Long time dynamics of stochastic fractionally dissipative quasi-geostrophic equations with stochastic damping *

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Abstract A stochastic fractionally dissipative quasi-geostrophic equation with stochastic damping is considered in this paper. First, we show that the null solution is exponentially stable in the sense of q^{-}-th moment of $\|\cdot\|_{L^q}$, where $q > 2/(2\alpha-1)$ and q^{-} denotes the number strictly less than q but close to it, and from this fact we further prove that the sample paths of solutions converge to zero almost surely in L^q as time goes to infinity. In particular, a simple example is used to interpret the intuition. Then the uniform boundedness of pathwise solutions in H^s with $s \geq 2-2\alpha$ and $\alpha \in (1/2, 1)$ is established, which implies the existence of non-trivial invariant measures of the quasi-geostrophic equation driven by nonlinear multiplicative noise.

Keywords: Stochastic quasi-geostrophic equation, stochastic damping, moment exponential stability, almost sure exponential stability, invariant measure.

1 Introduction

The quasi-geostrophic equation is an important model in geophysical sciences which describes a kind of dynamics of large-scale phenomena in the atmosphere and ocean, see [38] for more details. The mathematical study of the quasi-geostrophic equation was initiated by Constantin, Majda and Tabak in [7], where they pointed out that it shared many features with 3D Euler equations.

The quasi-geostrophic equation has attracted much attention from both scientists and mathematicians because of its mathematical importance and potential applications in

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meteorology and oceanography. The existence, uniqueness and regularity of solutions to the quasi-geostrophic equation have been considered in [1, 9, 22, 26, 28, 47], and the asymptotic behavior of the quasi-geostrophic equation has been studied in [8, 10, 13, 14, 15, 23, 41, 44]. The result of [11] has shown the nonlinear stability of steady states in $L^2(\mathbb{R}^2)$ for the quasi-geostrophic equation. In [18], the existence of steady states to the quasi-geostrophic equation has been proved, where the global solution were showed to converge to the steady states in $L^p(\mathbb{R}^2)$, $1 \leq p \leq 2$, as time goes to infinity. Recently, we established the existence and uniqueness of the stationary solution to the quasi-geostrophic equation with infinite delay, and used several different methods to analyze its stability in [30]. For other interesting results on the steady state to the quasi-geostrophic equation, we refer the reader to [4] and the references therein. See also the work [32] where the existence and nonuniqueness of steady-state weak solutions to the Navier-Stokes equation have been investigated in dimensions $d \geq 4$.

Despite an extensive literature on the quasi-geostrophic equation, the most results are obtained in the deterministic case. This paper is devoted to the study of a class of s-tochastic quasi-geostrophic equations with stochastic damping. The damping stems from the resistance to the motion of the flow, it describes various physical situations such as friction effects and some dissipative mechanisms [21]. Thanks to the surrounding environment and intrinsic uncertainties, the damping may be associated with hidden unresolved processes. It may be positive or negative (for example, cyclones and anticyclones). Such problem is very important in climate modeling and physical fluid dynamics; see, e.g. [17].

Consider the following quasi-geostrophic equation with stochastic damping on the periodic domain $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$\begin{cases} d\theta(t) + (\kappa(-\Delta)^\alpha \theta(t) + u(t) \cdot \nabla\theta(t) + \zeta(w(t))\theta(t))dt = G(t, \theta(t))dW(t), \\ dw(t) = h(w(t))dt + dB(t), \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1)$, $\kappa > 0$ is a diffusivity coefficient, θ represents the potential temperature, $W(t)$ is a Wiener process on a suitable probability space which will be given below, $B(t)$ is an n-dimensional Wiener process independent of $W(t)$, $\zeta : \mathbb{R}^n \to \mathbb{R}$ denotes the damping rate, $h : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function, and the velocity $u = (u_1, u_2)$ is determined by θ via the formula

