Submitted to
Journal of the Mathematical Society of Japan

A refined transversality theorem on linear perturbations and its applications

By Shunsuke Ichiki

Abstract. In this paper, we establish a refined transversality theorem on linear perturbations from a new perspective of Hausdorff measures. Furthermore, we give its applications not only to singularity theory but also to multiobjective optimization.

1. Introduction

Transversality theorems are fundamental tools for investigating generic mappings. In 1973, Mather gave a striking transversality theorem on generic projections as the main theorem of the celebrated paper [11]. Let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the space consisting of all linear mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$. In what follows, we will regard $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ as the Euclidean space $(\mathbb{R}^m)^\ell$ in the obvious way. Briefly, Mather's result is a transversality theorem for a composition $\pi \circ f : X \to \mathbb{R}^\ell$ of a C^∞ embedding f from a C^∞ manifold X into $\mathbb{R}^m$ and a projection $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \setminus \Sigma$, where Σ is a subset of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero. The theorem yields important applications on a composition of a C^∞ embedding and a projection (e.g. Theorems 2 and 3 of [11]).

After that, in [6], for a C^∞ immersion f from a C^∞ manifold X into an open set V of $\mathbb{R}^m$ and an arbitrary C^∞ mapping $g : V \to \mathbb{R}^\ell$, a transversality theorem on the 1-jet extension of a composition of f and a mapping obtained by generically linearly perturbing g, that is $(g + \pi) \circ f : X \to \mathbb{R}^\ell$ ($\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \setminus \Sigma$), is given, where Σ is a subset of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero.

Moreover, in [7], the transversality theorem of [6] on generic linear perturbations described above had been improved so that it works even in the case where manifolds and mappings are not necessarily of class C^∞ (see Proposition 2.2 of this paper). The theorem also yields applications on a composition of an immersion and a generically linearly perturbed mapping (as in Propositions 5.1 and 5.6 of this paper). Moreover, it also gives an application not only to singularity theory but also to multiobjective optimization (see Proposition 6.3 of this paper). Namely, it has been a useful tool to yield various applications on generically linearly perturbed mappings.

However, the transversality theorem (Proposition 2.2) is still in the stage of Lebesgue measures and therefore are its applications since Thom's parametric transversality theorem is used as a lemma in its proof. On the other hand, in [8], Thom's parametric transversality theorem had been already improved from a new perspective of Hausdorff measures which generalize Lebesgue measures. Thus, the purpose of this paper is to give a refined version of the transversality theorem from the viewpoint of Hausdorff measures and its various applications.

2010 Mathematics Subject Classification. Primary 58K30; Secondary 57R45.
Key Words and Phrases. transversality theorem, genericity, Hausdorff measure, Pareto set.

In [6, 7], not only immersions but also injections are investigated. More precisely, in [6], for a C^∞ injection f from a C^∞ manifold X into an open set V of $\mathbb{R}^m$ and a C^∞ mapping $g : V \to \mathbb{R}^\ell$, a specialized transversality theorem on crossings of a composition of f and a mapping obtained by generically linearly perturbing g and its applications are also given from the viewpoint of Lebesgue measures. Furthermore, in [7], the specialized transversality theorem and some of its applications have been improved so that they work even in the case where manifolds and mappings are not necessarily of class C^∞. On the other hand, for refined versions of these injective cases from the viewpoint of Hausdorff measures, there are still some supplementary problems that remain unsolved (for details, see Remark 2.4 (6)). Thus, in this paper, we do not treat injective cases, but we give an application to multiobjective optimization from the viewpoint of singularity theory and differential topology, which is a refined version of a result obtained in [4] (Proposition 6.3 in this paper) from a new perspective of Hausdorff measures.

The remainder of this paper is organized as follows. In Section 2, we state the main theorem. In Section 3, we review the definition of Hausdorff measures and prepare an essential tool for the proof of the main theorem, and in Section 4, we show the main theorem. In Section 5, we give applications of the main theorem. In Section 6, we also give an application of the main theorem to multiobjective optimization from the viewpoint of singularity theory and differential topology, and in Section 7, we give the proof of the application.

2. The main theorem

In this paper, unless otherwise stated, all manifolds are without boundary and assumed to have a countable basis. In this section, we prepare some notations and state the main theorem. First, we review the definition of transversality.

Definition 2.1. Let X and Y be C^r manifolds, and Z a C^r submanifold of Y ($r \geq 1$). Let $f : X \to Y$ be a C^1 mapping.

(1) We say that $f : X \to Y$ is *transverse* to Z *at* $x \in X$ if $f(x) \notin Z$ or in the case of $f(x) \in Z$, the following holds:
$$df_x(T_xX) + T_{f(x)}Z = T_{f(x)}Y.$$

(2) We say that $f : X \to Y$ is *transverse* to Z if for any $x \in X$, the mapping f is transverse to Z at x.

Let X be a C^r manifold ($r \geq 2$) of dimension n, and $J^1(X, \mathbb{R}^\ell)$ the space of 1-jets of mappings of X into $\mathbb{R}^\ell$. Then, note that $J^1(X, \mathbb{R}^\ell)$ is a C^{r-1} manifold. For a given C^r mapping $f : X \to \mathbb{R}^\ell$, the 1-jet extension $j^1f : X \to J^1(X, \mathbb{R}^\ell)$ is defined by $q \mapsto j^1f(q)$. Then, notice that j^1f is of class C^{r-1}. For details on $J^1(X, \mathbb{R}^\ell)$ and j^1f, see for example, [3].

Now, let $\{(V_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ be a coordinate neighborhood system of X. Let $\Pi : J^1(X, \mathbb{R}^\ell) \to X \times \mathbb{R}^\ell$ be the natural projection defined by $\Pi(j^1f(q)) = (q, f(q))$. Let $\Phi_\lambda : \Pi^{-1}(V_\lambda \times \mathbb{R}^\ell) \to \varphi_\lambda(V_\lambda) \times \mathbb{R}^\ell \times J^1(n, \ell)$ be the homeomorphism defined by
$$\Phi_\lambda \left(j^1f(q)\right) = \left(\varphi_\lambda(q), f(q), j^1(\psi_\lambda \circ f \circ \varphi_\lambda^{-1} \circ \widetilde{\varphi}_\lambda)(0)\right),$$

where $J^1(n,\ell)=\{\, j^1f(0) \mid f:(\mathbb{R}^n,0)\to(\mathbb{R}^\ell,0)\,\}$ and $\widetilde{\varphi}_\lambda:\mathbb{R}^n\to\mathbb{R}^n$ (resp., $\psi_\lambda:\mathbb{R}^\ell\to\mathbb{R}^\ell$) is the parallel translation satisfying $\widetilde{\varphi}_\lambda(0)=\varphi_\lambda(q)$ (resp., $\psi_\lambda(f(q))=0$). Then, $\{\,(\Pi^{-1}(V_\lambda\times\mathbb{R}^\ell),\Phi_\lambda)\,\}_{\lambda\in\Lambda}$ is a coordinate neighborhood system of $J^1(X,\mathbb{R}^\ell)$.

