connection on a compact Riemannian manifold, a *standard map* is defined as the induced map by a subspace of the space of sections on the bundle with L^2 scalar product. We show that any harmonic map with gauge and Einstein-Hermitian conditions is realized as a deformation of a standard map and corresponds to a pair (W, T), where W is an eigenspace with L^2 scalar product of the Laplacian acting on sections of the vector bundle and T is a positive semi-definite Hermitian endomorphism on W satisfying *the MC equations* (Theorem 5.12). One of the important features in our theory is that every harmonic map can be realized as an induced map with a bundle isomorphism called *the natural identification* (Definition 5.9). To establish the correspondence, we introduce the notion of *gauge equivalence* of maps, while the equivalence relation used in the original do Carmo-Wallach theory is called *image equivalence* : the isometry (sub)group of a Grassmannian acts on the space of harmonic maps into the Grassmannian via the composite, of which the orbits give the image equivalence relation of maps. An isometry of Grassmannian also induces an isomorphism of the universal quotient bundle which covers the isometry. Since any harmonic map is realized along with the natural identification, the induced bundle isomorphism enters into our theory to define the gauge equivalence relation of maps (Definitions 5.7 and 5.9). From the point of view of gauge theory, even in the original do Carmo-Wallach theory, we can find an isomorphism of the pull-back of the universal quotient bundle and a flat line bundle preserving metrics and connections for each isometric minimal immersion, which is fulfilled with the position vector. This means that the gauge condition is automatically satisfied in the original theory. On the other hand, the mean curvature operator is regarded as a function in the case where the target is the sphere or the complex projective space, since the universal quotient bundle is of real or complex rank one in each case. Thus, the Einstein-Hermitian condition is also automatically satisfied in the original theory. These could explain why the gauge theoretic side of the theory might have been overlooked in the literature. However the situation changes: we could have various admissible pairs when considering harmonic maps into Grassmannians of higher rank. For this, we adopt the gauge and Einstein-Hermitian conditions in a generalization of do Carmo-Wallach theory at the first stage.

Next, by relaxing the Einstein-Hermitian condition, we develop a considerable generalization of do Carmo-Wallach theory which is needed in the case where the target is a Grassmannian of higher rank. Instead of eigenspaces of the Laplacian, we consider the solution space of the "generalized Laplace equation" $(\Delta + A)\,t = 0$ for sections t, where A is a bundle endomorphism, which is eventually recognized as the mean curvature operator of the induced map. In the generalization of do Carmo-Wallach theory, any harmonic map in question is realized as an induced map with the natural identification and parametrized in an analogous way by a pair (W_A, T) modulo gauge equivalence, where W_A is the solution space of the generalized Laplace equation (Theorem 5.15).

To construct the moduli space, we need a relative version of a generalization of do Carmo-Wallach theorem (Theorem 5.20). Then the moduli space $\mathcal{M}$ modulo gauge equivalence is described in Theorem 5.24. Using the L^2 scalar product on W_A, we provide $\mathcal{M}$ with a natural topology and derive topological properties of it. We will interpret the geometric meaning of the compactification of $\mathcal{M}$ (see Remark after Corollary 5.25). The compactification involves totally geodesic submanifolds of the target manifold and harmonic maps into the submanifolds. Such a totally geodesic submanifold emerges as the zero set of the sections of the universal quotient bundle which belong to $\operatorname{Ker} T \subset W_A$ and is thus a Grassmann manifold whose universal quotient bundle has the same rank as that of the target. The compactification of $\mathcal{M}$ leads to the notion of *terminal* harmonic map (Definition 5.26), which is implicit in the original theory. To proceed further, we utilise the connectedness of the compactified moduli space to conclude that the terminal harmonic map is rigid (Corollary 5.27). We also exploit the connectedness of the compactified moduli space to obtain various rigidity results (Theorems 6.24, 6.28 , 6.29, 6.30 and Corollary 6.25). In any case, it plays a significant role that totally geodesic embeddings of low dimensional Grassmannians are involved in our compactification of the moduli spaces.

When considering the moduli space by image equivalence, we encounter the action on $\mathcal{M}$ of a Lie subgroup of the structure group of the pull-back of the universal quotient bundle. The Lie subgroup is indeed the centralizer of the holonomy group of the induced connection on the pull-back bundle. In the original do Carmo-Wallach theory, both the holonomy group and the structure group are trivial. It follows that the moduli spaces modulo image equivalence coincide with those modulo gauge equivalence (Lemma 5.28). After reviewing the original do Carmo-Wallach theorem briefly, we present a direct generalization of it as Theorem 5.29. Our theory thus includes the original theory. On the contrary, we will give an example in which the moduli space by gauge equivalence is different from that by image equivalence (see Theorems 6.21 and 6.22). This example could justify to bring a bit of complicated gauge equivalence relation of maps into our theory, because the moduli space $\mathcal{M}$ by gauge equivalence is easily described as a connected, convex and open subset of some Euclidean space (Theorem 5.24), while the moduli space by image equivalence might be a quotient space of $\mathcal{M}$ by a Lie group (Theorem 5.32).

Next, we focus our attention on a homogeneous vector bundle with a canonical connection over a compact reductive Riemannian homogeneous space G/K . Then, each invariant subspace of the eigenspace of the Laplace operator on the vector bundle induces a G-equivariant map from G/K into a Grassmannian, which is also called a standard map. We give a sufficient condition for a standard map being a harmonic map with gauge and Einstein-Hermitian conditions (Lemma 5.36). Since the standard map is G-equivariant, the energy density is constant, its value being expressible through its eigenvalue. In this subsection a few examples of standard maps are displayed, some of which are related to Kähler or quaternion-Kähler moment maps. We shall modify the MC equations to obtain a classification of harmonic maps in Theorem 5.37. As a result, the description of moduli spaces is connected with the representation theory of compact Lie groups. This gives a straightforward generalization of Toth-D'ambra theory [35] and Wallach [36] and will be of use in §6.

In section 6, we apply our generalization of do Carmo-Wallach theory to obtain various rigidity theorems and moduli spaces. We give an alternative proof of the Theorem of Bando-Ohnita [2], J.Bolton-G.R.Jensen-M.Rigoli-L.M.Woodward [3] and Ohnita [28], which states the rigidity of isometric minimal immersions of the complex projective line $\mathbf{C}P^1$ into complex projective spaces. The same method yields rigidity of holomorphic isometric embeddings between complex projective spaces, which is a part of well-known theorem of Calabi [5]. Toth defines the notion of polynomial minimal immersion between complex projective spaces [34]. In this notion, the gauge condition is implicitly supposed to be satisfied. Then, we show that every harmonic map between complex projective spaces satisfying the gauge condition for the canonical connection on a complex line bundle is automatically a polynomial map in the sense of Toth by Theorem 3.5. We give a generalization of Calabi's rigidity theorem on holomorphic isometric embeddings between complex projective spaces as Theorem 6.14, in which we show the rigidity of Einstein-Hermitian holomorphic embeddings from irreducible Hermitian symmetric spaces of compact type into complex Grassmannians with the gauge condition for a direct sum of r-copies of an irreducible homogeneous bundle and the canonical connections.

Next, we use Theorems 5.15, 5.24, 5.32 and 5.37 to describe the moduli spaces of holomorphic isometric embeddings of $\mathbf{C}P^1$ into complex hyperquadrics Q^N, which are also Einstein-Hermitian holomorphic embeddings. At this stage, these examples manifest the difference of gauge equivalence and image equivalence. We see from the description of the compactified moduli space that the real standard map (defined in §6) is the unique representative in the homotopy class of maps of degree 2 of $\mathbf{C}P^1$ into Q^N

which is the Einstein-Hermitian holomorphic terminal map with the pull-back connection being a Hermitian Yang-Mills connection. Finally, we give an equivariant version of Theorem 5.20 (Theorem 6.23) to obtain classification theorems on equivariant harmonic maps of the complex projective spaces into $Gr_p(\mathbf{C}^N)$ or Q^N. Here Theorem 5.15 is needed and we eventually meet harmonic maps into complex Grassmannians which do not satisfy the Einstein-Hermitian condition.

In the final section, we compare the generalized do Carmo-Wallach construction with the well-known ADHM-construction of instantons on S^4. Though both the harmonic map equation and the anti-self-dual equation are non-linear, linear equations naturally emerge in our geometric setting, which lead us to a description of moduli spaces in linear algebraic terms.

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2. Preliminaries

Throughout this paper, a manifold is supposed to be connected. We review some standard material, mostly in order to fix our notation.

2.1. **A harmonic map.** Let M and N be Riemannian manifolds and $f : M \to N$ be a (smooth is always understood) map. We define *the energy density* $e(f) : M \to \mathbf{R}$ of f as

$$e(f)(x) := |df|^2 = \sum_{i=1}^{\dim M} |df(e_i)|^2,$$

where we use both Riemannian metrics on M and N and $e_1, \cdots, e_{\dim M}$ denotes an orthonormal basis of the tangent space T_xM at x. Then, *the tension field* $\tau(f)$ of f is defined to be

$$\tau(f)_x := \operatorname{trace} \nabla df = \sum_{i=1}^{\dim M} (\nabla_{e_i} df)(e_i),$$

which is a section of the pull-back bundle $f^*TN \to M$ of the tangent bundle $TN \to N$. The definition of harmonic maps is due to Eells and Sampson [11].

Definition 2.1. A map $f : M \to N$ is called a *harmonic map* if the tension field vanishes ($\tau(f) \equiv 0$).

The symmetric form ∇df with values in $f^*TN \to M$ is called *the second fundamental form.* We say that a map $f : M \to N$ is a *totally geodesic map* if $\nabla df \equiv 0$. By definition, a totally geodesic map is a harmonic map.

If we suppose that $f : M \to N$ is an isometric immersion, then the tension field is a mean curvature vector, the second fundamental form is the same as that in submanifold geometry and a harmonic map is nothing but a minimal immersion.

2.2. **Geometry of Grassmann manifolds.** First of all, we focus our attention on a *real* Grassmann manifold.

Let W be a real N-dimensional vector space with an orientation. Let $Gr_p(W)$ be a Grassmann manifold of (oriented) p-planes in W and $S \to Gr_p(W)$ the tautological vector bundle. Then we have an exact sequence of vector bundles:

$$0 \to S \xrightarrow{i} \underline{W} \xrightarrow{\pi} Q \to 0,$$

where $\underline{W} \to Gr_p(W)$ is a trivial vector bundle of fiber W, and $Q \to Gr_p(W)$ is the quotient bundle which is called the *universal quotient bundle*.

When W is equipped with an inner product $(\cdot,\cdot)$ on W, we can define a homogeneous Riemannian metric g_{Gr} induced by $(\cdot,\cdot)$ such that $Gr_p(W)$ is a Riemannian symmetric space. To define g_{Gr} more precisely, we notice that $(\cdot,\cdot)$ induces the fibre metrics g_S on $S \to Gr_p(W)$ and g_Q on $Q \to Gr_p(W)$. Since the tangent bundle $T = TGr_p(W)$ of $Gr_p(W)$ is identified with $S^* \otimes Q \cong S \otimes Q$, the Riemannian metric g_{Gr} is induced as the tensor product of g_S and g_Q : $g_{Gr} = g_S \otimes g_Q$. We call g_{Gr} a Riemannian metric of Fubini-Study type. To emphasize the role of the inner product $(\cdot,\cdot)$, we denote by $(Gr_p(W), (\cdot,\cdot))$ a Grassmannian with a metric of Fubini-Study type induced by $(\cdot,\cdot)$.

We fix an orthonormal basis $w_1, \cdots, w_N$ of W which is compatible with its orientation. We denote by $\mathbf{R}^p$ the subspace spanned by $w_1, \cdots, w_p$ and by $\mathbf{R}^q$ the orthogonal complementary subspace. The orthogonal projection to $\mathbf{R}^p$ is denoted by π_p and the orthogonal projection to $\mathbf{R}^q$ by π_q. Using the orthogonal projection π_q, we can explicitly write a bundle map $\pi : \underline{W} \to Q$:

$$\pi_{[g]}(w) := \left[g, \pi_q(g^{-1}w)\right] \in Q = G \times_{K_0} \mathbf{R}^q, \; w \in W, \; g \in G,$$

where $G = \mathrm{SO}(N)$ and $K_0 = \mathrm{SO}(p) \times \mathrm{SO}(q)$. The inner product $(\cdot,\cdot)$ gives a bundle injection $\pi^* : Q \to \underline{W}$: $\pi^*([g,v]) = ([g], gv)$, $v \in \mathbf{R}^q$, which is the adjoint bundle map of π. Hence, $Q \to Gr_p(W)$ is also regarded as the orthogonal complementary bundle $S^\perp \to Gr_p(W)$ to $S \to Gr_p(W)$. We can define a connection ∇^Q on $Q \to Gr_p(W)$. If t is a section of $Q \to Gr_p(W)$, then we have

$$\nabla^Q t = \pi d\left(\pi^*(t)\right).$$

The connection ∇^Q is called the canonical connection.

In a similar way, we can use $i : S \to \underline{W}$: $i([g,u]) = ([g], gu)$, $u \in \mathbf{R}^p$ and its adjoint bundle map $i^* : \underline{W} \to S$:$i^*(w) := \left[g, \pi_p(g^{-1}w)\right] \in G \times_{K_0} \mathbf{R}^p$ to define the connection ∇^S:

$$\nabla^S s = i^* d\left(i(s)\right), \quad s \in \Gamma(S),$$

which is also called the canonical connection. Using the identification $T = S^* \otimes Q$, we can see that the Levi-Civita connection coincides with one induced by the canonical connections.

In this context, since $S \to Gr_p(W)$ is a subbundle of $\underline{W} \to Gr_p(W)$, it is natural to introduce the second fundamental form H (for example, see [16] and [18]), which is a 1-form with values in $\mathrm{Hom}(S,Q) \cong S^* \otimes Q$:

$$dis = i\nabla^S s + \pi^* Hs, \quad Hs = \pi d\left(i(s)\right).$$

If $s = i^*(w)$, then we can compute

$$Hs = \left[g, \pi_{\mathfrak{m}}(g^{-1}dg)\pi_p(g^{-1}w)\right], \tag{2.1}$$

where we use an orthogonal decomposition of Lie algebra $\mathfrak{g}$ of G: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $\mathfrak{k} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$, with respect to a G-invariant inner product on $\mathfrak{g}$ and the orthogonal projection $\pi_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$.

Since T is identified with $S^* \otimes Q$, the cotangent bundle T^* is considered as $T^* = S \otimes Q^*$. Hence the second fundamental form $H \in \Omega^1(S^* \otimes Q)$ can also be regarded as a section of $T^* \otimes T = S \otimes S^* \otimes Q \otimes Q^*$.

Lemma 2.2. *The second fundamental form H can be regarded as the identity transformation of the tangent bundle T.*

Proof. Since H is an invariant form, we may evaluate at the reference point $[e]$ of $Gr_p(W)$, where e is the unit element of G. From (2.1), using $T = G \times_{K_0} \mathfrak{m}$ and $T_{[e]} = \mathfrak{m}$, we have for $X \in \mathfrak{m}$ that $H_X = [e, \pi_{\mathfrak{m}}(X)] = X$. $\square$

Since the canonical connections preserves the irreducible decomposition of $T^* \otimes T$, we can see that

Corollary 2.3. *The second fundamental form H is parallel.*

We can also define the second fundamental form $K \in \Omega^1(Q^* \otimes S)$ of a subbundle $\pi^* : Q \to \underline{W}$:

$$di_Q t = \pi^* \nabla^Q t + iKt, \quad Kt = i^* d\left(\pi^*(t)\right)$$

For a fixed vector $w \in W$, assigning $x \in Gr_p(W)$ to $i_x^*(w)$ and $\pi_x(w)$ for $w \in W$, we obtain two sections $s = i^*(w)$ of S and $t = \pi(w)$ of Q, each of which is called *the section corresponding to w*. Thus W can be considered as a subspace of sections of $S, Q \to Gr_p(W)$.

Proposition 2.4. *If s and t are the sections corresponding to $w \in W$, then*

$$\nabla^S s = -Kt, \quad \nabla^Q t = -Hs.$$

Proof. By definition, we have that $w = i(s) + \pi^*(t)$. Then,

$$\nabla^S s = i^* di(s) = i^* d\left(w - \pi^*(t)\right) = -i^* d\pi^*(t) = -Kt.$$

We get the other equation in a similar way. $\square$

Lemma 2.5. *The second fundamental forms H and K satisfy*

$$g_Q(Hu, v) = -g_S(u, Kv), \quad u \in S_x, \text{ and } \quad v \in Q_x,$$

where S_x and Q_x are the fibers of S and Q over $x \in Gr_p(W)$, respectively.

Proof. If we take sections s of S and t of Q such that $s(x) = u$ and $t(x) = v$, then $g_Q(Hs, t) = (di(s), \pi^* t)_W = -(is, d\pi^* t)_W = -g_S(s, Kt)$. $\square$

The second fundamental form $K \in \Omega^1(Q^* \otimes S)$ can also be regarded as a section of $Q \otimes S \otimes Q \otimes S$. Under the irreducible decomposition, K corresponds to a constant section of $\mathbf{R} \subset Q \otimes S \otimes Q \otimes S$. Obviously, we have

Lemma 2.6. *The second fundamental form K is also parallel.*

The orthonormal basis $w_1, \cdots, w_N$ of W provides us with the corresponding sections $s_A = i^*(w_A)$ and $t_A = \pi_Q(w_A)$.

Proposition 2.7. *For arbitrary tangent vectors X and Y on a real Grassmannian, we have*

$$
\begin{aligned}
(2.2) \qquad g_{Gr}(X,Y) &= \sum_A g_{S^*\otimes Q}(H_X, H_Y) = \sum_A g_{Q^*\otimes S}(K_X, K_Y) \\
&= \sum_A g_Q(H_X s_A, H_Y s_A) = \sum_A g_S(K_X t_A, K_Y t_A).
\end{aligned}
$$

Proof. The key fact is that the second fundamental form H is considered as an isomorphism of bundles from T to $S^* \otimes Q$ preserving the metrics and the connections (Lemma 2.2). Consequently, (2.2) follows. $\square$

Remark. Lemma 2.5 and Proposition 2.7 give us

$$g_{Gr} = -\mathrm{trace}_Q\, HK = -\mathrm{trace}_S\, KH.$$

Next, we consider a *complex* Grassmann manifold. The main difference of a complex Grassmannian from a real Grassmannian is that we can use the Hodge decomposition, since a complex Grassmannian is a Kähler manifold. More precisely, let W be a complex vector space with a Hermitian inner product $(\cdot,\cdot)_W$ and $Gr_p(W)$ a complex Grassmannian of p-planes in W. We can define homogeneous vector bundles $S \to Gr_p(W)$ and $Q \to Gr_p(W)$ with induced Hermitian metrics h_S and h_Q by W, respectively. The canonical connections equip S and $Q \to Gr_p(W)$ with holomorphic vector bundle structures. Then, W is regarded as the space of holomorphic sections of $Q \to Gr_p(W)$. The holomorphic tangent bundle is identified with $S^* \otimes Q$ and the holomorphic cotangent bundle is $S \otimes Q^*$. The identification is compatible with Hermitian metrics and connections. The second fundamental form $H \in \Omega^1(\mathrm{Hom}(S,Q))$ is of type $(1,0)$. It follows that H is considered as a section of $S \otimes Q^* \otimes S^* \otimes Q$ and we obtain $H = id_S \otimes id_Q$ in a similar way. The second fundamental form $K \in \Omega^1(\mathrm{Hom}(Q,S))$ is of type $(0,1)$. Since the Hermitian metric on the holomorphic tangent bundle is induced from that on $S^* \otimes Q \to Gr_p(W)$, the Riemannian metric g_{Gr} satisfies

$$
\begin{aligned}
(2.3) \qquad g_{Gr}(X,Y) &= -\,\mathrm{trace}_Q\, H_X K_Y - \mathrm{trace}_S\, K_X H_Y \\
&= -\,2\mathrm{Re}\left(\mathrm{trace}_Q\, H_X K_Y\right),
\end{aligned}
$$

for arbitrary real tangent vectors X and Y on the complex Grassmannian.

3. Harmonic maps into Grassmannians

In this section, we shall prove the main theorems. We denote by $\underline{W} \to M$ a trivial vector bundle over M with a vector space W as its fiber: $M \times W \to M$. For a vector bundle $V \to M$, $\Gamma(V)$ denotes the space of (smooth) sections of $V \to M$. Then for each $x \in M$, we have a linear map $ev_x : \Gamma(V) \to V_x$ called the *evaluation map* as assigning $t \in \Gamma(V)$ to $t(x)$, where V_x is the fiber of $V \to M$ over $x \in M$ (see, for example, [4, p.298]). Thus the evaluation map is considered as a bundle map $ev : M \times \Gamma(V) \to V$ defined by $ev(x,t) = ev_x(t) = t(x)$ for $x \in M$ and $t \in \Gamma(V)$. If a (finite-dimensional) subspace $W \subset \Gamma(V)$ is given, then the restriction of ev to $M \times W$ is also called the evaluation map which is denoted by the same symbol $ev : \underline{W} \to V$ and so, we have a linear map $ev_x : W \to V_x$ for each $x \in M$ as $ev_x(w) = ev(x,w)$ for

$w \in W$. We denote by $\operatorname{End} V$ the bundle of endomorphisms of $V \to M$. For a connection on $V \to M$, its curvature 2-form is denoted by $R^V \in \Omega^2(\operatorname{End} V)$.

Definition 3.1. Let $V \to M$ be a vector bundle and W a space of sections of $V \to M$. The vector bundle $V \to M$ is called to be *globally generated by* W if the evaluation map $ev : \underline{W} \to V$ is surjective.

Since $\pi : \underline{W} \to Q$ provides the corresponding section of $Q \to Gr_p(W)$ to each $w \in W$, π is also recognized as the evaluation map $ev : \underline{W} \to Q$: $ev(x, w) = \pi_x(w)$. We always regard W as a space of sections of $Q \to Gr_p(W)$ in this way. Then $Q \to Gr_p(W)$ is globally generated by W.

Let $f : M \to Gr_p(W)$ be a smooth map. We fix an inner product or a Hermitian inner product $(\cdot, \cdot)$ on a vector space W according to a ground field. We call $(\cdot, \cdot)$ a *scalar product*. Then, as explained in §2, the scalar product on W provides the Grassmannian $Gr_p(W)$ with a Riemannian structure and the vector bundles $S \to Gr_p(W)$ and $Q \to Gr_p(W)$ can be regarded as homogeneous vector bundles on $Gr_p(W)$ with fiber metrics and canonical connections. Pulling back $Q \to Gr_p(W)$ to M by f, we obtain a vector bundle $f^*Q \to M$. Since W can be regarded as a subspace of $\Gamma(Q)$, we have a linear map $F : W \to \Gamma(f^*Q)$ defined by

$$F(w)(x) = (x, ev(f(x), w)) = (f^*t_w)(x) \in f^*Q \subset M \times Q, \tag{3.1}$$

for $x \in M$ and $w \in W$, which is the pull-back of the section t_w of $Q \to Gr_p(W)$ corresponding to $w \in W$. However, it might not be an injection. Even in such a case, W is still called a space of sections of $f^*Q \to M$ and we obtain the evaluation map denoted by the same symbol $ev : M \times W \to f^*Q$ as

$$ev(x, w) = F(w)(x), \tag{3.2}$$

which is the pull-back of the evaluation map $ev : Gr_p(W) \times W \to Q$. Then the pull-back bundle $f^*Q \to M$ is also globally generated by W. We also pull back a fiber metric and a connection on $Q \to Gr_p(W)$ to obtain a fiber metric g_{f^*Q} and a connection ∇^{f^*Q} on $f^*Q \to M$.

In a similar way, the pull-back bundle $f^*S \to M$ has the pull-back fiber metric g_{f^*S} and the pull-back connection ∇^{f^*S}. When we fix $w \in W$ and take the sections corresponding to w denoted by $s \in \Gamma(S)$ and $t \in \Gamma(Q)$, the pull-back of sections $f^*s \in W \subset \Gamma(f^*S)$ and $f^*t \in W \subset \Gamma(f^*Q)$ are also called the *sections corresponding to* w. We abbreviate f^*s and f^*t to s and t, respectively.

The second fundamental forms are also pulled back and denoted by the same symbols $H \in \Gamma(f^*T^* \otimes (f^*S)^* \otimes f^*Q)$ and $K \in \Gamma(f^*T^* \otimes (f^*Q)^* \otimes f^*S)$. If we restrict bundle-valued linear forms H and K on the pull-back bundle $f^*T^* \to M$ to linear forms on M, H and K are nothing but the second fundamental forms of the exact sequence: $0 \to f^*S \to \underline{W} \to f^*Q \to 0$.

From now on, we assume that (M, g) is a Riemannian manifold with a metric g. Then, we use the Riemannian structure on M and the pull-back connection on $f^*Q \to M$ to define the Laplace operator $\Delta^{f^*Q} = \Delta = -\sum_{i=1}^{n} \nabla^{f^*Q}_{e_i} \left(\nabla^{f^*Q}\right)(e_i)$ acting on sections of $f^*Q \to M$ and an endomorphism A of the bundle $f^*Q \to M$ is defined as the trace of the composition of the second fundamental forms H and K:

$$A_x := \sum_{i=1}^{m} H_{e_i} K_{e_i},$$

where m is the dimension of M and $\{e_i\}_{i=1,2,\cdots m}$ is an orthonormal basis of the tangent space to M at x. The bundle endomorphism $A \in \Gamma\,(\mathrm{End}\, f^*Q)$ is called the *mean curvature operator* of f.

We now describe properties of $A \in \Gamma\,(\mathrm{End}\, f^*Q)$.

Lemma 3.2. *The mean curvature operator A is a negative semi-definite symmetric (or Hermitian) endomorphism of $f^*Q \to M$.*

Proof. It follows from Lemma 2.5 that ${K_X}^* = -H_X$. From the definition of A, we immediately obtain the result. $\square$

Lemma 3.3. *The energy density $e(f)$ is equal to $-\mathrm{trace}\, A$ in the case when W is a real vector space or $-2\mathrm{trace}\, A$ in the case when W is a complex vector space.*

Proof. We use Proposition 2.7 to obtain

$$e(f) = \sum_{i=1}^{m} g_{Gr}(df(e_i), df(e_i)) = -\sum_{i=1}^{m} \mathrm{trace}\, H_{e_i} K_{e_i} = -\mathrm{trace}\, A.$$

When W is a complex vector space, use (2.3). $\square$

Definition 3.4. Let $V \to M$ be a vector bundle and t a section of $V \to M$. We denote by Z_t the zero set of t: $Z_t := \{x \in M \,|\, t(x) = 0\}$. A space of sections W of a vector bundle $V \to M$ is called to have the *zero property* for the Laplacian if $Z_t \subset Z_{\Delta t}$ for an arbitrary $t \in W$.

Example. When W is an eigenspace for Δ, it has the zero property.

We now formulate one of main theorems in this section.

Theorem 3.5. *Let (M, g) be a Riemannian manifold and $f : M \to Gr_p(W)$ a smooth map. We fix a scalar product $(\cdot, \cdot)$ on W, which equips $Gr_p(W)$ with a Riemannian structure. We regard W as a space of sections of the pull-back bundle $f^*Q \to M$.*

Then, the following three conditions are equivalent.

1. *$f : (M, g) \to (Gr_p(W), (\cdot, \cdot))$ is a harmonic map.*
2. *W has the zero property for the Laplacian acting on sections of the pull-back bundle of the universal quotient bundle.*
3. *There exists an endomorphism $\tilde{A}$ of the pull-back bundle of the universal quotient bundle such that $\Delta t + \tilde{A} t = 0$ for an arbitrary $t \in W$.*

Under these conditions, $\tilde{A} = A$, where A is the mean curvature operator of $f : M \to Gr_p(W)$ and

$$e(f) = -\mathrm{trace}\, A \ (\textit{when}\, W = \mathbf{R}^N), \quad e(f) = -2\mathrm{trace}\, A \ (\textit{when}\, W = \mathbf{C}^N).$$

Proof. Let X and Y be tangent vectors of M and $t \in \Gamma(f^*Q)$. We consider the second fundamental form $K \in \Omega^1\left((f^*Q)^* \otimes f^*S\right)$. Since $\nabla K = 0$ on

$Gr_p(W)$, we have

$$\begin{aligned}(\nabla_X K)(Y;t) &= \nabla_X^{f^*S}(K_Y t) - K_{\nabla_X Y} t - K_Y(\nabla_X^{f^*Q} t)\\ &= \nabla_X^{f^*S}(K_Y t) - K_{\tilde{\nabla}_X Y - (\nabla_X df)(Y)} t - K_Y(\nabla_X^{f^*Q} t)\\ &= K_{(\nabla_X df)(Y)} t,\end{aligned}$$

where ∇ is the Levi-Civita connection on M, $\tilde{\nabla}$ is the Levi-Civita connection on $Gr_p(W)$ and K is also regarded as a section of $f^*T^* \otimes (f^*Q)^* \otimes f^*S$. In particular, we obtain

$$-\delta^{\nabla} K = K_{\tau(f)}, \tag{3.3}$$

where $\tau(f)$ is the tension field of $f : M \to Gr_p(W)$.

Next we fix a vector $w \in W$ and take the sections $s \in W \subset \Gamma(f^*S)$ and $t \in W \subset \Gamma(f^*Q)$ corresponding to w. Then it follows from Proposition 2.4 that

$$\begin{aligned}\nabla_X^{f^*Q}\left(\nabla^{f^*Q} t\right)(Y) =&\nabla_X^{f^*Q}\left(\nabla_Y^{f^*Q} t\right) - \nabla_{\nabla_X Y}^{f^*Q} t\\ =&\nabla_{df(X)}^{Q}\left(\nabla_{df(Y)}^{Q} t\right) - \nabla_{\tilde{\nabla}_X Y - (\nabla_X df)(Y)}^{Q} t\\ =&\nabla_{df(X)}^{Q}\left(\nabla^{Q} t\right)(df(Y)) + \nabla_{(\nabla_X df)(Y)}^{Q} t\\ =&H_Y K_X t - H_{(\nabla_X df)(Y)} s.\end{aligned}$$

In particular, we obtain

$$\Delta t - H_{\tau(f)} s + \sum_{i=1}^{m} H_{e_i} K_{e_i} t = \Delta t - H_{\tau(f)} s + At = 0. \tag{3.4}$$

First, we assume that the condition (1) is satisfied. The assumption that $f : M \to Gr_p(W)$ is harmonic yields that the equation (3.4) reduces to

$$\Delta t + At = 0. \tag{3.5}$$

We conclude that W has the zero property and the condition (3).

Conversely, suppose that we have the condition (2). For an arbitrary vector $u \in (f^*S)_x$, $x \in M$, we can find an element $w \in W$ such that the corresponding sections $s \in \Gamma(f^*S)$ and $t \in \Gamma(f^*Q)$ satisfy

$$s(x) = u, \quad \text{and} \quad t(x) = 0.$$

The equation (3.4) gives us

$$H_{\tau(f)} s = \Delta t + At.$$

Since W has the zero property for the Laplacian and $t(x) = 0$, it follows that $\Delta t(x) = 0$. Hence we have

$$H_{\tau(f)} u = 0,$$

and Lemma 2.2 yields $\tau(f) = 0$, which means that f is a harmonic map.

It is clear that (3) yields (2).

Finally, we suppose that the conditions (1), (2) and (3) are satisfied. It follows from (3.5) that $\Delta t + At = 0$ for any $t \in W$. On the other hand, (3) yields that $\Delta t + \tilde{A} t = 0$ for an arbitrary $t \in W$. Since W globally generates $V \to M$, we can deduce that $\tilde{A} = A$. □

Corollary 3.6. *Let $f : (M, g) \to (Gr_p(W), (\cdot,\cdot))$ be a smooth map. Then f is a totally geodesic map if and only if the second fundamental form K of vector bundles is covariant constant.*

Proof. In the proof of Theorem 3.5, we show that $\nabla K = K_{\nabla df}$. From the definition of totally geodesic map, we get the result. $\square$

Remark. Totally geodesic maps into Grassmannians will be discussed in detail and Theorem of Takahashi will be generalized in the case where the target is any symmetric space of compact type [25].

Next, suppose that $f : M \to Gr_p(W)$ is an isometric immersion. Instead of the tension field, we use the mean curvature vector to obtain a similar result. When f is an isometric immersion, the energy density is a constant function: $e(f) = m$. In particular, we can state a straightforward generalization of Theorem of Takahashi [32] by replacing a harmonic map by minimal immersion.

Corollary 3.7. *Let (M, g) be an m-dimensional Riemannian manifold and $f : (M, g) \to (Gr_p(W), (\cdot,\cdot))$ an isometric immersion.*
Then, the following three conditions are equivalent.

(1) *$f : M \to Gr_p(W)$ is a minimal immersion.*
(2) *W has the zero property for the Laplacian acting on sections of the pull-back bundle of the universal quotient bundle.*
(3) *There exists a bundle endomorphism $\tilde{A}$ of the pull-back of the universal quotient bundle such that*

$$\Delta t + \tilde{A}t = 0 \quad \textit{for an arbitrary } t \in W.$$

Moreover, under the above conditions, we have $\tilde{A} = A$, where A is the mean curvature operator of f and

$$m = -\mathrm{trace}\, A \ (\textit{when}\, W = \mathbf{R}^N), \quad m = -2\mathrm{trace}\, A \ (\textit{when}\, W = \mathbf{C}^N).$$

Remark. This gives us the original form of Theorem of Takahashi [32]. To this end, we regard the standard sphere as the Grassmannian of oriented hyperplanes in $\mathbf{R}^{n+1}$: $Gr_n(\mathbf{R}^{n+1})$. Since $Q \to Gr_n(\mathbf{R}^{n+1})$ is of rank 1, which can also be regarded as the normal bundle of the unit sphere S^n in $\mathbf{R}^{n+1}$, the mean curvature operator A can always be considered as a function $-e(f)$ from Lemma 3.3.