$$u = (u_1, u_2) = (-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}), \quad \text{where} \quad (-\Delta)^{\frac{1}{2}}\psi = -\theta, \tag{1.2}$$

or, in a more explicit way

$$u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) \equiv \mathcal{R}^\perp\theta, \tag{1.3}$$

where $\mathcal{R}_j$, $j = 1, 2$, denote the standard 2D Riesz transforms (see, e.g. [35, p.299]). Clearly, the velocity $u = (u_1, u_2)$ is divergence-free. Without loss of generality we may restrict the discussion to flows which have zero average, i.e.,

$$\int_{\mathbb{T}^2} \theta(t, x) dx = 0, \quad \forall t \geq 0.$$

This formulation can be found in many nonlinear models, such as stochastic parameterization Kalman filter, Lagrangian floater and turbulent passive tracer [2, 3, 33]. We also refer to [43] for the two-dimensional advection-diffusion equation with random transport velocity, and [19, 45] for the stochastic lattice model with random viscosity.

Stochastic stability has been one of the most active areas in stochastic analysis and many mathematicians have devoted their interests to it. The reader may find a systematic presentation in the books [20, 31]. First we aim to analyze the exponential stability of solutions to Eq. (1.1). In general, dissipative mechanisms may affect the asymptotic behavior as well as the smoothness of solutions. For instance, the positive damping may produce a stabilization effect on unstable systems. While we wonder how the stochastic damping affects the stability of the system. The main difficulty lies in dealing with the stochastic damping in Eq. (1.1) since some of standard techniques for verifying the exponential stability of the system can not be directly applied to this type of equation. Following the idea of [34], we construct a Lyapunov function that is the product of two parts, one part is a potential that depends on w, the other part is roughly the moment of θ. Here we succeed in generalizing the analysis framework related to stochastic ordinary differential equations to stochastic partial differential equations, which can be regarded as an extension of [34].

From the fact that the solution to Eq. (1.1) converges to the null solution exponentially, it is easy to obtain that the system (1.1) has an invariant measure degenerate at zero. Another purpose of this work is to show the existence of non-trivial invariant measures of the system (1.1). The invariant measure of the quasi-geostrophic equation has been investigated; see, e.g. [40] for the case of non-degenerate additive noise and [6, 46] for the case of degenerate additive noise. However, the invariant measure of the quasi-geostrophic equation driven by multiplicative noise as in (1.1) has never been studied before. This paper contributes to this issue by establishing a close relationship between the uniform boundedness of pathwise solutions and the existence of non-trivial invariant measures.

The rest of paper is organized as follows. In Section 2, we introduce some notations, and briefly recall some necessary estimates and preliminaries related to functional analysis and probability theory. The moment exponential stability and almost sure exponential stability of solutions to Eq. (1.1) is established in Section 3. In Section 4, we first show the uniform boundedness of pathwise solutions in H^s with $s \geq 2-2\alpha$ and $\alpha \in (1/2, 1)$. Then we study the Feller property of solutions, and prove the existence of non-trivial invariant measures for the corresponding Feller semigroup.

2 Preliminaries and notations

2.1 The functional framework

Denote $\Lambda \equiv (-\Delta)^{\frac{1}{2}}$. The fractional Laplacian Λ^s can be defined for $s \in \mathbb{R}$ by

$$\widehat{\Lambda^s f}(k) = |k|^s \widehat{f}(k),$$

where $\widehat{f}$ denotes the Fourier transform of f. Let L^p denote the Banach space of Lebesgue integrable functions and l^p denote the space of sequences. The following standard notations are used:

$$\|f\|_{L^p}^p = \int_{\mathbb{T}^2} |f(x)|^p dx, \quad \|f\|_{L^\infty} = \operatorname{ess}\sup_{x\in\mathbb{T}^2} |f(x)|,$$
$$\|x\|_{l^p}^p = \sum_{j=1}^{\infty} |x_j|^p, \quad \|x\|_{l^\infty} = \sup_{j\in\mathbb{Z}^+} |x_j|.$$