Set

$$S^k=\{\, j^1f(0)\in J^1(n,\ell) \mid \operatorname{corank} Jf(0)=k\,\},$$

where $\operatorname{corank} Jf(0)=\min\{\,n,\ell\,\}-\operatorname{rank} Jf(0)$ and $k=1,2,\ldots,\min\{\,n,\ell\,\}$. Set

$$S^k(X,\mathbb{R}^\ell)=\bigcup_{\lambda\in\Lambda}\Phi_\lambda^{-1}\left(\varphi_\lambda(V_\lambda)\times\mathbb{R}^\ell\times S^k\right).$$

Then, the set $S^k(X,\mathbb{R}^\ell)$ is a submanifold of $J^1(X,\mathbb{R}^\ell)$ satisfying

$$\operatorname{codim} S^k(X,\mathbb{R}^\ell)=\dim J^1(X,\mathbb{R}^\ell)-\dim S^k(X,\mathbb{R}^\ell)=(n-v+k)(\ell-v+k),$$

where $v=\min\{\,n,\ell\,\}$. (For details on S^k and $S^k(X,\mathbb{R}^\ell)$, see [3, pp. 60–61]).

Proposition 2.2 ([7]). *Let $f:X\to V$ be a C^r immersion, and $g:V\to\mathbb{R}^\ell$ a C^r mapping, where r is an integer satisfying $r\geq 2$, X is a C^r manifold and V is an open subset of $\mathbb{R}^m$. Let k be an integer satisfying $1\leq k\leq\min\{\,\dim X,\ell\,\}$. If*

$$r\geq\max\{\,\dim X-\operatorname{codim} S^k(X,\mathbb{R}^\ell),0\,\}+2,$$

then the set

$$\Sigma_k=\{\,\pi\in\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) \mid j^1((g+\pi)\circ f) \text{ is not transverse to } S^k(X,\mathbb{R}^\ell)\,\}$$

has Lebesgue measure zero in $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$.

As a side note, [6, Theorem 1] is Proposition 2.2 in the case where all manifolds and mappings are of class C^∞. Namely, Proposition 2.2 is an improvement of [6, Theorem 1]. The following is the main theorem of this paper, which is a refined version of Proposition 2.2 from a new perspective of Hausdorff measures.

Theorem 2.3. *Let $f:X\to V$ be a C^r immersion, and $g:V\to\mathbb{R}^\ell$ a C^r mapping, where r is an integer satisfying $r\geq 2$, X is a C^r manifold and V is an open subset of $\mathbb{R}^m$. Let k be an integer satisfying $1\leq k\leq\min\{\,\dim X,\ell\,\}$. Set*

$$\Sigma_k=\{\,\pi\in\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) \mid j^1((g+\pi)\circ f) \text{ is not transverse to } S^k(X,\mathbb{R}^\ell)\,\}.$$

Then, the following hold:

(1) *Suppose $\dim X-\operatorname{codim} S^k(X,\mathbb{R}^\ell)\geq 0$. Then, for any real number s satisfying*

$$s\geq m\ell-1+\frac{\dim X-\operatorname{codim} S^k(X,\mathbb{R}^\ell)+1}{r-1},\tag{2.1}$$

the set Σ_k has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$.

(2) *Suppose $\dim X-\operatorname{codim} S^k(X,\mathbb{R}^\ell)<0$. Then, we have the following:*

(2a) *For any real number s satisfying*

$$s > m\ell + \dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell), \tag{2.2}$$

the set Σ_k has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

(2b) *For any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \setminus \Sigma_k$, we have $j^1((g+\pi)\circ f)(X) \cap S^k(X, \mathbb{R}^\ell) = \varnothing$.*

REMARK 2.4. We give the following remarks on Theorem 2.3.

(1) We will show that Theorem 2.3 implies Proposition 2.2. Let f and g (resp. k and r) be mappings (resp. integers) satisfying the assumption of Proposition 2.2. First, we consider the case $\dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell) \geq 0$. Since $r \geq \dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell) + 2$, we can set $s = m\ell$ in (2.1). Thus, since Σ_k has $m\ell$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ by Theorem 2.3 (1), Σ_k also has Lebesgue measure zero. In the case $\dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell) < 0$, since we can set $s = m\ell$ in (2.2), Σ_k has $m\ell$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ by Theorem 2.3 (2a), which implies that Σ_k has Lebesgue measure zero.

(2) In Theorem 2.3 (1), if all manifolds and mappings are of class C^∞, then for any real number s such that $s > m\ell - 1$, there exists a positive integer r satisfying (2.1). Thus, in the C^∞ case, we can replace (2.1) by

$$s > m\ell - 1.$$

(3) In Theorem 2.3, since f is an immersion, we have $n \leq m$, where $n = \dim X$. Thus, in Theorem 2.3 (2a), since

$$m\ell + \dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell) \geq m\ell + n - n\ell = (m-n)\ell + n \geq n,$$

it is not necessary to assume that s is non-negative.

(4) In Theorem 2.3, there is an advantage that the domain of $g : V \to \mathbb{R}^\ell$ is not $\mathbb{R}^m$ but an arbitrary open subset V of $\mathbb{R}^m$. Suppose $V = \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = |x|$. Since g is not differentiable at $x = 0$, we cannot apply Theorem 2.3 to $g : \mathbb{R} \to \mathbb{R}$. On the other hand, if $V = \mathbb{R} \setminus \{0\}$, then we can apply the theorem to $g|_V$.

(5) The assumptions (2.1) and (2.2) cannot be improved in general (see Remark 5.5 and Remark 5.9, respectively), which implies that these are the best evaluations in general.

(6) As explained in Section 1, in [6, 7], not only the case where f is an immersion but also the case where f is an injection is investigated from the viewpoint of Lebesgue measures. By using the refined version of Thom's parametric transversality theorem obtained in [8] (Theorem 3.3 in this paper), we can certainly update some results on injections obtained in [6, 7] from the new perspective of Hausdorff measures. However, for those updated results, it is unsolved whether the evaluations on Hausdorff measures (such as (2.1) and (2.2) in immersion cases) are the best or not in general. Thus, in this paper, we do not deal with injective cases, but we give an application of Theorem 2.3 to multiobjective optimization in Sections 6 and 7.