4. Functionals

4.1. **The equations of Gauss-Codazzi.** Let $f : (M, g) \to (Gr_p(W), (\cdot,\cdot))$ be a smooth map. Since the connection on $\underline{W} \to M$ is flat, it follows from the Gauss-Codazzi equations [18, p.23 (6.12)] for vector bundles that

$$R^{f^*Q}(X, Y) = H_Y K_X - H_X K_Y, \quad (\nabla_X K)(Y) = (\nabla_Y K)(X). \tag{4.1}$$

We have

$$\begin{aligned}(\nabla_X R^{f^*Q})(Y, Z) =& H_{(\nabla_X df)(Z)} K_Y + H_Z K_{(\nabla_X df)(Y)} \\ &- H_{(\nabla_X df)(Y)} K_Z - H_Y K_{(\nabla_X df)(Z)}.\end{aligned}$$

In particular,

$$(4.2)\qquad \begin{aligned}(\delta^{\nabla} R^{f^*Q})(X) &= -(\nabla_{e_i} R^{f^*Q})(e_i, X)\\ &= - H_{(\nabla_{e_i} df)(X)} K_{e_i} - H_X K_{\tau(f)} + H_{\tau(f)} K_X + H_{e_i} K_{(\nabla_{e_i} df)(X)}.\end{aligned}$$

On the other hand, we obtain

$$(4.3)\qquad \nabla_X A = H_{(\nabla_{e_i} df)(X)} K_{e_i} + H_{e_i} K_{(\nabla_{e_i} df)(X)}.$$

First of all, we have

Lemma 4.1. *Let $f : (M, g) \to (Gr_p(W), (\cdot, \cdot))$ be a smooth map. If f is a totally geodesic map, then the pull-back connection on $f^*Q \to M$ is a Yang-Mills connection and A is parallel.*

Proof. The result follows from Corollary 3.6 with (4.2) and (4.3). □

We shall present another occurrence in which the pull-back connection is a Yang-Mills connection and A is covariant constant. Let $V \to M$ be a holomorphic vector bundle with a Hermitian metric h over a Kähler manifold M, which is denoted by (V, h) and called a Hermitian holomorphic bundle. The unique connection on (V, h) compatible with the Hermitian metric and the holomorphic vector bundle structure is called the *Hermitian* connection (we follow Kobayashi [18]). We denote by $H^0(M; V)$ the space of holomorphic sections on $V \to M$.

Lemma 4.2. *Let M be a Kähler manifold and $V \to M$ a holomorphic vector bundle with a Hermitian metric. We take the Hermitian connection ∇ on $V \to M$. Then, for any holomorphic section $t \in H^0(M; V)$, we have*

$$\Delta t = K_{EH} t,$$

where K_{EH} is the mean curvature in the sense of Kobayashi [18]:

$$(4.4)\qquad K_{EH} = \sqrt{-1} \sum_{i=1}^{m} R(e_i, Je_i)$$

where J is the complex structure of M, $e_1, Je_1, \cdots, e_m, Je_m$ is an orthonormal basis and R is the curvature of the Hermitian connection.

Proof. We put $Z_i = \frac{1}{\sqrt{2}}\left(e_i - \sqrt{-1} Je_i\right)$. We can extend the vectors Z_i locally to get a local holomorphic frame field denoted by the same symbols. On the one hand, since t is a holomorphic section, we have that

$$\nabla_{Z_i} (\nabla t) (\overline{Z_i}) = \nabla_{Z_i} \left(\nabla_{\overline{Z_i}} t\right) - \nabla_{D_{Z_i} \overline{Z_i}} t = 0.$$

On the other hand, we get

$$2 \sum_{i=1}^{m} \nabla_{Z_i} (\nabla t) (\overline{Z_i}) = -\Delta t + \sum_{i=1}^{m} \sqrt{-1} R(e_i, Je_i) t,$$

and the result follows. □

Though it is well-known that any holomorphic map between Kähler manifolds is a harmonic map, we can use Theorem 3.5 to see

Proposition 4.3. *Let M be a Kähler manifold and $Gr_p(W)$ a complex Grassmannian or a complex quadric with the Fubini-Study metric. Suppose that $f : M \to Gr_p(W)$ is a holomorphic map. Then f is a harmonic map and the mean curvature K_{EH} of the pull-back bundle $f^*Q \to M$ of the universal quotient bundle equals minus the mean curvature operator A of f.*

Proof. We can regard W as a space of holomorphic sections of the pull-back of the universal quotient bundle. It follows from Theorem 3.5 and Lemma 4.2 that f is a harmonic map and $A = -K_{EH}$. □

Remark. When the target is a complex Grassmannian, the formula $A = -K_{EH}$ can also be obtained as a straightforward application of the equation of Gauss (4.1). Since the second fundamental forms H and K are of type $(1,0)$ and $(0,1)$, respectively, (4.1) and (4.4) yield the result.

If the mean curvature K_{EH} of (V,h) satisfies $K_{EH} = \mu Id_V$ for some constant μ, then (V,h) is called an *Einstein-Hermitian* vector bundle. If (V,h) is an Einstein-Hermitian vector bundle, then the Hermitian connection is called a *Hermitian Yang-Mills* connection (see, for example, [9] and [18]).

We have from Lemma 4.2 that

Corollary 4.4. *Under the hypothesis of Lemma 4.2, suppose that $V \to M$ is globally generated by $H^0(M;V)$. Then (V,h) is an Einstein-Hermitian vector bundle if and only if $H^0(M;V)$ is an eigenspace of the Laplacian.*

Corollary 4.5. *Let M be a compact Kähler manifold and $Gr_p(W)$ a complex Grassmannian or a complex quadric with the Fubini-Study metric.*

If $f : M \to Gr_p(W)$ is a holomorphic map such that the pull-back bundle $f^*Q \to M$ with the induced metric is an Einstein-Hermitian vector bundle, then the Hermitian connection on $f^*Q \to M$ is a Yang-Mills connection and the mean curvature operator A is equal to $-\mu Id_{f^*Q}$ for some non-negative constant μ.

Proof. Since any Hermitian Yang-Mills connection minimizes the Yang-Mills functional, the pull-back connection is a Yang-Mills connection. Proposition 4.3 yields that $A = -\mu Id_{f^*Q}$ for some non-negative constant μ. □

4.2. **Functionals.** When M is compact, we naturally have three functionals on the space of mappings $f : M \to Gr_p(W)$:

$$\int_M |R^{f^*Q}|^2 dv_M, \quad \int_M |H|^2 dv_M, \quad \int_M |A|^2 dv_M,$$

where dv_M is the Riemannian volume form on M. If the tangent space to the space of mappings $C^\infty(M, Gr_p(W))$ at f is identified with $\Gamma((f^*S)^* \otimes f^*Q)$, the Euler-Lagrange equation for the first functional takes the form:

$$\delta^\nabla \left(C\left(R^{f^*Q}H\right)\right) = -\sum_{i=1}^m \nabla_{e_i}\left(C\left(R^{f^*Q}H\right)\right)(e_i) = 0, \tag{4.5}$$

where C is the contraction operator: $C\left(R^{f^*Q}H\right) = \sum_{i=1}^m R^{f^*Q}(e_i,\cdot)H_{e_i}$, and $e_1,\cdots,e_m$ is an orthonormal basis of TM. Using the equation of Codazzi (4.1), we see that (4.5) is equivalent to the equation

$$C\left((\delta^\nabla R^{f^*Q})H\right) = \sum_{i=1}^m (\delta^\nabla R^{f^*Q})(e_i)H_{e_i} = 0.$$

When we define the total energy $E(f)$ of the map $f : M \to Gr_p(W)$ by

$$E(f) = \int_M e(f)dv_M,$$

Lemma 3.3 and the definition of A yield that

$$\int_M |H|^2 dv_M = \int_M |K|^2 dv_M = \begin{cases} E(f), & \text{when } W = \mathbf{R}^N \\ \frac{1}{2}E(f), & \text{when } W = \mathbf{C}^N, \end{cases}$$

and we can see that $\delta^\nabla H = -\tau(f)$ from Lemma 2.5 and (3.3). Finally, we have

$$\delta^\nabla(AH) = -\sum_{i=1}^m \nabla_{e_i}(AH)(e_i) = 0$$

as the Euler-Lagrange equation for the third functional.

If M is a Kähler manifold and $f : M \to Gr_p(W)$ is a holomorphic map into a complex Grassmannian or a complex quadric, then Proposition 4.3 yields that

$$\int_M |A|^2 dv_M = \int_M |K_{EH}|^2 dv_M.$$

In this case, it is the same as the Yang-Mills functional up to a topological constant [18, p.111], (though the functional is defined on $C^\infty(M, Gr_p(W))$).

Hence if $f : M \to Gr_p(W)$ is a totally geodesic map or if $f : M \to Gr_p(W)$ is a holomorphic map with the pull-back bundle being an Einstein-Hermitian vector bundle, then f is an extremal of all the three functionals.

4.3. **The Einstein-Hermitian condition.** We consider a set of harmonic maps f from a compact Riemannian manifold M into a real Grassmannian $Gr_p(\mathbf{R}^N)$ with the fixed energy $E(f)$. Let μ be a non-negative constant determined by $q\mu\mathrm{Vol}(M) = E(f)$, where $q = N - p$ and $\mathrm{Vol}(M) = \int_M dv_M$. Then

$$0 \leqq |A + \mu Id_{f^*Q}|^2 = |A|^2 + 2\mu\,\mathrm{trace}A + \mu^2 q = |A|^2 - 2\mu e(f) + \mu^2 q.$$

Integration and the definition of μ yield that

$$q\mu^2\mathrm{Vol}(M) = \mu E(f) \leqq \int_M |A|^2 dv_M,$$

where the equality holds if and only if $A = -\mu Id_{f^*Q}$.

When f is a map into a complex Grassmannian $Gr_p(\mathbf{C}^N)$, μ is defined by $2q\mu\mathrm{Vol}(M) = E(f)$. Then we have that

$$\frac{1}{2}\mu E(f) \leqq \int_M |A|^2 dv_M, \tag{4.6}$$

where the equality holds if and only if $A = -\mu Id_{f^*Q}$.

If the mean curvature operator A of $f : M \to Gr_p(W)$ is expressed as $A = -\mu Id_{f^*Q}$ for some non-negative constant μ, then f is said to satisfy *Einstein-Hermitian condition* or *EH condition* for short. A map which satisfies the Einstein-Hermitian condition is called an *Einstein-Hermitian (EH) map.* If f is an Einstein-Hermitian map with $A = -\mu Id_{f^*Q}$, then f is called to have an *Einstein-Hermitian (EH) constant* $-\mu$.

4.4. **Maps of Kähler manifolds into complex Grassmannians.** Let $f : M \to Gr_p(\mathbf{C}^{p+q})$ be a smooth map of compact Kähler manifold M with the second fundamental form K of vector bundles which is a one-form with values in $\mathrm{Hom}(f^*Q, f^*S)$. We denote by m the complex dimension of M and by ω_M the Kähler form on M. According to the bi-degree, the components of K are denoted by $K^{1,0} \in \Omega^{1,0}\left(\mathrm{Hom}(f^*Q, f^*S)\right)$ and $K^{0,1} \in \Omega^{0,1}\left(\mathrm{Hom}(f^*Q, f^*S)\right)$, respectively. With this understood,

Lemma 4.6. *Let $f : M \to Gr_p(\mathbf{C}^{p+q})$ be a smooth map of complex m-dimensional Kähler manifold M. Then we have that*

$$\sqrt{-1}\mathrm{tr}\, R^{f^*Q} \wedge \omega_M^{m-1} = \frac{1}{m}\left(-|K^{1,0}|^2 + |K^{0,1}|^2\right)\omega_M^m. \tag{4.7}$$

Proof. When we denote by $\bigwedge$ the contraction of the curvature R^{f^*Q} with the Kähler form ω_M on M and $e_1, Je_1, e_2, Je_2, \cdots, e_m, Je_m$ is an orthonormal basis of TM, the equation of Gauss-Codazzi and Lemma 2.5 yield that

$$\begin{aligned}
\sqrt{-1}\bigwedge \mathrm{tr}\, R^{f^*Q} =& \mathrm{tr}\sum_{i=1}^m R^{f^*Q}(Z_i, \overline{Z_i}) = \sum_{i=1}^m h_{f^*Q}\left(R^{f^*Q}(Z_i, \overline{Z_i})v_\alpha, v_\alpha\right)\\
=& \sum_{i=1}^m h_{f^*Q}\left((H_{\overline{Z_i}}K_{Z_i} - H_{Z_i}K_{\overline{Z_i}})v_\alpha, v_\alpha\right)\\
=& -\sum_{i=1}^m h_{f^*Q}\left(K_{Z_i}v_\alpha, K_{Z_i}v_\alpha\right) + \sum_{i=1}^m h_{f^*Q}\left(K_{\overline{Z_i}}v_\alpha, K_{\overline{Z_i}}v_\alpha\right)\\
=& -|K^{1,0}|^2 + |K^{0,1}|^2,
\end{aligned}$$

where $Z_i = \frac{1}{\sqrt{2}}\left(e_i - \sqrt{-1}Je_i\right),\ i = 1, 2, \cdots, m$ and $v_1, \cdots, v_q$ is a unitary basis of f^*Q. From the identity

$$R^{f^*Q} \wedge \omega_M^{m-1} = \frac{1}{m}\left(\bigwedge R^{f^*Q}\right)\omega_M^m,$$

(4.7) is proved. □

Corollary 4.7. *If $f : M \to Gr_p(\mathbf{C}^{p+q})$ is a smooth map of a compact Kähler manifold, then the first Chern class $c_1(f^*Q)$ of $f^*Q \to M$ satisfies the inequality :*

$$\frac{4\pi}{(m-1)!}\left|\int_M c_1(f^*Q) \wedge \omega_M^{m-1}\right| \leqq E(f).$$

the equality holds if and only if f is holomorphic, i.e. $K^{1,0} = 0$, *or anti-holomorphic,* i.e. $K^{0,1} = 0$.

Proof. If $\int_M c_1(f^*Q) \wedge \omega_M^{m-1} \geqq 0$, then Lemma 4.6 and Chern-Weil theory yield that

$$
\begin{aligned}
2\pi m \int_M c_1(f^*Q) \wedge \omega_M^{m-1} &= \int_M \left(-|K^{1,0}|^2 + |K^{0,1}|^2\right) \omega_M^m \leqq \int_M |K^{0,1}|^2 \omega_M^m \\
&\leqq \int_M \left(|K^{1,0}|^2 + |K^{0,1}|^2\right) \omega_M^m = \int_M |K|^2 \omega_M^m = \frac{1}{2}\int_M |df|^2 \omega_M^m.
\end{aligned}
$$

We use $dv_M = \frac{1}{m!}\omega_M^m$ to obtain the result. The equality holds if and only if $K^{1,0} = 0$. In the other case, we use

$$
-2\pi m \int_M c_1(f^*Q) \wedge \omega_M^{m-1} = \int_M \left(|K^{1,0}|^2 - |K^{0,1}|^2\right) \omega_M^m \leqq \int_M |K^{1,0}|^2 \omega_M^m,
$$

to obtain the result. $\square$

Theorem 4.8. *If $f : M \to Gr_p(\mathbf{C}^{p+q})$ is a smooth map of a compact Kähler manifold, then*

$$
\left(\frac{2\pi}{(m-1)!}\int_M c_1(f^*Q) \wedge \omega_M^{m-1}\right)^2 \leqq q\mathrm{vol}(M)\int_M |A|^2 dv_M.
$$

The equality holds if and only if f is an Einstein-Hermitian holomorphic or anti-holomorphic map with EH constant $-|c|$, where

$$
c = \frac{2\pi}{q\mathrm{vol}(M)(m-1)!}\int_M c_1(f^*Q) \wedge \omega_M^{m-1}.
$$

Proof. The definition of μ with (4.6) yields that

$$
\frac{1}{4}E(f)^2 \leqq q\mathrm{vol}(M)\int_M |A|^2 dv.
$$

The inequality follows from Corollary 4.7.

The equality holds if and only if $A = -\mu Id$ and

$$
\frac{4\pi}{(m-1)!}\left|\int_M c_1(f^*Q) \wedge \omega_M^{m-1}\right| = E(f),
$$

in other words, f is holomorphic or anti-holomorphic.

If f is holomorphic, then Lemma 4.6 yields that

$$
c_1(f^*Q) \wedge \omega_M^{m-1} = \frac{1}{2\pi m}|K^{0,1}|^2 \omega_M^m = \frac{(m-1)!}{2\pi}|K|^2 dv_M.
$$

It follows from Lemma 3.3 that

$$
\mu = \frac{1}{2q\mathrm{vol}(M)}E(f) = \frac{1}{q\mathrm{vol}(M)}\int_M |K|^2 dv_M = c.
$$

When f is anti-holomorphic, we have $\mu = -c$ in a similar way. $\square$

Notice that, by definition, the constant c depends only on the homotopy class of f and the cohomology class of ω_M.

Remark. When the target is a quadric hypersurface of the complex projective space, we have a similar result:

$$
8\left(\frac{\pi}{(m-1)!}\int_M c_1(f^*Q) \wedge \omega_M^{m-1}\right)^2 \leqq \mathrm{vol}(M)\int_M |A|^2 dv_M,
$$

and

$$c = \frac{\pi}{\mathrm{vol}(M)(m-1)!}\int_M c_1(f^*Q)\wedge \omega_M^{m-1}.$$

Since a complex quadric can be identified with a real Grassmann manifold of codimension-2 oriented subspaces (see §6.3), this difference arises from the definition of μ in §4.3.

4.5. **An admissible pair.** We consider the complex projective space as $Gr_n(\mathbf{C}^{n+1}) = \mathbf{P}(\mathbf{C}^{n+1^*})$. Then the universal quotient bundle is the complex line bundle of degree 1. If $f : M \to Gr_n(\mathbf{C}^{n+1})$ is a holomorphic isometric immersion, then we have $K_{EH} = \mu Id_{f^*Q}$ for some positive constant μ from Lemma 3.3 and Proposition 4.3. We thus have only the Hermitian Yang-Mills connection as the pull-back connection and f is an Einstein-Hermitian holomorphic map with EH constant $-\mu$: $A = -\mu Id_{f^*Q}$.

For a similar reason, if $f : M \to Gr_n(\mathbf{R}^{n+1}) = S^n$ is an isometric minimal immersion, then we have only the product connection as the pull-back connection and a constant function as the mean curvature operator: f is an EH harmonic map.

However, when the target is a Grassmannian of higher rank, we could have various connections and bundle endomorphisms as the induced geometric structures, which are called *admissible connections* and *admissible endomorphisms* for harmonic maps, respectively. We abbreviate them to an *admissible pair*. Hence, to describe a set of harmonic maps, we need to detect admissible pairs as the first step. Even if we could succeed in specifying an admissible pair (∇, A), we could have various harmonic maps with (∇, A) as admissible pair, which is the subject of §5. Thus the set of harmonic maps could be a singular fiber bundle over the set of admissible pairs.

Example. For a holomorphic map, from Proposition 4.3 we may consider only an admissible connection instead of an admissible pair. It is shown in [20] that the set of admissible connections for equivariant holomorphic embeddings $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+2})$ coincides with the set of invariant connections with positive semi-definite curvatures on rank 2 homogeneous vector bundles modulo gauge equivalence (for the definition of an equivariant map, see §6.4). In addition, for each admissible connection ∇ modulo gauge equivalence, we have the *unique* equivariant holomorphic embedding $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+2})$ modulo congruence, which has ∇ as the admissible connection. This sort of rigidity result on holomorphic map with a fixed admissible connection into the complex Grassmannian could be regarded as a generalization of the well-known Calabi's rigidity theorem on holomorphic isometric immersions into the complex projective space [5], which is one of the main subjects in [26].

5. A generalization of Theory of do Carmo and Wallach

We will prove a series of main theorems in this section: a generalization of do Carmo-Wallach theory [7]. In the original theory, the space of functions is regarded as the representation space and they established the classification of minimal immersions of spheres into spheres modulo congruence using

Theorem of Takahashi and representation theory of Lie groups. Instead of representation theory, we employ geometry of vector bundles. As already stated, it is sufficient to consider only a product connection and a constant function as the mean curvature operator in the original theory. We will need to fix a connection and a bundle endomorphism in the sequel to develop a generalization.

Definition 5.1. Let $V \to M$ be a real or complex vector bundle of rank q which is globally generated by $W \subset \Gamma(V)$ of dimension N. Then $Gr_p(W)$ denotes a real or complex Grassmannian according to the coefficient field of V, where $p = N - q$. We regard $ev_x : W \to V_x$ as a surjective linear map for each $x \in M$. Then a map $f : M \to Gr_p(W)$ is defined as

$$f(x) := \operatorname{Ker} ev_x = \{t \in W \mid t(x) = 0\}.$$

We call $f : M \to Gr_p(W)$ the *map induced by* $(V \to M, W)$, or the *map induced by* W, if the vector bundle $V \to M$ is specified. (Such an f is also called a *classifying map* in [4, p.298]. As the choice of a space of sections is crucial in our theory, we use the term a *map induced by* (V, W).)

When V is a real vector bundle with an orientation, we also fix an orientation on W and suppose that $Gr_p(W)$ is an oriented real Grassmannian. To obtain a map into an oriented Grassmannian $Gr_p(W)$, we induce the orientation of $f(x)$ from those of $V \to M$ and W.

From the definition of the induced map $f : M \to Gr_p(W)$, the vector bundle $V \to M$ can naturally be identified with $f^*Q \to M$. To be more precise, let $\operatorname{Ker} ev \to M$ be a vector bundle obtained as the kernel of $ev : \underline{W} \to V$. Since $S_{f(x)} = \operatorname{Ker} ev_x$, we get a natural identification $i : \operatorname{Ker} ev \to f^*S$. Then the following diagram gives a bundle map $\phi : V \to f^*Q$, which is called the *natural identification* of $V \to M$ with $f^*Q \to M$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} ev & \longrightarrow & \underline{W} & \longrightarrow & V & \longrightarrow & 0 \\
 & & {\scriptstyle i}\downarrow & & \| & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & f^*S & \longrightarrow & \underline{W} & \longrightarrow & f^*Q & \longrightarrow & 0.
\end{array}$$

Conversely, if $f : M \to Gr_p(W)$ is a smooth map, then we obtain a vector bundle $f^*Q \to M$ which is globally generated by W, where W is regarded as a space of sections of the pull-back bundle. It is easily observed that the map induced by W is the same as the original map $f : M \to Gr_p(W)$. In this way, every map $f : M \to Gr_p(W)$ can be recognized as a map induced by $(f^*Q \to M, W)$.

Let (M, g) be a compact Riemannian manifold and $V \to M$ a vector bundle with a fiber metric h and a compatible connection ∇. Then the space of sections $\Gamma(V)$ of $V \to M$ has the L^2 scalar product induced by g and h. Moreover, using the Riemannian structure and ∇, we can define the Laplace operator Δ acting on $\Gamma(V)$. Since Δ is an elliptic operator, we can decompose $\Gamma(V)$ into the eigenspaces of the Laplacian in the L^2-sense:

$$\Gamma(V) = \oplus_\mu W_\mu, \quad W_\mu := \{t \in \Gamma(V) \mid \Delta t = \mu t\}.$$

It is well-known that W_μ is a finite-dimensional vector space equipped with the scalar product $(\cdot, \cdot)_\mu$ induced from the L^2 scalar product.

5.1. **Standard maps.** Suppose that a rank q vector bundle $V \to M$ is globally generated by a finite-dimensional subspace W of $\Gamma(V)$ with L^2 scalar product $(\cdot,\cdot)$. Then we define the map $f_0 : M \to Gr_p(W)$ induced by (V, W), where $p = \dim W - q$. We call f_0 *the standard map* by W. Thus the standard map means a map into $(Gr_p(W), (\cdot,\cdot))$ defined as $\operatorname{Ker} ev_x$ or more precisely, the term *standard* amounts to fixing the scalar product on W by L^2 scalar product and a bundle map $ev : M \times W \to V$ which is the restriction of the evaluation map $ev : M \times \Gamma(V) \to V$.

When a real vector bundle V has an orientation, to define the induced map into an oriented Grassmannian $Gr_p(\mathbf{R}^n)$, we need to fix an orientation of $\mathbf{R}^n \subset \Gamma(V)$. We denote by $\mathbf{R}^n_+$ the Euclidean space with the fixed orientation and by $\mathbf{R}^n_-$ that with the opposite orientation. Then we can consider two standard maps: those into $Gr_p(\mathbf{R}^n_+)$ and $Gr_p(\mathbf{R}^n_-)$, respectively. To distinguish between them, we define an isometry $\tau : Gr_p(\mathbf{R}^n) \to Gr_p(\mathbf{R}^n)$ as the map obtained by switching the orientation of p-dimensional subspaces of $\mathbf{R}^n$ and τ is called the *inversion* in this article. If f_0 is the standard map into $Gr_p(\mathbf{R}^n_+)$, then so is $\tau \circ f_0$ into $Gr_p(\mathbf{R}^n_-)$. Notice that τ might not be an element of the orthogonal group $\mathrm{O}(n)$ (e.g. when p and n are even), while in the original do Carmo-Wallach theory, τ does belong to $\mathrm{O}(n)$. This will affect our theory through the definition of *gauge* and *image equivalence*.

5.2. **A generalization of do Carmo-Wallach Theory.** In this section, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$. When $\mathbf{K}^n$ is a real or complex vector space with a scalar product $(\cdot,\cdot)$, self-adjoint endomorphisms on $\mathbf{K}^n$ are called Hermitian endomorphisms, for simplicity.

Definition 5.2. Let $f : M \to Gr_p(\mathbf{K}^n)$ be a smooth map and we consider the induced linear map $F : \mathbf{K}^n \to \Gamma(f^*Q)$, (see (3.1) for the definition of F). Then the map $f : M \to Gr_p(\mathbf{K}^n)$ is called a *full map* if $F : \mathbf{K}^n \to \Gamma(f^*Q)$ is injective.

To understand the definition of the fullness, we give a generalization of a result of Erbacher [12].

Let $f : M \to Gr_p(\mathbf{K}^n)$ be a smooth map. Assume that $\mathbf{K}^n$ has a scalar product. The pull-back bundles of $S \to Gr_p(\mathbf{K}^n)$ and $Q \to Gr_p(\mathbf{K}^n)$ are denoted by $U \to M$ and $V \to M$, respectively:

$$0 \to U \xrightarrow{i} \underline{\mathbf{K}^n} \xrightarrow{\pi} V \to 0.$$

Define $U^1_x \subset U_x$ as the image of the second fundamental form at $x \in M$:

$$U^1_x := \operatorname{Im} K_x,$$

where $K \in \Omega^1(V^* \otimes U)$ is the second fundamental form of vector bundles.

Suppose that the dimension of U^1_x is independent of $x \in M$ and so, we have a subbundle $U^1 \to U$. Then we obtain an orthogonal direct sum of vector bundles: $U = U^1 \oplus U^2$, where $U^2 \to M$ is the orthogonal complementary bundle of $U^1 \to M$ in $U \to M$. It follows from Lemma 2.5 that ${U^2}_x$ is characterized by ${U^2}_x = \{u \in U_x \,|\, Hu = 0\}$, where $H \in \Omega^1(U^* \otimes V)$ is the second fundamental form. With this understood, we have

Theorem 5.3. *Let $f : M \to Gr_p(\mathbf{K}^n)$ be a smooth map. If $U^1 \to M$ is a rank k vector bundle preserved by the induced connection ∇^U, then we* *have a totally geodesic submanifold* $Gr_k\left(\mathbf{K}^{n-(p-k)}\right)$ *of* $Gr_p(\mathbf{K}^n)$ *such that* $f(M) \subset Gr_k(\mathbf{K}^{n-(p-k)})$. *The subspace* $\mathbf{K}^{n-(p-k)}$ *of* $\mathbf{K}^n$ *is the orthogonal complement of the kernel of* $F : \mathbf{K}^n \to \Gamma(f^*Q)$.

Proof. Since ∇^U is compatible with the induced metric, $U^2 \to M$ is also a vector bundle preserved by ∇^U. Then the equation of Gauss (4.1) yields that $U^2 \to M$ is flat;

$$R^U(X,Y)|_{U^2}u = K_Y H_X u - K_X H_Y u = 0, \quad \text{for } u \in U^2.$$

Let ξ be a (local) parallel section of $U^2 \to M$. From the definition of ∇^U, we have that

$$d\left(i(\xi)\right) = i(\nabla^U \xi) + \pi^*(H\xi) = 0 + 0 = 0.$$

Hence the W-valued function $i(\xi)$ is constant, say $w = i(\xi) \in W$. It follows from the definition of $s_w \in \Gamma(U)$ corresponding to w that $s_w = i^*(w) = i^*i(\xi) = \xi$. Since $w = i(s_w) = i(\xi) = ii^*(w)$, we can see that $\pi^*\pi(w) = w - ii^*(w) = w - i(s_w) = 0$. Therefore $t_w = \pi(w) \in \Gamma(V)$ vanishes, Thus, for any $x \in M$,

$$i(U^2_x) = \left\{w \in \mathbf{K}^n \,|\, s_w(x) \in U^2_x\right\} = \left\{w \in \mathbf{K}^n \,|\, t_w(x) = 0\right\}.$$

Since M is connected, we obtain a subspace $i(U^2)$ of $\mathbf{K}^n$, which is the kernel of $F : \mathbf{K}^n \to \Gamma(f^*Q)$. It follows that $f(M)$ is a subset of $\{y \in Gr_p(\mathbf{K}^n) \,|\, t_w(y) = 0, w \in i(U^2)\}$. □

In the case when $p = n-1$, Theorem 5.3 is an Erbacher's result [12].

Notice that the notion of full map is the same as one in [7], [29] and [34] if the target space is the sphere or the complex projective space.

Let f be a full map of M into $Gr_p(\mathbf{K}^n)$ with a Fubini-Study metric. Suppose that the pull-back of the universal quotient bundle has a decomposition $V_0 \oplus V_1 \to M$, where $V_0 \to M$ is a trivial bundle with a flat connection. Suppose that $\mathbf{K}^n$ has a subspace W_0 which consists of parallel sections of $V_0 \to M$ and does not induce any sections of $V_1 \to M$ except the zero section. Since f is a full map, we have that $\dim W_0 = \operatorname{rank} V_0$. We take the orthogonal complementary subspace of W_0 in $\mathbf{K}^n$ denoted by W_1. Then we obtain a totally geodesic embedding $i : Gr_p(W_1) \to Gr_p(\mathbf{K}^n)$. Since each element of $W_0 \subset \mathbf{K}^n$ is a parallel section, we see that f is a composite of i and the induced map f_1 by $(V_1 \to M, W_1)$. Hence f is essentially the same as f_1 from the point of view of Riemannian geometry.

Definition 5.4. Let f be a full map of M into $Gr_p(\mathbf{K}^n)$ with a Fubini-Study metric.

Then $f : M \to Gr_p(\mathbf{K}^n)$ is called a full map with *trivial summand*, if
(1) the pull-back of the universal quotient bundle is decomposed into $V_0 \oplus V_1 \to M$, where $V_0 \to M$ is a trivial bundle with a flat connection, and
(2) $\mathbf{K}^n$ has a subspace W_0 which consists of parallel sections of $V_0 \to M$ and does not induce any sections of $V_1 \to M$ except the zero section, (hence we have that $\dim W_0 = \operatorname{rank} V_0$).

We call f a full map with *no trivial summand*, unless f is a full map with trivial summand.

When $\mathbf{K}^n$ has a scalar product $(\cdot,\cdot)$, we give two equivalence relations of maps. We denote by $\operatorname{Aut}(\mathbf{K}^n)$ the unitary group $\mathrm{U}(n)$ or the orthogonal group $\mathrm{O}(n)$ according to the coefficient field $\mathbf{K}$. In the sequel, $\mathrm{Aut}(\mathbf{K}^n)$ will also be regarded as a subgroup of the isometry group of $(Gr_p(\mathbf{K}^n), (\cdot,\cdot))$.

Definition 5.5. Let f_1 and $f_2 : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ be smooth maps. Then f_1 is called *image equivalent* to f_2, if there exists an isometry $\psi \in \mathrm{Aut}(\mathbf{K}^n)$ of $(Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ such that $f_2 = \psi \circ f_1$.

An isometry $\psi \in \mathrm{Aut}(\mathbf{K}^n)$ induces an isomorphism of the bundle $Q \to Gr_p(\mathbf{K}^n)$ denoted by $\tilde{\psi}$ which covers ψ. If we have a map $f : M \to Gr_p(\mathbf{K}^n)$, then $\psi \circ f : M \to Gr_p(\mathbf{K}^n)$ is also a map and $\tilde{\psi}$ induces an isomorphism of bundles denoted by the same symbol $\tilde{\psi} : f^*Q \to (\psi \circ f)^*Q = f^*\tilde{\psi}Q$. By definition, $\tilde{\psi}\left((x, v)\right) = (x, \tilde{\psi}(v))$, where $x \in M$ and $v \in Q_{f(x)}$. We also regard $\tilde{\psi}$ itself as an element of $\mathrm{Aut}(\mathbf{K}^n)$.

From now on, we assume that a vector bundle $V \to M$ has a fiber metric h and a connection ∇ compatible with the metric h, for which we write $(V \to M, h, \nabla)$ or (V, h, ∇). In this article, a vector bundle $V_1 \to M$ is called to be *isomorphic* to $V_2 \to M$ if there exists a bundle map $\phi : V_1 \to V_2$ such that ϕ is an isomorphism of vector bundles preserving the metrics and the connections. In the case when each vector bundle $V_i \to M$ $(i = 1, 2)$ defined over $\mathbf{R}$ has an orientation, ϕ is also supposed to preserve the orientation. Then ϕ is called a *bundle isomorphism.*

To define another equivalence relation for maps into Grassmannians with the Fubini-Study metric, we need to fix a vector bundle over the domain manifold.

Definition 5.6. We fix a vector bundle $(V \to M, h, \nabla)$ and let f be a smooth map of M into $Gr_p(\mathbf{K}^n)$ with the Fubini-Study metric. Then f is called to satisfy *the gauge condition* for $(V \to M, h, \nabla)$ if $f^*Q \to M$ is isomorphic to $V \to M$.

Remark. Let $\mathcal{G}(V)$ be the group of gauge transformations on $V \to M$ i.e. the group of automorphisms on $V \to M$ preserving the metric (and the orientation, if the structure group of V is $\mathrm{SO}(n-p)$) and covering the identity map on M. If $\phi : (V, h, \nabla) \to (f^*Q, h_{f^*Q}, \nabla^{f^*Q})$ is a bundle isomorphism, then so is $\phi \circ g : (V, h, g\nabla g^{-1}) \to (f^*Q, h_{f^*Q}, \nabla^{f^*Q})$ for any $g \in \mathcal{G}(V)$, where h_{f^*Q} and ∇^{f^*Q} are the pull-back metric and connection by f, respectively. Thus the gauge condition fixes a representative of the gauge equivalence class of connections.