For any tempered distribution f on $\mathbb{T}^2$ and $s \in \mathbb{R}$, we define

$$\|f\|_{H^s}^2 = \|\Lambda^s f\|_{L^2}^2 = \sum_{k\in\mathbb{Z}^2} |k|^{2s} |\widehat{f}(k)|^2,$$

and H^s denotes the Sobolev space of all f for which $\|f\|_{H^s}$ is finite. For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, the space $H^{s,p}$ is a subspace of L^p, consisting of all f which can be written in the form $f = \Lambda^{-s} g$, $g \in L^p$, and the $H^{s,p}$ norm of f is defined by

$$\|f\|_{H^{s,p}}^p = \|\Lambda^s f\|_{L^p}^p.$$

In a similar way, we can define these kinds of spaces for vector functions. For convenience, we will use the same notation to denote the norms for vector and scalar functions. For example, if $v(x) = (v_1(x), v_2(x), \ldots)$ is an l^2-valued measurable function on $\mathbb{T}^2$, then

$$\|v\|_{L^p}^p = \int_{\mathbb{T}^2} \|v(x)\|_{l^2}^p dx = \int_{\mathbb{T}^2} \left(\sum_{k=1}^{\infty} |v_k(x)|^2 \right)^{\frac{p}{2}} dx.$$

We denote by $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ the inner products of L^2 and l^2 respectively. Given a Banach space X and its dual X', we also denote the dual pairing between X and X' by $\langle \cdot, \cdot \rangle$, unless noted otherwise.

The following result can be obtained by the fact that the Riesz transforms commute with $(-\Delta)^l$ and the boundedness of the Riesz transforms in L^p; see [42, Chapter III] for more details.

Lemma 2.1. *Let $1 < p < \infty$ and $l \geq 0$. Then there exists a constant $C(l, p)$ such that*

$$\|(-\Delta)^l u\|_{L^p} \leq C(l,p)\|(-\Delta)^l \theta\|_{L^p}. \tag{2.1}$$

If $p = 2$, the inequality (2.1) *can be strengthened to*

$$\|(-\Delta)^l u\|_{L^2} = \|(-\Delta)^l \theta\|_{L^2}. \tag{2.2}$$

We recall some important estimates which will be used frequently in the sections below.

Lemma 2.2. *Suppose that $s > 0$ and $1 < p < \infty$. If $f, g \in \mathcal{S}$, the Schwartz class, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C_1 \left(\|\nabla f\|_{L^{p_1}} \|g\|_{H^{s-1,p_2}} + \|f\|_{H^{s,p_3}} \|g\|_{L^{p_4}}\right)$$

and

$$\|\Lambda^s(fg)\|_{L^p} \leq C_2 \left(\|f\|_{L^{p_1}} \|g\|_{H^{s,p_2}} + \|f\|_{H^{s,p_3}} \|g\|_{L^{p_4}}\right),$$

with $p_2, p_3 \in (1, \infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Lemma 2.3. *Suppose that $s \in [0, 2]$, $\theta, \Lambda^s\theta \in L^p$, where $p \geq 2$. Then*

$$\int_{\mathbb{T}^2} |\theta|^{p-2}\theta\Lambda^s\theta dx \geq \frac{2}{p}\int_{\mathbb{T}^2} \left(\Lambda^{\frac{s}{2}}|\theta|^{\frac{p}{2}}\right)^2 dx. \tag{2.3}$$

Lemma 2.2 is the famous commutator estimates which have been proved in [24, 25], and Lemma 2.3 is an improved version of the positivity lemma presented in [23]. When p is even, we can also refer to [9].

Let Y be a Banach space with the norm $\|\cdot\|$. Next we introduce the notions of the exponential stability.