3. Preliminaries for the proof of the main theorem

First, we review the definition of Hausdorff measures. Let s be an arbitrary non-negative real number. Then, the *s-dimensional Hausdorff outer measure* on $\mathbb{R}^n$ is defined as follows. Let B be a subset of $\mathbb{R}^n$. The 0-dimensional Hausdorff outer measure of B is the number of points in B. For $s > 0$, the s-dimensional Hausdorff outer measure of B is defined by

$$\lim_{\delta \to 0} \mathcal{H}^s_\delta(B),$$

where for each $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(B) = \inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} C_j)^s \;\middle|\; B \subset \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j \leq \delta \right\}.$$

Here, for a subset C of $\mathbb{R}^n$, we write

$$\operatorname{diam} C = \sup \{ \|x - y\| \mid x, y \in C \},$$

where $\|z\|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$. Note that the infimum in $\mathcal{H}^s_\delta(B)$ is over all coverings of B by subsets $C_1, C_2, \dots$ of $\mathbb{R}^n$ satisfying $\operatorname{diam} C_j \leq \delta$ for all positive integers j.

Let s be an arbitrary non-negative real number. Let N be a C^r manifold ($r \geq 1$) of dimension n, and $\{ (U_\lambda, \varphi_\lambda) \}_{\lambda \in \Lambda}$ a coordinate neighborhood system of N. Then, a subset Σ of N has *s-dimensional Hausdorff measure zero* in N if for any $\lambda \in \Lambda$, the set $\varphi_\lambda(\Sigma \cap U_\lambda)$ has s-dimensional Hausdorff (outer) measure zero in $\mathbb{R}^n$. Note that this definition does not depend on the choice of a coordinate neighborhood system of N. Moreover, for a subset Σ of N, set

$$\mathrm{HD}_N(\Sigma) = \inf \{ s \in [0, \infty) \mid \Sigma \text{ has } s\text{-dimensional Hausdorff measure zero in } N \},$$

which is called the *Hausdorff dimension of* Σ in N.

Next, we will prepare an essential tool (Theorem 3.3) for the proof of the main theorem, which is a refined version of Thom's parametric transversality theorem and its improvement which was given by Mather in [11]. In order to state Theorem 3.3, we prepare some definitions. Let X, A and Y be C^r manifolds ($r \geq 1$), and U an open set of $X \times A$. In what follows, by $\pi_1 : U \to X$ and $\pi_2 : U \to A$, we denote the natural projections defined by

$$\pi_1(x, a) = x, \quad \pi_2(x, a) = a.$$

Let $F : U \to Y$ be a C^1 mapping. For any element $a \in \pi_2(U)$, let

$$F_a : \pi_1(U \cap (X \times \{ a \})) \to Y$$

be the mapping defined by $F_a(x) = F(x, a)$. Here, note that $\pi_1(U \cap (X \times \{ a \}))$ is open in X. Let Z be a submanifold of Y. Set

$$\Sigma(F, Z) = \{ a \in \pi_2(U) \mid F_a \text{ is not transverse to } Z \}.$$

DEFINITION 3.1. Let X and Y be C^r manifolds, and Z a C^r submanifold of Y $(r \geq 1)$. Let $f : X \to Y$ be a C^1 mapping. For any $x \in X$, set

$$\delta(f, x, Z) = \begin{cases} 0 & \text{if } f(x) \notin Z, \\ \dim Y - \dim(df_x(T_x X) + T_{f(x)} Z) & \text{if } f(x) \in Z, \end{cases}$$
$$\delta(f, Z) = \sup \{ \delta(f, x, Z) \mid x \in X \} .$$

In the case that all manifolds and mappings are of class C^∞, Definition 3.1 is the definition of [11, p. 230]. As in [1], $\delta(f, x, Z)$ measures the extent to which f fails to be transverse to Z at x.

DEFINITION 3.2. Let X, A and Y be C^r manifolds, and Z a C^r submanifold of Y $(r \geq 1)$. Let $F : U \to Y$ be a C^1 mapping, where U is an open set of $X \times A$. Then, we define

$$\begin{aligned} W(F, Z) &= \{ (x, a) \in U \mid \delta(F_a, x, Z) = \delta(F, (x, a), Z) > 0 \} , \\ \delta^*(F, Z) &= \dim X + \dim Z - \dim Y + \delta(F, Z) \\ &= \dim X - \operatorname{codim} Z + \delta(F, Z), \end{aligned}$$

where $\operatorname{codim} Z = \dim Y - \dim Z$.

In what follows, we denote the image of a given mapping f by $\operatorname{Im} f$.

THEOREM 3.3 ([8]). *Let X, A and Y be C^r manifolds, Z a C^r submanifold of Y, and $F : U \to Y$ a C^r mapping, where U is an open set of $X \times A$ and r is a positive integer. Then, the following hold:*

(1) *Suppose $\delta^*(F, Z) \geq 0$. Then, for any real number s satisfying*

$$s \geq \dim A - 1 + \frac{\delta^*(F, Z) + 1}{r}, \tag{3.1}$$

the following (α) and (β) are equivalent.

  (α) *The set $\pi_2(W(F, Z))$ has s-dimensional Hausdorff measure zero in $\pi_2(U)$.*

  (β) *The set $\Sigma(F, Z)$ has s-dimensional Hausdorff measure zero in $\pi_2(U)$.*

(2) *Suppose $\delta^*(F, Z) < 0$. Then, the following hold:*

  (2a) *We have $W(F, Z) = \varnothing$.*

  (2b) *For any non-negative real number s satisfying $s > \dim A + \delta^*(F, Z)$, the set $\Sigma(F, Z)$ has s-dimensional Hausdorff measure zero in $\pi_2(U)$.*

  (2c) *For any $a \in \pi_2(U) \setminus \Sigma(F, Z)$, we have $\operatorname{Im} F_a \cap Z = \varnothing$.*

4. Proof of the main theorem

Let $\Gamma : X \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \to J^1(X, \mathbb{R}^\ell)$ be the C^{r-1} mapping defined by

$$\Gamma(q, \pi) = j^1((g + \pi) \circ f)(q).$$

The strategy of this proof is to apply Theorem 3.3 as $F = \Gamma$ and $Z = S^k(X, \mathbb{R}^\ell)$. First, by the same method as in the proofs of [6, Theorem 1] and Proposition 2.2, we can obtain $\delta(\Gamma, S^k(X, \mathbb{R}^\ell)) = 0$. Thus, we have

$$\delta^*(\Gamma, S^k(X, \mathbb{R}^\ell)) = \dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell).$$

Then, note that $\Sigma_k = \Sigma(\Gamma, S^k(X, \mathbb{R}^\ell))$.

Next, we will show Theorem 2.3 (1). Since $\dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell) \geq 0$, we obtain $\delta^*(\Gamma, S^k(X, \mathbb{R}^\ell)) \geq 0$. Notice that Γ is of class C^{r-1} $(r \geq 2)$ and we have

$$s \geq m\ell - 1 + \frac{\dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell) + 1}{r-1} = m\ell - 1 + \frac{\delta^*(\Gamma, S^k(X, \mathbb{R}^\ell)) + 1}{r-1}.$$

Since $\delta(\Gamma, S^k(X, \mathbb{R}^\ell)) = 0$, we obtain $W(\Gamma, S^k(X, \mathbb{R}^\ell)) = \varnothing$. Therefore, the set $\pi_2(W(\Gamma, S^k(X, \mathbb{R}^\ell)))$ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. Thus, by Theorem 3.3 (1), the set $\Sigma(\Gamma, S^k(X, \mathbb{R}^\ell))$ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. Since $\Sigma_k = \Sigma(\Gamma, S^k(X, \mathbb{R}^\ell))$, we have Theorem 2.3 (1).