Remark. The gauge condition might be close to the *balanced condition* (cf. [8] and [39]). However, the gauge condition only concerns the fiber metric and the connection induced by the scalar product on $\mathbf{K}^n$, while in the balanced condition we also pay attention on the L^2 scalar product induced by the fiber metric and the volume form. We will be concerned with the case when the scalar product on $\mathbf{K}^n$ is not an L^2 scalar product (see Theorem 5.20).

We would like to take account of bundle isomorphisms to introduce another equivalence relation of maps. To do so, we take two steps.

We consider a pair (f, ϕ), where $f : M \to Gr_p(\mathbf{K}^n)$ is a smooth map satisfying gauge condition for $(V \to M, h, \nabla)$ and $\phi : V \to f^*Q$ is a bundle isomorphism. With this understood,

Definition 5.7. Fix a vector bundle $(V \to M, h, \nabla)$ and let f be a map of M into $(Gr_p(\mathbf{K}^n), (\cdot, \cdot))$ satisfying the gauge condition for $(V \to M, h, \nabla)$. Then pairs (f_1, ϕ_1) and (f_2, ϕ_2) are called *gauge equivalent*, if there exists an isometry $\psi \in \mathrm{Aut}(\mathbf{K}^n)$ of $Gr_p(\mathbf{K}^n)$ such that $f_2 = \psi \circ f_1$ and $\phi_2 = \tilde{\psi} \circ \phi_1$.

We fix $(V \to M, h, \nabla)$ and consider a map f of M into a Grassmannian $Gr_p(\mathbf{K}^n)$ with the Fubini-Study metric induced by a scalar product on $\mathbf{K}^n$. Let $ev : \underline{\mathbf{K}^n} \to f^*Q$ denote the pull-back of the evaluation map (see (3.2)). Suppose that a bundle isomorphism $\phi : (V, h, \nabla) \to \left(f^*Q, h_{f^*Q}, \nabla^{f^*Q}\right)$ is given. Then ϕ induces a linear isomorphism of $\Gamma(V)$ onto $\Gamma(f^*Q)$ as assigning $t \in \Gamma(V)$ to $\phi(t) \in \Gamma(f^*Q)$. Thus, $\phi^{-1}(\mathbf{K}^n) \subset \Gamma(V)$ is a subspace of the space of sections of $V \to M$.

Definition 5.8. Fix a vector bundle $(V \to M, h, \nabla)$. When we have a linear monomorphism $i : \mathbf{K}^n \to \Gamma(V)$, we consider a linear map $ev_x \circ i : \mathbf{K}^n \to V_x$ for each $x \in M$. Thus we obtain a bundle map, with a slight abuse of notation, $ev \circ i : \underline{\mathbf{K}^n} \to V$ defined as $ev \circ i(x, w) = ev_x(i(w))$, for $w \in \mathbf{K}^n$. If $ev \circ i$ is surjective, then it is said that $\mathbf{K}^n$ *globally generates* $V \to M$. When $\mathbf{K}^n$ globally generates $V \to M$, then the map $f : M \to Gr_p(\mathbf{K}^n)$ defined as assigning $x \in M$ to a subspace $\mathrm{Ker}\,(ev_x \circ i)$ of $\mathbf{K}^n$:

$$f(x) = \mathrm{Ker}\,(ev_x \circ i)(\subset \mathbf{K}^n) = \{w \in \mathbf{K}^n \,|\, ev_x(i(w)) = 0\}\,, \quad x \in M$$

is called the *map induced by* a triple $(V, \mathbf{K}^n, i)$.

When a real vector bundle $V \to M$ has an orientation, to determine the map into an oriented Grassmannian induced by a triple $(V, \mathbf{K}^n, i)$, we suppose that $\mathbf{K}^n$ has an orientation.

Remark. Using this term, we can say that (one of) the standard map is the map induced by (V, W, Id_W) of M into $(Gr_p(W), (\cdot, \cdot))$, where $W \subset \Gamma(V)$ (with an orientation) and $(\cdot, \cdot)$ is the L^2 scalar product.

In our theory, linear monomorphisms of $\mathbf{K}^n$ into $\Gamma(V)$ will parametrize harmonic maps modulo *gauge equivalence* of maps.

Definition 5.9. Let $(V \to M, h, \nabla)$ be a vector bundle and $\mathbf{K}^n$ an n-dimensional vector space with a scalar product $(\cdot, \cdot)$. Let f_α $(\alpha = 1, 2)$ be maps induced by $(V \to M, \mathbf{K}^n, i_\alpha)$ from M into $(Gr_p(\mathbf{K}^n), (\cdot, \cdot))$. Then, we have a bundle map denoted by $ev_\alpha := ev \circ i_\alpha : \underline{\mathbf{K}^n} \to V$. We take the adjoint bundle map $ev_\alpha^* : V \to \underline{\mathbf{K}^n}$ of ev_α with respect to h and $(\cdot, \cdot)$. If $f_\alpha^*Q \to M$ are identified with the orthogonal complement of $f_\alpha^*S \to M$ in $M \times \mathbf{K}^n$ and we recognize ev_α^* as a bundle map onto the image, then ev_α^* can be considered as a bundle map onto f_α^*Q. Suppose that each $ev_\alpha^* : V \to f_\alpha^*Q$ is a bundle isomorphism.

Then, f_1 is called *gauge equivalent* to f_2 if (f_1, ev_1^*) is gauge equivalent to (f_2, ev_2^*). We call $ev_\alpha : \underline{\mathbf{K}^n} \to V$ an *evaluation map by* f_α and $ev_\alpha^* : V \to f_\alpha^*Q$ a *natural identification by* f_α.

Forgetting the bundle isomorphisms, we see that gauge equivalence yields image equivalence of maps.

For a vector space W, $\mathrm{End}\,(W)$ denotes the set of endomorphisms on W.

Definition 5.10. We fix a vector bundle $(V \to M, h, \nabla)$ over $\mathbf{K}$. Suppose that W is a subspace of $\Gamma(V)$ with a scalar product $(\cdot, \cdot)_W$ and $T \in \mathrm{End}\,(W)$

is a positive semi-definite Hermitian endomorphism on W. We denote by $\operatorname{Ker} T^{\perp}$ the orthogonal complement of $\operatorname{Ker} T$ in W.
(i) *The MC equations* for (V, h, ∇) to $T \in \operatorname{End}(W)$ is defined as:

$$ev \circ T^2 \circ ev^* = Id_V, \text{ and } ev \circ T^2 \circ (\nabla ev^*) = 0 \in \Omega^1(\operatorname{End} V), \tag{5.1}$$

where $ev^* : V \to \underline{W}$ is the adjoint bundle map of $ev : \underline{W} \to V$ and the connection ∇ on $\operatorname{Hom}(V, \underline{W})$ is induced by the product connection on $\underline{W} \to M$ and ∇ on $V \to M$.
(ii) Let $(\mathbf{K}^n, (\cdot,\cdot))$ be a vector space with a scalar product $(\cdot,\cdot)$ and $\iota : \mathbf{K}^n \to W$ a linear injection. We take the scalar product $\iota^*(\cdot,\cdot)_W$ on $\mathbf{K}^n$ induced from $(\cdot,\cdot)_W$ by ι. Then the adjoint linear map $\iota^* : W \to \mathbf{K}^n$ is defined with respect to $(\cdot,\cdot)_W$ and $\iota^*(\cdot,\cdot)_W$. Thus $\iota^* : W \to \mathbf{K}^n$ can be regarded as the orthogonal projection onto $\iota(\mathbf{K}^n)$. Then $(\mathbf{K}^n, (\cdot,\cdot), \iota)$ is said to be *compatible with* (W, T), if $\iota(\mathbf{K}^n) = \operatorname{Ker} T^{\perp}$ and

$$(\iota^* T \iota \cdot, \iota^* T \iota \cdot) = \iota^*(\cdot,\cdot)_W. \tag{5.2}$$

Notice that $\iota^* T \iota$ is a positive Hermitian endomorphism on $(\mathbf{K}^n, \iota^*(\cdot,\cdot)_W)$.

Remark. We will not distinguish $\mathbf{K}^n$ from $\operatorname{Ker} T^{\perp}$ by considering ι as the inclusion. Then the induced scalar product $\iota^*(\cdot,\cdot)_W$ by $\iota : \mathbf{K}^n \to W$ will be abbreviated to $(\cdot,\cdot)_W$, when no confusion may arise.

The role of the MC equations will be elucidated in Theorem 5.12, which is related to the gauge condition: a fiber metric and a connection. First of all, we mention consequences from the compatibility condition.

Lemma 5.11. *Let (M, g) be a Riemannian manifold and $V \to M$ a rank q vector bundle over the coefficient field $\mathbf{K}$ with a metric h. Suppose that W is a subspace of $\Gamma(V)$ with a scalar product $(\cdot,\cdot)_W$ and $T \in \operatorname{End}(W)$ is a positive semi-definite Hermitian endomorphism on W such that $V \to M$ is globally generated by $\operatorname{Ker} T^{\perp}$. When $V \to M$ is a real vector bundle with an orientation, we fix orientations of both W and $\operatorname{Ker} T$.*

If $(\mathbf{K}^n, (\cdot,\cdot), \iota)$ is compatible with (W, T), then
(i) *we have a unique full map $f : M \to Gr_p(\mathbf{K}^n)$ $(n = p + q)$ induced by a triple $(V, \mathbf{K}^n, \iota(\iota^* T \iota))$:*

$$f(x) = (\iota^* T \iota)^{-1} \operatorname{Ker}(ev_x \circ \iota) = \operatorname{Ker}(ev_x \circ \iota \circ (\iota^* T \iota)) \tag{5.3}$$

such that the natural identification by f is expressed as a bundle map $(\iota^* T \iota) \circ \iota^* \circ ev^* = (ev \circ \iota \circ (\iota^* T \iota))^* : V \to f^* Q$, when we regard $Gr_p(\mathbf{K}^n)$ as a Riemannian manifold $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_W)$ and
(ii) *T and ι induce an isometry $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)) \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_W)$ and a totally geodesic submanifold $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_W)$ of $\left(Gr_{p'}(W), (\cdot,\cdot)_W\right)$ as the common zero set of sections of $\operatorname{Ker} T \subset W$, where $p' = p + \dim \operatorname{Ker} T$.*

Proof. Since $\operatorname{Ker} T^{\perp}$ globally generates $V \to M$ by the hypothesis, the compatibility condition yields that we can define a map $f : M \to Gr_p(\mathbf{K}^n)$ as the map induced by $(V, \mathbf{K}^n, \iota(\iota^* T \iota))$ (Definition 5.8). When $V \to M$ is an oriented real vector bundle, since we fix orientations of W and $\operatorname{Ker} T$ by definition, the orientation of $\mathbf{R}^n$ is uniquely determined. Then f is also uniquely defined so that the restriction of ev_x to $\operatorname{Ker}(ev_x \circ \iota)^{\perp}$ onto V_x preserves the orientation.

It follows from $\operatorname{Ker} T \cap \iota(K^n) = \{0\}$ that f is a full map.

By Definition 5.9, we see that the natural identification by f takes the form: $(ev \circ \iota \circ (\iota^* T \iota))^* : V \to f^*Q$.

Since $\iota^* T \iota$ is invertible, we have a map $Gr_p(\mathbf{K}^n) \to Gr_p(\mathbf{K}^n)$ given by $U \mapsto (\iota^* T \iota)^{-1} U$, where U is a p-dimensional subspace of $\mathbf{K}^n$. It turns out to be an isometry $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)) \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_W)$ from (5.2) in Definition 5.10.

We will describe the common zero set Z of sections of $\operatorname{Ker} T \subset W$ in $Gr_{p'}(W)$. From definition of the corresponding section, we have that

$$\begin{aligned} Z &= \left\{ U' \subset W \,|\, \dim U' = p' \text{ and } \operatorname{Ker} T \subset U' \right\} \\ &= \left\{ U \oplus \operatorname{Ker} T \subset W \,|\, U \subset \operatorname{Ker} T^{\perp} \text{ and } \dim U = p \right\}. \end{aligned}$$

Then we can define an embedding of $Gr_p(\mathbf{K}^n)$ into $Gr_{p'}(W)$ by identifying the zero set Z with its image. (When $V \to M$ is an oriented bundle over $\mathbf{R}$, since the orientation of $\operatorname{Ker} T$ is fixed, we can define the embedding.) It follows that $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_W)$ is a totally geodesic submanifold of $\big(Gr_{p'}(W), (\cdot,\cdot)_W\big)$. $\square$

Remark. Let $i_T : (Gr_p(\mathbf{K}^n), (\cdot,\cdot)) \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_W)$ denote the isometry obtained in Lemma 5.11. Thus, for $f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$, we can not distinguish between the composition $i_T \circ f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_W)$ and f itself from the point of view of Riemannian geometry. If f is a harmonic map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot))$, then so is $i_T \circ f$ as a map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_W)$. When $i_T \circ f$ can be expressed as the map induced by $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$, f is said to *be realized as the map induced by* $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$ into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_W)$.

We are now in a position to state one of the main theorems in this section.

Theorem 5.12. *Let (M, g) be a compact Riemannian manifold. We fix a vector bundle $(V \to M, h, \nabla)$ of rank q and denote by $(W_\lambda, (\cdot,\cdot)_\lambda) \subset \Gamma(V)$ the eigenspace with eigenvalue λ of the Laplacian acting on $\Gamma(V)$ with the L^2 scalar product $(\cdot,\cdot)_\lambda$.*

Let $f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ be a full harmonic map into a Grassmannian with the Fubuni-Study metric satisfying the following two properties.
(i) *f satisfies the gauge condition for (V, h, ∇). (Hence, $q = n - p$.)*
(ii) *f is an Einstein-Hermitian map with EH-constant $-\mu$ for some positive real number μ, and so $e(f) = \mu q$ $(\mathbf{K} = \mathbf{R})$ or $e(f) = 2\mu q$ $(\mathbf{K} = \mathbf{C})$.*

Then μ is an eigenvalue of the Laplacian acting on $\Gamma(V)$. If we fix a bundle isomorphism between (V, h, ∇) and $(f^*Q, f^*h_Q, \nabla^{f^*Q})$, then there exist a unique linear injection $\iota : \mathbf{K}^n \to W_\mu$ and a positive semi-definite Hermitian endomorphism $T \in \operatorname{End}(W_\mu)$ satisfying the MC equations for (V, h, ∇) such that
(a) *$(\mathbf{K}^n, (\cdot,\cdot), \iota)$ is compatible with (W_μ, T) (In particular, $n \leqq \dim W_\mu$).*
(b) *$f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ is realized as the map induced by a triple $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$ into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_\mu)$.*

Conversely, for any eigenvalue μ of the Laplacian acting on $\Gamma(V)$, if a positive semi-definite Hermitian endomorphism $T \in \operatorname{End}(W_\mu)$ satisfies the MC equations for (V, h, ∇) and $\mathbf{K}^n := (\operatorname{Ker} T)^{\perp}$ globally generates $V \to M$, then the map induced by a triple $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$ is a full harmonic map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_\mu)$ satisfying (i) *and* (ii)*, where $\iota : \mathbf{K}^n \to W_\mu$ is the inclusion.*

Suppose that $f_i : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_\mu)$ be maps induced by those triples $(V, \mathbf{K}^n, \iota(\iota^* T_i \iota))$ with the inclusion $\iota : \mathbf{K}^n \to W_\mu$ such that $\iota(\mathbf{K}^n)^\perp = \operatorname{Ker} T_i$, $(i = 1, 2)$. Then, f_1 and f_2 are gauge equivalent (Definition 5.9) *if and only if $T_1 = T_2$.*

Remark. One of the important features is the following: any harmonic map with properties (i) and (ii) can be realized as an induced map with the natural identification which is a bundle isomorphism.

We begin with a lemma needed for the proof of Theorem 5.12.

Lemma 5.13. *Let W be a vector space with a scalar product and $\mathbf{K}^n$ a subspace of W. The orthogonal projection is denoted by $\pi : W \to \mathbf{K}^n$. Let V be a vector space with a scalar product and suppose that we have a surjective linear map $ev : W \to V$.*

If the restriction of ev to $\mathbf{K}^n$ denoted by ev_K is also surjective, then,

$$\pi\left(\operatorname{Ker} ev^\perp\right) = (\operatorname{Ker} ev_K)^\perp,$$

where $\operatorname{Ker} ev^\perp$ (resp. $(\operatorname{Ker} ev_K)^\perp$) denotes the orthogonal complement of $\operatorname{Ker} ev$ of W (resp. $\operatorname{Ker} ev_K$ of $\mathbf{K}^n$ endowed with the induced scalar product).

Proof. Using the scalar product, we have adjoint homomorphisms ev^* and ev_K^* of ev and ev_K, respectively. From the hypothesis, we have $ev \circ \iota = ev_K$ on $\mathbf{K}^n$, where $\iota : \mathbf{K}^n \to W$ is the inclusion map. The adjoint of ι is nothing but the projection π. It follows that $ev_K^* = \iota^* \circ ev^*$ and so, $ev_K^* = \pi \circ ev^*$. □

Proof of Theorem 5.12 . Let $f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ be a full harmonic map satisfying properties (i) and (ii).

From the gauge condition (i), we fix a bundle isomorphism $\phi : (V, h, \nabla) \to \left(f^*Q, f^*h_Q, \nabla^{f^*Q}\right)$. Then ϕ induces a linear isomorphism $\Gamma(V) \to \Gamma(f^*Q)$ denoted by the same symbol. It follows from Theorem 3.5 together with the Einstein-Hermitian condition (ii) that $\Delta t = \mu t$, for an arbitrary $t \in \phi^{-1}(\mathbf{K}^n) \subset \Gamma(V)$. Since f is a full map, $\phi^{-1}(\mathbf{K}^n)$ is a subspace of $W = W_\mu$ and so, $n \leqq \dim W$. Thus the restriction of $\phi^{-1} : \Gamma(f^*Q) \to \Gamma(V)$ to $\mathbf{K}^n$ provides a unique injective linear map $\iota : \mathbf{K}^n \to \phi^{-1}(\mathbf{K}^n) \subset W$. Since $\phi^{-1}(\mathbf{K}^n)$ globally generates $V \to M$, so does W. Hence we have a surjective evaluation map $ev_x : W \to V_x$ for each $x \in M$, which is the restriction of $ev_x : \Gamma(V) \to V_x$. Therefore the composition of ι and ev_x defines a surjective bundle map $ev \circ \iota : \underline{\mathbf{K}^n} \to V$. Then the map f is expressed as

$$f(x) = \operatorname{Ker}(ev_x \circ \iota) \subset \mathbf{K}^n. \tag{5.4}$$

Alternatively, $f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ is realized as the map induced by $(V, \mathbf{K}^n, \iota)$. (In the case where $Gr_p(\mathbf{R}^n)$ is an oriented Grassmannian, notice that $\mathbf{R}^n$ is supposed to possess an orientation from our definition.) Hence, we need to identify the scalar product on $\mathbf{K}^n$ to describe a full harmonic map f into $Gr_p(\mathbf{K}^n)$ with (i) and (ii).

To do so, we recognize $\mathbf{K}^n$ as a subspace of W by ι and induce another scalar product on $\mathbf{K}^n$ by the restriction of $(\cdot,\cdot)_\mu$ by ι, which is denoted by the same symbol. To describe the difference of two scalar products on $\mathbf{K}^n$, we use a positive Hermitian endomorphism $\underline{T} : \mathbf{K}^n \to \mathbf{K}^n$ such that

$$(\underline{T}\cdot, \underline{T}\cdot) = (\cdot,\cdot)_\mu. \tag{5.5}$$

As in Lemma 5.11, we have an isometry denoted by $\underline{T}^{-1} : (Gr_p(\mathbf{K}^n), (\cdot,\cdot)) \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_\mu)$ is given by $U \mapsto \underline{T}^{-1}U$, where U is a p-dimensional subspace of $\mathbf{K}^n$. Then the composition $\underline{T}^{-1} \circ f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_\mu)$ is also a full harmonic map satisfying properties (i) and (ii). Thus, from now on, we consider $\underline{T}^{-1} \circ f$ which is referred to simply as $f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_\mu)$. Then we have

$$f(x) = \underline{T}^{-1}\mathrm{Ker}\,(ev_x \circ \iota) = \mathrm{Ker}\,(ev_x \circ \iota \circ \underline{T}) \subset \mathbf{K}^n. \tag{5.6}$$

In this way, a full harmonic map f is specified by the positive Hermitian endomorphism $\underline{T}$ on $\mathbf{K}^n$ with the fixed scalar product $(\cdot,\cdot)_\mu$.

Since

$$\left(\underline{T}^{-1}\mathrm{Ker}\, ev_x \circ \iota\right)^{\perp} = \underline{T}\,(\mathrm{Ker}\, ev_x \circ \iota)^{\perp},$$

we apply lemma 5.13 to obtain

$$\left(\underline{T}^{-1}\mathrm{Ker}\, ev_x \circ \iota\right)^{\perp} = \underline{T}\iota^* \left(\mathrm{Ker}\, ev_x^{\perp}\right),$$

where $\iota^* : W \to \mathbf{K}^n$ is the adjoint of ι with respect to $(\cdot,\cdot)_\mu$ on $\mathbf{K}^n$ and W (i.e. ι^* is the orthogonal projection). Let ${\mathbf{K}^n}^{\perp}$ be the orthogonal complement of $\mathbf{K}^n$ in W. We define a positive semi-definite Hermitian endomorphism $T : W \to W$ in such a way that ${\mathbf{K}^n}^{\perp}$ is the kernel of T and $T|_{\mathbf{K}^n} = \underline{T}$. More precisely, the latter condition means that $\iota^* T \iota = \underline{T}$. Thus the definition of T with (5.5) yields that $(\mathbf{K}^n, (\cdot,\cdot), \iota)$ is compatible with (W, T).

The definition of T and (5.6) yields that f is expressed as

$$f\,(x) = (\iota^* T \iota)^{-1}\,\mathrm{Ker}\,(ev_x \circ \iota) = \mathrm{Ker}\,\left(ev_x \circ \iota\,(\iota^* T \iota)\right),$$

which is the map induced by a triple $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$. When $f^*Q \to M$ is identified with the orthogonal complement of $f^*S \to M$, it follows from Lemma 5.11 that the natural identification $\phi : V \to f^*Q$ by f takes the form:

$$\phi_x\,(v) = \left(x, (\iota^* T \iota)\,\iota^* ev_x^*(v)\right), \tag{5.7}$$

where $v \in V_x$ and $(\iota^* T \iota)\,\iota^* ev_x^*$ is considered as a map onto the image.

Since the metric h_{f^*Q} on $f^*Q \to M$ is induced by the scalar product on $\mathbf{K}^n \subset W$, it follows from the gauge condition (i) and (5.7) that

$$\begin{aligned} h(v, v') =& h_{f^*Q}\left(\phi_x(v), \phi_x(v')\right) = \left((\iota^* T \iota)\,\iota^* ev_x^*(v), (\iota^* T \iota)\,\iota^* ev_x^*(v')\right)_\mu \\ =& h\left(ev_x \circ \iota\,(\iota^* T \iota)^2\,\iota^* \circ ev_x^*(v), v'\right) \end{aligned}$$

for arbitrary $v, v' \in V$. From the definition of T, we have that $(\iota^* T \iota)^2 = \iota^* T^2 \iota$ and $\iota\iota^* T^2 \iota\iota^* = T^2$. Hence

$$ev \circ \iota\,(\iota^* T \iota)^2\,\iota^* \circ ev^*(v) = ev \circ \iota\iota^* T^2(\iota\iota^* ev^*(v)) = ev \circ T^2 \circ ev^*(v)$$

for $v \in V$. Thus

$$h(v, v') = h\left(ev \circ T^2 \circ ev^*(v), v'\right), \tag{5.8}$$

and we obtain

$$ev \circ T^2 \circ ev^* = Id_V. \tag{5.9}$$

Next, we compare the original connection ∇ with the pull-back connection ∇^{f^*Q} by $f : M \to Gr_p(\mathbf{K}^n)$ on $V \to M$. Using the natural identification, we do not distinguish V from f^*Q. Since the orthogonal projection $\underline{\mathbf{K}^n} \to V$ is $ev \circ \iota\,(\iota^* T \iota)$, which is the adjoint homomorphism of ϕ, the induced connection is expressed as follows:

$$\begin{aligned}
&\nabla^{f^*Q} t = ev \circ \iota\,(\iota^* T \iota) \circ d\phi(t) = ev \circ \iota\,(\iota^* T \iota)\,(x, (\iota^* T \iota)\,\iota^* d\,(ev^*(t))) \\
=&ev \circ \iota\,(\iota^* T \iota)\,(x, (\iota^* T \iota)\,\iota^*\,(\nabla ev^*)\,(t)) + ev \circ \iota\,(\iota^* T \iota)\,(x, (\iota^* T \iota)\,\iota^* \circ ev^*(\nabla t)) \\
=&ev \circ \iota\,(\iota^* T \iota)^2\,\iota^* \circ (\nabla ev^*)\,(t) + \phi^* \phi(\nabla t) \\
=&ev \circ T^2 \circ (\nabla ev^*)\,(t) + \phi^* \phi(\nabla t)
\end{aligned}$$

for an arbitrary section $t \in \Gamma(V)$. It follows from (5.9) that $\phi^* \phi = Id_V$ and we derive

$$\nabla^{f^*Q} - \nabla = ev \circ T^2 \circ (\nabla ev^*)\,. \tag{5.10}$$

The gauge condition (i) with (5.9) and (5.10) yields that T satisfies the MC equations for (V, h, ∇).

Let W be an eigenspace with an eigenvalue μ and $T \in \mathrm{End}\,(W)$ a positive semi-definite Hermitian endomorphism satisfying the MC equations for (V, h, ∇). Suppose that $\mathbf{K}^n := (\mathrm{Ker}\,T)^\perp$ globally generates $V \to M$. Then (after fixing the orientation of $\mathbf{K}^n$, if necessary), we can define a full map $f : M \to Gr_p(\mathbf{K}^n)$ induced by $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$. As a map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_\mu)$, we obtain property (i), because the natural identification by f preserves the metrics and the connections from (5.8) and (5.10). Therefore, we can apply Theorem 3.5 to conclude that f is an Einstein-Hermitian full harmonic map with EH-constant $-\mu$.

Let f_1 and f_2 be the maps induced by $(V, \mathbf{K}^n, \iota\,(\iota^* T_i \iota))$ with the inclusion $\iota : \mathbf{K}^n \to W_\mu$, where $T_i \in \mathrm{End}\,(W_\mu)$ is a positive semi-definite Hermitian endomorphism satisfying the MC equations for (V, h, ∇) and $\iota\,(\mathbf{K}^n) = \mathrm{Ker}\,T_1^\perp = \mathrm{Ker}\,T_2^\perp$ globally generates $V \to M$, $(i = 1, 2)$. Suppose that f_1 and f_2 are gauge equivalent as induced maps into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_\mu)$. By definition of gauge equivalence, we can see from (5.7) that for the natural identification ϕ_i by f_i, there exists a $\psi \in \mathrm{Aut}(\mathbf{K}^n)$ such that

$$\tilde{\psi}\phi_1(v) = \phi_2(v) \Longleftrightarrow \tilde{\psi}\,(\iota^* T_1 \iota)\,\iota^* ev^*(v) = (\iota^* T_2 \iota)\,\iota^* ev^*(v), \tag{5.11}$$

for an arbitrary $v \in V$. Since f_1 and f_2 are full maps, (5.11) gives

$$\tilde{\psi}\iota^* T_1 \iota = \iota^* T_2 \iota.$$

Then the uniqueness of the polar decomposition yields that $\tilde{\psi} = Id_{\mathbf{K}^n}$ and $\iota^* T_1 \iota = \iota^* T_2 \iota$. Then, from $\mathrm{Ker}\,T_1 = \mathrm{Ker}\,T_2$, we obtain $T_1 = T_2$.

Conversely, suppose that $T_1 = T_2$. A problem may arise only when the target is an oriented Grassmannian, in other words, a real vector bundle $V \to M$ has an orientation. However, since $\mathbf{K}^n$ is supposed to have an orientation from our definition of the induced map, the gauge condition (i) yields that $(V, \mathbf{K}^n, \iota\,(\iota^* T_i \iota))$ do induce the same maps.

Remark. When the sphere S^{N-1} is identified with $Gr_{N-1}(\mathbf{R}^N)$ a Grassmannian of oriented hyperplanes of $\mathbf{R}^N$, the position vector gives a trivialization of $Q \to Gr_{N-1}(\mathbf{R}^N)$ and so, a product connection. Thus, this connection gives the usual derivative of functions. When we have a map $f : M \to Gr_{N-1}(\mathbf{R}^N)$, f can also be recognized as a trivialization of $f^*Q \to M$. Then every section of the pull-back bundle can be recognized as a function on M, and the pull-back connection gives the usual derivative of functions. Consequently, when the target is the standard sphere, we do not need the gauge condition (i) in Theorem 5.12.

Remark. When the target is a symmetric space of rank 1, the universal quotient bundle is of rank 1. Hence the mean curvature operator can be considered as a function. Thus, in this case, the EH condition in Theorem 5.12 is equivalent to the condition that f has constant energy density from Lemma 3.3.

Hence, the properties (i) and (ii) are automatically satisfied in the original do Carmo-Wallach theory. In addition, when the target is neither the sphere nor the projective space, it turns out to be more involved to describe the moduli space by do Carmo-Wallach theory from the viewpoint of the theory of connections. To see this, we give a remark.

Remark. At the beginning of the proof of Theorem 5.12, we have fixed a bundle isomorphism $\phi : (V, h, \nabla) \to (f^*Q, f^*h_Q, \nabla^{f^*Q})$ for a given f. Then the linear injection $\iota : \mathbf{K}^n \to W_\mu$ is uniquely determined as the restriction of $\phi^{-1} : \Gamma(f^*Q) \to \Gamma(V)$ to $\mathbf{K}^n \subset \Gamma(f^*Q)$. We would like to discuss the effect when we take another bundle isomorphism ϕ_1 between them. To do so, let $C(\nabla)$ denote the centralizer of the holonomy group of ∇ in the structure group of V. Since $\phi_1^{-1}\phi$ is a gauge transformation preserving the metric and the connection on (V, h, ∇) and the base manifold is connected, we can find $c \in C(\nabla)$ such that $\phi = \phi_1 \circ c$, where $C(\nabla)$ is now regarded as a subgroup of $\mathcal{G}(V)$ the group of gauge transformations on $V \to M$. Thus $\phi_1^{-1}(\mathbf{K}^n) = c\phi^{-1}(\mathbf{K}^n)$. Notice that any $g \in \mathcal{G}(V)$ induces a linear automorphism on $\Gamma(V)$ denoted by the same symbol: $(g(t))\,(x) = g(t(x))$ for $t \in \Gamma(V)$ and $x \in M$. In other words, $ev \circ g = g \circ ev$. Since c preserves $W_\mu \subset \Gamma(V)$, this means that f can also be realized as the map induced by $(V, \mathbf{K}^n, c\iota((c\iota)^*(cTc^*)c\iota)) = (V, \mathbf{K}^n, c\iota(\iota^*T\iota))$. In the case where the target is the sphere, $C(\nabla)$ is a trivial group (cf. Lemma 5.28). When the target is the projective space, $C(\nabla)$ consists of scalars of unit length (cf. Proposition 5.30). Hence the map induced by $(V, \mathbf{K}^n, c\iota(\iota^*T\iota))$ is the same induced by $(V, \mathbf{K}^n, \iota(\iota^*T\iota))$ in those cases. On the contrary, when we consider a map into a Grassmannian of higher rank, $C(\nabla)$ might provide a non-trivial action on the set of the map (see Proposition 6.21 for a concrete example). We will encounter $C(\nabla)$ again when considering the image equivalence relation in the next subsection.

In order to proceed further for a generalization of do Carmo-Wallach theory, we relax the EH condition and so, we do not need the eigenspaces of the Laplace operator. Indeed we have examples of harmonic maps which do not satisfy the EH condition in the case that the target is a Grassmannian of higher rank (see Theorem 6.28).

To state the generalization, we fix (V, h, ∇) over a compact Riemannian manifold M, and let $A \in \Gamma(\operatorname{End} V)$ be a negative semi-definite Hermitian endomorphism of $V \to M$. Since $\Delta + A$ is an elliptic operator on a compact manifold M, the solution space of $(\Delta + A)\,t = 0$ for $t \in \Gamma(V)$ is a finite-dimensional vector subspace of $\Gamma(V)$. The equation $(\Delta + A)\,t = 0$ is called the *generalized Laplace equation with* A and its solution space is denoted by

W_A throughout this paper. If W_A is non-trivial, then W_A has the induced L^2 scalar product $(\cdot,\cdot)_{W_A}$ and the evaluation map $ev : \underline{W_A} \to V$. When W_A globally generates $V \to M$, we also have the standard map $f : M \to (Gr_p(W_A), (\cdot,\cdot)_{W_A})$ defined as $f(x) = \operatorname{Ker} ev_x \subset W_A$.

Definition 5.14. We fix a vector bundle $(V \to M, h, \nabla)$ and a negative semi-definite Hermitian endomorphism $A \in \Gamma(\operatorname{End} V)$. Then a harmonic map $f : M \to Gr_p(W)$ is called to *have an admissible pair* $((V, h, \nabla), A)$ if f satisfies the gauge condition for (V, h, ∇) and has $A \in \Gamma(\operatorname{End} V)$ as the mean curvature operator.

Remark. Different from the Einstein-Hermitian case, the gauge group $\mathcal{G}(V)$ could act on A in a non-trivial way. Thus it acts on the set of admissible pairs: $((V, h, \nabla), A) \mapsto \left((V, h, g\nabla g^{-1}), gAg^{-1}\right)$ for any $g \in \mathcal{G}(V)$. Hence by fixing the admissible pair, we exclude the redundancy due to the gauge group.

With this understood, Theorem 5.12 can be generalized.