Definition 2.4. The null solution to Eq. (1.1) is said to be exponentially stable in p-th moment with $p \geq 2$, if there exist some constants $a > 0$ and $M > 0$ such that

$$\mathbb{E}\|\theta(t)\|_Y^p \leq Me^{-at}, \quad t \geq 0.$$

Definition 2.5. The null solution to Eq. (1.1) is said to be almost surely exponentially stable in Y, if there exists $\tau > 0$ such that

$$\lim_{t\to\infty} \frac{1}{t}\log\|\theta(t)\|_Y \leq -\tau, \quad \text{almost surely.}$$

2.2 The stochastic framework

Given a filtered probability space $(\Omega_W, \mathcal{F}_W, \{\mathcal{F}_{Wt}\}_{t\geq 0}, P_W, \{W_k\}_{k\geq 1})$, where $\{W_k\}_{k\geq 1}$ is a sequence of mutually independent standard one dimensional Brownian motions adapted to a complete and right continuous filtration $\{\mathcal{F}_{Wt}\}_{t\geq 0}$. Let $(\Omega_B, \mathcal{F}_B, \{\mathcal{F}_{Bt}\}_{t\geq 0}, P_B, B)$ be another filtered probability space, where B is an n-dimensional Wiener process independent of $\{W_k\}_{k\geq 1}$. Define the product probability space by

$$(\Omega, \mathcal{F}, P) := (\Omega_W \times \Omega_B, \mathcal{F}_W \times \mathcal{F}_B, P_W \times P_B).$$

Let $\mathbb{E}$ denote the expectation under the probability measure P. We denote by X a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_X$ and norm $\| \cdot \|_X$. Let $\mathcal{U}$ be a separable Hilbert space with an orthonormal basis $\{e_k\}_{k\geq 1}$. Define W by taking

$$W(t) = \sum_{k=1}^{\infty} W_k(t) e_k, \quad t \geq 0.$$

Such $W(t)$ is a cylindrical Wiener process evolving over $\mathcal{U}$. Let $L_2(\mathcal{U}, X)$ be the collection of all Hilbert-Schmidt operators from $\mathcal{U}$ to X, endowed with the norm

$$\|\phi\|^2_{L_2(\mathcal{U},X)} = \sum_{k=1}^{\infty} \|\phi e_k\|^2_X.$$

Similarly we adopt for any $p \geq 2$,

$$\|\phi\|^p_{L_p(\mathcal{U},L^p)} = \int_{\mathbb{T}^2} \left(\sum_{k=1}^{\infty} |\phi(x) e_k|^2 \right)^{\frac{p}{2}} dx.$$

For any $T > 0$, given an X-valued adapted process $G \in L^2(\Omega_W \times [0,T]; L_2(\mathcal{U}, X))$, we can define the Itô stochastic integral

$$M_t := \int_0^t G dW = \sum_{k=1}^{\infty} \int_0^t G_k dW_k, \quad \text{where } G_k = G e_k,\ t \in [0,T].$$

Clearly $\{M_t\}_{t\geq 0}$ is an X-valued square integrable martingale (cf. [39]). Particularly, the Burkholder-Davis-Gundy inequality holds which in the present context takes the form

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left\| \int_0^t G dW \right\|_X^r \right) \leq C \mathbb{E}\left(\int_0^T \|G\|^2_{L_2(\mathcal{U},X)} dt \right)^{\frac{r}{2}}, \tag{2.4}$$

where $r \geq 1$ and C is a positive constant depending only on r.

In the following lemma, we present a general integration by parts formula [37, p.55] and the general Itô formula [37, p.48].

Lemma 2.6. *Let $X(t)$ and $Y(t)$ be Itô processes in $\mathbb{R}$. Then*

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t) \cdot dY(t). \tag{2.5}$$

In order to study the stability of the system (1.1), we shall also need the notion of asymptotical contraction.