Finally, we will show Theorem 2.3 (2). Since $\dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell) < 0$, we obtain $\delta^*(\Gamma, S^k(X, \mathbb{R}^\ell)) < 0$. Since Γ is of class C^{r-1} $(r \geq 2)$ and we have

$$s > m\ell + \dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell) = m\ell + \delta^*(\Gamma, S^k(X, \mathbb{R}^\ell)),$$

the set $\Sigma(\Gamma, S^k(X, \mathbb{R}^\ell))$ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ by Theorem 3.3 (2b). By Theorem 3.3 (2c), for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \setminus \Sigma(\Gamma, S^k(X, \mathbb{R}^\ell))$, we have $\operatorname{Im} \Gamma_\pi \cap S^k(X, \mathbb{R}^\ell) = \varnothing$. Since $\Sigma_k = \Sigma(\Gamma, S^k(X, \mathbb{R}^\ell))$, we obtain Theorem 2.3 (2). □

5. Applications of the main theorem

In this section, we give applications and their examples of the main theorem in the two cases $\ell = 1$ and $\ell \geq 2 \dim X$.

First, we consider the case $\ell = 1$. Let X be a C^r manifold $(r \geq 1)$, and let $f : X \to \mathbb{R}$ be a C^1 mapping. A point $x \in X$ is called a *critical point* of f if $\operatorname{rank} df_x = 0$. We say that a point of $\mathbb{R}$ is a *critical value* if it is the image of a critical point. A C^r function $f : X \to \mathbb{R}$ $(r \geq 2)$ is called a *Morse function* if all of the critical points of f are nondegenerate, where X is a C^r manifold. For details on Morse functions, see for example, [3, p. 63]. In [7], the following result is obtained as an application of Proposition 2.2.

Proposition 5.1 ([7]). *Let f be a C^r immersion of a C^r manifold X into an open subset V of $\mathbb{R}^m$, and $g : V \to \mathbb{R}$ a C^r function, where r is an integer satisfying $r \geq 2$. Then, the set*

$$\Sigma = \{\, \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) \mid (g + \pi) \circ f : X \to \mathbb{R} \textit{ is not a Morse function} \}$$

has Lebesgue measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$.

As a side note, Corollary 1 of [6] is Proposition 5.1 in the case where all manifolds and mappings are of class C^∞. Namely, Proposition 5.1 is an improvement of Corollary 1 of [6]. In this paper, by using the main theorem, we further upgrade Proposition 5.1 from a new perspective of Hausdorff measures as follows:

THEOREM 5.2. *Let f be a C^r immersion of a C^r manifold X into an open subset V of $\mathbb{R}^m$, and $g : V \to \mathbb{R}$ a C^r function, where r is an integer satisfying $r \geq 2$. Set*

$$\Sigma = \{\, \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) \mid (g + \pi) \circ f : X \to \mathbb{R} \textit{ is not a Morse function} \,\}.$$

Then, for any real number s satisfying

$$s \geq m - 1 + \frac{1}{r-1}, \tag{5.1}$$

the set Σ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$.

REMARK 5.3. In Theorem 5.2, if all manifolds and mappings are of class C^∞, then we can replace (5.1) by $s > m - 1$ by the same argument as in Remark 2.4 (2) .

Proof of Theorem 5.2. It is clearly seen that Σ is the set consisting of all elements $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ satisfying that $j^1((g + \pi) \circ f)$ is not transverse to $S^1(X, \mathbb{R})$. Since $\dim X - \operatorname{codim} S^1(X, \mathbb{R}) = 0$, by Theorem 2.3 (1), for any real number s satisfying (5.1), the set Σ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$. □

As in the following example (Example 5.4), there exists an example such that (5.1) in Theorem 5.2 cannot be improved. Namely, (5.1) is the best evaluation in general. In Example 5.4, we also explain an advantage of Theorem 5.2 from a new perspective of Hausdorff measures compared to Proposition 5.1 from the viewpoint of Lebesgue measures.

EXAMPLE 5.4 (An example of Theorem 5.2). Set $X = V = \mathbb{R}$ and $f(x) = x$ in Theorem 5.2. Let $g : \mathbb{R} \to \mathbb{R}$ be the C^r function defined by $g(x) = -\int_0^x \eta(x)dx$, where r is an integer satisfying $r \geq 2$ and $\eta : \mathbb{R} \to \mathbb{R}$ is a C^{r-1} function such that the Hausdorff dimension of the set consisting of all critical values of η is $\frac{1}{r-1}$. Note that the existence of such a function is guaranteed in [2, Example 4.2][1]. Set

$$\Sigma = \{\, a \in \mathbb{R} \mid g + \pi : \mathbb{R} \to \mathbb{R} \text{ is not a Morse function, where } \pi(x) = ax \,\}.$$

By noting that a linear mapping $\pi \in \mathcal{L}(\mathbb{R}, \mathbb{R})$ can be expressed by $\pi(x) = ax$ $(a \in \mathbb{R})$ and thus that it can be identified with $a \in \mathbb{R}$, for any real number s satisfying

$$s \geq \frac{1}{r-1}, \tag{5.2}$$

the set Σ has s-dimensional Hausdorff measure zero in $\mathbb{R}$ by Theorem 5.2. Now, we will show that (5.2) cannot be improved. Let $\mathrm{cv}(\eta)$ be the set consisting of all critical values of η. Since $a \in \Sigma$ if and only if there exists $x \in \mathbb{R}$ such that $\frac{d(g+\pi)}{dx}(x) = 0$ and $\frac{d^2(g+\pi)}{dx^2}(x) = 0$ (i.e. $a = \eta(x)$ and $\frac{d\eta}{dx}(x) = 0$) if and only if $a \in \mathrm{cv}(\eta)$, we obtain $\Sigma = \mathrm{cv}(\eta)$, which implies that $\mathrm{HD}_{\mathbb{R}}(\Sigma) = \frac{1}{r-1}$. Namely, we cannot improve the assumption (5.2), which implies

[1]In [2, Example 4.2], in general, there exists a C^r mapping $\widetilde{\eta} : \mathbb{R}^m \to \mathbb{R}^m$ such that the Hausdorff dimension of the set consisting of all critical values of $\widetilde{\eta}$ (i.e. the set of all $\widetilde{\eta}(x) \in \mathbb{R}^m$ such that $x \in \mathbb{R}^m$ satisfies $\operatorname{rank} d\widetilde{\eta}_x < m$) is equal to $m - 1 + \frac{1}{r}$, where r is a positive integer.

that (5.1) is the best evaluation in general.