Theorem 5.15. *Let M be a compact Riemannian manifold. We fix a vector bundle $(V \to M, h, \nabla)$ and a negative semi-definite Hermitian endomorphism $A \in \Gamma(\operatorname{End} V)$. Let $(W_A, (\cdot,\cdot)_{W_A}) \subset \Gamma(V)$ be the solution space of the generalized Laplace equation with A with the L^2 scalar product $(\cdot,\cdot)_{W_A}$.*

If $f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ is a full harmonic map into a Grassmannian with the Fubuni-Study metric which has an admissible pair $((V, h, \nabla), A)$ and we fix a bundle isomorphism between (V, h, ∇) and $(f^*Q, f^*h_Q, \nabla^{f^*Q})$, then there exist a unique linear injection $\iota : \mathbf{K}^n \to W_A$ and a positive semi-definite Hermitian endomorphism T on W_A satisfying the MC equations for (V, h, ∇) such that
(a) $(\mathbf{K}^n, (\cdot,\cdot), \iota)$ *is compatible with* (W_A, T)*. In particular,* $n \leqq \dim W_A$.
(b) $f : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ *is realized as the map induced by a triple* $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$ *into* $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_{W_A})$.

Conversely, if a positive semi-definite Hermitian endomorphism T on W_A satisfies the MC equations for (V, h, ∇) and $\mathbf{K}^n := (\operatorname{Ker} T)^\perp$ globally generates $V \to M$, then the map induced by a triple $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$ is a full harmonic map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_{W_A})$ with an admissible pair $((V, h, \nabla), A)$, where $\iota : \mathbf{K}^n \to W_A$ is the inclusion.

Let $f_i : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_{W_A})$ be the maps induced by those triples $(V, \mathbf{K}^n, \iota\,(\iota^* T_i \iota))$ with the inclusion $\iota : \mathbf{K}^n \to W_A$ such that $\iota\,(\mathbf{K}^n)^\perp = \operatorname{Ker} T_i$, $(i = 1, 2)$. Then, f_1 and f_2 are gauge equivalent if and only if $T_1 = T_2$.

Proof. From Theorem 3.5, we may replace an eigenspace W_μ in the proof of Theorem 5.12 by the solution space W_A. □

Remark. When f is an Einstein-Hermitian harmonic map or more generally, the mean curvature operator A is covariant constant, the centralizer $C(\nabla)$ of the holonomy group of ∇ in the structure group of V acts trivially on A. Otherwise, viewing $C(\nabla)$ as a subgroup of $\mathcal{G}(V)$, we can see that $c(W_A)$ is the solution space of the generalized Laplace equation with cAc^{-1}. Therefore the moduli space of harmonic maps with an admissible pair $((V, h, \nabla), A)$ can naturally be identified with that of harmonic maps with an admissible pair $\left((V,h,\nabla),cAc^{-1}\right)$. Thus $C(\nabla)$ also acts on the set of admissible pairs for harmonic maps, even if (V,h,∇) is fixed.

Definition 5.16. Let $(V\to M,h,\nabla)$ be a vector bundle on a Riemmanian manifold M and $A\in\Gamma(\operatorname{End}V)$ a negative semi-definite Hermitian endomorphism. Let $W_A\subset\Gamma(V)$ denote the solution space of the generalized Laplace equation with A. Suppose that $f:M\to(Gr_p(\mathbf{K}^n),(\cdot,\cdot))$ is a full harmonic map with $((V,h,\nabla),A)$ as an admissible pair.

If $n=\dim W_A$, then f is called a *maximal* harmonic map with an admissible pair $((V,h,\nabla),A)$.

Using L^2 scalar product, we can show the following

Lemma 5.17. *Let M be a compact Riemannian manifold. We fix a vector bundle $(V\to M,h,\nabla)$ of rank q and a negative semi-definite Hermitian endomorphism $A\in\Gamma(\operatorname{End}V)$. Let W_A be the kernel of $\Delta+A$ equipped with the L^2 scalar product $(\cdot,\cdot)_{W_A}$.*

If T is a positive semi-definite Hermitian endomorphism on W_A satisfying the MC equation, then we have that

$$\operatorname{trace}T^2=q\operatorname{Vol}(M), \tag{5.12}$$

where, $\operatorname{Vol}(M)$ *denotes the volume of M.*

Proof. Let $w_1,\cdots,w_N$ be a unitary basis of W_A. We claim that

$$\sum_{B=1}^{N}(Tw_B,Tw_B)_{W_A}=q\operatorname{Vol}(M).$$

To see this, we use the definition of L^2 scalar product to get

$$\begin{aligned}\sum_{B=1}^{N}(Tw_B,Tw_B)_{W_A}&=\sum_{B=1}^{N}\int_M h\left(ev(Tw_B),ev(Tw_B)\right)dv\\&=\int_M\sum_{B=1}^{N}h\left(ev(Tw_B),ev(Tw_B)\right)dv,\end{aligned} \tag{5.13}$$

where dv denotes the volume form on M. Fix a point $x\in M$ and let $v_1,\cdots,v_q$ be a unitary basis of V_x. Since T satisfies the MC equation by assumption, we can derive that $(w_i,w_j)_{W_A}=h\left(ev(Tw_i),ev(Tw_j)\right)$, for $w_i,w_j\in f^*Q_x$. If $w\in W$ is perpendicular to f^*Q_x, then

$$0=(Tev^*(v_i),w)_{W_A}=(ev^*(v_i),Tw)_{W_A}=h(v_i,ev(Tw)).$$

Hence, if necessary, we can take another unitary basis of W such that $ev(Tw_i)=v_i$ for $i=1,\cdots,q$ and $ev(Tw_{q+j})=0$ for $j=1,\cdots N-q$. However, the sum $\sum_{B=1}^{N}h_{V_x}\left(ev(Tw_B),ev(Tw_B)\right)$ does not change, because $h_x\left(evT,evT\right)$ can be considered as a Hermitian form on W_A. Therefore,

$$\sum_{B=1}^{N}h\left(ev(Tw_B),ev(Tw_B)\right)=\sum_{i=1}^{q}h\left(v_i,v_i\right)=q.$$

Combining this with (5.13), we obtain the desired formula. □

Remark. Notice that we use only $evT^2ev^*=Id_V$ in the MC equations in the proof.

Corollary 5.18. *Under the hypothesis of Lemma 5.17, suppose that the standard map $f_0(x) = \operatorname{Ker} ev_x$ by $(V \to M, W_A)$ has an admissible pair $((V, h, \nabla), A)$. If T is a positive semi-definite Hermitian endomorphism on W_A satisfying the MC equation, then we have that*

$$\operatorname{trace} T^2 = \dim W_A. \tag{5.14}$$

Proof. Since the identity transformation Id on W_A corresponds to the standard map in Theorem 5.15, we see that Id satisfies the MC equations. We can apply Lemma 5.17 to Id to get

$$\dim W_A = \operatorname{trace} Id^2 = q\mathrm{Vol}(M).$$

The result follows from (5.12). □

5.3. **The moduli spaces.** Now we can discuss a topological property on the moduli space. Before proceeding further, we give a remark.

Remark. In the sequel, when we consider a map into an oriented Grassmannian $Gr_p(\mathbf{R}^n)$, we will not distinguish a map $f : M \to Gr_p(\mathbf{R}^n)$ from a map $\tau \circ f : M \to Gr_p(\mathbf{R}^n)$. In particular, two standard maps f_0 and $\tau \circ f_0$ are identified. This means that τ could induce non-trivial action on the sets of harmonic maps into $Gr_p(\mathbf{R}^n)$ modulo gauge and image equivalence. Then we take the quotient of them which are referred to the *moduli spaces.*

For a vector space W with a scalar product, $\mathrm{H}(W)$ denotes the set of Hermitian endomorphisms on W. We equip $\mathrm{H}(W)$ with the induced inner product $(\cdot, \cdot)_H$; $(A, B)_H := \operatorname{trace} AB$, for $A, B \in \mathrm{H}(W)$.

Under the hypothesis in Theorem 5.15, let $\mathcal{M}$ be the set of gauge equivalence classes of harmonic maps of M into $(Gr_p(W_A), (\cdot, \cdot)_{W_A})$ with admissible pair $((V, h, \nabla), A)$. Theorem 5.15 yields that $\mathcal{M}$ is identified with a subset of $\mathrm{H}(W_A)$. Hence $\mathcal{M}$ has the topology induced from that of $\mathrm{H}(W_A)$.

Proposition 5.19. *Suppose that M is a compact Riemannian manifold, $(V \to M, h, \nabla)$ is a vector bundle and $A \in \Gamma(\operatorname{End} V)$ is a negative semi-definite Hermitian endomorphism. Let W_A be the kernel of $\Delta + A$ acting on $\Gamma(V)$ equipped with the L^2 scalar product up to a positive constant multiple.*

If $\mathcal{M}$ denotes the set of gauge equivalence classes of harmonic maps of M into $(Gr_p(W_A), (\cdot, \cdot)_{W_A})$ with $((V, h, \nabla), A)$ as admissible pairs, then $\mathcal{M}$ is a bounded set in $\mathrm{H}(W_A)$.

Proof. From Theorem 5.15, $f \in \mathcal{M}$ corresponds to $T \in \mathrm{H}(W_A)$ which is positive semi-definite and satisfies the MC equations. Then the result follows from Lemma 5.17 by definition of the inner product on $\mathrm{H}(W_A)$. □

To obtain more topolological properties of the moduli space, we need a *relative* version of Theorem 5.15 without mentioning the L^2 scalar product. For this, we regard any harmonic map into Grassmannian as an induced map.

When we have a positive semi-definite Hermitian endomorphism S on a vector space W with a scalar product, we can define a unique positive semi-definite Hermitian endomorphism T on W such that $T^2 = S$ and the kernel of T coincides with that of S. Then T is denoted by $S^{\frac{1}{2}}$.

Theorem 5.20. *Let (V, h, ∇) be a vector bundle over a compact Riemannian manifold M and A a negative semi-definite Hermitian endomorphism on V.*

Suppose that f_0 is a full harmonic map of M into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_0)$ with an admissible pair $((V, h, \nabla), A)$. We realize f_0 as an induced map with $ev_0 : \underline{\mathbf{K}^n} \to V$ and $ev_0^* : V \to f^*Q$ as the evaluation map and the natural identification by f_0, respectively.

If $m \leqq n$ and $f_1 : M \to \big(Gr_{p'}(\mathbf{K}^m), (\cdot,\cdot)_1\big)$ is a full harmonic map with an admissible pair $((V, h, \nabla), A)$ (hence, $n - p = m - p'$), then there exist a linear injection $\iota : \mathbf{K}^m \to \mathbf{K}^n$ and a Hermitian endomorphism C on $\mathbf{K}^n$ which is neither positive nor negative semi-definite (possibly $C = O$) such that :
(i) *C satisfies*

$$ev_0 C ev_0^* = 0, \quad ev_0 C \nabla ev_0^* = 0. \tag{5.15}$$

(ii) *$Id + C$ is positive semi-definite.*
(iii) *$(\mathbf{K}^m, (\cdot,\cdot)_1, \iota)$ is compatible with $(\mathbf{K}^n, T)$, where $T = (Id + C)^{\frac{1}{2}}$.*
(iv) *$f_1 : M \to \big(Gr_{p'}(\mathbf{K}^m), (\cdot,\cdot)_1\big)$ is realized as the map induced by a triple $(V, \mathbf{K}^m, \iota(\iota^* T \iota))$ into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_0)$.*

Conversely, if a Hermitian endomorphism C on $(\mathbf{K}^n, (\cdot,\cdot)_0)$ satisfies the conditions (i) *and* (ii)*, then*
(a) *$\mathbf{K}^m := \mathrm{Ker}\,(Id + C)^{\perp}$ globally generates $V \to M$, and*
(b) *the map f into $\big(Gr_{p'}(\mathbf{K}^m), (\cdot,\cdot)_0\big)$ induced by $\Big(V, \mathbf{K}^m, \iota\iota^*(Id + C)^{\frac{1}{2}}\iota\Big)$ is a full harmonic map with $((V, h, \nabla), A)$ as an admissible pair, where $\iota : \mathbf{K}^m \to \mathbf{K}^n$ is the inclusion.*

Let $f_i : M \to (Gr_p(\mathbf{K}^m), (\cdot,\cdot)_0)$ be the maps induced by those triples $\Big(V, \mathbf{K}^m, \iota\iota^*(Id + C_i)^{\frac{1}{2}}\iota\Big)$ with the inclusion $\iota : \mathbf{K}^m \to \mathbf{K}^n$ such that $\iota(\mathbf{K}^m) = \mathrm{Ker}\,(Id + C_i)^{\perp}$, $(i = 1, 2)$. Then, f_1 and f_2 are gauge equivalent if and only if $C_1 = C_2$.

If we can take the standard map into $(Gr_p(W_A), (\cdot,\cdot))$ as f_0, where W_A is the solution space of the generalized Laplace equation with A and $(\cdot,\cdot)$ is the L^2 scalar product up to a positive constant multiple, then C is trace-free.

Proof. The set of scalar products on $\mathbf{K}^n$ is identified with the symmetric space $\mathrm{GL}(\mathbf{K}^n)/\mathrm{Aut}(\mathbf{K}^n)$. Now we may replace the L^2 scalar product by $(\cdot,\cdot)_0$ as a reference point of $\mathrm{GL}(\mathbf{K}^n)/\mathrm{Aut}(\mathbf{K}^n)$. Then the proof proceeds in almost the same way as one in the proof of Theorem 5.12 and so, we recognize $\mathbf{K}^m$ and $\mathbf{K}^n$ as subspaces of W_A.

Let $f_1 : M \to \big(Gr_{p'}(\mathbf{K}^m), (\cdot,\cdot)_1\big)$ be a full harmonic map with an admissible pair $((V, h, \nabla), A)$ for $m \leqq n$. Then we can find an inclusion $\iota : \mathbf{K}^m \to \mathbf{K}^n$ and a positive semi-definite Hermitian endomorphism T on $\mathbf{K}^n$ such that the kernel of T is the orthogonal complement of $\mathbf{K}^m$ in $\mathbf{K}^n$, the restriction of T to $\mathbf{K}^m$ is positive and $(\iota^* T \iota \cdot, \iota^* T \iota \cdot)_1 = \iota^*(\cdot,\cdot)_0$. Thus $T^{-1} \circ f_1 : M \to \big(Gr_{p'}(\mathbf{K}^n), (\cdot,\cdot)_0\big)$ is also a harmonic map with an admissible pair $((V, h, \nabla), A)$, where the inverse of T is taken on the orthogonal complement of $\mathrm{Ker}\, T$. Hence, we can realize f_1 as the induced map:

$$f_1(x) = \mathrm{Ker}\ (ev_{0_x} \iota\iota^* T \iota) = \mathrm{Ker}\ (ev_{0_x} T \iota)\,, \quad x \in M.$$

Since f_0 and f_1 satisfy the gauge condition for (V, h, ∇), we have that

$$ev_0 ev_0^* = Id_V, \quad ev_0 \nabla ev_0^* = 0, \tag{5.16}$$

$$ev_0 T^2 ev_0^* = Id_V, \quad ev_0 T^2 \nabla ev_0^* = 0. \tag{5.17}$$

Hence, if we express $T^2 = Id + C$, then C is a Hermitian endomorphism on $\mathbf{K}^n$ satisfying (5.15) from (5.16) and (5.17). The first condition in (5.15) yields that C is neither positive nor negative semi-definite on the image of $ev_0^* V_x$ for each $x \in M$. Since f_0 is a full map, C is neither positive nor negative semi-definite.

Since each geometric meaning is now transparent, we can see in a similar way in proofs of Theorems 5.12 and 5.15 that the converse holds except the condition (a): the linear map $ev_{0_x} \circ (Id + C)^{\frac{1}{2}} \iota : \mathbf{K}^m \to V_x$ is surjective for each $x \in M$. This will be shown in the next proposition.

If f_0 is the standard map into $(Gr_p(W_A), (\cdot,\cdot))$, then Corollary 5.18 gives $\operatorname{trace} T^2 = \operatorname{trace} I$. Thus C is trace-free. □

Remark. The equation (5.15) is referred to as *derived MC equations* for $ev_0 : \underline{\mathbf{K}^n} \to V$ or *dMC equations* for short.

Compared with the conclusion of Theorems 5.12 and 5.15, the key ingredient is that (a) is obtained as a consequence under the hypothsis that C satisfies the dMC equations and $Id + C$ is positive semi-definite. This will clarify the geometric meaning of the compactification of the moduli spaces.

Proposition 5.21. *Let (V, h, ∇) be a vector bundle over a compact Riemannian manifold M and A a negative semi-definite Hermitian endomorphism on $V \to M$.*

Suppose that f is a full harmonic map of M into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ with an admissible pair $((V, h, \nabla), A)$. We realize f as an induced map with $ev_0 : \underline{\mathbf{K}^n} \to V$ as the evaluation map by f.

Suppose that there exists a Hermitian endomorphism C on $(\mathbf{K}^n, (\cdot,\cdot))$ such that C satisfies dMC equations (5.15) and $Id + C$ is positive semi-definite. We denote by $\mathbf{K}^m$ the orthogonal complement of the kernel of $Id + C$.

Then $Id + tC$ for an arbitrary $t \in [0, 1)$ induces a full harmonic map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ with the same admissible pair as that of f. When $t = 1$, $Id + C$ induces a full harmonic map into $\big(Gr_{p'}(\mathbf{K}^m), (\cdot,\cdot)\big)$, which has the same admissible pair as that of f. Here $n - p = m - p'$.

Proof. We must show that $\mathbf{K}^m$ *globally generates* $V \to M$ by the bundle map $ev_1 := ev_0 (Id + C)^{\frac{1}{2}} : \underline{\mathbf{K}^m} \to V$.

For an arbitrary $k \in \mathbf{N}$, we can choose $t_k \in (1 - \frac{1}{k}, 1)$ such that $ev_{t_k} := ev_0 (Id + t_k C)^{\frac{1}{2}} : \underline{\mathbf{K}^n} \to V$ globally generates $V \to M$. For any $x \in M$, fix $v \in V_x$ satisfying $|v| = 1$. Then by the first equation of dMC equations (5.15) for $t_k C$ we can find $w_k \in \mathbf{K}^n$ such that $ev_x(w_k) = v$ and $|w_k| = 1$. Since $\mathbf{K}^n$ is a *finite*-dimensional vector space, we can choose a subsequence which is convergent to $w \in \mathbf{K}^n$ with $|w| = 1$ denoted by the same symbol $\{w_k\}_{k \in \mathbf{N}}$. Using an operator norm $|| \cdot ||$ on a finite-dimensional vector space

$\mathbf{K}^{n*} \otimes V_x$, we obtain the inequality:

$$\begin{aligned} |ev_{1x}(w) - v| &= \left|ev_{1x}(w) - ev_{t_k x}(w) + ev_{t_k x}(w) - ev_{t_k x}(w_k)\right| \\ &\leqq \left|\left|ev_{1x} - ev_{t_k x}\right|\right| |w| + \left|ev_{t_k x}(w - w_k)\right|. \end{aligned}$$

Then we can deduce that a unit vector w belongs to $\mathbf{K}^m$ and $ev_{1_x}(w) = v$.

We can therefore define a map f_t for $t \in [0,1]$ as

$$f_t(x) = \operatorname{Ker}\left(ev_{0_x}(Id + tC)^{\frac{1}{2}}\right), \quad x \in M.$$

Since dMC equations are linear in C, tC also satisfies (5.15) for $t \in [0,1]$. Hence the pull-back bundle with the pull-back metric and the connection is isomorphic to (V, h, ∇). Theorem 3.5 yields that f_t is also a harmonic map with A as the mean curvature operator. $\square$

The vector bundle (V, h, ∇) is called *holonomy irreducible* if the holonomy group of ∇ acts irreducibly on the fiber V_x for each $x \in M$. When (V, h, ∇) is holonomy irreducible in Theorem 5.15, then the MC equations (5.1) reduce to a single equation, which will be useful to describe moduli spaces in later chapter (in particular, see the proof of Theorems 6.20 and 6.21).

Proposition 5.22. *Let (V, h, ∇) be a rank q vector bundle over a compact Riemannian manifold M and $A \in \Gamma(\operatorname{End} V)$ a negative semi-definite Hermitian endomorphism. Suppose that (V, h, ∇) is holonomy irreducible. We denote by $W_A \subset \Gamma(V)$ the kernel of $\Delta + A$ equipped with the L^2 scalar product $(\cdot,\cdot)_{W_A}$ and by $ev : \underline{W_A} \to V$ the evaluation map.*

If $T \in \mathrm{H}(W_A)$ is positive semi-definite and satisfies the equation

$$ev \circ T^2 \circ (\nabla ev^*) = 0 \in \Omega^1(\operatorname{End} V), \tag{5.18}$$

and $\mathbf{K}^n := \operatorname{Ker} T^\perp$ globally generates $V \to M$, then we have a positive number s such that the map induced by $(V, \mathbf{K}^n, s\iota\iota^* T\iota)$ is a full harmonic map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_{W_A})$ with an admissible pair $((V, h, \nabla), A)$, where $\iota : \mathbf{K}^n \to W_A$ is the inclusion.

Proof. Since T is a Hermitian endomorphism on W_A, it follows from (5.18)

$$\begin{aligned} 0 =& h\left(ev \circ T^2 \circ (\nabla ev^*)(v_1), v_2\right) = \left(T^2 \circ (\nabla ev^*)(v_1), ev^*(v_2)\right)_{W_A} \\ =& \left((\nabla ev^*)(v_1), T^2 \circ ev^*(v_2)\right)_{W_A} = h\left(v_1, (\nabla ev) \circ T^2 \circ ev^*(v_2)\right), \end{aligned}$$

for arbitrary $v_1, v_2 \in V$. Hence an endomorphism of the bundle $V \to M$ defined by $ev \circ T^2 \circ ev^*$ is covariant constant with respect to the induced connection on $\operatorname{End} V \to M$ from ∇. From the assumption that (V, h, ∇) is holonomy irreducible, there exists a real number $\tilde{s}$ such that

$$ev \circ T^2 \circ ev^* = \tilde{s} Id_V.$$

Since $\mathbf{K}^n$ globally generates $V \to M$ and T is positive on $\mathbf{K}^n$, $\tilde{s}$ is a positive number. When we put $s = \tilde{s}^{-\frac{1}{2}}$ and $\tilde{T} := sT$, $\tilde{T}$ satisfies the MC equations, since (5.18) also holds for $\tilde{T}$. We may apply Theorem 5.15 for $\tilde{T}$ to obtain the result. $\square$

Remark. This proposition interprets the role of a fiber metric in the definition of the gauge condition. Indeed, we can naturally identify the moduli space of harmonic maps with an admissible pair $((V, h, \nabla), A)$ with that of harmonic maps with an admissible pair $((V, sh, \nabla), A)$ for $s > 0$. Fixing the fiber metric removes an ambiguity of $\mathbf{R}_{>0}$ in the choice of a triple in the gauge condition.

Corollary 5.23. *Let (V, h, ∇) be a vector bundle over a compact Riemannian manifold M and $A \in \Gamma(\operatorname{End} V)$ a negative semi-definite Hermitian endomorphism. Suppose that (V, h, ∇) is holonomy irreducible. Let W_A be the kernel of $\Delta + A$ equipped with the L^2 scalar product $(\cdot,\cdot)_{W_A}$ and $ev : \underline{W_A} \to V$ the evaluation map. Suppose that the standard map by $(V \to M, W_A)$ is a harmonic map with an admissible pair $((V, h, \nabla), A)$.*

If a trace-free Hermitian endomorphism C on W_A satisfies the equation

$$ev \circ C \circ (\nabla ev^*) = 0 \in \Omega^1(\operatorname{End} V), \tag{5.19}$$

and $Id + C$ is positive semi-definite, then $\mathbf{K}^n := \operatorname{Ker}(Id + C)^{\perp}$ globally generates $V \to M$ and the map induced by $\left(V, \mathbf{K}^n, \iota\iota^*(Id + C)^{\frac{1}{2}}\iota\right)$ is a full harmonic map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_{W_A})$ with an admissible pair $((V, h, \nabla), A)$, where $\iota : \mathbf{K}^n \to W_A$ is the inclusion.

Proof. Define a positive semi-definite Hermitian endomorphism T on W_A as $T := (Id + C)^{\frac{1}{2}}$. Since the standard map is a harmonic map with an admissible pair $((V, h, \nabla), A)$ by hypothesis, Theorem 5.15 implies that Id satisfies the MC equations:

$$ev \circ Id \circ ev^* = Id_V, \text{ and } ev \circ Id \circ \nabla (ev^*) = 0.$$

Then Proposition 5.22 and (5.19) yield that there exists a positive number s such that $\tilde{T} := s^{-\frac{1}{2}}T$ satisfies the MC equations and is positive semi-definite. By definition of T, we get

$$\tilde{T}^2 = s^{-1}Id + s^{-1}C.$$

Since $\operatorname{trace} C = 0$, we can derive from Corollary 5.18 that

$$\dim W_A = \operatorname{trace} \tilde{T}^2 = s^{-1} \dim W_A,$$

and so, $s = 1$. Hence C satisfies the dMC equations (5.15) for $ev : \underline{W_A} \to V$. Theorem 5.20 yields the result. ☐

For a vector space W with a scalar product and $T \in \mathrm{H}(W)$, we write $T > 0$ to indicate that T is positive definite and $T \geqq 0$ means that T is positive semi-definite.

5.3.1. *Gauge equivalence relation.* We will describe the moduli space.

Theorem 5.24. *Let (V, h, ∇) be a vector bundle over a compact Riemannian manifold M and A a negative semi-definite Hermitian endomorphism on V.*

Suppose that f_0 is a full harmonic map of M into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_0)$ with $((V, h, \nabla), A)$ as an admissible pair and is realized as an induced map with its evaluation map $ev_0 : \underline{\mathbf{K}^n} \to V$ and its natural identification $ev_0^* : V \to f^*Q$.

Then we can define a subspace $\mathrm{M}(\mathbf{K}^n)$ of $\mathrm{H}(\mathbf{K}^n)$ as :

$$\mathrm{M}(\mathbf{K}^n) = \{C \in \mathrm{H}(\mathbf{K}^n) \mid ev_0 C ev_0^* = 0, \text{ and } ev_0 C (\nabla ev_0^*) = 0\}.$$

Let $\mathcal{M}$ denote the set of gauge equivalence classes of full harmonic maps of M into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_0)$ with $((V, h, \nabla), A)$ as admissible pairs. Then $\mathcal{M}$ is identified with a subset of $\mathrm{M}(\mathbf{K}^n)$:

$$\mathcal{M} = \{C \in \mathrm{M}(\mathbf{K}^n) \,|\, Id + C > 0\}.$$

If $\mathcal{M}$ is equipped with the relative topology from that on $\mathrm{H}(\mathbf{K}^n)$ *induced by $(\cdot,\cdot)_0$, then it is a bounded connected convex open set in* $\mathrm{M}(\mathbf{K}^n)$. *The harmonic map parametrized by $C \in \mathcal{M}$ is expressed as $T^{-1}f_0(x)$ for $x \in M$, where $f_0(x)$ stands for a p-dimensional subspace of $\mathbf{K}^n$ and $T = (Id + C)^{\frac{1}{2}}$.*

Under the topology of $\mathrm{H}(\mathbf{K}^n)$, *the natural compactification of the moduli space is provided by taking the closure of $\mathcal{M}$:*

$$\overline{\mathcal{M}} = \{C \in \mathrm{M}(\mathbf{K}^n) \,|\, Id + C \geqq 0\},$$

which is a compact connected convex body in $\mathrm{M}(\mathbf{K}^n)$.

Proof. From Theorem 5.15, $\mathbf{K}^n$ is regarded as a subspace of W_A and f_0 is realized as the induced map with $ev_0 : \mathbf{K}^n \to V$. Since both equations in the dMC equations for $ev_0 : \underline{\mathbf{K}^n} \to V$ are linear in C, $\mathrm{M}(\mathbf{K}^n)$ is a subspace of $\mathrm{H}(\mathbf{K}^n)$. Then Theorem 5.20 implies the desired identification.

We denote by $(\cdot,\cdot)$ the induced scalar product on $\mathbf{K}^n$ from the L^2 scalar product. If $\tilde{\mathcal{M}}$ denotes the set of gauge equivalence classes of harmonic maps of M into $(Gr_p(W_A), (\cdot,\cdot))$ with the same admissible pairs, then Theorem 5.15 yields that $\mathcal{M}$ can be regarded as a subset of $\tilde{\mathcal{M}}$. Since $\mathrm{H}(\mathbf{K}^n)$ is finite-dimensional, the norm of $\mathrm{H}(\mathbf{K}^n)$ induced by $(\cdot,\cdot)_0$ is equivalent to one induced by $(\cdot,\cdot)$ as a subset of $\mathrm{End}\,(\mathbf{K}^n)$. Thus the topology of $\mathcal{M}$ is the same as that induced from $\tilde{\mathcal{M}}$. Poposition 5.19 yields that $\mathcal{M}$ itself is bounded.

If $\mathrm{M}(\mathbf{K}^n)$ is non-trivial and $C \in \mathcal{M}$, then $Id + C + C_1$ is positive for $C_1 \in \mathrm{M}(\mathbf{K}^n)$ small enough. Hence, we can deduce that the moduli space $\mathcal{M}$ is open set in $\mathrm{M}(\mathbf{K}^n)$.

It follows from Theorem 5.20 that $\mathcal{M}$ is connected and convex.

Thus $\overline{\mathcal{M}}$ is a compact connected convex body in $\mathrm{M}(\mathbf{K}^n)$, which is considered as a compactification of $\mathcal{M}$. □

Remark. If $\mathrm{M}(\mathbf{K}^n) = \{0\}$, then f_0 is the unique full harmonic map into $Gr_p(\mathbf{K}^n)$ which has an admissible pair $((V, h, \nabla), A)$ up to gauge equivalence. This motivates Definition 5.26 below.

Let us discuss the moduli space $\mathcal{M}$ modulo gauge equivalence when (one of) the standard map is in $\mathcal{M}$. This is relevant in the original do Carmo-Wallach theory and in later chapter. To do so, for a vector space W with a scalar product, we denote by $\mathrm{H}_0(W)$ the set of *trace-free* Hermitian endomorphisms on W.

Corollary 5.25. *Under the same hypothesis as in Theorem 5.24, suppose further that f_0 is the standard map, $(\cdot,\cdot)$ is the L^2 scalar product up to a constant multiple on W_A and $ev : \underline{W_A} \to V$ is the evaluation map. We identify f_0 with $\tau \circ f_0$ when $V \to M$ is an oriented real bundle.*

We define a subspace $\mathrm{M}_0(W_A)$ *of* $\mathrm{H}_0(W_A)$ *as :*

$$\mathrm{M}_0(W_A) = \{C \in \mathrm{H}_0(W_A) \,|\, evCev^* = 0, \text{ and } evC\,(\nabla ev^*) = 0\}.$$

When $\mathcal{M}$ denotes the moduli space modulo gauge equivalence of full harmonic maps of M into $(Gr_p(W_A), (\cdot,\cdot))$ with $((V, h, \nabla), A)$ as admissible pairs, $\mathcal{M}$ is identified with a bounded connected open convex set in $\mathrm{M}_0(W_A)$:

$$\mathcal{M} = \{C \in \mathrm{M}_0(W_A) \,|\, Id + C > 0\},$$

and the compactified moduli space $\overline{\mathcal{M}}$ is a compact connected convex body in $\mathrm{M}_0(W_A)$:

$$\overline{\mathcal{M}} = \{C \in \mathrm{M}_0(W_A) \,|\, Id + C \geqq 0\}.$$

Proof. Theorem 5.24 yields the description of $\mathcal{M}$ and $\overline{\mathcal{M}}$. Since $\operatorname{trace} C = 0$ from Theorem 5.20, C is in $\mathrm{H}_0(W_A)$ by definition. □

Remark. We can interpret the geometric meaning of the compactification of the moduli space.

Since f_1 is realized as a map induced by a triple $(V, \mathbf{K}^m, \iota^* T \iota)$, where $T = (Id + C)^{\frac{1}{2}}$, in Theorem 5.20, Lemma 5.11 yields that we have a totally geodesic embedding of $Gr_{p'}(\mathbf{K}^m)$ into $Gr_p(\mathbf{K}^n)$ by $\mathbf{K}^m = \operatorname{Ker} T^{\perp}$, where $n - p = m - p'$, and a bundle isomorphism $(ev_0 \iota \iota^* T \iota)^* : V \to f^* Q$ as the natural identification by f_1. Thus each boundary point $C \in \overline{\mathcal{M}}$ (where, $Id + C$ is not positive, but positive semi-definite,) determines a triplet:
(1) a *totally geodesic embedding* of $Gr_{p'}(\operatorname{Ker}(Id + C)^{\perp})$ into $Gr_p(\mathbf{K}^n)$ as the zero set of sections of the universal quotient bundle which belong to $\operatorname{Ker}(Id + C) \subset \Gamma(Q)$ $(p = p' + \dim \operatorname{Ker}(Id + C))$,
(2) a *full harmonic map* into the zero set with an admissible pair $((V, h, \nabla), A)$ and
(3) a distinguished *bundle isomorphism* of $V \to M$ to the pull-back of the universal quotient bundle as the natural identification.

Thus some maps represented as points of the boundary of the compactified moduli space could be identical.

When f_0 is a map into an oriented Grassmannian $Gr_p(\mathbf{R}^n)$, though the orientation of $\mathbf{R}^n$ is fixed by our definition, we need an orientation of $\operatorname{Ker} T$ to determine (1) and (2) uniquely. Viewing the induced maps as maps into an oriented Grassmannian $Gr_{p'}(\operatorname{Ker}(Id+C)^{\perp})$, we easily see that one is the composition of the inversion τ and the other. However, from our convention, we disregard an orientation. Hence we can still say that a point of the boundary represents a totally geodesic embedding of $Gr_{p'}(\operatorname{Ker}(Id + C)^{\perp})$ into $Gr_p(\mathbf{R}^n)$ and an induced map into $Gr_{p'}(\operatorname{Ker}(Id + C)^{\perp})$.

However, in the case where the target is the sphere, since the antipodal map τ belongs to the orthogonal group, f and $\tau \circ f$ are gauge equivalent as maps. Thus in the original do Carmo-Wallach theory, we do not need to take account of the orientation of $\operatorname{Ker} T$ and f can be identified with $\tau \circ f$ *without* our convention.

In any case, the moduli space could be regarded as a subset of the symmetric space $\mathrm{GL}(\mathbf{K}^n)/\mathrm{Aut}(\mathbf{K}^n)$.