Definition 2.7. Given two probability measures μ and ν on $\mathbb{R}^n$, we use $d(\mu, \nu)$ to denote the Wasserstein-1 distance between μ and ν, generated by the $\mathbb{R}^n$ norm. Let $\mathcal{P}_t^w$ denote the distribution of $w(t)$ with initial value w. We say $w(t)$ is asymptotically contractive if there are constants $C_\gamma, \gamma > 0$ such that

$$d(\mathcal{P}_t^{w_1}, \mathcal{P}_t^{w_2}) \leq C_\gamma e^{-\gamma t} \|w_1 - w_2\|_{\mathbb{R}^n}$$

holds for all w_1, w_2 and $t \geq 0$.

The following lemma will provide an example to show that a stable OU process is asymptotically contractive.

Lemma 2.8. *Consider the one-dimensional OU process $dw_t = -\gamma w_t dt + dB_t$ with initial datum $w_0 = w$, where B_t is one-dimensional Wiener process. If $\gamma > 0$, then w_t is asymptotically contractive.*

Proof. Consider another one-dimensional OU process $dv_t = -\gamma v_t dt + dB_t$ with initial datum $v_0 = v$, where B_t is the same as the one in the SDE of w_t. We observe that

$$d(w_t - v_t) = -\gamma(w_t - v_t)dt \Rightarrow |w_t - v_t| = e^{-\gamma t}|w - v|.$$

We denote by $\mathcal{P}_t^w$ and $\mathcal{P}_t^v$ the distributions of w_t and v_t, respectively. Then

$$d(\mathcal{P}_t^w, \mathcal{P}_t^v) \leq e^{-\gamma t}|w - v|,$$

which implies that w_t is asymptotically contractive. □

To describe the conditions imposed for the systems in this paper, we introduce some notations. For any pair of Banach spaces $\mathscr{X}, \mathscr{Y}$, we denote by $Bnd_u(\mathscr{X}, \mathscr{Y})$ the collection of all mappings $g = g(t, x) : [0, \infty) \times \mathscr{X} \to \mathscr{Y}$ which are essentially bounded in time, continuous in x and $\{\mathcal{F}_t\}_{t \geq 0}$ adapted such that

$$\|g(t, x)\|_{\mathscr{Y}} \leq \|g\|_{Bnd}(1 + \|x\|_{\mathscr{X}}), \quad \text{for any } t \geq 0 \text{ and all } x \in \mathscr{X},$$

where $\|g\|_{Bnd}$ is a positive constant, which is independent of t so that g is uniformly bounded in t. If, in addition, $g \in Bnd_u(\mathscr{X}, \mathscr{Y})$ satisfies

$$\|g(t, x) - g(t, y)\|_{\mathscr{Y}} \leq \|g\|_{Lip}\|x - y\|_{\mathscr{X}}, \quad \text{for any } t \geq 0 \text{ and all } x, y \in \mathscr{X},$$

we say that g is in $Lip_u(\mathscr{X},\mathscr{Y})$. Here and below, $\|g\|_{Lip}$ denotes the Lipschitz constant of any function $g \in Lip_u(\mathscr{X},\mathscr{Y})$.

Let C denote a real positive constant which can vary from a line to another and even in the same line. If the constant C depends on some variable x, we denote it by C_x. For $r \in \mathbb{R}$, we denote by r^- the number strictly less than r but close to it.

3 Exponential stability of the solution

The global existence of pathwise solutions to Eq. (1.1) can be established by a similar method to the one in the proof of [29, Theorem 3.4].