Now, by using this example, we simply explain an advantage of Theorem 5.2 compared to Proposition 5.1. By the above argument, in the case $r \geq 3$, we have

$$\mathrm{HD}_{\mathbb{R}}(\Sigma) < \mathrm{HD}_{\mathbb{R}}(K) = \frac{\log 2}{\log 3} = 0.63\cdots,$$

where K is the Cantor set in $\mathbb{R}$. Thus, in the case $r \geq 3$. the set Σ is never equal to K.

On the other hand, in the case $r = 2$, there exists an example such that the bad set is equal to K as follows. Let $h : \mathbb{R} \to \mathbb{R}$ be the C^2 function defined by $h(x) = -\int_0^x \xi(x)dx$, where $\xi : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that the set consisting of all critical values of ξ is the Cantor set K. Note that the existence of the mapping ξ can be easily shown from [12, Proposition 2 (p. 1485)][2]. Set

$$\Sigma' = \{\, a \in \mathbb{R} \mid h + \pi : \mathbb{R} \to \mathbb{R} \text{ is not a Morse function, where } \pi(x) = ax \,\}.$$

Since we have $a \in \Sigma'$ if and only if there exists $x \in \mathbb{R}$ such that $\frac{d(h+\pi)}{dx}(x) = 0$ and $\frac{d^2(h+\pi)}{dx^2}(x) = 0$ (i.e. $a = \xi(x)$ and $\frac{d\xi}{dx}(x) = 0$), which is equivalent to that $a \in K$, we obtain $\Sigma' = K$.

Since any subset of $\mathbb{R}$ whose Hausdorff dimension is strictly smaller than 1 has Lebesgue measure zero in $\mathbb{R}$, we cannot investigate whether the bad set is equal to K or not by Proposition 5.1. On the other hand, as in the case $r \geq 3$ of this example, we can see that the bad set Σ is never equal to K by Theorem 5.2.

Here, we give a remark on the assumption (2.1) of Theorem 2.3.

Remark 5.5. In Theorem 2.3, let $X = V = \mathbb{R}$, and let f and g be the functions defined in Example 5.4. Then, Σ_1 in Theorem 2.3 is expressed as follows:

$$\Sigma_1 = \{\, \pi \in \mathcal{L}(\mathbb{R}, \mathbb{R}) \mid j^1(g + \pi) \text{ is not transverse to } S^1(\mathbb{R}, \mathbb{R}) \,\}.$$

Since $\dim \mathbb{R} - \operatorname{codim} S^1(\mathbb{R}, \mathbb{R}) = 0$, for any real number s satisfying $s \geq \frac{1}{r-1}$, the set Σ_1 has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}, \mathbb{R})$ by Theorem 2.3 (1). Since Σ_1 can be identified with the set Σ in Example 5.4, we have $\mathrm{HD}_{\mathcal{L}(\mathbb{R},\mathbb{R})}(\Sigma_1) = \frac{1}{r-1}$. Thus, the assumption $s \geq \frac{1}{r-1}$ cannot be improved, which implies that (2.1) is the best evaluation in general.

Next, we consider the case $\ell \geq 2 \dim X$. In [7], the following result is also obtained as an application of Proposition 2.2.

Proposition 5.6 ([7]). *Let f be a C^r immersion of an n-dimensional C^r manifold X into an open subset V of $\mathbb{R}^m$, and $g : V \to \mathbb{R}^\ell$ a C^r mapping, where $\ell \geq 2n$ and $r \geq 2$. Then, the following set*

$$\Sigma = \{\, \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid (g + \pi) \circ f : X \to \mathbb{R}^\ell \text{ is not an immersion} \,\}$$

[2]Proposition 2 of [12] is as follows: If $\widetilde{K}$ is a compact subset of the closed interval [0,1], then $\widetilde{K}$ has Lebesgue measure zero if and only if the set consisting of all critical values of $\widetilde{\xi}$ is equal to $\widetilde{K}$ for some C^1 function $\widetilde{\xi} : \mathbb{R} \to \mathbb{R}$. Since the Cantor set K is a compact subset of $[0, 1]$ with Lebesgue measure zero, we can guarantee the existence of the above function $\xi : \mathbb{R} \to \mathbb{R}$.

has Lebesgue measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

As a side note, Corollary 3 of [6] is Proposition 5.6 in the case where all manifolds and mappings are of class C^∞. Namely, Proposition 5.6 is an improvement of Corollary 3 of [6]. In this paper, by using the main theorem, we further upgrade Proposition 5.6 from a new perspective of Hausdorff measures as follows:

THEOREM 5.7. *Let f be a C^r immersion of an n-dimensional C^r manifold X into an open subset V of $\mathbb{R}^m$, and $g : V \to \mathbb{R}^\ell$ a C^r mapping, where $\ell \geq 2n$ and $r \geq 2$. Set*

$$\Sigma = \{\, \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid (g + \pi) \circ f : X \to \mathbb{R}^\ell \text{ is not an immersion} \,\}.$$

Then, for any real number s satisfying

$$s > m\ell + (2n - \ell - 1), \tag{5.3}$$

the set Σ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

Proof of Theorem 5.7. Let k be an integer satisfying $1 \leq k \leq n$. As in Theorem 2.3, set

$$\Sigma_k = \{\, \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid j^1((g + \pi) \circ f) \text{ is not transverse to } S^k(X, \mathbb{R}^\ell) \,\}.$$

Since $\ell \geq 2n$, we have

$$\dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell) \leq \dim X - \operatorname{codim} S^1(X, \mathbb{R}^\ell) = 2n - \ell - 1 < 0.$$

Hence, since

$$s > m\ell + (2n - \ell - 1) \geq m\ell + (\dim X - \operatorname{codim} S^k(X, \mathbb{R}^\ell)),$$

by Theorem 2.3 (2), we have the following:

(a) The set Σ_k has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

(b) For any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \setminus \Sigma_k$, we have $j^1((g + \pi) \circ f)(X) \cap S^k(X, \mathbb{R}^\ell) = \varnothing$.

By (b), we can easily obtain $\Sigma = \bigcup_{k=1}^n \Sigma_k$. By (a), the set Σ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. □

As in the following example (Example 5.8), there exists an example such that (5.3) in Theorem 5.7 cannot be improved. Namely, (5.3) is the best evaluation in general. In Example 5.8, we also explain an advantage of Theorem 5.7 from a new perspective of Hausdorff measures compared to Proposition 5.6 from the viewpoint of Lebesgue measures.