We can characterize the case when the moduli space modulo gauge equivalence consists of a *one point* as in the Remark after Theorem 5.24. To state the result, we need

Definition 5.26. We fix $(V \to M, h, \nabla)$. Let $f : M \to Gr_p(W)$ be a harmonic map satisfying the gauge condition for (V, h, ∇). Then f is called *terminal* if there does not exist any harmonic map $f' : M \to Gr_{p'}(W')$ which has the same admissible pair as that of f with $\dim W' < \dim W$.

Corollary 5.27. *Let (V, h, ∇) be a vector bundle over a compact Riemannian manifold M and $f : M \to Gr_p(W)$ a full harmonic map satisfying the gauge condition for (V, h, ∇) with $A \in \Gamma(\operatorname{End} V)$ as the mean curvature operator.*

Then f is terminal if and only if an arbitrary harmonic map into $Gr_p(W)$ with $((V, h, \nabla), A)$ as the admissible pair is gauge equivalent to f as maps.

Proof. We realize f as an induced map with $ev : \underline{W} \to V$ and $ev^* : V \to f^*Q$ as the evaluation map and the natural identification by f, respectively.

Suppose that $f : M \to Gr_p(W)$ is a terminal map and there exists another terminal harmonic map $f_1 : M \to Gr_p(W)$ with an admissible pair $((V, h, \nabla), A)$ which is not gauge equivalent to f. Theorem 5.20 yields that there exists a non-trivial Hermitian endomorphism C which is neither positive nor negative semi-definite on W such that C satisfies dMC equations for $ev : \underline{W} \to V$:

$$evCev^* = 0, \quad evC\nabla ev^* = 0, \tag{5.20}$$

$Id + C$ is positive and f_1 is expressed as the induced map:

$$f_1(x) = \operatorname{Ker}\left(ev_{0_x}(Id + C)^{\frac{1}{2}}\right), \quad x \in M.$$

Let $U \subset W$ be the eigenspace of C corresponding to the smallest eigenvalue, say $-\lambda$, $\lambda > 0$. The orthogonal complement of U in W is denoted by $U^{\perp}$. Then we have a positive semi-definite Hermitian endomorphism $Id + \lambda^{-1}C$ with $\operatorname{Ker}\left(Id + \lambda^{-1}C\right) = U$. Theorem 5.20 with (5.20) implies that $Id + \lambda^{-1}C$ corresponds to a harmonic map $f_{\lambda^{-1}} : M \to Gr_{p'}(U^{\perp})$ with $((V, h, \nabla), A)$ as an admissible pair, which is a contradiction to the assumption that f is terminal.

Suppose that an arbitrary harmonic map into $Gr_p(W)$ with an admissible pair $((V, h, \nabla), A)$ is gauge equivalent to f. If f is not terminal, then by definition we have a terminal harmonic map $f' : M \to Gr_{p'}(W')$ with $((V, h, \nabla), A)$ as an admissible pair. Hence we can regard $Gr_{p'}(W')$ as totally geodesic submanifold of $Gr_p(W)$ and denote by ι the inclusion of W' in W. From Theorem 5.20 and Proposition 5.21, we have a Hermitian endomorphism C on W such that C satisfies dMC equations (5.20), $Id + C$ is positive on $W' = \operatorname{Ker}(Id + C)^{\perp}$ and f' is realized as a map induced by $(V, W', \iota\iota^*(Id + C)^{\frac{1}{2}}\iota)$.

Since $Id + tC$ is positive for small $t > 0$, it follows from (5.20) that $Id + tC$ induces a full harmonic map $f_t : M \to Gr_p(W)$ with an admissible pair $((V, h, \nabla), A)$, which is not gauge equivalent to f. This contradicts our hypothesis. $\square$

5.3.2. *Image equivalence relation.* Suppose that two full harmonic maps f_1 and f_2 are image equivalent as maps into $(Gr_p(\mathbf{K}^n), (\cdot, \cdot))$ with a fixed admissible pair $((V, h, \nabla), A)$. By definition of image equivalence, we have an isometry $\psi \in \operatorname{Aut}(\mathbf{K}^n)$ such that $f_2 = \psi \circ f_1$. We realize f_1 as an induced map into $(Gr_p(\mathbf{K}^n), (\cdot, \cdot))$ with $ev_1 : \underline{\mathbf{K}^n} \to V$ and $ev_1^* : V \to f_1^*Q$ as the evaluation map and the natural identification by f_1, respectively.

From Theorem 5.20, we can find a positive Hermitian endomorphism T_2 on $\mathbf{K}^n$ such that $f_2 : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ is realized as the map induced by $(V, \mathbf{K}^n, T_2)$: $f_2(x) = T_2^{-1}\mathrm{Ker}\, ev_{1_x}$. The bundle isomorphism $\tilde{\psi}$ gives $f_2^*Q = f_1^*\tilde{\psi}Q$. Using the natural identifications $\phi_1 := ev_1^*$ by f_1 and $\phi_2 := T_2 ev_1^*$ by f_2 obtained in Theorem 5.11, we get two bundle isomorphisms $\tilde{\psi} \circ \phi_1$ and $\phi_2 : V \to f_2^*Q$. Hence, as in the third remark after Theorem 5.12, there exists $c \in C(\nabla) \subset \mathrm{Aut}\, V$ such that

$$\phi_2^{-1}\tilde{\psi}\phi_1(v) = cv, \quad v \in V. \tag{5.21}$$

Then (5.11) and (5.21) yield that

$$\tilde{\psi} ev_1^*(v) = T_2 ev_1^*(cv). \tag{5.22}$$

Therefore, to obtain the moduli space modulo image equivalence, we need to take the action of $C(\nabla)$ on the moduli space modulo gauge equivalence into account.

Lemma 5.28. *Let f_1 and f_2 be full harmonic maps from a compact Riemannian manifold (M, g) into the sphere $Gr_n(\mathbf{R}^{n+1})$. If f_1 and f_2 are image equivalent, then both are gauge equivalent as maps.*

Proof. When the target is the sphere, the universal quotient bundle is a trivial bundle of real rank one with an orientation and so, the structure group is trivial. Thus the centralizer of the holonomy group is also trivial. It follows from (5.22) that

$$\tilde{\psi} ev_1^*(v) = T_2 ev_1^*(cv) = T_2 ev_1^*(v).$$

The fullness of f_i implies that $\tilde{\psi} = T_2$. Uniqueness of the polar decomposition yields that $\tilde{\psi} = T_2 = Id_{\mathbf{R}^{n+1}}$. □

As we have already pointed out, f and $\tau \circ f$ are image equivalent as maps in this case. Therefore in the original do Carmo-Wallach theory, we do not need to bear the inversion τ in mind and can consider that the standard map is unique when describing the moduli space. Furthermore, though our theory adopt the gauge equivalence relation of maps instead of image equivalence for a generalization of do Carmo-Wallach theoy, we can conclude that Theorem 5.12 and Theorem 5.20 include the original do Carmo-Wallach theory from Lemma 5.28 with the Remarks after Theorem 5.12.

To make this point clarify, we review their theory briefly on isometric minimal immersions of the sphere into spheres. More generally, we will construct the moduli space of harmonic maps with constant energy density μ. First of all, those maps are Einstein-Hermitian harmonic maps from the second Remark after Theorem 5.12. Let W_μ be the eigenspace corresponding to μ with the L^2 inner product in the function space on the sphere. Then we can show that the standard map is an Einstein-Hermitian harmonic map (see Lemma 5.36 below). We define

$$L = \{C \in \mathrm{H}_0(W_\mu) \,|\, ev \circ C \circ ev^* = 0,\ I + C > 0\}.$$

(We do not need the equation $ev \circ C \circ \nabla ev^* = 0$ in this case from the first Remark after Theorem 5.12. Or differentiating both sides of $ev \circ C \circ ev^* = 0$, we obtain $g_{f^*Q}\left(ev \circ C \circ \nabla ev^*(v), v\right) = 0$ for any $v \in f^*Q$. Since f^*Q is of rank one, we can deduce that $ev \circ C \circ \nabla ev^* = 0$.) Hence Corollary 5.25

yields that L is the moduli space of those maps modulo gauge equivalence and so, modulo image equivalence by Lemma 5.28. In this argument, we used the hypothesis that the the domain manifold is the sphere only once to deduce that the standard map is an EH-map. Since we can replace W_μ by the solution space of the generalized Laplace equation, we have established a direct generalization of do Carmo-Wallach theorem:

Theorem 5.29. *Let M be a compact Riemannian manifold. For a non-positive function A on M, we denote by W_A the kernel of $\Delta + A$ with the L^2 inner product in the function space on M. Using the evaluation map $ev : \underline{W_A} \to M \times \mathbf{R}$, we define a subspace $\mathrm{M}_0(W_A)$ of $\mathrm{H}_0(W_A)$ as :*

$$\mathrm{M}_0(W_A) = \{C \in \mathrm{H}_0(W_A) \,|\, ev \circ C \circ ev^* = 0\}.$$

Let $\mathbf{M}$ be the set of image equivalence classes of full harmonic maps from M into the hypersphere in W_A with energy density $-A$. If the standard map induced by $(M \times \mathbf{R} \to M, W_A)$ is a harmonic map with energy density $-A$, then $\mathbf{M}$ is identified with M_+ defined as :

$$M_+ = \{C \in \mathrm{M}_0(W_A) \,|\, I + C > 0\}.$$

Remark. Let f_0 denote the standard map. In our theory, f_0 is considered as a map into a Grassmannian of oriented hyperplanes of W_A. When we take the orthogonal unit vector $w \in W_A$ for each oriented hyperplane S of W_A such that the orientation given by w and S is compatible with that of W_A, f_0 can be regarded as a map into the hypersphere of W_A. Then $C \in M_+$ corresponds to a map $Tf_0(x)$ for $x \in M$, where $f_0(x)$ represents a unit vector in W_A and $T = (Id + C)^{\frac{1}{2}}$ (see [7]).

The compactification of $\overline{M_+}$ has the same geometric meaning as in Remark after Corollary 5.25:

$$\overline{M_+} = \{C \in \mathrm{M}_0(W_A) \,|\, I + C \geqq 0\}.$$

Though this is a generalization of do Carmo-Wallach [7] and Toth-D'ambra's result [35], the geometric meaning of $\overline{M_+}$ is more clarified.

Proposition 5.30. *Let f_1 and f_2 be full harmonic maps from a compact Riemannian manifold (M, g) into the complex Grassmannian $Gr_p(\mathbf{C}^n)$ which satisfy the gauge conditions for (V, h, ∇) and have the same mean curvature operators. Suppose that (V, h, ∇) is holonomy irreducible.*

If f_1 and f_2 are image equivalent, then both are gauge equivalent as maps.

Proof. Since the holonomy group acts irreducibly, Schur's lemma yields that c in (5.22) is regarded as a scalar multiplication. Since c gives a unitary transformation on $V \to M$, c is regarded as complex number a with $|a| = 1$. Then (5.22) yields that

$$\tilde{\psi} ev_1^*(v) = T_2 ev_1^*(cv) = aT_2 ev_1^*(v),$$

and so, $\tilde{\psi} = aT_2$ by fullness of maps. The polar decomposition yields that $\tilde{\psi} = aId_{\mathbf{C}^n}$ and $T_2 = Id_{\mathbf{C}^n}$. Thus we obtain the desired result. □

When the target is the complex projective space, we can show the following theorem in the same spirit of Theorem 5.29 with Corollary 5.23.

Theorem 5.31. *Suppose that M is a compact Riemannian manifold and $(L \to M, h, \nabla)$ is a complex line bundle. For a non-positive function A on M, we denote by $W_A \subset \Gamma(L)$ the kernel of $\Delta + A$ with the L^2 Hermitian inner product. Using the evaluation map $ev : \underline{W_A} \to L$, we define a subspace $\mathrm{M}_0(W_A)$ of $\mathrm{H}_0(W_A)$ as :*

$$\mathrm{M}_0(W_A) = \{C \in \mathrm{H}_0(W_A) \mid ev \circ C \circ (\nabla ev^*) = 0\}.$$

Let $\mathbf{M}$ be the set of image equivalence classes of full harmonic maps from M into the complex projective space $\mathbf{P}(W_A^*)$ with $((L, h, \nabla), A)$ as their admissible pair. If the standard map induced by (L, W_A) is a harmonic map with an admissible pair $((L, h, \nabla), A)$, then $\mathbf{M}$ is identified with M_+ defined as :

$$M_+ = \{C \in \mathrm{M}_0(W_A) \mid I + C > 0\}.$$

Suppose that the scalar product on $\mathbf{K}^n \subset \Gamma(V)$ is induced from the L^2 scalar product up to a positive constant multiple. When $C(\nabla)$ is regarded as a subgroup of $\mathcal{G}(V)$, $c \in C(\nabla)$ induces a unitary transformation on $\Gamma(V)$, which is denoted by the same symbol. Thus we have $ev \circ c = c \circ ev$.

Theorem 5.32. *Let M be a compact Riemannian manifold. Fix a vector bundle $(V \to M, h, \nabla)$ and a negative semi-definite Hermitian endomorphism $A \in \Gamma(\operatorname{End} V)$. Let W_A be the solution space of the generalized Laplace equation with A equipped with the L^2 scalar product $(\cdot,\cdot)$ up to a positive constant multiple. We take a subspace W_1 of W_A, For W_1, we define a subgroup $C_{W_1}(\nabla)$ of $C(\nabla)$ as :*

$$C_{W_1}(\nabla) = \left\{c \in C(\nabla) \mid c(W_1) = W_1 \text{ and } cAc^{-1} = A\right\}.$$

Suppose that there exists a full harmonic map $f : M \to (Gr_p(W_1), (\cdot,\cdot))$ with an admissible pair $((V, h, \nabla), A)$ and $ev : \underline{W_1} \to V$ as its evaluation map.

Let $\mathcal{M}$ be the set of gauge equivalence classes of full harmonic maps of M into $(Gr_p(W_1), (\cdot,\cdot))$ with $((V, h, \nabla), A)$ as their admissible pairs. When $\overline{\mathcal{M}}$ is identified with :

$$\overline{\mathcal{M}} = \{C \in \mathrm{H}(W_1) \mid evCev^* = 0,\ evC\nabla ev^* = 0,\ Id + C \geqq 0\},$$

as in Theorem 5.24 and we put $T = (Id + C)^{\frac{1}{2}}$, $C_{W_1}(\nabla)$ acts on $\overline{\mathcal{M}}$ as

$$T \mapsto cTc^{-1}, \quad c \in C_{W_1}(\nabla).$$

Let $\mathbf{M}$ be the moduli space modulo image equivalence of those full harmonic maps into $(Gr_p(W_1), (\cdot,\cdot))$. Then $\mathbf{M}$ is identified with the space of orbits of $C_{W_1}(\nabla)$ in $\mathcal{M}$:

$$\mathbf{M} = \mathcal{M}/C_{W_1}(\nabla).$$

Proof. If a triple (V, W_1, T) induces a harmonic map f_1 into $Gr_p(W_1)$ with an admissible pair $((V, h, \nabla), A)$, then for any $c \in C_{W_1}(\nabla)$, $\left(V, W_1, cTc^{-1}\right)$ induces a map f_2 into $Gr_p(W_1)$. Then f_2 is also a harmonic map with the same admissible pair. To see this, we show that cTc^{-1} satisfies the MC equations. It follows from $evT^2ev^* = Id_V$ and $ev \circ c = c \circ ev$ that

$$ev \circ cT^2c^* \circ ev^* = cev \circ T^2 \circ ev^*c^* = cId_Vc^* = Id_V.$$

Since c is covariant constant and $ev \circ T^2(\nabla ev^*) = 0$, we obtain

$$\begin{aligned} &ev \circ cT^2c^* \circ (\nabla ev^*) = cev \circ T^2\nabla\left(c^* \circ ev^*\right) = cev \circ T^2\nabla\left(ev^*c^*\right) \\ =&cev \circ T^2\left\{(\nabla ev^*)\, c^* + ev^*\left(\nabla c^*\right)\right\} = 0. \end{aligned}$$

Using $cAc^{-1} = A$, we can deduce from Theorem 5.15 and Theorem 5.20 that f_2 has the desired properties. We thus conclude that $C_{W_1}(\nabla)$ acts on $\overline{\mathcal{M}}$.

To describe $\mathbf{M}$, suppose further that f_1 and f_2 are full maps into $Gr_p(W_1)$. By definition of the induced map, f_2 is expressed as

$$f_2(x) = \mathrm{Ker}\,\left(ev_x \circ cTc^{-1})\right).$$

Since any $c \in C_{W_1}(\nabla)$ is a unitary transformation on $V \to M$ and $ev \circ c = c \circ ev$, we have that

$$f_2(x) = \mathrm{Ker}\,\left(cev_x \circ Tc^{-1})\right) = \mathrm{Ker}\,\left((ev_x \circ T)c^{-1}\right) = c\mathrm{Ker}\,(ev_x \circ T)\,.$$

By definition, $C_{W_1}(\nabla)$ preserves $W_1 \subset \Gamma(V)$ and so, $f_2(x) = cf_1(x)$ for any $x \in M$, where $f_i(x)$ represent subspaces of W_1 $(i = 1, 2)$. Since $C_{W_1}(\nabla)$ can be considered as a subgroup of the isometry group of $(Gr_p(W_1), (\cdot,\cdot))$, f_2 is image equivalent to f_1. Thus all points in any orbit of $C_{W_1}(\nabla)$ in $\mathcal{M}$ represent image equivalent maps as maps into $(Gr_p(W_1), (\cdot,\cdot))$.

Next, we assume that the maps f_i induced by (V, W_1, T_i) for some $T_i \in \mathrm{H}(W_1)$ with $T_i > 0$ $(i = 1, 2)$ are image equivalent harmonic maps into $(Gr_p(W_1), (\cdot,\cdot))$ with the same admissible pair $((V, h, \nabla), A)$. By definition, we have an isometry $\psi \in \mathrm{Aut}\,(W_1)$ of $Gr_p(W_1)$ such that $f_2 = \psi \circ f_1$.

Let ϕ_i be the corresponding natural identifications by f_i. Since $\phi_2^{-1}\tilde{\psi}\phi_1 \in \mathcal{G}(V)$ preserves the connection, viewing $C(\nabla)$ as a subgroup of $\mathcal{G}(V)$, we have $c \in C(\nabla)$ satisfying

$$\phi_2^{-1}\tilde{\psi}\phi_1 = c. \tag{5.23}$$

It follows that

$$c(W_1) = W_1. \tag{5.24}$$

Then, from Theorem 3.5, $(\Delta + A)c^{-1}(t) = 0$ for any $t \in W_1$. Hence we get

$$0 = c(\Delta + A)c^{-1}(t) = (c\Delta c^{-1} + cAc^{-1})t = (\Delta + cAc^{-1})t,$$

and so, we can deduce from Theorem 3.5 that $cAc^{-1} = A$. Together with (5.24), this yields that $c \in C_{W_1}(\nabla)$.

Since $ev \circ c = c \circ ev$ and so, $\tilde{\psi}T_1ev^* = T_2ev^*c = T_2cev^*$ from (5.11) and (5.23), using fullness of f_i, we obtain

$$\tilde{\psi}T_1 = T_2c \iff \tilde{\psi}T_1\tilde{\psi}^{-1}\tilde{\psi} = T_2c \iff \tilde{\psi}T_1\tilde{\psi}^{-1}\tilde{\psi}c^{-1} = T_2. \tag{5.25}$$

The polar decomposition yields that

$$\tilde{\psi}T_1\tilde{\psi}^{-1} = T_2, \quad \tilde{\psi} = c.$$

Thus we can take $c \in C_{W_1}(\nabla)$ as ψ, in other words, $f_2 = cf_1$ and the Proposition is proved. □

Remark. We consider the compactification of $\mathbf{M}$ by taking the quotient topology. Thus it should be the closure of $\mathbf{M}$ denoted by $\overline{\mathbf{M}}$:

$$\overline{\mathbf{M}} = \overline{\mathcal{M}}/C_{W_1}(\nabla).$$

Hence each boundary point $C \in \overline{\mathbf{M}}$ (where, $Id + C$ is not positive, but positive semi-definite,) determines a pair:
(1) a *totally geodesic embedding* of $Gr_{p'}(\mathrm{Ker}\,(Id+C)^{\perp})$ into $Gr_p(W_1)$ modulo $C_{W_1}(\nabla)$-action.
(2) a *full harmonic map* into $Gr_{p'}(\mathrm{Ker}\,(Id+C)^{\perp})$ with an admissible pair $((V,h,\nabla),A)$. In the case when the target is an oriented Grassmannian, C defines two maps, say f and $\tau \circ f$. These two maps should be identified from our convention.

Corollary 5.33. *Under the same hypothesis as in Theorem 5.32, suppose further that the standard map f_0 into $(Gr_p(W_A),(\cdot,\cdot))$ is a full harmonic map with an admissible pair $((V,h,\nabla),A)$.*

We define a subgroup $C_A(\nabla)$ of $C(\nabla)$ as :

$$C_A(\nabla) = \left\{ c \in C(\nabla) \,|\, cAc^{-1} = A \right\}.$$

Let $\mathcal{M}$ be the set of gauge equivalence classes of full harmonic maps of M into $(Gr_p(W_A),(\cdot,\cdot))$ with $((V,h,\nabla),A)$ as their admissible pair. As in Corollary 5.25, we identify $\overline{\mathcal{M}}$ with :

$$\overline{\mathcal{M}} = \{ C \in \mathrm{M}_0(W_A) \,|\, Id + C \geqq 0 \}.$$

When we put $T = (Id+C)^{\frac{1}{2}}$, $C_A(\nabla)$ acts on $\overline{\mathcal{M}}$ as

$$T \mapsto cTc^{-1}, \quad c \in C_A(\nabla).$$

Let $\mathbf{M}$ be the moduli space modulo image equivalence of those full harmonic maps into $(Gr_p(W_A),(\cdot,\cdot))$. Then

$$\mathbf{M} = \mathcal{M}/C_A(\nabla).$$

Proof. Since $cAc^{-1} = A$ for any $c \in C_A(\nabla)$ by definition, we obtain $(\Delta + A)c(t) = c\Delta c^{-1}c(t) + cAc^{-1}c(t) = c(\Delta + A)t = 0$, for any $t \in W_A$. Thus $c(W_A) = W_A$. The result follows from Theorem 5.32. □

Remark. Notice that Corollary 5.33 is a direct generalization of Theorems 5.29 and 5.31.

5.4. **Homogeneous cases: a generalization of Toth-D'ambra theory.** Let $M = G/K_0$ be a compact reductive Riemannian homogeneous space with K_0-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where G is a compact Lie group, K_0 is a closed subgroup of G and $\mathfrak{g}$ and $\mathfrak{k}$ are the corresponding Lie algebras, respectively (see, for example, [19, p.190]). By Riemannian homogeneous space, we indicate that a G-invariant metric on G/K_0 is fixed.

Let V_0 be a q-dimensional orthogonal or unitary K_0-representation space with a K_0-invariant scalar product. We can construct a homogeneous vector bundle $V \to M$, $V := G \times_{K_0} V_0$ with an invariant fiber metric g_V induced by the scalar product on V_0. Moreover $V \to M$ has a canonical connection ∇ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. (This means that the horizontal distribution is defined as $\{L_g\mathfrak{m} \subset TG_g \,|\, g \in G\}$ on the principal fiber bundle $\pi : G \to M$, where L_g denotes the left translation on G.)

The Lie group G naturally acts on the space of sections $\Gamma(V)$ of $V \to M$, which has a G-invariant L^2 scalar product.

Using the Levi-Civita connection and ∇, we can decompose the space of sections of $V \to M$ into the eigenspaces of the Laplacian: $\Gamma(V) = \oplus_\mu W_\mu$.

It is well-known that W_μ is a finite-dimensional G-module equipped with a G-invariant scalar product induced from the L^2 scalar product.

Lemma 5.34. *Let $V = G \times_{K_0} V_0 \to G/K_0$ be a homogeneous vector bundle and W a G-subspace of $\Gamma(V)$. By the restriction of the action to the subgroup K_0, we also consider W as a K_0-module. If W globally generates $V \to G/K_0$, then V_0 can be realized as a K_0-invariant subspace of W.*

Proof. We identify V_0 with the fiber $V_{[e]}$ of $V \to M$ at $[e] \in M$, where e is the unit element of G. Since the evaluation map $ev : \underline{W} \to V$ is G-equivariant and the scalar product and the fiber metric are G-invariant, the adjoint bundle map $ev^* : V \to \underline{W}$ is also G-equivariant. Thus the image of $ev^*_{[e]}$ is a subspace of W equivalent to V_0 as K_0-representation, since W globally generates $V \to M$. $\square$

From Lemma 5.34, when we consider a homogeneous vector bundle $V = G \times_{K_0} V_0 \to G/K_0$ and a G-subspace $W \subset \Gamma(V)$ which globally generates $V \to G/K$, K_0-module V_0 is supposed to be realized as a K_0-invariant subspace of W by ev^*.

5.4.1. *Standard maps.* Suppose that an eigenspace W_μ globally generates $V \to M$. Then we have the standard map $f_0 : M \to Gr_p(W_\mu)$ by W_μ, where $p = \dim W_\mu - q$. In general, W_μ is not irreducible as G-representation.

More generally, let W be a G-subspace of $\Gamma(V)$ and suppose that W globally generates $V \to M$. Then the map induced by W is also called the standard map by W. Since V_0 can be realized as a subspace of W by Lemma 5.34, we have the orthogonal complement of V_0 denoted by U_0. Then the standard map $f_0 : M \to Gr_p(W)$ is expressed as

$$f_0([g]) = gU_0 \subset W,$$

which is G-equivariant. When we regard $f^*Q \to M$ as a subbundle of $\underline{W} \to M$, the adjoint of the evaluation map $ev^* : V \to f^*Q$ is given by

$$ev^*\left([g, v]\right) = \left([g], gv\right), \tag{5.26}$$

where $g \in G$ and $v \in V_0 \subset W$.

Next, we consider the pull-back connection ∇^V and the gauge condition for the standard map by W. We denote by $\mathfrak{m}V_0$ the subspace of W generated by $\xi v \in W$, where $\xi \in \mathfrak{m}$ and $v \in V_0 \subset W$.

Lemma 5.35. *Let $V = G \times_{K_0} V_0 \to G/K_0$ be a homogeneous vector bundle and W a G-subspace of $\Gamma(V)$ which globally generates $V \to G/K$. Let $f_0 : M \to Gr(W)$ be the map induced by W. From Lemma 5.34, we realize V_0 as a subspace of W. Then the pull-back connection ∇^V is gauge equivalent to the canonical connection if and only if $\mathfrak{m}V_0 \subset U_0$.*

Proof. We use sections $t[g] = \left[g, \pi_0(g^{-1}w)\right] \in \Gamma(V)$ corresponding to $w \in W$, where $\pi_0 : W \to V_0$ denotes the orthogonal projection. The canonical connection ∇^0 is computed as follows:

$$\nabla^0_{d\pi L_g \xi} t = \left[g, -\pi_0\left(\xi g^{-1} w\right)\right].$$

Next, we use the adjoint of the evaluation map to compute the pull-back connection:

$$\nabla^V_{d\pi L_g\xi} t = \pi_V d_{d\pi L_g \xi} ev^*(t) = \left[g, \pi_0\left(\xi\pi_0(g^{-1}w)\right) - \pi_0\left(\xi g^{-1}w\right)\right].$$

Since W globally generates $V \to M$, the result follows. $\square$

Lemma 5.36. *Let $V = G\times_{K_0} V_0 \to G/K_0$ be a homogeneous vector bundle and W a G-subspace of $W_\mu \subset \Gamma(V)$. Suppose that W globally generates $V \to G/K_0$. We use Lemma 5.34 to realize V_0 as a subspace of W. If $\mathfrak{m}V_0$ is orthogonal to V_0, then the standard map $f_0 : M \to Gr_p(W)$ is an Einstein-Hermitian harmonic map with*

$$e(f_0) = q\mu,\ (or\ 2q\mu), \quad A = -\mu Id_V.$$

Proof. Since Lemma 5.35 yields that the pull-back connection is the canonical connection, W is also a subspace of the eigenspace of the Laplacian induced by the pull-back connection. Then we apply Theorem 3.5 to get the result. $\square$

Example. Let $\mathbf{C}P^1 = \mathrm{SU}(2)/\mathrm{U}(1)$ be a complex projective line and $\mathcal{O}(1) \to \mathbf{C}P^1$ a holomorphic line bundle of degree 1 with the canonical connection. The symmetric power $S^n\mathbf{C}^2$ ($n \in \mathbf{Z}_{\geqq 0}$) of the standard representation $\mathbf{C}^2$ is an irreducible representation of SU(2) denoted by $(\varrho_n, S^n\mathbf{C}^2)$. Frobenius reciprocity yields that $S^{2n+1}\mathbf{C}^2$ ($n \in \mathbf{Z}_{\geqq 0}$) is an SU(2)-invariant space of sections of $\mathcal{O}(1) \to \mathbf{C}P^1$. Moreover, $S^{2n+1}\mathbf{C}^2$ is an eigenspace of the Laplacian (see [37]). We denote by $\mathbf{C}_k$ ($k \in \mathbf{Z}$) an irreducible U(1)-module with weight k. As homogeneous vector bundle, $\mathcal{O}(1) \to \mathbf{C}P^1$ is regarded as $\mathrm{SU}(2)\times_{\mathrm{U}(1)} \mathbf{C}_{-1}$. Then we can realize $\mathbf{C}_{-1}$ as a subspace of $S^{2n+1}\mathbf{C}^2$ by Lemma 5.34. Since the complexification of $\mathfrak{m}$ is identified with $\mathbf{C}_2 \oplus \mathbf{C}_{-2}$, we have that

$$\varrho_{2n+1}(\mathfrak{m})\mathbf{C}_{-1} \subset \mathbf{C}_{-3} \oplus \mathbf{C}_1.$$

Consequently, the standard map $f_0 : \mathbf{C}P^1 \to \mathbf{C}P^{2n} = \mathbf{P}(S^{2n+1}\mathbf{C}^2)$ is an Einstein-Hermitian harmonic map from Lemma 5.36.

See also Ohnita [29] about an equivariant harmonic map into a complex projective space (see §6.4 for the definition of an equivariant map and Theorem 6.28 for a generalization of this example).

Example. Let $M = \mathbf{H}P^1 = \mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ be the quaternion projective line. To distinguish two copies of Sp(1) in the isotropy subgroup, we write the isotropy subgroup as $\mathrm{Sp}_+(1) \times \mathrm{Sp}_-(1)$. Let $\mathbf{H} \cong \mathbf{C}^2$ be the standard representation of $\mathrm{Sp}_+(1)$ and $\mathbf{E}$ the standard representation of $\mathrm{Sp}_-(1)$. Then the associated complex homogeneous vector bundles are denoted by the same symbols $\mathbf{H} \to M$ and $\mathbf{E} \to M$, respectively. We suppose that $\mathbf{H} \to M$ is the tautological vector bundle and $\mathbf{E} \to M$ is the orthogonal complement in a trivial bundle $\underline{\mathbf{H}^2} \cong \underline{\mathbf{C}^4} \to M$.

We take the symmetric power $\mathbf{S}^k\mathbf{H} \to M$ of $\mathbf{H} \to M$ and $\mathbf{S}^l\mathbf{E} \to M$ of $\mathbf{E} \to M$. When k (resp.l) is even, $S^k\mathbf{H}$ (resp.$S^l\mathbf{E}$) has a real structure, which is a conjugate-linear involution. If $k+l$ are even, then $S^k\mathbf{H}\otimes S^l\mathbf{E}$ has a real structure. In those cases, for example, $S^k\mathbf{H}$ is supposed to represent a real representation or the associated real vector bundle.

Since the Lie algebra $\mathfrak{sp}(2)$ has the standard decomposition as the symmetric pair $(\mathrm{Sp}(2), \mathrm{Sp}(1) \times \mathrm{Sp}(1))$:

$$\mathfrak{sp}(2) = S^2\mathbf{H} \oplus S^2\mathbf{E} \oplus (\mathbf{H} \otimes \mathbf{E}),$$

$\mathfrak{sp}(2)$ can be regarded as an eigenspace of the Laplacian acting on sections of $S^2\mathbf{H} \to M$. Using again that $(\mathrm{Sp}(2), \mathrm{Sp}(1) \times \mathrm{Sp}(1))$ is a symmetric pair, we have

$$[\mathbf{H} \otimes \mathbf{E}, S^2\mathbf{H}] \subset \mathbf{H} \otimes \mathbf{E}.$$

Lemma 5.36 implies that the standard map $f_0 : \mathbf{H}P^1 \to Gr_7(\mathfrak{sp}(2)) = Gr_7(\mathbf{R}^{10})$ is an Einstein-Hermitian harmonic map.

Now the standard map has another interpretation (see also Swann [31] and Gambioli [15]). Let $\mu : \mathbf{H}P^1 \to \mathfrak{sp}(2)^* \otimes S^2\mathbf{H}$ be a quaternion moment map [14]. By definition of a moment map, for an arbitrary $X \in \mathfrak{sp}(2)$, we have

$$\mu_X([g]) = \left[g, \pi_{S^2\mathbf{H}}(g^{-1}Xg)\right], \quad g \in \mathrm{Sp}(2),$$

where $\pi_{S^2\mathbf{H}} : \mathfrak{sp}(2) \to S^2\mathbf{H}$ is the orthogonal projection. It follows that $\mathfrak{sp}(2)$ is a subspace of sections of $S^2\mathbf{H} \to M$ by the moment map μ. It is clear that $\mathfrak{sp}(2)$ globally generates $S^2\mathbf{H} \to M$. We can define the induced map $f_\mu : \mathbf{H}P^1 \to Gr_7(\mathbf{R}^{10})$. By definition of the induced map, we have

$$\begin{aligned} f_\mu([g]) =& \left\{X \in \mathfrak{sp}(2) \mid \mathrm{Ad}(g^{-1})X \in S^2\mathbf{E} \oplus (\mathbf{H} \otimes \mathbf{E})\right\} \\ =& \mathrm{Ad}(g) \left(S^2\mathbf{E} \oplus (\mathbf{H} \otimes \mathbf{E})\right) \subset \mathfrak{sp}(2), \end{aligned}$$

which is the same as the standard map f_0.