Theorem 3.1. *Fix $\alpha \in (\frac{1}{2},1)$, $s \geq 2-2\alpha$ and a stochastic basis $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P,W)$. Suppose that $G \in Bnd_u(L^q, L_q(\mathcal{U},L^q)) \cap Lip_u(H^s, L_2(\mathcal{U},H^s))$ with $q > \frac{2}{2\alpha-1}$, and $\zeta(w) \in [-L,L]$ for some positive constant L. Then for any initial valve $\theta(0) \in H^s \cap L^q$, there exists a unique global pathwise solution θ to Eq.* (1.1) *such that for any $T>0$,*

$$\theta(t) \in L^2(\Omega; C(0,T;H^s)) \cap L^2(\Omega; L^2(0,T;H^{s+\alpha})). \tag{3.1}$$

It is clear that the null solution is a stationary solution to Eq. (1.1). In the following, we analyze the exponential stability of the null solution to Eq. (1.1). Hence in the rest of this work, we assume that the conditions in Theorem 3.1 are always satisfied, and let θ be the unique global pathwise solution to Eq. (1.1).

First we present a result on the stability of the system (1.1) when the damping term is a positive constant, i.e., the stability of the following equation

$$d\theta(t) + (\kappa(-\Delta)^\alpha\theta(t) + u(t)\cdot\nabla\theta(t) + \zeta_0\theta(t))dt = G(t,\theta)dW(t), \tag{3.2}$$

with the initial datum $\theta(0)$, where ζ_0 is a positive constant.

Lemma 3.2. *Let $q > \frac{2}{2\alpha-1}$ with $\alpha \in (\frac{1}{2},1)$. Assume $G \in Lip_u(L^q, L_q(\mathcal{U},L^q))$ satisfying $G(t,0)=0$ for all $t \geq 0$. If*

$$2\zeta_0 > (q-1)\|G\|^2_{Lip}, \tag{3.3}$$

then the solution θ to Eq. (3.2) *converges to zero exponentially in q-th moment, namely, there exists a constant $a_0 > 0$ such that*

$$\mathbb{E}\|\theta(t)\|^q_{L^q} \leq e^{-a_0 t}\mathbb{E}\|\theta(0)\|^q_{L^q}, \quad t \geq 0.$$

Proof. Let $\phi \geq 0$ be a smooth function with supp$\phi \subset [1,2]$ and $\int_0^\infty \phi(t)dt = 1$. For $\varepsilon > 0$, define

$$U_\varepsilon[\theta](t) := \int_0^\infty \phi(\tau)(\rho_\varepsilon * \mathcal{R}^\perp\theta)(t-\varepsilon\tau)d\tau,$$

where ρ_ε is the periodic Poisson kernel in $\mathbb{T}^2$ given by $\widehat{\rho_\varepsilon}(\xi) = e^{-\varepsilon|\xi|}$, $\xi \in \mathbb{Z}^2$, and we set $\theta(t) = 0$ for $t < 0$. Take a sequence $\varepsilon_n \to 0$ and consider the equation

$$d\theta_n(t) + (\kappa(-\Delta)^\alpha \theta_n(t) + u_n(t)\cdot\nabla\theta_n(t) + \zeta_0\theta_n(t))dt = \rho_{\varepsilon_n} * G(t,\theta_n)dW(t), \tag{3.4}$$

with initial data $\theta_n(0) = \rho_{\varepsilon_n} * \theta(0)$ and $u_n = U_{\varepsilon_n}[\theta_n]$, where $\rho_{\varepsilon_n} * G(t,\theta_n)$ means for $y \in U$, $\rho_{\varepsilon_n} * G(t,\theta_n)(y) := \rho_{\varepsilon_n} * (G(t,\theta_n)(y))$. Let $a_0 > 0$ be a fixed constant which will be determined later on. By Lemma 5.1 in [27], we have