EXAMPLE 5.8 (An example of Theorem 5.7). Set $X = V = \mathbb{R}$ and $f(x) = x$ in Theorem 5.7. Let $g : \mathbb{R} \to \mathbb{R}^\ell$ ($\ell \geq 2$) be the C^∞ mapping defined by $g(x) = (x^2, \ldots, x^2)$. As in Theorem 5.7, set

$$\Sigma = \{\, \pi \in \mathcal{L}(\mathbb{R}, \mathbb{R}^\ell) \mid g + \pi : \mathbb{R} \to \mathbb{R}^\ell \text{ is not an immersion} \,\}.$$

Then, for any real number s satisfying $s > 1$, the set Σ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)$ by Theorem 5.7. On the other hand, by the following direct calculation, we obtain $\Sigma = B$, where

$$B = \{ \pi = (\pi_1, \ldots, \pi_\ell) \in \mathcal{L}(\mathbb{R}, \mathbb{R}^\ell) \mid \pi_1 = \cdots = \pi_\ell \}.$$

Since B does not have 1-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)$, we cannot improve the assumption $s > 1$, which means that in general, (5.3) cannot be improved.

Now, we prove $\Sigma = B$. First, we show $\Sigma \subset B$. Let $\pi = (\pi_1, \ldots, \pi_\ell) \in \Sigma$ be an arbitrary element. Set $\pi_i(x) = a_i x$ ($a_i \in \mathbb{R}$) for $i = 1, \ldots, \ell$. Then, there exists $\widetilde{x} \in \mathbb{R}$ such that $2\widetilde{x} + a_i = 0$ for all $i = 1, \ldots, \ell$. Since $a_1 = \cdots = a_\ell$, we have $\pi \in B$.

Next, we show $B \subset \Sigma$. Let $\pi \in B$ be an arbitrary element. Then, we can express $\pi_i(x) = ax$ ($a \in \mathbb{R}$) for all $i = 1, \ldots, \ell$. Set $\widetilde{x} = -\frac{a}{2}$. Since $d(g + \pi)_{\widetilde{x}} = 0$, we obtain $\pi \in \Sigma$.

Now, by using this example, we explain an advantage of Theorem 5.7 compared to Proposition 5.6. Since any subset of $\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)$ whose Hausdorff dimension is strictly smaller than ℓ has Lebesgue measure zero in $\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)$, we cannot estimate the Hausdorff dimension of the bad set Σ by Proposition 5.6. On the other hand, by Theorem 5.7, we have an estimate of the Hausdorff dimension of the bad set, such as $\mathrm{HD}_{\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)}(\Sigma) \leq 1$. For example, we consider the case of $\ell \geq 3$. Since a “surface” such as a 2-dimensional sphere has Lebesgue measure zero in $\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)$, we cannot exclude the possibility that the bad set is such a 2-dimensional set by Proposition 5.6. On the other hand, by using Theorem 5.7, we can conclude that the bad set is never equal to a “surface” such as a 2-dimensional sphere, since $\mathrm{HD}_{\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)}(\Sigma) \leq 1$.

Here, we give a remark on the assumption (2.2) of Theorem 2.3.

REMARK 5.9. In Theorem 2.3, let $X = V = \mathbb{R}$ and let f and $g : \mathbb{R} \to \mathbb{R}^\ell$ ($\ell \geq 2$) be the mappings defined in Example 5.8. Then, Σ_1 in Theorem 2.3 is expressed as follows:

$$\Sigma_1 = \{ \pi \in \mathcal{L}(\mathbb{R}, \mathbb{R}^\ell) \mid j^1(g + \pi) \text{ is not transverse to } S^1(\mathbb{R}, \mathbb{R}^\ell) \}.$$

Since $\dim \mathbb{R} - \operatorname{codim} S^1(\mathbb{R}, \mathbb{R}^\ell) = 1 - \ell < 0$, for any real number s satisfying $s > 1$, the set Σ_1 has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)$ by Theorem 2.3 (2a). Since Σ_1 is equal to the set Σ in Example 5.8, we cannot improve the assumption $s > 1$, which implies that (2.2) is the best evaluation in general.

6. An application of the main theorem to multiobjective optimization

The purpose of this section is to give an application of the main theorem to multiobjective optimization from the viewpoint of differential topology and singularity theory (see Theorem 6.4) . For a positive integer ℓ, set

$$L = \{ 1, \ldots, \ell \}.$$

We consider the problem of optimizing several functions simultaneously. More precisely, let $f : X \to \mathbb{R}^\ell$ be a mapping, where X is a given arbitrary set. A point $x \in X$

is called a *Pareto solution* of f if there does not exist another point $y \in X$ such that $f_i(y) \leq f_i(x)$ for all $i \in L$ and $f_j(y) < f_j(x)$ for at least one index $j \in L$. We denote the set consisting of all Pareto solutions of f by $X^*(f)$, which is called the *Pareto set* of f. The set $f(X^*(f))$ is called the *Pareto front* of f. The problem of determining $X^*(f)$ is called the *problem of minimizing* f.

Let $f = (f_1, \ldots, f_\ell) : X \to \mathbb{R}^\ell$ be a mapping, where X is a given arbitrary set. For a non-empty subset $I = \{ i_1, \ldots, i_k \}$ of L such that $i_1 < \cdots < i_k$, set

$$f_I = (f_{i_1}, \ldots, f_{i_k}).$$

The problem of determining $X^*(f_I)$ is called a *subproblem* of the problem of minimizing f. Set

$$\Delta^{\ell-1} = \left\{ (w_1, \ldots, w_\ell) \in \mathbb{R}^\ell \;\middle|\; \sum_{i=1}^{\ell} w_i = 1,\ w_i \geq 0 \right\}.$$

We also denote a face of $\Delta^{\ell-1}$ for a non-empty subset I of L by

$$\Delta_I = \{ (w_1, \ldots, w_\ell) \in \Delta^{\ell-1} \mid w_i = 0 \ (i \notin I) \}.$$

In this section, for a C^r manifold N (possibly with corners) and a subset V of $\mathbb{R}^\ell$, a mapping $g : N \to V$ is called a C^r *mapping* (resp., a C^r *diffeomorphism*) if $g : N \to \mathbb{R}^\ell$ is of class C^r (resp., if $g : N \to \mathbb{R}^\ell$ is a C^r immersion and $g : N \to V$ is a homeomorphism), where r is a positive integer or $r = \infty$. Here, C^0 mappings and C^0 diffeomorphisms are continuous mappings and homeomorphisms, respectively.

DEFINITION 6.1 ([4, 5]). Let $f = (f_1, \ldots, f_\ell) : X \to \mathbb{R}^\ell$ be a mapping, where X is a subset of $\mathbb{R}^m$. Let r be an integer satisfying $r \geq 0$ or $r = \infty$. The problem of minimizing f is C^r *simplicial* if there exists a C^r mapping $\Phi : \Delta^{\ell-1} \to X^*(f)$ such that both the mappings $\Phi|_{\Delta_I} : \Delta_I \to X^*(f_I)$ and $f|_{X^*(f_I)} : X^*(f_I) \to f(X^*(f_I))$ are C^r diffeomorphisms for any non-empty subset I of L. The problem of minimizing f is C^r *weakly simplicial* if there exists a C^r mapping $\phi : \Delta^{\ell-1} \to X^*(f)$ such that $\phi(\Delta_I) = X^*(f_I)$ for any non-empty subset I of L.