The standard map induced by the pair $\left(S^2\mathbf{H} \to \mathbf{H}P^1, \mathfrak{sp}(2)\right)$ can be generalized on any compact quaternion symmetric space. It is induced by a quaternion moment map for an isometry group in the same way.

Using a moment map, we obtain a result of Takeuchi-Kobayashi [33]:

Example. Let (G, K) be an irreducible Hermitian symmetric pair of compact type and consider a moment map $\mu : G/K \to \mathfrak{g}^*$. In this situation, $\mu_X : G/K \to \mathbf{R}$ for an arbitrary $X \in \mathfrak{g}$ is an eigenfunction of the Laplacian. Then Theorem of Takahashi [32] yields that the induced map $f : G/K \to S^N \subset \mathfrak{g}$ is a harmonic map, where S^N is a hypersphere of $\mathfrak{g}$.

5.4.2. *A generalization of do Carmo-Wallach Theory in homogeneous cases.* Let G be a compact Lie group and W an orthogonal or a unitary representation of G with an invariant scalar product $(\cdot,\cdot)_W$. Then G naturally acts on $\mathrm{H}(W)$. If we equip $\mathrm{H}(W)$ with an inner product $(\cdot,\cdot)_H$, then it is easily seen that $(\cdot,\cdot)_H$ is G-invariant. We define a symmetric or Hermitian operator $H(u, v)$ for u, $v \in W$ as

$$H(u, v) := \frac{1}{2}\left\{u \otimes (\cdot, v)_W + v \otimes (\cdot, u)_W\right\}.$$

Then it follows that for an arbitrary $B \in \mathrm{H}(W)$

$$(B, H(u, v))_H = \frac{1}{2}\left\{(Bu, v)_W + (Bv, u)_W\right\}. \tag{5.27}$$

If U and V are subspaces of W, we define a real subspace $H(U, V) \subset \mathrm{H}(W)$ spanned by $H(u, v)$ where $u \in U$ and $v \in V$. In a similar fashion, $GH(U, V)$

denotes the subspace of $\mathrm{H}(W)$ spanned by $gH(u,v)$, where $g \in G$, and so $GH(U,V)$ is a G-submodule of $\mathrm{H}(W)$.

We can now formulate a generalization of do Carmo-Wallach theory in homogeneous cases. Though the difference from Theorem 5.15 is only the condition imposed on T, we state the theorem in its complete form for readers' convenience.

Theorem 5.37. *Let G/K_0 be a compact reductive Riemannian homogeneous space with K_0-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Fix a rank q homogeneous vector bundle $\left(V = G \times_{K_0} V_0, h, \nabla^0\right)$ with an invariant metric h and the canonical connection ∇^0 and a G-invariant negative semi-definite Hermitian endomorphism $A \in \Gamma(\mathrm{End}\, V)$. Let $W_A \subset \Gamma(V)$ be the solution space of the generalized Laplace equation with A with the L^2 scalar product $(\cdot,\cdot)_{W_A}$. Since ∇^0 and A are G-invariant, W_A can be regarded as $\mathfrak{g}$-representation $\varrho : \mathfrak{g} \to \mathrm{End}\,(W_A)$. By Lemma 5.34, we realize V_0 as a subspace of W_A.*

If $f : G/K_0 \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ $(n = p+q)$ is a full harmonic map with an admissible pair $\left((V,h,\nabla^0), A\right)$ and we fix a bundle isomorphism between (V,h,∇^0) and $(f^*Q, f^*h_Q, \nabla^{f^*Q})$, then there exist a unique linear injection $\iota : \mathbf{K}^n \to W_A$ and a positive semi-definite Hermitian endomorphism T on W_A such that
(a) *T satisfies*

$$(5.28) \qquad \left(T^2 - Id_W, GH(V_0,V_0)\right)_H = 0, \ \left(T^2, GH(\varrho(\mathfrak{m})V_0, V_0)\right)_H = 0,$$

(b) *$(\mathbf{K}^n, (\cdot,\cdot), \iota)$ is compatible with (W_A, T). In particular, $n \leqq \dim W_A$,*
(c) *$f : G/K_0 \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot))$ is realized as the map induced by a triple $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$ into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_{W_A})$.*

Conversely, if a positive semi-definite Hermitian endomorphism T on W_A satisfies condition (a) *and $\mathbf{K}^n := (\mathrm{Ker}\, T)^\perp$ globally generates $V \to M$, then the map induced by $(V, \mathbf{K}^n, \iota\,(\iota^* T \iota))$ is a full harmonic map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_{W_A})$ with an admissible pair $\left((V,h,\nabla^0), A\right)$, where $\iota : \mathbf{K}^n \to W_A$ is the inclusion.*

Let $f_i : M \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_{W_A})$ be the maps induced by those triples $(V, \mathbf{K}^n, \iota\,(\iota^* T_i \iota))$ such that $\iota\,(\mathbf{K}^n)^\perp = \mathrm{Ker}\, T_1 = \mathrm{Ker}\, T_2$, where $\iota : \mathbf{K}^n \to W_A$ is the inclusion. Then, f_1 and f_2 are gauge equivalent if and only if $T_1 = T_2$.

Proof. We follow the notation in the proof of Theorem 5.12 and only pay attention to the role played by the condition (a), which is the substitution for the MC equations.

When $f^*Q \to M$ is identified with the orthogonal complement of $f^*S \to M$, it follows from (5.7) and (5.26) that the natural identification $\phi : V \to f^*Q$ is expressed as

$$(5.29) \qquad \phi\,([g,v]) = ([g], Tgv)\,,$$

where $g \in G$ and $v \in V_0 \subset W$.

Since the metric on $f^*Q \to M$ is induced by the scalar product on W, it follows from the gauge condition and (5.29) that

$$(5.30) \qquad (Tgv, Tgv')_W = (v,v')_W,$$

for arbitrary $v, v' \in V_0$. From the definition of the scalar product on $\mathrm{H}(W)$, we have

$$\begin{aligned}(5.31)\qquad \mathrm{Re}(Tgv, Tgv')_W =&(T^2, gH(v,v'))_H \quad \text{and}\\ \mathrm{Im}(Tgv, Tgv')_W =&(T^2, gH(v, \sqrt{-1}v'))_H.\end{aligned}$$

Together with $(Id, H(v,v'))_H = \mathrm{Re}(v,v')_W$, (5.30) and (5.31) yield that

$$(T^2 - Id, gH(v,v'))_H = 0$$

for an arbitrary $g \in G$ and arbitrary v, $v' \in V_0$, which is equivalent to

$$(T^2 - Id, GH(V_0, V_0))_H = 0. \tag{5.32}$$

Since the equation (5.30) is equivalent to

$$(g^{-1}T^2 gv, v')_W = (v, v')_W,$$

we obtain

$$\pi_0(g^{-1}T^2 g)i_0 = Id_{V_0}, \tag{5.33}$$

where $i_0 : V_0 \to W$ is the natural inclusion and $\pi_0 : W \to V_0$ is the orthogonal projection to V_0.

Next, we compare the canonical connection ∇^0 with the induced connection ∇ on $V \to M$ by $f : M \to Gr_p(\mathbf{K}^n)$. To describe the orthogonal projection $\pi_V : \underline{W} \to V$, notice that π_V is recognized as the adjoint of ϕ. Thus $\pi_V : \underline{W} \to V$ is expressed as:

$$\pi_V\left([g], w\right) = \left[g, \pi_0(g^{-1}Tw)\right].$$

If we use a section $t[g] = \left[g, \pi_0(g^{-1}w)\right]$ corresponding to $w \in W$, then the canonical connection is calculated as follows:

$$\nabla^0_{d\pi L_g \xi} t = \left[g, -\pi_0\left(\varrho(\xi)g^{-1}w\right)\right]. \tag{5.34}$$

Next the induced connection is calculated as follows:

$$\begin{aligned}\nabla_{d\pi L_g \xi} t =& \pi_V d_{d\pi L_g \xi}\phi(t) = \pi_V d_{d\pi L_g \xi}\left([g], Tg\pi_0(g^{-1}w)\right)\\ =& \pi_V\left([g], Tg\varrho(\xi)\pi_0(g^{-1}w) - Tg\pi_0(\varrho(\xi)g^{-1}w)\right)\\ =& \left[g, \pi_0\left(g^{-1}T^2 g\varrho(\xi)\pi_0(g^{-1}w)\right) - \pi_0\left(g^{-1}T^2 g\pi_0(\varrho(\xi)g^{-1}w)\right)\right].\end{aligned}$$

It follows from (5.33) and (5.34) that

$$\nabla_{d\pi L_g \xi} t - \nabla^0_{d\pi L_g \xi} t = \left[g, \pi_0\left(g^{-1}T^2 g\varrho(\xi)\pi_0(g^{-1}w)\right)\right].$$

Since W globally generates $V \to M$, the gauge condition yields that

$$\pi_0(g^{-1}T^2 g\varrho(\xi))i_0 = 0, \tag{5.35}$$

for an arbitrary $g \in G$ and $\xi \in \mathfrak{m}$, which is equivalent to

$$(T^2, GH(\varrho(\mathfrak{m})V_0, V_0))_H = 0. \tag{5.36}$$

It follows from (5.32) and (5.36) that condition (a) holds. □

6. Applications

6.1. **Algebraic preliminaries.** First of all, we shall give an algebraic result which is a slight modification of do Carmo and Wallach [7].

Let G/K_0 be a compact reductive Riemannian homogeneous space with K_0-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where G is a compact Lie group. Let V_0 be a K_0-representation with an invariant scalar product and $V = G \times_{K_0} V_0 \to G/K_0$ an associated homogeneous vector bundle with the induced invariant metric. Let W be a G-submodule of $\Gamma(V)$ endowed with a G-invariant scalar product induced by L^2 scalar product. If W globally generates $V \to G/K_0$, then we realize V_0 as a subspace of W as in Lemma 5.34. We denote by $\pi_0 : W \to V_0$ the orthogonal projection. Let U_0 be the orthogonal complement of V_0 in W with the orthogonal projection $\pi_1 : W \to U_0$.

We would like to describe the decomposition of W as K_0-module when regarding W as K_0-module by restriction of the representation. We follow an idea of do Carmo and Wallach [7].

Definition 6.1. A linear map $B_1 : \mathfrak{m} \otimes V_0 \to U_0$ is defined as:

$$B_1 (\xi \otimes v) = \pi_1 (\xi v), \quad \text{for} \quad \xi \in \mathfrak{m}, v \in V_0.$$

Then we also define

$$N_2 := (V_0 \oplus \operatorname{Im} B_1)^{\perp} \subset W,$$

and the orthogonal projection $\pi_2 : W \to N_2$. For simplicity, U_0 is also denoted by N_1.

Since π_1 is K_0-equivariant, we have

Lemma 6.2. *A linear map $B_1 : \mathfrak{m} \otimes V_0 \to U_0$ is K_0-equivariant.*

Corollary 6.3. $\operatorname{Im} B_1$ *is a K_0-module.*

The n-th symmetric power of $\mathfrak{m}$ is denoted by $S^n\mathfrak{m}$. Let S_n denote the permutation group of order n and define an element of $S^n\mathfrak{m}$ as

$$\xi_1\xi_2 \cdots \xi_n := \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)},$$

for $\xi_1, \xi_2, \cdots, \xi_n \in \mathfrak{m}$.

Definition 6.4. Inductively, we define the subspace $\left(V_0 \bigoplus \oplus_{p=1}^{n-1} \operatorname{Im} B_p\right)^{\perp}$ of W denoted by N_n with the orthogonal projection $\pi_n : W \to N_n$. Then a linear map $B_n : S^n\mathfrak{m} \otimes V_0 \to N_n$ is defined as

$$B_n ((\xi_1 \cdots \xi_n) \otimes v) = \pi_n ((\xi_1 \cdots \xi_n)v),$$

where $(\xi_1 \cdots \xi_n)v = \frac{1}{n!} \sum_{\sigma \in S_n} \left(\xi_{\sigma(1)} \left(\xi_{\sigma(2)} \cdots \left(\xi_{\sigma(n-1)} \left(\xi_{\sigma(n)} v\right)\right) \cdots \right)\right)$.

Since $B_n : S^n\mathfrak{m} \otimes V_0 \to N_n$ is K_0-equivariant, $\operatorname{Im} B_n$ is a K_0-module.

Lemma 6.5. *If (G, K_0) is a symmetric pair with the corresponding orthogonal decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, then*

$$B_n ((\xi_1 \cdots \xi_n) \otimes v) = \pi_n (\xi_1 (\xi_2 (\cdots (\xi_n v) \cdots))).$$

Proof. For $\xi_1, \cdots, \xi_n \in \mathfrak{m}$ and $v \in V_0 \subset W$, we have

$$\xi_1\left(\xi_2\left(\xi_3 \cdots \xi_n\right) v\right) - \xi_2\left(\xi_1\left(\xi_3 \cdots \xi_n\right) v\right) = \left[\xi_1, \xi_2\right]\left(\left(\xi_3 \xi_4 \cdots \xi_n\right) v\right).$$

By definition, it follows that $\left(\xi_3 \cdots \xi_n\right) v \in V_0 \oplus \operatorname{Im} B_1 \oplus \cdots \oplus \operatorname{Im} B_{n-2}$. From the hypothesis that (G, K_0) is a symmetric pair, we get $[\xi_1, \xi_2] \in \mathfrak{k}$. Since $V_0 \oplus \operatorname{Im} B_1 \oplus \cdots \oplus \operatorname{Im} B_{n-2}$ is a K_0-module, we have

$$\left[\xi_1, \xi_2\right]\left(\left(\xi_3 \xi_4 \cdots \xi_n\right) v\right) \in V_0 \oplus \operatorname{Im} B_1 \oplus \cdots \oplus \operatorname{Im} B_{n-2},$$

and so,

$$\pi_n\left(\xi_1 \xi_2\left(\xi_3 \cdots \xi_n\right) v\right) = \pi_n\left(\xi_2 \xi_1\left(\xi_3 \cdots \xi_n\right) v\right).$$

In a similar way, we obtain

$$\begin{aligned} &\pi_n\left(\left(\xi_1 \cdots \xi_{i-1}\right)\left(\xi_i\left(\xi_{i+1}\left(\xi_{i+2} \cdots \xi_n\right) v\right)\right)\right) \\ =&\pi_n\left(\left(\xi_1 \cdots \xi_{i-1}\right)\left(\xi_{i+1}\left(\xi_i\left(\xi_{i+2} \cdots \xi_n\right) v\right)\right)\right), \end{aligned}$$

and the result follows. □

Definition 6.6. Let W be a G-module with an invariant scalar product and $V_0 \subset W$ a K_0-module when we regard W as K_0-module by restriction of the representation. If we can decompose W as

$$W = V_0 \oplus \operatorname{Im} B_1 \oplus \cdots \oplus \operatorname{Im} B_n, \tag{6.1}$$

then (W, V_0) is said to have a *normal* decomposition (into K_0-modules). We sometimes denote V_0 by $\operatorname{Im} B_0$ in the normal decomposition.

Proposition 6.7. *If W is an irreducible G-module, then for any K_0-module $V_0 \subset W$ there exists a positive integer n such that*

$$W = V_0 \oplus \operatorname{Im} B_1 \oplus \cdots \oplus \operatorname{Im} B_n,$$

which is a normal decomposition of (W, V_0).

Proof. Since W is irreducible, the result follows. □

We have a modification of Lemma 4.2 in [7].

Proposition 6.8. *Let W be a G-module with an invariant scalar product $(\cdot,\cdot)$ and V_0 a K_0-subspace of W when W is regarded as K_0-module by restriction of the representation. Suppose that (W, V_0) has a normal decomposition* (6.1). *Consider a K_0-irreducible decomposition of $\operatorname{Im} B_i$ for each $i = 0, \cdots, n$. Assume that $\operatorname{Im} B_i$ and $\operatorname{Im} B_j$ have no common irreducible K_0-submodules in the K_0-irreducible decomposition of them, if $i \neq j = 0, \cdots, n$.*

If T is a K_0-equivariant positive semi-definite Hermitian endomorphism on W satisfying $(Tgv_1, Tgv_2) = (v_1, v_2)$ for arbitrary $g \in G$ and v_1, $v_2 \in V_0$, then $T = Id_W$.

Proof. Since $\operatorname{Im} B_i$ and $\operatorname{Im} B_j$ has no common irreducible K_0-submodules when $i \neq j$, Schur's lemma yields that only the zero map is a K_0-equivariant linear map from $\operatorname{Im} B_i$ to $\operatorname{Im} B_j$. Hence we have that $T\operatorname{Im} B_i \subset \operatorname{Im} B_i$, where $i = 0, \cdots, n$.

Combining $T\operatorname{Im} B_0 \subset \operatorname{Im} B_0$ with the assumption $(Tgv_1, Tgv_2) = (v_1, v_2)$, we can recognize $T|_{V_0}$ as a positive semi-definite Hermitian endomorphism preserving the induced scalar product on V_0. We thus deduce that $T|_{V_0} = Id_{V_0}$.

From the hypothesis, it follows that for an arbitrary $\xi \in \mathfrak{m}$

$$(Tge^{t\xi}v_1, Tge^{t\xi}v_2) = (v_1, v_2),$$

where $t \in \mathbf{R}$ and v_1, $v_2 \in V_0$. Then we get

$$0 = \frac{d^{2p}}{dt^{2p}}\Big|_{t=0}(Te^{t\xi}v_1, Te^{t\xi}v_2) = \sum_{r=0}^{2p}\binom{2p}{r}(T\xi^r v_1, T\xi^{2p-r}v_2),$$

and so,

$$\begin{aligned}(6.2)\qquad &\binom{2p}{p}(T\xi^p v_1, T\xi^p v_2)\\ &= -\sum_{r=0}^{p-1}\binom{2p}{r}(T\xi^r v_1, T\xi^{2p-r}v_2) - \sum_{r=p+1}^{2p}\binom{2p}{r}(T\xi^r v_1, T\xi^{2p-r}v_2)\\ &= -\sum_{r=0}^{p-1}\binom{2p}{r}(T\xi^r v_1, T\xi^{2p-r}v_2) - \sum_{r=0}^{p-1}\binom{2p}{r}(T\xi^{2p-r} v_1, T\xi^{r}v_2).\end{aligned}$$

Suppose that T is the identity on $V_0 \oplus \operatorname{Im} B_1 \oplus \cdots \oplus \operatorname{Im} B_{p-1}$. From the inductive hypothesis and the condition that T is a Hermitian endomorphism, when $r < p$, we can deduce that

$$(T\xi^r v_1, T\xi^{2p-r}v_2) = (\xi^r v_1, T\xi^{2p-r}v_2) = (T\xi^r v_1, \xi^{2p-r}v_2) = (\xi^r v_1, \xi^{2p-r}v_2).$$

Consequently, it follows from (6.2) that $(T\xi^p v_1, T\xi^p v_2) = (\xi^p v_1, \xi^p v_2)$. Hence $T|_{\operatorname{Im} B_p}$ is a positive semi-definite Hermitian endomorphism preserving the scalar product and we can derive $T|_{\operatorname{Im} B_p} = Id_{\operatorname{Im} B_p}$. □

Since we would like to consider an orthogonal direct sum of r-copies of irreducible module, let $\tilde{W}$ denote a G-module and $\tilde{V}_0 \subset \tilde{W}$ a K_0-module, respectively. Suppose that the K_0-module $H(\tilde{V}_0, \tilde{V}_0)$ is decomposed into K_0-irreducible modules of the form:

$$H(\tilde{V}_0, \tilde{V}_0) = \oplus_{i=1}^{l} H_i,$$

where H_0 denotes the one-dimensional trivial representation corresponding to the identity transformation of $\tilde{V}_0$. Let GH_0 be the subspace of $\mathrm{H}(\tilde{W})$ generated by G and H_0.

A G-irreducible representation is said to be a *class one representation* of (G, K_0), if it contains non-zero K_0-invariant elements. In a similar way to the proof of Lemma 4.4 in [7], we can show

Proposition 6.9. *Let W be an irreducible G-module and $V_0 \subset W$ a K_0-submodule. Suppose that* $\operatorname{Im} B_i$ *and* $\operatorname{Im} B_j$ *in the normal decomposition* (6.1) *of (W, V_0) have no common K_0-irreducible submodules, if $i \neq j$.*

Let $\tilde{W}$ be an orthogonal direct sum of r-copies of W and $\tilde{V}_0$ an orthogonal direct sum of r-copies of V_0 which is regarded as a subspace of $\tilde{W}$ in a natural way.

Then $GH_0 \subset \mathrm{H}(\tilde{W})$ consists of class one submodules of (G, K_0). An arbitrary class one representations of (G, K_0) in $\mathrm{H}(\tilde{W})$ is a submodule of $GH(\tilde{V}_0, \tilde{V}_0)$.

Proof. First of all, we take an orthogonal decomposition of GH_0 into G-irreducible modules:

$$GH_0 = \oplus_p W_p.$$

The orthogonal projection is denoted by $\pi_p : GH_0 \to W_p$, for each p, which is a G-equivariant map. Then $\pi_p(H_0) \neq \{0\}$ for an arbitrary p, by definition of GH_0 and so, W_p is a class one representation since $\pi_p : GH_0 \to W_p$ is also K_0-equivariant.

Next, suppose that H is a class one subrepresentation of (G, K_0) in $\mathrm{H}(\tilde{W})$ such that $H \cap GH(\tilde{V}_0, \tilde{V}_0) = \{0\}$. Then, by standard arguments, we can assume that $H \perp GH(\tilde{V}_0, \tilde{V}_0)$ without loss of generality.

Since H is a class one representation, there exists a non-zero $C \in H$ such that $kCk^{-1} = C$ for any $k \in K_0$. It follows from $H \perp GH(\tilde{V}_0, \tilde{V}_0)$ that

$$0 = (C, gH(v_1, v_2))_H = (C, H(gv_1, gv_2))_H = \frac{1}{2}\left\{(Cgv_1, gv_2) + (Cgv_2, gv_1)\right\},$$

for arbitrary $g \in G$ and $v_1,\ v_2 \in V_0$. Thus we get

$$0 = (Cgv_1, gv_2), \quad g \in G,\ v_1, v_2 \in V_0.$$

If C is sufficiently small, then $Id + C > 0$ and so, we can define a positive Hermitian endomorphism T satisfying $T^2 = Id + C$. Then we have

$$(Tgv_1, Tgv_2) = (v_1, v_2) \quad g \in G,\ v_1, v_2 \in V_0.$$

Since C is K_0-equivariant, so is T. Since $(\tilde{W}, \tilde{V}_0)$ has the normal decomposition $\tilde{W} = \tilde{V}_0 \oplus \operatorname{Im}\tilde{B}_1 \oplus \cdots \oplus \operatorname{Im}\tilde{B}_n$ induced from that of (W, V_0) and T satisfies $T\operatorname{Im}\tilde{B}_i \subset \operatorname{Im}\tilde{B}_i$ for each i from the hypothesis, Lemma 6.8 yields that $T = Id$ and so, $C = 0$, which is a contradiction. $\square$

We use Proposition 6.9 to obtain a rigidity result on harmonic maps. To do so, let $\oplus_r V \to G/K_0$ be the orthogonal direct sum of r-copies of the homogeneous vector bundle $V \to G/K_0$ with the canonical connection. Then we can induce an invariant connection on $\oplus_r V \to G/K_0$ from the canonical connection, which is also called the canonical connection.

Theorem 6.10. *Let G/K_0 be a compact reductive Riemannian homogeneous space and $V = G \times_{K_0} V_0$ a homogeneous vector bundle with an invariant metric and the canonical connection, where V_0 is an irreducible K_0-module.*

Suppose that the eigenspace W with an eigenvalue μ of the Laplacian acting on $\Gamma(V)$ is an irreducible representation of G. We realize V_0 as a subspace of W as in Lemma 5.34. Furthermore, W is supposed to satisfy
(1) $\operatorname{Im} B_i$ *and* $\operatorname{Im} B_j$ *in the normal decomposition* (6.1) *of* (W, V_0) *has no common K_0-irreducible submodules, if $i \neq j$, and*
(2) *any G-irreducible submodule of* $\mathrm{H}(W)$ *is a class one representation of* (G, K_0).

We denote by $\tilde{W}$ the orthogonal direct sum of r-copies of W and by $\tilde{V} = \oplus_r V \to G/K_0$ the orthogonal direct sum of r-copies of vector bundle V with the induced invariant metric h and the canonical connection ∇.

Then the standard map by $(\tilde{V}, \tilde{W})$ is the unique Einstein-Hermitian full harmonic map with an EH constant $-\mu$ satisfying the gauge condition for $(\tilde{V}, h, \nabla)$ up to gauge equivalence.

Proof. It follows from Frobenius reciprocity and irreducibility of V_0 that W globally generates V and so, $\tilde{W}$ does $\tilde{V}$. From condition (1) and the definition of the normal decomposition, Lemma 5.36 yields that the standard map is the desired map.

If f is not the standard map, from Theorems 5.12 and 5.37, we have a positive semi-definite endomorphism T of $\tilde{W}$ satisfying

$$\left(T^2 - Id_{\tilde{W}}, GH(\oplus_r V_0, \oplus_r V_0)\right)_H = 0.$$

On the other hand, by Proposition 6.9, the conditions (1) and (2) imply that $\mathrm{H}(\tilde{W}) = GH(\oplus_r V_0, \oplus_r V_0)$. Thus $T^2 = Id_{\tilde{W}}$, which means that f is the standard map. Hence we have a contradiction. □

6.2. **Complex projective spaces.** We introduce two theorems which are proved in independent ways. A unified proof can be given in the light of Theorems 5.37 and 6.10.

The following theorem is shown independently by S.Bando-Y.Ohnita [2], J.Bolton-G.R.Jensen-M.Rigoli-L.M.Woodward [3] and Y.Ohnita [28].

Theorem 6.11. *Let $f : \mathbf{C}P^1 \to \mathbf{C}P^n$ be a full harmonic map with constant energy density and a constant Kähler angle. Then f is the standard map up to gauge equivalence.*

We have a rigidity theorem of holomorphic isometric embeddings between complex projective spaces by Calabi [5].

Theorem 6.12. *Let $f : \mathbf{C}P^m \to \mathbf{C}P^n$ be a full holomorphic map with constant energy density. Then f is the standard map up to gauge equivalence.*

Remark. Since we can classify those maps up to *gauge* equivalence, our results are slightly stronger than the previous results. However our claims are essentially the same as theirs by Proposition 5.30. Notice that Theorem 6.12 is indeed slightly stronger than the Calabi's rigidity on holomorphic *isometric* embeddings of $\mathbf{C}P^m$ into $\mathbf{C}P^n$.

Before giving a proof, we fix notation used throughout this section. First of all, we begin with standard representation theory of SU(2) treated in the Remark after Lemma 5.36. For an irreducible representation $S^k\mathbf{C}^2$ of SU(2), let

$$S^k\mathbf{C}^2 = \mathbf{C}_k \oplus \mathbf{C}_{k-2} \oplus \cdots \oplus \mathbf{C}_{-(k-2)} \oplus \mathbf{C}_{-k} \tag{6.3}$$

be a weight decomposition with respect to a standard diagonal subgroup U(1), where $\mathbf{C}_l$ is an irreducible representation of U(1) with weight l. We consider a symmetric pair $(\mathrm{SU}(2), \mathrm{U}(1))$ and the holomorphic line bundle $\mathcal{O}(k) \to \mathbf{C}P^1$ with the canonical connection, which is regarded as a homogeneous bundle $\mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathbf{C}_{-k}$. Using the theory of spherical harmonics, we have a decomposition of $\Gamma(\mathcal{O}(k))$ in the L^2-sense:

$$\Gamma(\mathcal{O}(k)) = \sum_{l=0}^{\infty} S^{|k|+2l}\mathbf{C}^2. \tag{6.4}$$

Moreover, $S^{|k|+2l}\mathbf{C}^2$ is an eigenspace of the Laplacian induced by the canonical connection. The Bott-Borel-Weil theorem yields that $H^0\left(\mathbf{C}P^1; \mathcal{O}(k)\right) \cong S^k\mathbf{C}^2$, if $k \geqq 0$.

We have an analogous theory for a symmetric pair $(\mathrm{SU}(m+1), \mathrm{U}(m))$, where an element in the isotropy subgroup $\mathrm{U}(m)$ is of the form:

$$\begin{pmatrix} |A| & O \\ O & A^{-1} \end{pmatrix}, \quad A \in \mathrm{U}(m).$$

Let $\mathcal{H}_n^{k,l}$ be the complex vector space of harmonic polynomials on $\mathbf{C}^n$ of bi-degree (k,l) which is an irreducible representation of $\mathrm{SU}(n)$. In particular, the space $\mathcal{H}_{m+1}^{k,0}$ of holomorphic polynomials has the following irreducible decomposition as $\mathrm{U}(m)$-module:

$$\mathcal{H}_{m+1}^{k,0} = \bigoplus_{p=0}^{k} \mathbf{C}_{-k+p} \otimes \mathcal{H}_m^{p,0}. \tag{6.5}$$

Here $\mathbf{C}_l$ denotes an irreducible one-dimensional representation (the determinant representation) of $\mathrm{U}(m)$ with weight l. Let $\mathcal{O}(k) \to \mathbf{C}P^m$ be a holomorphic line bundle of degree k, which is regarded as a homogeneous bundle $\mathrm{SU}(m+1) \times_{\mathrm{U}(m)} \mathbf{C}_{-k}$ with the canonical connection. Analogously, we have an irreducible decomposition of $\Gamma(\mathcal{O}(k))$ in the L^2-sense:

$$\Gamma(\mathcal{O}(k)) = \begin{cases} \sum_{l=0}^{\infty} \mathcal{H}_{m+1}^{k+l,l}, & k \geqq 0, \\ \sum_{l=0}^{\infty} \mathcal{H}_{m+1}^{l,|k|+l}, & k \leqq 0. \end{cases} \tag{6.6}$$

Furthermore, each representation space appeared in the decomposition is an eigenspace of the Laplacian induced by the canonical connection. It follows from Bott-Borel-Weil that $H^0(\mathbf{C}P^m; \mathcal{O}(k)) \cong \mathcal{H}_{m+1}^{k,0}$, if $k \geqq 0$.

We can easily see from (2.3) and (4.1) that the curvature of the canonical connection on $\mathcal{O}(k) \to \mathbf{C}P^m$ is the Kähler form on $\mathbf{C}P^m$ up to a constant multiple and so, it is a Hermitian Yang-Mills connection. Since any line bundle over a compact Kähler manifold is a stable vector bundle, the canonical connection on $\mathcal{O}(k) \to \mathbf{C}P^m$ is the unique Hermitian Yang-Mills connection modulo gauge equivalence.

Proof of Theorems 6.11 and 6.12 . We regard $\mathbf{C}P^n$ as a complex Grassmannian $Gr_n(\mathbf{C}^{n+1}) = \mathbf{P}(\mathbf{C}^{n+1^*})$ in both cases. Then the universal quotient bundle $Q \to Gr_n(\mathbf{C}^{n+1})$ is of rank 1.

Let f be a harmonic map from $\mathbf{C}P^1$ to $Gr_n(\mathbf{C}^{n+1})$ with constant energy density and constant Kähler angle or a holomorphic map from $\mathbf{C}P^m \to Gr_n(\mathbf{C}^{n+1})$ with constant energy density. Since $f^*Q \to \mathbf{C}P^m$ is of rank 1 and f has constant energy density, f is an Einstein-Hermitian map (see the second Remark after the proof of Theorem 5.12).

In case of $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+1})$, since every harmonic map of $\mathbf{C}P^1$ is conformal (see, for example, [10]) and f has a constant energy density, we can deduce that f is a homothety. Combining this with the assumption that f has a constant Kähler angle, we see that the pull-back of the Kähler form on $Gr_n(\mathbf{C}^{n+1})$ is a constant multiple of the Kähler form on $\mathbf{C}P^1$. Thus the pull-back connection on $f^*Q \to \mathbf{C}P^1$ is the canonical connection.

When $f : \mathbf{C}P^m \to Gr_n(\mathbf{C}^{n+1})$ is an EH holomorphic map, Proposition 4.3 yields that $f^*Q \to \mathbf{C}P^m$ is an Einstein-Hermitian vector bundle. We thus have the canonical connection as the induced connection on the pull-back of the universal quotient bundle.

Therefore each f is an Einstein-Hermitian harmonic or holomorphic map with the gauge condition for the canonical connection on a line bundle.

Suppose that the pull-back bundle is a holomorphic line bundle $\mathcal{O}(k) \to \mathbf{C}P^m$ with the canonical connection. It follows from (6.4) and (6.6) that $\mathbf{C}^{n+1} \subset S^{|k|+2l}\mathbf{C}^2$ for some non-negative integer l in the former case and $\mathbf{C}^{n+1} \subset \mathcal{H}^{k,0}$ in the latter case. The relevant representation space is denoted by W. In each case, the $\mathrm{U}(m)$-decomposition of W is regarded as the normal decomposition of $(W, \mathbf{C}_{-k})$ from Proposition 6.7. Then, it follows from (6.3) and (6.5) that $\operatorname{Im} B_i$ and $\operatorname{Im} B_j$ in the normal decomposition of (W, V_0) has no common K_0-irreducible submodules, if $i \neq j$.

The representation $\mathrm{H}(W)$ has an irreducible decomposition as follows:

$$\mathrm{H}(W) = \begin{cases} \oplus_{i=0}^{2|k|+4l} S^{2|k|+4l-2i}\mathbf{C}^2 \\ \oplus_{l=0}^{k} \mathcal{H}_{n+1}^{k-l,k-l} \end{cases}.$$

(Though we must take an invariant real vector space in the decomposition, we omit it.) Since all representations appeared in the decomposition of $\mathrm{H}(W)$ are class one representations, Theorem 6.10 implies that the standard maps by $(\mathcal{O}(k) \to \mathbf{C}P^m, W)$ are the desired maps and f itself is the standard map up to gauge equivalence of maps. □

As a result, those maps are isometric immersions up to a constant multiple of the metric.

Remark. Using a technique of [2] and [3], we can conclude that any harmonic map from $\mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+1}) = \mathbf{P}(\mathbf{C}^{n+1^*})$ with constant energy density *has* a constant Kähler angle.

Remark. We can see from Theorems 6.11 and 6.12 that the standard map is the unique representative in each indicated homotopy class of maps into complex projective spaces, which is the Einstein-Hermitian harmonic or holomorphic map with the pull-back connection being a Hermitian Yang-Mills connection.