$$\begin{aligned}
e^{a_0 t}\|\theta_n(t)\|_{L^q}^q = {} & \|\rho_{\varepsilon_n} * \theta(0)\|_{L^q}^q + (a_0 - q\zeta_0)\int_0^t e^{a_0 r}\|\theta_n(r)\|_{L^q}^q dr \\
& - q\kappa \int_0^t \int_{\mathbb{T}^2} e^{a_0 r}|\theta_n(r)|^{q-2}\theta_n(r)(-\Delta)^\alpha\theta_n(r)dxdr \\
& - q\int_0^t\int_{\mathbb{T}^2} e^{a_0 r}|\theta_n(r)|^{q-2}\theta_n(r)(u_n(r)\cdot\nabla\theta_n(r))dxdr \\
& + \frac{q(q-1)}{2}\int_0^t\int_{\mathbb{T}^2} e^{a_0 r}|\theta_n(r)|^{q-2}\sum_{k=1}^\infty |\rho_{\varepsilon_n} * G(r,\theta_n)(e_k)|^2 dxdr \\
& + q\int_0^t\int_{\mathbb{T}^2} e^{a_0 r}|\theta_n(r)|^{q-2}\theta_n(r)\rho_{\varepsilon_n} * G(r,\theta_n)dxdW(r).
\end{aligned} \tag{3.5}$$

For the third term on the right-hand side of (3.5), it can be deduced from Lemma 2.3 that

$$\begin{aligned}
& - q\kappa \int_0^t \int_{\mathbb{T}^2} e^{a_0 r}|\theta_n(r)|^{q-2}\theta_n(r)(-\Delta)^\alpha\theta_n(r)dxdr \\
& \le -2\kappa\int_0^t\int_{\mathbb{T}^2} e^{a_0 r}\left((-\Delta)^{\frac{\alpha}{2}}|\theta_n(r)|^{\frac{q}{2}}\right)^2 dxdr \le 0.
\end{aligned} \tag{3.6}$$

Note that $\nabla\cdot u_n = 0$. Then we infer that

$$\begin{aligned}
q\int_{\mathbb{T}^2}|\theta_n(r)|^{q-2}\theta_n(r)(u_n(r)\cdot\nabla\theta_n(r))dx &= q\int_{\mathbb{T}^2}|\theta_n(r)|^{q-2}\theta_n(r)\sum_{i=1}^2 u_{n,i}(r)\frac{\partial\theta_n}{\partial x_i}(r)dx \\
&= \int_{\mathbb{T}^2}\sum_{i=1}^2 u_{n,i}(r)\frac{\partial|\theta_n|^q}{\partial x_i}(r)dx \\
&= -\int_{\mathbb{T}^2}|\theta_n(r)|^q\sum_{i=1}^2\frac{\partial u_{n,i}}{\partial x_i}(r)dx = 0,
\end{aligned} \tag{3.7}$$

where $u_{n,i}$ denotes the i-th component of u_n, $i = 1, 2$. Combining (3.5)-(3.7) and taking expectation, we obtain

$$e^{a_0 t}\mathbb{E}\|\theta_n(t)\|_{L^q}^q \le \mathbb{E}\|\rho_{\varepsilon_n} * \theta(0)\|_{L^q}^q + (a_0 - q\zeta_0)\mathbb{E}\int_0^t e^{a_0 r}\|\theta_n(r)\|_{L^q}^q dr$$

$$+\frac{q(q-1)}{2}\mathbb{E}\int_0^t\int_{\mathbb{T}^2} e^{a_0 r}|\theta_n(r)|^{q-2}\sum_{k=1}^{\infty}|\rho_{\varepsilon_n} * G(r,\theta_n)(e_k)|^2 dxdr. \quad (3.8)$$