As described in [4], simpliciality is an important property, which can be seen in several practical problems ranging from the facility location problem studied half a century ago [10] to sparse modeling actively developed today [4]. If a problem is simplicial, then we can efficiently compute a parametric-surface approximation of the entire Pareto set with few sample points [9].

A subset X of $\mathbb{R}^m$ is *convex* if $tx + (1-t)y \in X$ for all $x, y \in X$ and all $t \in [0, 1]$. Let X be a convex set in $\mathbb{R}^m$. A function $f : X \to \mathbb{R}$ is *strongly convex* if there exists $\alpha > 0$ such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}\alpha t(1-t) \|x - y\|^2$$

for all $x, y \in X$ and all $t \in [0, 1]$, where $\|z\|$ is the Euclidean norm of $z \in \mathbb{R}^m$. The constant α is called a *convexity parameter* of the function f. A mapping $f = (f_1, \ldots, f_\ell) : X \to \mathbb{R}^\ell$ is *strongly convex* if f_i is strongly convex for any $i \in L$.

THEOREM 6.2 ([4, 5]). *Let $f : \mathbb{R}^m \to \mathbb{R}^\ell$ be a strongly convex C^r mapping, where r is a positive integer or $r = \infty$. Then, the problem of minimizing f is C^{r-1} weakly simplicial. Moreover, this problem is C^{r-1} simplicial if the rank of the differential df_x is equal to $\ell - 1$ for any $x \in X^*(f)$.*

Moreover, in [4], the following result is obtained as an application of Proposition 2.2.

PROPOSITION 6.3 ([4]). *Let $f : \mathbb{R}^m \to \mathbb{R}^\ell$ $(m \geq \ell)$ be a strongly convex C^r mapping, where r is an integer satisfying $r \geq 2$ or $r = \infty$. Set*

$$\Sigma = \{\, \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid \textit{The problem of minimizing } f + \pi \textit{ is not } C^{r-1} \textit{ simplicial} \,\}.$$

If $m - 2\ell + 4 > 0$, then Σ has Lebesgue measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

By using the main theorem, we can also upgrade Proposition 6.3 as follows:

THEOREM 6.4. *Let $f : \mathbb{R}^m \to \mathbb{R}^\ell$ $(m \geq \ell)$ be a strongly convex C^r mapping, where r is an integer satisfying $r \geq 2$ or $r = \infty$. Set*

$$\Sigma = \{\, \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid \textit{The problem of minimizing } f + \pi \textit{ is not } C^{r-1} \textit{ simplicial} \,\}.$$

If $m - 2\ell + 4 > 0$, then for any non-negative real number s satisfying

$$s > m\ell - (m - 2\ell + 4), \tag{6.1}$$

the set Σ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

As in the following example (Example 6.6), there exists an example such that (6.1) in Theorem 6.4 cannot be improved. Namely, (6.1) is the best evaluation in general. In Example 6.6, we also explain an advantage of Theorem 6.4 from a new perspective of Hausdorff measures compared to Proposition 6.3 from the viewpoint of Lebesgue measures. Now, in order to show that a given mapping in Example 6.6 is strongly convex, we prepare Lemma 6.5, which is a well-known result (for the proof, for example, see [5]). Let X be a convex subset of $\mathbb{R}^m$. A function $f : X \to \mathbb{R}$ is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in X$ and all $t \in [0, 1]$.

LEMMA 6.5. *Let X be a convex subset of $\mathbb{R}^m$. Then, a function $f : X \to \mathbb{R}$ is strongly convex with a convexity parameter $\alpha > 0$ if and only if the function $g : X \to \mathbb{R}$ defined by $g(x) = f(x) - \frac{\alpha}{2}\|x\|^2$ is convex.*

EXAMPLE 6.6 (An example of Theorem 6.4). Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by $f_i(x_1, x_2) = x_1^2 + x_2^2$ for $i = 1, 2$. Since $g(x) = f_i(x) - \frac{2}{2}\|x\|^2 = 0$ is convex, f is strongly convex by Lemma 6.5, where $x = (x_1, x_2)$. As in Theorem 6.4, set

$$\Sigma = \{\, \pi \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) \mid \text{The problem of minimizing } f + \pi \text{ is not } C^\infty \text{ simplicial} \,\}.$$

Then, for any real number s satisfying $s > 2$, the set Σ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)$ by Theorem 6.4.

On the other hand, by the following direct calculation, we obtain $\Sigma = B$, where

$$B = \{\, \pi = (\pi_1,\pi_2) \in \mathcal{L}(\mathbb{R}^2,\mathbb{R}^2) \mid \pi_1 = \pi_2 \,\}.$$

Since B does not have 2-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)$, we cannot improve the assumption $s > 2$.

Now, we show $\Sigma = B$. First, in order to show that $\Sigma \subset B$, we will show that $\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2) \setminus B \subset \mathcal{L}(\mathbb{R}^2,\mathbb{R}^2) \setminus \Sigma$. Let $\pi \in \mathcal{L}(\mathbb{R}^2,\mathbb{R}^2) \setminus B$ be an arbitrary element. Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism defined by $H(X_1,X_2) = (X_1 - X_2, X_2)$. As $f_1 = f_2$, we obtain $H \circ (f+\pi) = (\pi_1 - \pi_2, f_2 + \pi_2)$. Since $\pi_1 - \pi_2$ is a linear function satisfying $\pi_1 - \pi_2 \neq 0$, it follows that $\operatorname{rank} d(H \circ (f+\pi))_x \geq 1$ for any $x \in \mathbb{R}^2$. As H is a diffeomorphism, we have that $\operatorname{rank} d(f+\pi)_x \geq 1$ for any $x \in \mathbb{R}^2$. By Theorem 6.2, the problem of minimizing $f+\pi$ is C^∞ simplicial. Namely, we obtain $\pi \in \mathcal{L}(\mathbb{R}^2,\mathbb{R}^2) \setminus \Sigma$.

Next, we will show that $B \subset \Sigma$. Let $\pi = (\pi_1,\pi_2) \in B$ be an arbitrary element. Set $\pi_1(x_1,x_2) = \pi_2(x_1,x_2) = a_1x_1 + a_2x_2$, where $a_1, a_2 \in \mathbb{R}$. Since

$$\begin{aligned}(f_i + \pi_i)(x_1,x_2) &= x_1^2 + x_2^2 + a_1x_1 + a_2x_2 \\ &= \left(x_1 + \frac{a_1}{2}\right)^2 + \left(x_2 + \frac{a_2}{2}\right)^2 - \frac{a_1^2 + a_2^2}{4}\end{aligned}$$

for $i = 1,2$, we obtain $X^*(f+\pi) = \left\{\, (-\frac{a_1}{2}, -\frac{a_2}{2}) \,\right\}$ $(\subset \mathbb{R}^2)$. Hence, the problem of minimizing $f+\pi$ is not C^0 simplicial (and hence, not C^∞ simplicial). Namely, we obtain $\pi \in \Sigma$.