Toth gives a conception of *polynomial minimal immersion* between complex projective spaces [34]. In the definition of polynomial minimal immersions, Toth makes use of $\mathcal{H}_{n+1}^{k,l}$ to define *polynomial maps* between spheres and uses the Hopf fibration to get a map between complex projective spaces. This enables us to apply $\mathrm{U}(n+1)$-representation theory instead of $\mathrm{SO}(2n+2)$-representation theory in the original do Carmo-Wallach theory. In addition, Toth implicitly requires the gauge condition for the canonical connection on a line bundle as *horizontality.* Theorem 3.5 implies that the notion of polynomial maps is unnecessary to develop the theory. We replace a polynomial minimal immersion by polynomial harmonic map with constant energy density.

Lemma 6.13. *Let $f : \mathbf{C}P^m \to \mathbf{C}P^n$ $(m \geqq 2)$ be a full harmonic map with constant energy density. Then f is a polynomial harmonic map in the sense of Toth if and only if f satisfies the gauge condition for a complex line bundle with the canonical connection.*

Proof. Since the universal quotient bundle is of rank 1 and f has constant energy density, the mean curvature operator is proportional to the identity up to constant (see the second Remark after the proof of Theorem 5.12).

The sufficient condition holds by definition of polynomial harmonic map (see condition (3) (horizontality) in [34]).

Suppose that the pull-back bundle of the universal quotient bundle with the induced connection is isomorphic to a complex line bundle with the canonical connection. Then Theorem 3.5 and the decomposition (6.6) imply that $\mathbf{C}^{n+1} \subset \mathcal{H}^{k,l}_{m+1}$ for some non-negative integers k and l. This yields that f is a polynomial harmonic map. $\square$

Toth also gives an estimate of the dimension of the moduli space by the image equivalence relation in [34]. Lemma 5.36 and the weight decomposition yields that the standard map is a harmonic map with a constant energy density. Then Theorem 5.31 provides the description of the moduli space, in which M is $\mathbf{C}P^m$, A is a constant function and ∇ is the canonical connection on a line bundle. Toth estimates the dimension of $\mathrm{M}_0(\mathcal{H}^{k,l}_{m+1})$ using representation theory of $\mathrm{U}(m+1)$.

As a generalization of Theorem 6.12, we will have the rigidity of a special class of Einstein-Hermitian holomorphic embeddings of a compact Hermitian symmetric space.

Let (G, K_0) be an irreducible Hermitian symmetric pair of compact type with the corresponding orthogonal decomposition of Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Moreover, since $\mathfrak{k}$ has a center $\mathfrak{u}(1)$, $\mathfrak{k}$ is decomposed into $\mathfrak{k} = \mathfrak{u}(1) \oplus \mathfrak{k}_1$. The Lie subgroups corresponding to $\mathfrak{u}(1)$ and $\mathfrak{k}_1$ are denoted by $\mathrm{U}(1)$ and K_1, respectively.

Any irreducible homogeneous vector bundle on $M := G/K_0$ is expressed as $V := G \times_{K_0} (V_1 \otimes \mathbf{C}_k) \to M$, where V_1 is an irreducible representation of $\mathfrak{k}_1$ and $\mathbf{C}_k$ is a one-dimensional representation of $\mathfrak{u}(1)$ with weight k. Then $\mathfrak{m}^{\mathbf{C}}$, which is the complexification of $\mathfrak{m}$, is regarded as $(\mathbf{C}_{-l} \otimes T_1^*) \oplus (\mathbf{C}_l \otimes T_1)$. Here l is a positive integer and T_1 is an appropriate non-trivial irreducible representation space of $\mathfrak{k}_1$, if $\mathfrak{k}_1 \neq \{0\}$. If we denote by $\mathfrak{m}_{(0,1)}$ the set of tangent vectors of type $(0,1)$, then we have $\mathfrak{m}_{(0,1)} = \mathbf{C}_l \otimes T_1$ in the decomposition.

It follows from [18, p.121 Proposition (6.2)] and the uniqueness of the Einstein-Hermitian structure that the canonical connection is the unique Hermitian Yang-Mills connection on any irreducible homogeneous vector bundle $V \to G/K_0$.

Theorem 6.14. *Let (G, K_0) be an irreducible Hermitian symmetric pair of compact type and $V \to G/K_0$ an irreducible holomorphic homogeneous vector bundle. We denote by W the space of holomorphic sections of $V \to G/K_0$ with the L^2 Hermitian inner product up to constant multiple. Assume that any G-irreducible submodule of* $\mathrm{H}(W)$ *is a class one representation of (G, K_0).*

We denote by $\oplus_r V$ the orthogonal direct sum of r-copies of the bundle $V \to G/K_0$ and by $\oplus_r W$ the orthogonal direct sum of r-copies of W.

If f is an Einstein-Hermitian full holomorphic embedding of G/K_0 into a complex Grassmannian with the pull-back bundle of the universal quotient *bundle being holomorphically isomorphic to a direct sum of r-copies of $V \to G/K_0$, then f is the standard map by $(\oplus_r V, \oplus_r W)$ up to gauge equivalence.*

Proof. Let $V_0 = \mathbf{C}_{-m} \otimes V_1$ be an irreducible complex module of K_0 associated with $V \to G/K_0$, where m is an integer and V_1 is an irreducible representation of $\mathfrak{k}_1$. If f is an embedding into $Gr_p(\mathbf{C}^N)$, Theorem 3.5 yields that $\mathbf{C}^N$ is a subspace of a direct sum of r-copies of W and V has non-trivial holomorphic sections. Then Bott-Borel-Weil theorem implies that m is positive and (ϱ, W) is an irreducible G-representation. By Frobenius reciprocity and Lemma 5.34, V_0 is realized as a subspace of W. Then Proposition 6.7 yields that (W, V_0) has a normal decomposition (6.1).

Since the second fundamental form K is of type $(0,1)$, we get $\varrho(\mathfrak{m})V_0 = \varrho(\mathfrak{m}_{(0,1)})V_0$ and

$$\varrho(\mathfrak{m}_{(1,0)})V_0 = \{0\}. \tag{6.7}$$

From lemma 6.5 and (6.7), B_n can be regarded as a map from $S^n\mathfrak{m}_{(0,1)} \otimes V_0$ to N_n (see Definition 6.4). Since $\mathfrak{m}_{(0,1)} = \mathbf{C}_l \otimes T_1$, we see that $S^n\mathfrak{m}_{(0,1)} = \mathbf{C}_{nl} \otimes S^n T_1$. Then we deduce that $\operatorname{Im} B_i$ is a submodule of $\mathbf{C}_{-m+il} \otimes U_i$, where U_i is a representation of $\mathfrak{k}_1$ $(i \geqq 1)$. It follows that $\operatorname{Im} B_i$ and $\operatorname{Im} B_j$ has no common K_0-irreducible submodules, if $i \neq j$.

Since f is an EH holomorphic embedding, Proposition 4.3 yields that $f^*Q \to G/K_0$, the pull-back bundle of the universal quotient bundle, is an Einstein-Hermitian vector bundle. From [18, p.177 Theorem (8.3)], we can see that $f^*Q \to G/K_0$ is an orthogonal direct sum of holonomy irreducible Einstein-Hermitian vector bundles, say $\oplus_i(E_i, h_i)$. By the assumption that $f^*Q \to G/K_0$ is holomorphically isomorphic to a direct sum $\oplus_r V \to G/K_0$, we have a non-trivial bundle map $E_i \to V$ for each i. Since V and E_i are holonomy irreducible, we can deduce from [18, p.101 Proposition (1.7)] that V is holomorphically isomorphic to E_i. Thus the induced connection on V is the Hermitian Yang-Mills connection. As we already stated, the canonical connection is the unique Hermitian Yang-Mills connection on $V \to G/K_0$. Consequently, we can conclude that f satisfies the gauge condition for an orthogonal direct sum $\oplus_r V \to G/K_0$ with the induced invariant metric and the canonical connection.

Hence, Theorem 6.10 with the assumption on $\mathrm{H}(W)$ having only class one submodules yields the result. □

We give examples of pairs (G, K_0) and G-representation spaces W such that $\mathrm{H}(W)$ consists of class one representations.

Theorem 6.15. *Let G/K_0 be a compact simply-connected homogeneous Kähler manifold, where G is a compact connected semisimple Lie group, and $L \to G/K_0$ a holomorphic line bundle. We denote by W the space of holomorphic sections of $L \to G/K_0$. If $L \to G/K_0$ is a positive line bundle, then* $\operatorname{End}(W)$ *consists of class one representations of (G, K_0).*

Proof. It is well-known that every holomorphic line bundle on G/K_0 is a homogeneous vector bundle. Then, by Bott-Borel-Weil theorem, W is an irreducible representation space of G. The Kähler form may be chosen in $2\pi c_1(L)$ from the positivity of the line bundle. Since $L \to G/K_0$ is a line bundle on a Kähler manifold, $L \to G/K_0$ has a unique Einstein-Hermitian connection ∇. We adopt the curvature form Ω of ∇ as the Kähler form ω on G/K_0: $\omega = \sqrt{-1}\Omega$.

Let $f_0 : G/K_0 \to \mathbf{P}(W^*)$ be the standard map. We fix a G-invariant Hermitian structure h on $L \to G/K_0$ and equip W with a G-invariant Hermitian inner product $(\cdot,\cdot)_W$. Since f_0 is G-equivariant holomorphic map and $L \to G/K_0$ is of rank one, the mean curvature operator A can be regarded as a negative constant from Lemma 3.3. Then Proposition 4.3 yields that the pull-back connection on $f^*\mathcal{O}(1) \to G/K_0$ is the Hermitian Yang-Mills connection ∇ on $L \to G/K_0$. In particular, since the pull-back metric is a G-invariant metric on $L \to G/K_0$ by G-equivariance of f_0, we can assume that h coincides with the pull-back metric. Thus f_0 can be regarded as a holomorphic isometric embedding.

Let V_0 be a complex 1-dimensional representation of K_0 to which the associated bundle is $L \to G/K_0$. By Lemma 5.34, we realize V_0 as a subspace of W and take a unit vector $v_0 \in V_0 \subset W$. Let $\mathrm{H}_0(W)$ denote the set of trace-free Hermitian endomorphisms on W. Then, for $C \in \mathrm{H}_0(W)$, we define a real valued function $f_C : G/K_0 \to \mathbf{R}$ as

$$f_C([g]) := (Cgv_0, gv_0)_W. \tag{6.8}$$

It follows that the correspondence $C \mapsto f_C$ gives a G-equivariant homomorphism $F : \mathrm{H}_0(W) \to C^\infty(G/K_0)$, where $C^\infty(G/K_0)$ is the space of real valued smooth functions on G/K_0.

Suppose that $F(C) = 0$. If C is small enough, $Id + C$ is positive. Consequently, we can use $Id + C$ to define a full holomorphic map $f : G/K_0 \to \mathbf{P}(W^*)$ as $(Id + C)^{-\frac{1}{2}} f_0$. Then (6.8) gives $((Id + C)gv_0, gv_0)_W = 1$ and so, the pull-back metric by f coincides with h. Since the pull-back bundle by f is holomorphically isomorphic to $L \to G/K_0$, the uniqueness of the Hermitian connection implies that the pull-back connection also coincides with ∇. From the definition of the Kähler form, f also turns out to be a holomorphic isometric embedding. From Calabi's rigidity theorem [5], f must be image equivalent to f_0. Proposition 5.30 implies that f and f_0 are also gauge equivalent. It follows from Theorem 5.12 that $C = 0$. Thus $\mathrm{H}_0(W)$ can be regarded as G-submodule of $C^\infty(G/K_0)$.

Since every irreducible submodule of $C^\infty(G/K_0)$ is a class one representation of (G, K_0) and $\mathrm{End}(W)$ is the complexification of $\mathrm{H}(W)$, we obtain the desired result. □

Theorems 6.14 and 6.15 yield

Corollary 6.16. *Let (G, K_0) be an irreducible Hermitian symmetric pair of compact type, $L \to G/K_0$ a positive holomorphic line bundle and W the space of holomorphic sections of $L \to G/K_0$. We denote by $\oplus_r L$ the orthogonal direct sum of r-copies of the line bundle $L \to G/K_0$ and by $\oplus_r W$ the orthogonal direct sum of r-copies of W.*

If f is an Einstein-Hermitian full holomorphic embedding of G/K_0 into a complex Grassmannian with the pull-back bundle of the universal quotient bundle being holomorphically isomorphic to a direct sum of r-copies of the line bundle $L \to G/K_0$, then f is the standard map by $(\oplus_r L, \oplus W_r)$ up to gauge equivalence.

6.3. **Complex quadrics.** We consider Einstein-Hermitian holomorphic embeddings of the projective line into complex quadrics $Gr_n(\mathbf{R}^{n+2})$. Such embeddings turn out to be holomorphic isometric embeddings up to constant multiples of the metrics. Though research on harmonic maps from the projective line into quadrics has been approached from various viewpoints (for example, [6], [13], [21], [38] and [40]), we would like to describe the moduli space by a generalization of do Carmo-Wallach theory (Corollaries 5.25 and 5.33 and Theorem 5.37).

Remark. We need to fix an orientation of $\mathbf{R}^{n+2}$ to define a complex structure of $Gr_n(\mathbf{R}^{n+2})$. When we use notation in §5.1, the inversion $\tau : Gr_n(\mathbf{R}^{n+2}_+) \to Gr_n(\mathbf{R}^{n+2}_-)$ is a *holomorphic* isometry. In addition, τ does not belong to $\mathrm{O}(n)$, if n is even. Following our convention, we do not distinguish a map $f : M \to Gr_n(\mathbf{R}^{n+2})$ from a map $\tau \circ f : M \to Gr_n(\mathbf{R}^{n+2})$. In particular, two standard maps f_0 and $\tau \circ f_0$ are identified.

The curvature form R of the canonical connection on the universal quotient bundle is related to the fundamental 2-form ω_Q on $Gr_n(\mathbf{R}^{n+2})$ in such a way that $R = -\sqrt{-1}\omega_Q$. Denote by ω_0 the fundamental 2-form on $\mathbf{C}P^1$ satisfying $R_{\mathcal{O}(1)} = -\sqrt{-1}\omega_0$, where $R_{\mathcal{O}(1)}$ is the curvature form of the canonical connection on the hyperplane bundle over $\mathbf{C}P^1$.

Definition 6.17. Let $f : \mathbf{C}P^1 \to Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is called an isometric embedding of degree k if $f^*\omega_Q = k\omega_0$ (and so, k must be a positive integer).

Lemma 6.18. *Let $f : \mathbf{C}P^1 \to Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is an isometric embedding of degree k if and only if the induced connection on the pull-back of the universal quotient bundle is gauge equivalent to the canonical connection on $\mathcal{O}(k) \to \mathbf{C}P^1$. Under these conditions, f is an Einstein-Hermitian map.*

Proof. Since the holomorphic bundle structure of any line bundle on $\mathbf{C}P^1$ is unique, there exists a non-negative integer k such that $f^*Q \to \mathbf{C}P^1$ is holomorphically isomorphic to $\mathcal{O}(k) \to \mathbf{C}P^1$.

Since $f^*\omega_Q$ is the curvature form of the pull-back connection and $k\omega_0$ is that of the canonical connection on $\mathcal{O}(k) \to \mathbf{C}P^1$, it follows that f is an isometric embedding of degree k and so, $f^*\omega_Q = k\omega_0$, if and only if the pull-back connection is the canonical connection on $\mathcal{O}(k) \to \mathbf{C}P^1$.

Under these conditions, Proposition 4.3 yields that the mean curvature operator is considered as a complex endomorphism of $\mathcal{O}(k) \to \mathbf{C}P^1$. Since the canonical connection is the unique Hermitian Yang-Mills connection, the mean curvature operator is proportional to the identity on $\mathcal{O}(k) \to \mathbf{C}P^1$ and the same is true as a real endomorphism of $\mathcal{O}(k) \to \mathbf{C}P^1$. $\square$

Since the holonomy group of the canonical connection on any (positive) line bundle is irreducible, the pull-back fiber metric also coincides with the invariant fiber metric up to a positive constant multiple. If we change the inner product on $\mathbf{R}^{n+2}$ or the invariant metric on the line bundle by a positive constant multiple, then we can assume that the pull-back metric also coincides with the invariant metric from the beginning.

From this observation with Lemma 6.18, we can apply Theorem 5.37 to obtain the moduli space $\mathcal{M}_k$ of holomorphic isometric embeddings of degree k modulo gauge equivalence of maps. We also use the Hermitian Yang-Mills connection to obtain that any holomorphic section of $\mathcal{O}(k) \to \mathbf{C}P^1$ is an eigensection (Lemma 4.2).

Let $(\mathrm{SU}(2), \mathrm{U}(1))$ be the symmetric pair corresponding to $\mathbf{C}P^1$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ the corresponding orthogonal decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$, where $\mathfrak{k} = \mathfrak{u}(1)$. We notice that $\mathfrak{m}^{\mathbf{C}} = \mathbf{C}_2 \oplus \mathbf{C}_{-2}$.

When considering the standard map by $\left(\mathcal{O}(k), H^0\left(\mathbf{C}P^1; \mathcal{O}(k)\right)\right)$ into a *quadric*, we recognize $H^0\left(\mathbf{C}P^1; \mathcal{O}(k)\right)$ as a real vector space with the induced orientation by the complex structure. In general, by regarding a complex vector space $\mathbf{C}^N$ with a Hermitian inner product as a real vector space $\mathbf{R}^{2N}$ with the induced orientation and inner product, we obtain a totally geodesic embedding into an oriented Grassmannian $i : Gr_p(\mathbf{C}^N) \to Gr_{2p}(\mathbf{R}^{2N})$.

Lemma 6.19. *The standard map by* $\left(\mathcal{O}(k), H^0\left(\mathbf{C}P^1, \mathcal{O}(k)\right) \cong S^k\mathbf{C}^2\right)$ *is a holomorphic isometric embedding of degree* k *from the complex projective line into* $Gr_{2k}(\mathbf{R}^{2k+2})$.

Proof. Since $i : Gr_k\left(S^k\mathbf{C}^2\right) \to Gr_{2k}(\mathbf{R}^{2k+2})$ is a holomorphic totally geodesic embedding and the standard map in question is the composition of the standard map into $Gr_k\left(S^k\mathbf{C}^2\right)$ and i, Theorem 6.12 yields the result. $\square$

Notice that the L^2 inner product on $\mathbf{R}^{2k+2}$ is an SU(2)-invariant inner product. Thus, for the description of the moduli space, Corollary 5.25 is available. First of all, we consider the case that $k = 1$.

Theorem 6.20. *If* f *is a holomorphic isometric embedding of degree* 1 *from the complex projective line into a complex quadric, then* f *is the standard map by* $\left(\mathcal{O}(1) \to \mathbf{C}P^1, H^0\left(\mathbf{C}P^1; \mathcal{O}(1)\right)\right)$ *up to gauge equivalence.*

Proof. Let f be a holomorphic isometric embedding into $Gr_n(\mathbf{R}^{n+2})$ of degree 1. Then, Lemma 6.18 yields that f is an Einstein-Hermitian holomorphic embedding. Since f is of degree one and $H^0\left(\mathbf{C}P^1; \mathcal{O}(1)\right)$ is equivalent to the standard representation $\mathbf{C}^2$ as SU(2)-module by Bott-Borel-Weil theorem, Theorem 5.37 implies that $n + 2 \leqq 4$. Hence we consider an Einstein-Hermitian holomorphic embedding $f : \mathbf{C}P^1 \to Gr_2(\mathbf{R}^4)$: the maximal case. Since the Hermitian Yang-Mills connection is the canonical connection on $\mathcal{O}(1) \to \mathbf{C}P^1$, we can apply Theorem 5.37 to describe the moduli space.

Since $Gr_2(\mathbf{R}^4)$ is a real Grassmannian, we must regard $\mathbf{C}^2$ as a real vector space $\mathbf{R}^4$ with the complex structure J when applying Theorem 5.37. Consequently, $\mathrm{H}(\mathbf{R}^4)$ denotes the set of symmetric endomorphism on $\mathbf{R}^4$ in our convention. Since the standard map is an Einstein-Hermitian holomorphic map from Lemma 6.19, we only need to consider the equation

$$ev \circ C \circ \nabla ev^* = 0 \tag{6.9}$$

by Corollary 5.23, where $C \in \mathrm{H}_0(\mathbf{R}^4)$ which denotes the set of trace-free symmetric endomorphisms on $\mathbf{R}^4$.

Let $\mathbf{C}^2 = \mathbf{C}_1 \oplus \mathbf{C}_{-1}$ be the weight decomposition as U(1)-module. The line bundle $\mathcal{O}(1) \to \mathbf{C}P^1$ is expressed as $\mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathbf{C}_{-1}$ and we can identify the representation space $\mathbf{C}_{-1}$ of U(1) with the subspace denoted by the same symbol of the weight decomposition of $\mathbf{C}^2$ by Lemma 5.34. Then we can see from $\mathfrak{m}\mathbf{C}_{-1} = \mathbf{C}_1$ and Theorem 5.37 that (6.9) is equivalent to the condition that

$$(C, GH(\mathfrak{m}\mathbf{C}_{-1}, \mathbf{C}_{-1})) = (C, GH(\mathbf{C}_1, \mathbf{C}_{-1})) = 0,$$

where $(\cdot,\cdot)$ is the induced SU(2)-invariant inner product on $\mathrm{H}_0(\mathbf{R}^4)$.

We can irreducibly decompose an SU(2)-module $\mathrm{H}(\mathbf{R}^4)$ as:

$$\mathrm{H}(\mathbf{R}^4) = 3\mathbf{R}^3 \oplus \mathbf{R}, \tag{6.10}$$

and so $\mathrm{H}_0(\mathbf{R}^4) = 3\mathbf{R}^3$.

To apply Corollary 5.25, we need to understand the decomposition (6.10) in detail. To do so, let j be an SU(2)-invariant quaternion structure on $\mathbf{C}^2$, which is a conjugate-linear map such that $j^2 = -Id$. Then,

$$\mathbf{R}^3 = \sqrt{-1}\rho(\mathfrak{g}),\ \mathbf{R}^3 = j\sqrt{-1}\rho(\mathfrak{g}),\ \mathbf{R}^3 = Jj\sqrt{-1}\rho(\mathfrak{g}),$$

where $\rho : \mathfrak{su}(2) \to \mathrm{End}(\mathbf{C}^2)$ denotes the standard representation. Notice that $\sqrt{-1}\rho(\mathfrak{g})$ is the set of Hermitian endomorphisms on $(\mathbf{R}^4, J)$.

We take an orthonormal basis $\{v_1, Jv_1, v_{-1}, Jv_{-1}\}$ of $\mathbf{R}^4$ as $v_1, Jv_1 \in \mathbf{C}_1$, $v_{-1}, Jv_{-1} \in \mathbf{C}_{-1}$ and $jv_1 = v_{-1}$. Then, $H(\mathbf{C}_1, \mathbf{C}_{-1})$ is spanned by

$$H(v_1, v_{-1}), H(Jv_1, v_{-1}), H(v_1, Jv_{-1}) \text{ and } H(Jv_1, Jv_{-1}).$$

In general, we have $H(Ju, Jv) = -JH(u,v)J$ for arbitrary $u, v \in \mathbf{R}^4$. Hence $H(u,v) + H(Ju, Jv)$ is a Hermitian endomorphism on $(\mathbf{R}^4, J)$, because it commutes with J. In particular, $H(v_1, v_{-1}) + H(Jv_1, Jv_{-1})$ and $H(v_1, Jv_{-1}) - H(Jv_1, v_{-1})$ are Hermitian endomorphisms on $(\mathbf{R}^4, J)$. Thus

$$\sqrt{-1}\rho(\mathfrak{g}) \subset GH(\mathbf{C}_1, \mathbf{C}_{-1}).$$

Next, we see that

$$\begin{aligned} &2H(v_1, v_{-1}) - 2H(Jv_1, Jv_{-1}) \\ =&j\left\{H(v_1, v_1) - H(v_{-1}, v_{-1}) + H(Jv_1, Jv_1) - H(Jv_{-1}, Jv_{-1})\right\}, \end{aligned} \tag{6.11}$$

and so,

$$j\sqrt{-1}\rho(\mathfrak{g}) \subset GH(\mathbf{C}_1, \mathbf{C}_{-1}).$$

Finally, we get

$$H(v_1, Jv_{-1}) + H(Jv_1, v_{-1}) = J\left\{H(v_1, v_{-1}) - H(Jv_1, Jv_{-1})\right\}.$$

It follows from (6.11) that

$$Jj\sqrt{-1}\rho(\mathfrak{g}) \subset GH(\mathbf{C}_1, \mathbf{C}_{-1}).$$

Therefore, $\mathrm{H}_0(\mathbf{R}^4) = GH(\mathbf{C}_1, \mathbf{C}_{-1})$. Corollary 5.25 yields the result. $\square$

Next, we consider holomorphic isometric embeddings of degree 2. When the degree is even, say $2l$, $H^0\left(\mathbf{C}P^1; \mathcal{O}(2l)\right)$ has an invariant real subspace denoted by $W^l_{\mathbf{R}}$ of real dimension $2l+1$. Since $W^l_{\mathbf{R}}$ also globally generates $\mathcal{O}(2l) \to \mathbf{C}P^1$, we have a standard map by $W^l_{\mathbf{R}}$ which turns out to be a holomorphic isometric embedding of degree $2l$ by Lemma 5.36 and weight decomposition. We call the standard map by $\left(\mathcal{O}(2l) \to \mathbf{C}P^1, W^l_{\mathbf{R}}\right)$ the *real standard map*, which is a map into $Gr_{2l-1}(\mathbf{R}^{2l+1})$.

Theorem 6.21. *If $f : \mathbf{C}P^1 \to Gr_n(\mathbf{R}^{n+2})$ is a full holomorphic isometric embedding of degree 2 into a complex quadric, then $n \leqq 4$.*

When $\mathcal{M}_2$ denotes the moduli space of full holomorphic isometric embeddings of the complex projective line into $Gr_4(\mathbf{R}^6)$ of degree 2 modulo the gauge equivalence of maps, $\mathcal{M}_2$ is identified with an open unit disk in $\mathbf{C}$.

If we take $\overline{\mathcal{M}_2}$ as the compactification of $\mathcal{M}_2$ by the topology induced by L^2 scalar product on $\Gamma(\mathcal{O}(2))$, each boundary point of $\overline{\mathcal{M}_2}$ corresponds to a real standard map whose image is included in a totally geodesic submanifold $Gr_1(\mathbf{R}^3)$ of $Gr_4(\mathbf{R}^6)$. Each totally geodesic submanifold $Gr_1(\mathbf{R}^3)$ is specified as the common zero set of sections of the universal quotient bundle $Q \to Gr_4(\mathbf{R}^6)$, where these sections belong to the orthogonal complement of $\mathbf{R}^3$ in $\mathbf{R}^6$. The real standard map is a terminal harmonic map.

Proof. We use the same notation as in the proof of Theorem 6.20 and begin with a representation theory of SU(2) and U(1). Let $S^2\mathbf{C}^2$ be the complexification of the Lie algebra of SU(2) with a real structure σ which is an SU(2)-invariant conjugate-linear involution. A weight decomposition of $S^2\mathbf{C}^2$ is $\mathbf{C}_2 \oplus \mathbf{C}_0 \oplus \mathbf{C}_{-2}$. The associated line bundle $\mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathbf{C}_{-2}$ is a holomorphic line bundle $\mathcal{O}(2) \to \mathbf{C}P^1$ and $H^0\left(\mathbf{C}P^1; \mathcal{O}(2)\right)$ is identified with $S^2\mathbf{C}^2$ by Bott-Borel-Weil theorem. From Lemma 5.34, the representation space $\mathbf{C}_{-2}$ of U(1) is realized as a subspace of $S^2\mathbf{C}^2$ denoted by the same symbol. We can see that $\mathfrak{m}\mathbf{C}_{-2} = \mathbf{C}_0$.

Let $f : \mathbf{C}P^1 \to Gr_n(\mathbf{R}^{n+2})$ be a holomorphic isometric embedding of degree 2. Then Lemma 6.18 implies that f is an Einstein-Hermitian holomorphic embedding and so, $n \leqq 4$ by Theorem 5.37.

To apply Theorem 5.37, $S^2\mathbf{C}^2$ must be regarded as a real vector space $\mathbf{R}^6$ with the complex structure J. Let $\mathrm{H}(\mathbf{R}^6)$ be the set of symmetric endomorphisms on $\mathbf{R}^6$. Notice that $\mathrm{H}(\mathbf{R}^6, J)$, the set of Hermitian endomorphisms on $(\mathbf{R}^6, J)$, is a real subspace of $\mathrm{H}(\mathbf{R}^6)$. The Clebsh-Gordan formula yields that the complexification of $\mathrm{H}(\mathbf{R}^6, J)$ is decomposed as $S^4\mathbf{C}^2 \oplus S^2\mathbf{C}^2 \oplus \mathbf{C}$. Since these three spaces have invariant real structures, we denote by D_4, D_2 and $\mathbf{R}$ the corresponding real subspaces of $\mathrm{H}(\mathbf{R}^6, J)$, respectively. We claim that

$$\mathrm{H}(\mathbf{R}^6) = (D_4 \oplus \sigma D_4 \oplus J\sigma D_4) \oplus D_2 \oplus (\mathbf{R} \oplus \mathbf{R}\sigma \oplus \mathbf{R}J\sigma)\,. \tag{6.12}$$

We fix an orthonormal basis $\{v_2, v_0, v_{-2}, Jv_2, Jv_0, Jv_{-2}\}$ of $\mathbf{R}^6$ in such a way that $v_i \in \mathbf{C}_i$ and $\sigma(v_i) = v_{-i}$ $(i = 2, 0, -2)$. Using matrix representation and the block decomposition according to $\mathbf{R}^6 = \mathrm{Span}(v_2, v_0, v_{-2}) \oplus \mathrm{Span}(Jv_2, Jv_0, Jv_{-2})$, we have

$$D_4 = \left\{ \begin{pmatrix} D & O \\ O & D \end{pmatrix} \;\middle|\; {}^tD = D,\ \mathrm{trace}\, D = 0 \right\},$$

and so,

$$\sigma D_4 = \left\{ \begin{pmatrix} D & O \\ O & -D \end{pmatrix} \;\middle|\; {}^tD = D,\ \mathrm{trace}\, D = 0 \right\},$$

and

$$J\sigma D_4 = \left\{ \begin{pmatrix} O & D \\ D & O \end{pmatrix} \;\middle|\; {}^tD = D,\ \mathrm{trace}\, D = 0 \right\}.$$

Moreover, we have

$$D_2 = \left\{ \begin{pmatrix} O & -C \\ C & O \end{pmatrix} \,\middle|\, {}^tC = -C \right\}.$$

By the same reason (Corollary 5.23 and Lemma 6.19) as in the proof of Theorem 6.20, we need to specify $GH(\mathbf{C}_0, \mathbf{C}_{-2})$ as a subspace of $\mathrm{H}_0(\mathbf{R}^6)$ which is the set of trace-free symmetric endomorphisms on $\mathbf{R}^6$. From the definition, $H(\mathbf{C}_0, \mathbf{C}_{-2})$ is spanned by

$$H(v_0, v_{-2}), H(Jv_0, v_{-2}), H(v_0, Jv_{-2}) \,\text{and}\, H(Jv_0, Jv_{-2}).$$

The characterization of the decomposition of $\mathrm{H}_0(\mathbf{R}^6)$ yields that

$$H(v_0, v_{-2}) + H(Jv_0, Jv_{-2}) \in D_4, \quad H(Jv_0, v_{-2}) - H(v_0, Jv_{-2}) \in D_2,$$
$$H(v_0, v_{-2}) - H(Jv_0, Jv_{-2}) \in \sigma D_4, \quad H(Jv_0, v_{-2}) + H(v_0, Jv_{-2}) \in J\sigma D_4.$$

Thus $\mathbf{R}\sigma \oplus \mathbf{R}J\sigma$ is the orthogonal complement of $GH(\mathbf{C}_0, \mathbf{C}_{-2})$ in $\mathrm{H}_0(\mathbf{R}^6)$. Notice that the complex structure J on $\mathbf{R}^6$ gives a complex structure on $\mathbf{R}\sigma \oplus \mathbf{R}J\sigma$. From Corollary 5.25, the moduli space $\mathcal{M}_2$ can be regarded as a bounded connected convex open set in $\mathbf{R}\sigma \oplus \mathbf{R}J\sigma$. Indeed, a symmetric transformation $Id + (a\sigma + bJ\sigma)$ is positive, where $a, b \in \mathbf{R}$ if and only if $a^2 + b^2 < 1$. Thus $\mathcal{M}_2 = \left\{ z \in \mathbf{C} \,|\, |z|^2 < 1 \right\}$.

Next we consider a natural compactification $\overline{\mathcal{M}_2}$ of $\mathcal{M}_2$. Suppose that $a^2 + b^2 = 1$. Then $(a + bJ)\sigma$ is also an invariant real structure on $S^2\mathbf{C}^2$. Hence we may consider only the case that $a = 1$ and $b = 0$. Since the kernel of $Id+\sigma$ is $\mathrm{Span}(Jv_2, Jv_0, Jv_{-2})$, Lemma 5.11 and Theorem 5.12 imply that $Id+\sigma$ determines a totally geodesic submanifold $Gr_1(\mathbf{R}^3)$ of $Gr_4(\mathbf{R}^6)$ and a holomorphic isometric embedding into the submanifold $Gr_1(\mathbf{R}^3)$ represented by $2Id_3$. This map is nothing but a real standard map by $\mathbf{R}^3 = \left(S^2\mathbf{C}^2\right)_{\mathbf{R}}$ which is an invariant real subspace of $S^2\mathbf{C}^2$. Finally, since the standard map corresponding to Id is a maximal holomorphic map, $\overline{\mathcal{M}_2}$ parametrizes all holomorphic isometric embeddings of degree 2 (see Corollary 5.25 and the Remark after it). Therefore Corollary 5.27 yields that the real standard map is a terminal harmonic map. □

Since $Id + (a\sigma + bJ\sigma)$ is invariant under the $\mathrm{SU}(2)$-action, we can deduce that all holomorphic isometric embeddings of $\mathbf{C}P^1$ into $Gr_4(\mathbf{R}^6)$ of degree 2 are $\mathrm{SU}(2)$-equivariant. This is a result of [13].

Next, we consider the image equivalence of maps. The holonomy group of the canonical connection on $\mathcal{O}(2) \to \mathbf{C}P^1$ is the structure group $\mathrm{U}(1)$ of the bundle. Therefore the centralizer of the holonomy group is also the structure group $S^1 = \mathrm{U}(1)$. Then Theorem 6.21 with Corollary 5.33 yields the result by [6] and [21].

Theorem 6.22. *Let $\mathbf{M}_2$ be the moduli space of holomorphic isometric embeddings of degree 2 from the complex projective line into $Gr_4(\mathbf{R}^6)$ modulo the image equivalence. Then $\mathbf{M}_2 = \overline{\mathcal{M}_2}/S^1 = [0, 1]$; 0 (resp. 1) represents the standard map (resp. the real standard map).*

Proof. From Lemma 6.18, the centralizer S^1 of the holonomy group of ∇ preserves the mean curvature operator of any holomorphic isometric embedding. From Lemma 6.19, we can apply Corollary 5.33 to conclude that S^1 acts on $\overline{\mathcal{M}_2}$ and $\mathbf{M}_2 = \overline{\mathcal{M}_2}/S^1$.

To be more precise, let J_L be the complex structure of $\mathcal{O}(2) \to \mathbf{C}P^1$ and J the induced holomorphic structure of $H^0\left(\mathbf{C}P^1; \mathcal{O}(2)\right)$. This means that $J_L ev = evJ$. By Corollary 5.33, the S^1-action on $\mathcal{M}_2$ takes $C = a\sigma + bJ\sigma$ to $e^{i\theta}Ce^{-i\theta} = C(\cos 2\theta Id + \sin 2\theta J) = (\cos 2\theta Id - \sin 2\theta J)C$. We thus have the standard action of S^1 with weight 2 on the unit disk in $\mathbf{C}$ as the S^1 action on $\mathcal{M}_2$.

At the boundary of $\mathcal{M}_2$, the S^1-action takes an invariant subspace of $a\sigma+bJ\sigma$ $(a^2+b^2=1)$ to another invariant subspace. Accordingly, we obtain a totally geodesic submanifold $Gr_1(\mathbf{R}^3)$ and the real standard map into $Gr_1(\mathbf{R}^3)$. Since the S^1 acts transitively on the boundary, we can conclude that $\mathbf{M}_2 = \overline{\mathcal{M}_2}/S^1$. □

Remark. We can derive from Theorem 6.22 that the real standard map is the unique representative in the homotopy class of maps of degree 2 from the complex projective line into complex quadrics, which is the Einstein-Hermitian terminal holomorphic map with the pull-back connection being a Hermitian Yang-Mills connection.

We have a geometric interpretation of the existence of the complex structure on the moduli space $\mathcal{M}_2$. Let (f, ϕ) be a full holomorphic isometric embedding f of $\mathbf{C}P^1$ into $Gr_4(\mathbf{R}^6)$ of degree 2 with a natural identification $\phi : \mathcal{O}(2) \cong f^*Q$, which corresponds to $C \in \mathcal{M}_2$. Then, a tangent vector to $\mathcal{M}_2$ at (f, ϕ) may be identified with $D \in \mathbf{R}\sigma \oplus \mathbf{R}J\sigma$. As we have already seen, the complex structure J_L on the bundle induces a complex structure J on the space of the sections and $H^0(\mathbf{C}P^1; \mathcal{O}(2))$ is a *complex* subspace of $\Gamma(\mathcal{O}(2))$. Then JD is a tangent vector to the curve (f_t, ϕ_t) in $\mathcal{M}_2$ defined by

$$f_t = \operatorname{Ker} ev(Id + C + tJD)^{\frac{1}{2}}, \text{ and } \phi_t = (Id + C + tJD)^{\frac{1}{2}} ev^*$$

from Theorem 6.21, where $ev : \underline{H^0(\mathbf{C}P^1; \mathcal{O}(2))} \to \mathcal{O}(2)$ is the evaluation map. Thus we obtain a complex structure on $\mathcal{M}_2$ (see [26] for more general argument of the existence of the complex structure of the moduli space modulo gauge equivalence of maps).

In [30], Suyama obtains an analogous description of moduli spaces by image equivalence of holomorphic isometric immersions into quadrics using diastasis of Calabi [5] and establishes that the moduli space is the quotient of a subset of complex Euclidean space by the S^1-action. Our theory interprets the geometric meaning of $\mathcal{M}$ and the reason that the complex structure and the S^1-action on $\mathcal{M}$ emerge in the description of the moduli space modulo image equivalence from the point of view of differential geometry of vector bundles and connections.

The moduli space of holomorphic isometric embeddings of the complex projective line into quadrics of higher degree can be described by the same method. To do so, more detailed analysis of SU(2)-representations is required. This subject is discussed in [22]. Furthermore, when the domain manifold is the complex projective space, or the complex Grassmannian, all holomorphic isometric embeddings into quadrics will be completely classified in the framework of our theory ([24] and [27]).

6.4. **Equivariant harmonic maps.** In this subsection, our main task will be to classify equivariant harmonic maps of the complex projective spaces into complex Grassmannians of low rank and quadrics.

Let G/K_0 be a compact reductive Riemannian homogeneous space, where G is a compact Lie group. Then $f : G/K_0 \to Gr_p(\mathbf{K}^n)$ is an equivariant map if we have a unitary (or an orthogonal) representation $\varrho : G \to \mathrm{Aut}(\mathbf{K}^n)$ such that $f(gx) = \varrho(g)f(x)$, where $g \in G$, $x \in G/K_0$. Here, the image $f(x)$ of $x \in G/K_0$ is considered to represent a subspace of $\mathbf{K}^n$ and $\mathrm{Aut}(\mathbf{K}^n)$ denotes $\mathrm{U}(n)$ or $\mathrm{O}(n)$ depending on $\mathbf{K}$.

Let $f : G/K_0 \to Gr_p(\mathbf{K}^n)$ be an equivariant harmonic map. Then $f^*Q \to G/K_0$ is a homogeneous vector bundle with an invariant metric and an invariant connection under the action of G. The mean curvature operator is an equivariant endomorphism of $f^*Q \to G/K_0$.

Let $V \to G/K_0$ be a homogeneous vector bundle with an invariant fiber metric h and an invariant connection ∇ preserving h. Then an equivariant map $f : G/K_0 \to Gr_p(\mathbf{K}^n)$ is called to satisfy the gauge condition for (V, h, ∇) as *homogeneous bundles* if there exists a G-equivariant bundle isomorphism $\phi : V \to f^*Q$. In this case, the induced linear map $\Gamma(V) \to \Gamma(f^*Q)$ by ϕ is also G-equivariant. In addition, suppose that we have an *equivariant* negative semi-definite Hermitian endomorphism $A \in \Gamma(\mathrm{End}\, V)$. An equivariant map $f : G/K_0 \to Gr_p(\mathbf{K}^n)$ is said to *have an admissible pair* $((V, h, \nabla), A)$ as *homogeneous bundles*, if f satisfies the gauge condition for (V, h, ∇) as homogeneous bundles and has $A \in \Gamma(\mathrm{End}\, V)$ as its mean curvature operator. When W_A denotes the solution space of the generalized Laplace equation with A, W_A is a G-representation induced from that on $\Gamma(V)$. Since the action on $\mathbf{K}^n$ can also be considered to be induced from that on $\Gamma(V)$ by Theorem 3.5, $\mathbf{K}^n$ is a G-submodule of W_A.

With this understood, we have an equivariant version of Theorem 5.20.

Theorem 6.23. *Let $(V \to G/K_0, h, \nabla)$ be a homogeneous vector bundle with an invariant metric h and an invariant connection ∇ and $A \in \Gamma(\mathrm{End}\, V)$ an equivariant negative semi-definite Hermitian endomorphism. Let $f_0 : G/K_0 \to (Gr_p(\mathbf{K}^n), (\cdot,\cdot)_0)$ be an equivariant full harmonic map into $Gr_p(\mathbf{K}^n)$ with an admissible pair $((V, h, \nabla), A)$ as homogeneous bundles. We realize f_0 as an induced map into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_0)$ with $ev_0 : \underline{\mathbf{K}^n} \to V$ as its evaluation map and $ev_0^* : V \to f^*Q$ as its natural identification, both of which are equivariant maps.*

If there exists a G-equivariant linear injection $\iota : \mathbf{K}^m \to \mathbf{K}^n$ and $f_1 : G/K_0 \to \big(Gr_{p'}(\mathbf{K}^m), (\cdot,\cdot)_1\big)$ is an equivariant full harmonic map with an admissible pair $((V, h, \nabla), A)$ as homogeneous bundles (hence, $n - p = m - p'$), then we have an equivariant Hermitian endomorphism C on $\mathbf{K}^n$ which is neither positive nor negative semi-definite (possibly $C = O$) such that
(i) *C satisfies derived MC equations for $ev_0 : \underline{\mathbf{K}^n} \to V$.*
(ii) *$Id + C$ is positive semi-definite,*
(iii) *$(\mathbf{K}^m, (\cdot,\cdot)_1, \iota)$ is compatible with (W, T), where $T = (Id + C)^{\frac{1}{2}}$ and*
(iv) *$f_1 : M \to \big(Gr_{p'}(\mathbf{K}^m), (\cdot,\cdot)_1\big)$ is realized as the map induced by a triple $(V, \mathbf{K}^m, \iota(\iota^* T \iota))$ into $(Gr_p(\mathbf{K}^n), (\cdot,\cdot)_0)$.*

Conversely, if an equivariant Hermitian endomorphism C on $(\mathbf{K}^n, (\cdot,\cdot)_0)$ satisfies the conditions (i) *and* (ii)*, then $\mathbf{K}^m := \mathrm{Ker}\,(Id+C)^{\perp}$ globally generates $V \to G/K_0$ and the map $f : G/K_0 \to \left(Gr_{p'}(\mathbf{K}^m), (\cdot,\cdot)_0\right)$ induced by a triple $\left(V, \mathbf{K}^m, \iota\iota^*(Id+C)^{\frac{1}{2}}\iota\right)$, where $\iota : \mathbf{K}^m \subset \mathbf{K}^n$ is the inclusion, is an equivariant full harmonic map with an admissible pair $((V,h,\nabla), A)$ as homogeneous bundles.*

Let $f_i : M \to (Gr_p(\mathbf{K}^m), (\cdot,\cdot)_0)$ be the maps induced by those triples $\left(V, \mathbf{K}^m, \iota\iota^*(Id+C_i)^{\frac{1}{2}}\iota\right)$ with the inclusion $\iota : \mathbf{K}^m \to \mathbf{K}^n$ such that $\iota(\mathbf{K}^m) = \mathrm{Ker}\,(Id+C_i)^{\perp}$, $(i=1,2)$. Then, f_1 and f_2 are gauge equivalent if and only if $C_1 = C_2$.

If we can take the standard map into $(Gr_p(W_A), (\cdot,\cdot))$ as f_0, where $(\cdot,\cdot)$ is the L^2 scalar product on W_A up to a positive constant multiple, then C is trace-free.

Proof. The key issue is a fact that C is G*-equivariant*.

We take a positive semi-definite Hermitian endomorphism T on $\mathbf{K}^n$ such that $(T\cdot, T\cdot)_1 = \iota^*(\cdot,\cdot)_0$ and the kernel of T is $\mathbf{K}^{m\perp}$. Since both of the scalar products are invariant under the action of G and ι is an equivariant linear map, T^2 is an equivariant linear map:

$$(Tgw_1, Tgw_2)_1 = (gw_1, gw_2)_0 = (w_1, w_2)_0 = (Tw_1, Tw_2)_1.$$

Hence, $C = Id - T^2$ is also an equivariant map. Then the proof proceeds in the same way as in Theorem 5.20. □

From our convention, a map f is identified with $\tau \circ f$, when $V \to M$ is an oriented real vector bundle and the target is an oriented Grassmannian in the following theorems.

Theorem 6.24. *Let $(V \to G/K_0, h, \nabla)$ be a homogeneous vector bundle with an invariant metric and an invariant connection and $A \in \Gamma(\mathrm{End}\,V)$ an equivariant negative semi-definite Hermitian endomorphism. We denote by W_A the solution space of the generalized Laplace equation with A. Let $f_0 : G/K_0 \to Gr_p(\mathbf{K}^n)$ be an equivariant full harmonic map into $Gr_p(\mathbf{K}^n)$ with an admissible pair $((V,h,\nabla), A)$ as homogeneous bundles.*

If W_A is an irreducible representation of G, then $\mathbf{K}^n = W_A$ as G-module and f_0 is the unique equivariant full harmonic map into $Gr_p(W_A)$ with an admissible pair $((V,h,\nabla), A)$ as homogeneous bundles modulo gauge equivalence.

Proof. Since the induced linear map $\Gamma(V) \to \Gamma(f^*Q)$ is G-equivariant for any equivariant full harmonic map f of G/K_0 into $Gr_p(\mathbf{K}^n)$ with an admissible pair $((V,h,\nabla), A)$ as homogeneous bundles, Theorem 3.5 implies that $\mathbf{K}^n$ is a non-trivial G-subspace of W_A. The irreducibility of W_A yields that $\mathbf{K}^n = W_A$ and so, f is a maximal map.

Let $f_1 : G/K_0 \to Gr_p(W_A)$ be an equivariant full harmonic map with an admissible pair $((V,h,\nabla), A)$ as homogeneous bundles. Since f_0 is a maximal map, from Theorem 6.23, we have an equivariant Hermitian endomorphism C on W_A which is neither positive nor negative semi-definite on W_A such that f_1 is realized as a map induced by $\left(V, W_A, (Id+C)^{\frac{1}{2}}\right)$. Schur's lemma yields that $C = \lambda Id_{W_A}$, where $\lambda \in \mathbf{R}$. Since C is neither positive nor negative semi-definite, $\lambda = 0$. □

Corollary 6.25. *Let G/K_0 be a compact simply-connected homogeneous Kähler manifold, where G is a compact connected semisimple Lie group. Suppose that $(V \to G/K_0, h)$ is an irreducible homogeneous holomorphic vector bundle with an invariant Hermitian metric. Then its Hermitian connection ∇ is G-invariant.*

If $f_0 : G/K_0 \to Gr_p(\mathbf{C}^n)$ is an equivariant full holomorphic map with the gauge condition for (V, h, ∇) as homogeneous bundles, then f_0 is the unique Einstein-Hermitian equivariant full holomorphic map into $Gr_p(\mathbf{C}^n)$ with the gauge condition for (V, h, ∇) as homogeneous bundles up to gauge equivalence.

Proof. It follows from Bott-Borel-Weil theorem that $H^0(G/K_0; V)$ is an irreducible unitary G-module. From [18, p.121 Theorem 6.4], we see that (V, h) is an Einstein-Hermitian vector bundle. From Lemma 4.3, f_0 is an EH map and any equivariant holomorphic map with the gauge condition for (V, h, ∇) as homogeneous bundles has the same mean curvature operator as that of f_0. Theorem 6.24 yields the result. □

Let (G, K_0) be an irreducible Hermitian symmetric pair of compact type. Notice that any line bundle over G/K_0 is a homogeneous vector bundle and so, G acts on any orthogonal direct sum of line bundles $\oplus_{i=1} L_i$. Then using the canonical connection on each line bundle, we can induce an invariant connection on $\oplus_{i=1} L_i$, which is also called the canonical connection. We use the same notation as in §6.2.

Theorem 6.26. *Let (G, K_0) be an irreducible Hermitian symmetric pair of compact type except $(\mathrm{SU}(2), \mathrm{U}(1))$ and $V \to G/K_0$ an orthogonal direct sum of line bundles. Then the canonical connection on V is the unique invariant connection.*

Proof. Suppose that we have another invariant connection on $V \to G/K_0$. Then the difference of the two connections denoted by α can be regarded as an invariant 1-form with values in $\operatorname{End} V \to G/K_0$. Thus α is an invariant section of homogeneous vector bundle $T^*M^{\mathbf{C}} \otimes V^* \otimes V \to G/K_0$, where $T^*M^{\mathbf{C}}$ is the complexification of the cotangent bundle of G/K_0

Since $V \to G/K_0$ is an orthogonal direct sum of line bundles, so is $\operatorname{End} V$. Then the fiber of $T^*M^{\mathbf{C}} \otimes \operatorname{End} V \to M$ at o can be decomposed into

$$\bigoplus_{i=1} \left(T_1 \otimes \mathbf{C}_{l_i} \oplus T_1^* \otimes \mathbf{C}_{l_i} \right),$$

where T_1 is a non-trivial irreducible representation of K_1, since the pair (G, K_0) is not $(\mathrm{SU}(2), \mathrm{U}(1))$. Thus the fiber has no trivial summand. By Frobenius reciprocity, $T^*M^{\mathbf{C}} \otimes \operatorname{End} V \to G/K_0$ has no invariant section except the zero section. It follows that the canonical connection is the unique invariant connection on $V \to G/K_0$. □

We apply our results to obtain the rigidity theorems on equivariant holomorphic maps of the complex projective spaces. The corresponding symmetric pair is denoted by $(\mathrm{SU}(m+1), \mathrm{U}(m))$ as in §6.2.

Theorem 6.27. *Let $V \to \mathbf{C}P^m$ be a complex homogeneous vector bundle of rank $q < m$ and ∇ an invariant connection on $V \to \mathbf{C}P^m$.*

Then $V \to \mathbf{C}P^m$ is isomorphic to an orthogonal direct sum of line bundles as homogeneous vector bundle and ∇ is the canonical connection.

Proof. We denote by ϱ the action of $\mathrm{SU}(m+1)$ on $V \to \mathbf{C}P^m$. Since $\mathrm{U}(m)$ is the stabiliser subgroup at the reference point $o \in \mathbf{C}P^m$, ϱ restricted to $\mathrm{U}(m)$ is a representation of $\mathrm{U}(m)$ to the fiber V_o. It follows from the classification of $\mathrm{U}(m)$-representations and dimensional reason that V_o is an orthogonal direct sum of one-dimensional representations $\mathbf{C}_{k_1} \oplus \mathbf{C}_{k_2} \oplus \cdots \oplus \mathbf{C}_{k_q}$. Then we have a bundle isomorphism of $\mathrm{SU}(m+1) \times_{\mathrm{U}(m)} V_o$ to V as homogeneous vector bundles:

$$[g, v] \mapsto \varrho(g)(v).$$

Thus $V \to \mathbf{C}P^m$ is isomorphic to an orthogonal direct sum of line bundles. Then Theorem 6.26 yields the result. $\square$

To classify equivariant harmonic maps, we need the irreducible decomposition of $\mathcal{H}^{k+l,l}_{m+1}$ $(k \geqq 0)$ as $\mathrm{U}(m)$-module:

$$\mathcal{H}^{k+l,l}_{m+1} = \bigoplus_{0 \leqq p \leqq k+l,\, 0 \leqq q \leqq l} \mathcal{H}^{p,q}_m \otimes \mathbf{C}_{-k+(p-q)}. \tag{6.13}$$

If $m \geqq 2$, then $\mathbf{C}_{-k}$ is the only one-dimensional representation of $\mathrm{U}(m)$ in the decomposition (6.13). In the case of $\mathcal{H}^{l,|k|+l}_{m+1}$ $(k \leqq 0)$, we have a similar decomposition.

Theorem 6.28. *We denote by $\oplus_{i=1}^q \mathcal{O}(k_i)$ the orthogonal direct sum of line bundles $\mathcal{O}(k_i) \to \mathbf{C}P^m$ and by $\oplus W_i$ the orthogonal direct sum of vector spaces W_i.*

If $f : \mathbf{C}P^m \to Gr_p(\mathbf{C}^{p+q})$ is an equivariant full harmonic map with $q < m$, then f is the standard map by $(\oplus_{i=1}^q \mathcal{O}(k_i), \oplus_{i=1}^q W_i)$ $(k_i \in \mathbf{Z})$ up to gauge equivalence, where $W_i = \mathcal{H}^{k_i+l_i,l_i}_{m+1}$ $(k_i \geqq 0)$, or $\mathcal{H}^{l_i,|k_i|+l_i}_{m+1}$ $(k_i \leqq 0)$, is one of the eigenspaces of the Laplace operator induced by the canonical connection on $\mathcal{O}(k_i)$ for each $i = 1, \cdots, q$.

Proof. Let f be an equivariant full harmonic map of $\mathbf{C}P^m$ into $Gr_p(\mathbf{C}^{p+q})$ $(q < m)$. Then the pull-back bundle of the universal quotient bundle is homogeneous and the pull-back connection is an invariant connection. Theorem 6.27 yields that the pull-back bundle is $\oplus_{i=1}^q \mathcal{O}(k_i)$ with the canonical connection. Since the canonical connection reduces to $\mathrm{U}(1)$-connection on the direct sum of line bundles and the mean curvature operator A is an equivariant endomorphism of $\oplus_{i=1}^q \mathcal{O}(k_i)$, A is covariant constant and we can deduce that A is of the form $\mathrm{diag}(-\lambda_1, \cdots, -\lambda_q)$ according to the decomposition of the line bundles, where $\lambda_i \geqq 0$. From Theorem 3.5, λ_i is the eigenvalue of the Laplace operator acting on sections of $\mathcal{O}(k_i) \to \mathbf{C}P^m$ and together with fullness of f, we can see that $\mathbf{C}^{p+q}$ is a $\mathrm{SU}(m+1)$-subspace of $\oplus_{i=1}^q W_i$, where each W_i is the eigenspace corresponding to λ_i. Since we have a unique one-dimensional representation space $\mathbf{C}_{-k}$ of $\mathrm{U}(m)$ in the decomposition (6.13) of W_i, Lemma 5.34 with Frobenius reciprocity yields that $\oplus_{i=1}^q W_i \subset \mathbf{C}^{p+q}$ and so, we conclude that $\mathbf{C}^{p+q} = \oplus_{i=1}^q W_i$.

It follows from Lemma 5.36 and (6.13) that the standard map f_0 is an equivariant full harmonic map with $ev : \mathbf{C}P^m \times \oplus_{i=1}^q W_i \to \oplus_{i=1}^q \mathcal{O}(k_i)$.

Suppose that f is not the standard map. Since f_0 is a maximal map, from Theorem 6.23, there exists a non-trivial equivariant trace-free Hermitian endomorphism C on $\oplus_{i=1}^q W_i$ satisfying dMC equations for ev. Let $\mu < 0$ be the smallest eigenvalue of C. Thus $Id - \mu^{-1}C$ has the non-trivial kernel which is an $\mathrm{SU}(m+1)$-module and so, the orthogonal complement of the kernel in $\oplus_{i=1}^q W_i$ is a proper invariant subspace denoted by $\mathbf{C}^N$. Since $-\mu^{-1}C$ also satisfies the dMC equations, Theorem 6.23 implies that $Id - \mu^{-1}C$ induces an equivariant full harmonic map into $Gr_{N-q}(\mathbf{C}^N)$, which is a contradiction to the fact that $\mathbf{C}^N$must be equal to $\oplus_{i=1}^q W_i$. We thus conclude that $C = O$ and f is the standard map. $\square$

Remark. If $(k_i, l_i) = (0, 0)$, then f has a trivial summand and the image of f is a subset of a totally geodesic submanifold $Gr_p(\mathbf{C}^{p+q-1})$ of $Gr_p(\mathbf{C}^{p+q})$.

In the case where the target is a quadric, we need to consider real representations. If $m \geqq 2$, then $\mathcal{H}_{m+1}^{k+l,l}$ $(k > 0)$ and $\mathcal{H}_{m+1}^{l,|k|+l}$ $(k < 0)$ are irreducible as real modules. Since $\mathcal{H}_{m+1}^{l,l}$ has a real structure, $\mathcal{H}_{m+1}^{l,l}$ is decomposed into two equivalent irreducible real representation spaces. As usual, we do not distinguish two standard maps.

Theorem 6.29. *Let $f : \mathbf{C}P^m \to Gr_n(\mathbf{R}^{n+2})$ $(m \geqq 2)$ be an equivariant full harmonic map which is not totally real. Then f is the standard map by $(\mathcal{O}(k), \mathcal{H})$ for some $k \in \mathbf{Z}\backslash\{0\}$ modulo gauge equivalence, where $\mathcal{H} = \mathcal{H}_{m+1}^{k+l,l}$ $(k > 0)$, or $\mathcal{H}_{m+1}^{l,|k|+l}$ $(k < 0)$, is one of the eigenspaces of the Laplace operator induced by the canonical connection on $\mathcal{O}(k)$.*

Proof. Theorem 6.27 yields that the pull-back of the universal quotient bundle is a homogeneous vector bundle $\mathcal{O}(k)$ with the canonical connection. Since f is not totally real, we have $k \neq 0$. Since the mean curvature operator A is equivariant, $A = -\lambda Id$ $(\lambda \geqq 0)$. Theorem 3.5 implies that λ is the eigenvalue of the Laplace operator. The fullness of the map and irreducibility of $\mathcal{H}$ $(m \geqq 2)$ yield that $\mathbf{R}^{n+2} = \mathcal{H}$. It follows from Lemma 5.36 and (6.13) that the standard map f_0 is an equivariant full harmonic map. Theorem 6.24 yields the result. $\square$

Theorem 6.30. *Let $f : \mathbf{C}P^m \to Gr_n(\mathbf{R}^{n+2})$ be an Einstein-Hermitian equivariant full totally real harmonic map. Then f is the standard map by $\left(\mathcal{O}, \mathcal{H}_{m+1}^{l,l}\right)$ $(l \in \mathbf{Z}_{\geqq 0})$ up to gauge equivalence , where $\mathcal{H}_{m+1}^{l,l}$ is one of the eigenspaces of the Laplace operator acting on $C^\infty(\mathbf{C}P^m)$.*

Proof. Since f is totally real, Theorem 6.27 yields that the pull-back of the universal quotient bundle is a real trivial bundle $\mathcal{O}$ of rank 2 with the canonical connection (the product connection). The EH condition yields that $A = -\lambda Id$ $(\lambda \geqq 0)$. Theorem 3.5 implies that λ is the eigenvalue of the Laplace operator. By the fullness of the map, we see that $\mathbf{R}^{n+2} = \mathcal{H}^{l,l}$ or a real invariant subspace $\mathcal{K}$ of $\mathcal{H}^{l,l}$. However, in the latter case, it follows from (6.13) that the trivial module in the irreducible decomposition of $\mathcal{K}$ as $\mathrm{U}(m)$-module is a *real one*-dimensional subspace. Frobenius reciprocity yields that $\mathcal{K}$ does not globally generate $\mathcal{O} \to \mathbf{C}P^m$. Thus we deduce that $\mathbf{R}^{n+2} = \mathcal{H}^{l,l}$. It follows from Lemma 5.36 and (6.13) that the standard map f_0 is an EH equivariant full harmonic map. We finally get the result from Theorem 6.23 in a similar way to a proof of Theorem 6.28. □

Remark. The classification of the Einstein-Hermitian harmonic maps of $\mathbf{C}P^1$ into quadrics is the subject of [23]. We have an equivariant full totally real harmonic map of $\mathbf{C}P^1$ into quadrics which does not satisfy the EH-condition (see also [38]).

6.5. **Comparison with the ADHM-construction.** We denote by $S^4 = \mathbf{H}P^1$ the 4-dimensional sphere. We follow the notation of the Example after Lemma 5.36.

Let $\mathbf{H} \to \mathbf{H}P^1$ be the tautological (complex) vector bundle with the canonical connection which is a self-dual connection. The Penrose transform implies that the solution space of the twistor equation of $\mathbf{H} \to \mathbf{H}P^1$ is naturally identified with $H^0\left(\mathbf{C}P^3; \mathcal{O}(1)\right)$, where $\mathbf{C}P^3$ is the twistor space of S^4. The Bott-Borel-Weil theorem yields that $H^0\left(\mathbf{C}P^3; \mathcal{O}(1)\right)$ is regarded as the standard representation $\mathbf{C}^{4^*} \cong \mathbf{C}^4$ of Sp(2) with an invariant symplectic form ω on $\mathbf{C}^4$. Thus $\mathbf{C}^4$ is the solution space of the twistor equation and so, the eigenspace of the Laplace operator. Since $\mathbf{C}^4$ globally generates $\mathbf{H} \to \mathbf{H}P^1$, we can consider the induced map $f_0 : \mathbf{H}P^1 \to Gr_2(\mathbf{C}^4)$. This is nothing but a standard map by $(\mathbf{H} \to \mathbf{H}P^1, \mathbf{C}^4)$.

To apply Theorem 5.37, we observe an irreducible decomposition of the representation space $\mathrm{End}\,(\mathbf{C}^4)$ as Sp(2)-module:

$$\mathrm{End}\,(\mathbf{C}^4) = S^2\mathbf{C}^4 \oplus \wedge_0^2\mathbf{C}^4 \oplus \mathbf{C},$$

where, $\wedge_0^2\mathbf{C}^4$ is the orthogonal complement to $\mathbf{C}\omega$ in $\wedge^2\mathbf{C}^4$. As $\mathrm{Sp}_+(1) \times \mathrm{Sp}_-(1)$-module, we have that

$$\mathbf{C}^4 = \mathbf{H} \oplus \mathbf{E}, \tag{6.14}$$
$$S^2\mathbf{C}^4 = S^2\mathbf{H} \oplus \mathbf{H} \otimes \mathbf{E} \oplus S^2\mathbf{E},$$

and

$$\wedge^2\mathbf{C}^4 = \mathbf{C} \oplus \mathbf{H} \otimes \mathbf{E} \oplus \mathbf{C}. \tag{6.15}$$

The complexification of the tangent bundle $T^{\mathbf{C}}$ of $\mathbf{H}P^1$ is identified with $\mathbf{H} \otimes \mathbf{E}$. From $\mathbf{H} \otimes \mathbf{E} \otimes \mathbf{H} \cong (S^2\mathbf{H} \oplus \mathbf{C}) \otimes \mathbf{E}$, Lemma 5.36 yields that f_0 is an EH harmonic map. Since (6.14) is the normal decomposition of $(\mathbf{C}^4, \mathbf{H})$ by Proposition 6.7 and $\wedge_0^2\mathbf{C}^4$ is a class one representation of $(\mathrm{Sp}(2), \mathrm{Sp}(1) \times \mathrm{Sp}(1))$ from (6.15), Proposition 6.9 yields that $\wedge^2\mathbf{C}^4$ is included in the Sp(2)-submodule $GH(\mathbf{H}, \mathbf{H})$ of $H(\mathbf{C}^4)$ generated by $H(\mathbf{H}, \mathbf{H})$ and Sp(2). Since $H(h,h) - H(jh, jh)$ is in $S^2\mathbf{H}$ for an arbitrary $h \in \mathbf{H}$, where j is an invariant quaternion structure, $GH(\mathbf{H}, \mathbf{H})$ includes $S^2\mathbf{C}^4$. It follows from Theorem 5.37 that f_0 can not be deformed as a harmonic map satisfying the gauge condition for the canonical connection and the EH condition. Corollary 5.27 implies that f_0 is the terminal harmonic map. Since f_0 is an equivariant map, the second fundamental forms of $0 \to \mathbf{E} \to \underline{\mathbf{C}^4} \to \mathbf{H} \to 0$ are Sp(2)-equivariant. Replacing the role of $\mathbf{H} \to \mathbf{H}P^1$ by $\mathbf{E} \to \mathbf{H}P^1$, we can see that the induced connection on $\mathbf{E} \to \mathbf{H}P^1$ by f_0 is also the canonical connection. Since the canonical connections reduce to $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$-connection, we deduce that the second fundamental forms are covariant constant. Corollary 3.6 yields that f_0 is a totally geodesic map. As a conclusion, f_0 is the Einstein-Hermitian terminal totally geodesic map with the gauge condition for the canonical connection on $\mathbf{H} \to \mathbf{H}P^1$.

Next we consider the ADHM-construction of instantons [1]. For simplicity, we focus our attention on 1-instantons. Let $\alpha : \underline{\mathbf{C}^4} \to \mathbf{H}$ be a surjective bundle map satisfying the twistor equation [17]:

$$\mathcal{D}\alpha = 0,$$

where α is regarded as a section of $\underline{\mathbf{C}^{4^*}} \otimes \mathbf{H} \to \mathbf{H}P^1$. Suppose that $\mathbf{C}^4$ has an invariant Hermitian inner product and an invariant quaternion structure j under the action of $\mathrm{Sp}(2)$. Then we have the induced real structure of $\underline{\mathbf{C}^{4^*}} \otimes \mathbf{H} \cong \underline{\mathbf{C}^4} \otimes \mathbf{H}$. Then α is required to be a *real* section of $\underline{\mathbf{C}^{4^*}} \otimes \mathbf{H}$.

Using the twistor space and the Bott-Borel-Weil theorem, we know that α can be expressed as

$$\alpha_{[g]}(w) = \left[g, \pi(g^{-1}Tw)\right], \quad g \in \mathrm{Sp}(2),$$

where T is a positive Hermitian endomorphism of $\mathbf{C}^4$, and $\pi : \mathbf{C}^4 \to \mathbf{H}$ is the orthogonal projection. Since $\wedge_0^2\mathbf{C}^4$ has an invariant real structure induced by j, we can take a real representation $(\wedge_0^2\mathbf{C}^4)^{\mathbf{R}}$. Then, the ADHM-construction requires that T should satisfy

$$T^2 = Id + C, \quad C \in (\wedge_0^2\mathbf{C}^4)^{\mathbf{R}}.$$

If C is small enough, then $Id+C$ is positive, and so, $\mathrm{Ker}\,\alpha \subset \underline{\mathbf{C}^4}$ is an $\mathrm{SU}(2)$-bundle with the metric and an anti-self-dual connection induced from those on $\mathbf{C}^4$.

If we regard α as an evaluation homomorphism, then we obtain the induced map $f : \mathbf{H}P^1 \to Gr_2(\mathbf{C}^4)$:

$$f\left([g]\right) = Tg\mathbf{E}.$$

When T is the identity or equivalently, $C = O$, we recover the standard map f_0. In the case that $C \neq O$, the pull-back connection on the pull-back bundle $f^*Q \to M$ is not gauge equivalent to the canonical connection on $\mathbf{H} \to \mathbf{H}P^1$.

In both cases of the generalization of the do Carmo-Wallach construction and the ADHM-construction, the emergence of linear equations ($(\Delta + A)t = 0$ and $\mathcal{D}\alpha = 0$, respectively) makes it possible to describe moduli spaces in linear algebraic terms.

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