Finally, by using this example, we explain an advantage of Theorem 6.4 compared to Proposition 6.3. Since a set whose Hausdorff dimension is equal to 3, such as a 3-dimensional sphere, has Lebesgue measure zero in $\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)$, we cannot exclude the possibility that the bad set Σ is such a "3-dimensional set" by Proposition 6.3. On the other hand, by using Theorem 6.4, we can conclude that Σ is never equal to such a "3-dimensional set" since $\mathrm{HD}_{\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)}(\Sigma) \leq 2$.

7. Proof of Theorem 6.4

Since Theorem 6.4 clearly holds by combining the following two results (Lemmas 7.1 and 7.2) and Theorem 6.2, it is sufficient to prove Lemma 7.2.

LEMMA 7.1 ([4]). *Let $f : \mathbb{R}^m \to \mathbb{R}^\ell$ be a strongly convex mapping. Then, for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$, the mapping $f+\pi : \mathbb{R}^m \to \mathbb{R}^\ell$ is also strongly convex.*

LEMMA 7.2. *Let $f : \mathbb{R}^m \to \mathbb{R}^\ell$ $(m \geq \ell)$ be a C^r mapping $(r \geq 2)$. Set*

$$\Sigma = \{\, \pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) \mid \text{There exists } x \in \mathbb{R}^m \text{ such that } \operatorname{rank} d(f+\pi)_x \leq \ell - 2 \,\}.$$

If $m - 2\ell + 4 > 0$, then for any non-negative real number s satisfying

$$s > m\ell - (m - 2\ell + 4), \tag{7.1}$$

the set Σ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$.

REMARK 7.3. We give the following remarks on Lemma 7.2.

(1) In the case $\ell = 1$, note that $\Sigma = \varnothing$ and $m\ell - (m - 2\ell + 4) = -2$. Thus, in this case, since the set Σ $(= \varnothing)$ has 0-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$, Lemma 7.2 clearly holds.

(2) In the case $\ell \geq 2$, since $m \geq \ell$, we have $\operatorname{codim} S^2(\mathbb{R}^m, \mathbb{R}^\ell) = 2(m - \ell + 2)$. Thus, the inequality (7.1) implies that

$$s > m\ell - (m - 2\ell + 4) = m\ell + m - 2(m - \ell + 2) = m\ell + m - \operatorname{codim} S^2(\mathbb{R}^m, \mathbb{R}^\ell).$$

Proof of Lemma 7.2. By Remark 7.3 (1), it is sufficient to consider the case $\ell \geq 2$. As in Remark 7.3 (2), we have

$$\operatorname{codim} S^2(\mathbb{R}^m, \mathbb{R}^\ell) = 2(m - \ell + 2).$$

Since $m - 2\ell + 4 > 0$, we also have $\operatorname{codim} S^2(\mathbb{R}^m, \mathbb{R}^\ell) > m$.

Let k be an integer satisfying $2 \leq k \leq \ell$. As in Theorem 2.3, set

$$\Sigma_k = \{\, \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid j^1(f + \pi) \text{ is not transverse to } S^k(\mathbb{R}^m, \mathbb{R}^\ell) \,\}.$$

It follows that

$$m - \operatorname{codim} S^k(\mathbb{R}^m, \mathbb{R}^\ell) \leq m - \operatorname{codim} S^2(\mathbb{R}^m, \mathbb{R}^\ell) < 0.$$

By Remark 7.3 (2), note that the real number s in (7.1) satisfies that

$$s > m\ell + m - \operatorname{codim} S^k(\mathbb{R}^m, \mathbb{R}^\ell).$$

Since $r \geq 2$, by Theorem 2.3 (2), we have the following:

(a) The set Σ_k has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

(b) For any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \setminus \Sigma_k$, we have $j^1(f + \pi)(\mathbb{R}^m) \cap S^k(\mathbb{R}^m, \mathbb{R}^\ell) = \varnothing$.

By (b), it is clearly seen that $\Sigma = \bigcup_{k=2}^{\ell} \Sigma_k$. By (a), the set Σ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. $\square$

Acknowledgements

The author is most grateful to the anonymous reviewers for carefully reading the first manuscript of this paper and for giving invaluable suggestions. The author was supported by JSPS KAKENHI Grant Number JP21K13786. He is also grateful for the opportunity to give a talk on Sections 6 and 7 at 2021 IMI Joint Use Research Program, Short-term Joint Research "Mathematics for Evolutionary Computation" in Kyushu University.

References

[1] James William Bruce and Neil Kirk. Generic projections of stable mappings. *Bull. Lond. Math. Soc.*, 32(6):718–728, 2000.

[2] Carlos Gustavo Tamm de Araujo Moreira. Hausdorff measures and the Morse-Sard theorem. *Publ. Mat.*, 45(1):149–162, 2001.

[3] Martin Golubitsky and Victor Guillemin. *Stable Mappings and Their Singularities*, volume 14 of *Graduate Texts in Mathematics*. Springer, New York, 1973.

[4] Naoki Hamada, Kenta Hayano, Shunsuke Ichiki, Yutaro Kabata, and Hiroshi Teramoto. Topology of Pareto sets of strongly convex problems. *SIAM J. Optim.*, 30(3):2659–2686, 2020.

[5] Naoki Hamada and Shunsuke Ichiki. Simpliciality of strongly convex problems. *J. Math. Soc. Japan*, 73(3):965–982, 2021.

[6] Shunsuke Ichiki. Composing generic linearly perturbed mappings and immersions/injections. *J. Math. Soc. Japan*, 70(3):1165–1184, 2018.

[7] Shunsuke Ichiki. Transversality theorems on generic linearly perturbed mappings. *Methods Appl. Anal.*, 25(4):323–335, 2018.

[8] Shunsuke Ichiki. A refined version of parametric transversality theorems. *J. Geom. Anal.*, 32(9), 2022.

[9] Ken Kobayashi, Naoki Hamada, Akiyoshi Sannai, Akinori Tanaka, Kenichi Bannai, and Masashi Sugiyama. Bézier simplex fitting: Describing Pareto fronts of simplicial problems with small samples in multi-objective optimization. In *Proceedings of the Thirty-Third AAAI Conference on Artificial Intelligence*, volume 33 of *AAAI-19*, pages 2304–2313, 2019.

[10] Harold W Kuhn. On a pair of dual nonlinear programs. *Nonlinear Programming*, 1:37–54, 1967.

[11] John Norman Mather. Generic projections. *Ann. of Math. (2)*, 98:226–245, 1973.

[12] Alec Norton. A $C^{1\times\infty}$ function with an interval of critical values. *Indiana Univ. Math. J.*, 40(4):1483–1488, 1991.

Shunsuke ICHIKI

Department of Mathematical and Computing Science, School of Computing, Tokyo Institute of Technology, Tokyo 152-8552, Japan

E-mail: ichiki@c.titech.ac.jp