By the Hölder inequality, the last term on the right-hand side of (3.8) is bounded by

$$\begin{aligned}
&\frac{q(q-1)}{2}\mathbb{E}\int_0^t\int_{\mathbb{T}^2} e^{a_0 r}|\theta_n(r)|^{q-2}\sum_{k=1}^{\infty}|\rho_{\varepsilon_n} * G(r,\theta_n)(e_k)|^2 dxdr \\
&\le \frac{1}{2}q(q-1)\mathbb{E}\int_0^t e^{a_0 r}\|\theta_n(r)\|_{L^q}^{q-2}\|G(r,\theta_n)\|^2_{L_q(\mathcal{U},L^q)}dr \\
&\le \frac{1}{2}q(q-1)\|G\|^2_{Lip}\mathbb{E}\int_0^t e^{a_0 r}\|\theta_n(r)\|^q_{L^q}dr. \qquad (3.9)
\end{aligned}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned}
e^{a_0 t}\mathbb{E}\|\theta_n(t)\|^q_{L^q} &\le \mathbb{E}\|\rho_{\varepsilon_n} * \theta(0)\|^q_{L^q} + \left(a_0 - q\zeta_0 + \frac{1}{2}q(q-1)\|G\|^2_{Lip}\right)\mathbb{E}\int_0^t e^{a_0 r}\|\theta_n(r)\|^q_{L^q}dr \\
&\le \mathbb{E}\|\theta(0)\|^q_{L^q} + \left(a_0 - q\zeta_0 + \frac{1}{2}q(q-1)\|G\|^2_{Lip}\right)\mathbb{E}\int_0^t e^{a_0 r}\|\theta_n(r)\|^q_{L^q}dr.
\end{aligned}$$

In view of the condition (3.3), we choose $a_0 > 0$ sufficiently small such that

$$a_0 - q\zeta_0 + \frac{1}{2}q(q-1)\|G\|^2_{Lip} < 0.$$

Then we obtain that

$$\mathbb{E}\|\theta_n(t)\|^q_{L^q} \le e^{-a_0 t}\mathbb{E}\|\theta(0)\|^q_{L^q}, \quad t \ge 0.$$

Arguing as in the proof of [40, Theorem 3.3], we conclude that θ_n converge to the solution θ of Eq. (3.2). Thus we further obtain that

$$\mathbb{E}\|\theta(t)\|^q_{L^q} \le e^{-a_0 t}\mathbb{E}\|\theta(0)\|^q_{L^q}, \quad t \ge 0.$$

The proof is therefore complete. □

Remark 3.3. In the following estimates concerning the Krylov's L^q Itô formula, we use the same idea of approximation as in the proof of Lemma 3.2, since the solution θ_n to Eq. (3.4) satisfy the conditions of Lemma 5.1 in [27]. But for simplicity, we consider the solution θ to Eq. (1.1) instead of θ_n directly.

It can be seen from the above lemma that the system (1.1) with $\zeta(w(t)) \equiv \zeta_0$ is stable if the damping coefficient ζ_0 is sufficiently large. In the following, we investigate how the damping function affects the stability of the system if the damping function is considered as a random variable. Arguing as in the proof of Lemma 3.2, we obtain

$$e^{a_0 t}\mathbb{E}\|\theta(t)\|^q_{L^q} \le \mathbb{E}\|\theta(0)\|^q_{L^q} + \mathbb{E}\int_0^t e^{a_0 r}\left(a_0 - q\zeta(w(r)) + \frac{1}{2}q(q-1)\|G\|^2_{Lip}\right)\|\theta(r)\|^q_{L^q}dr.$$

Due to the appearance of $\zeta(w(r))$, the same method as in Lemma 3.2 can not be used to analyze the stability of Eq. (1.1). Inspired by [34], we shall look for a Lyapunov function that is the product of two parts, one part is a potential that depends on w, the other part is roughly the moment of θ. Before proceeding to this issue, we consider a function $\xi : \mathbb{R}^n \to \mathbb{R}$ as follows

$$\xi(w) = -\int_0^\infty \left(\mathbb{E}^w \zeta(w(t)) - \langle \zeta, \pi \rangle\right) dt, \tag{3.10}$$

where $\langle \zeta, \pi \rangle = \int_{\mathbb{R}^n} \zeta(w)\pi(dw)$ is the average of ζ under the equilibrium distribution π. It follows from Lemma A.2 in [34] that the function defined in (3.10) is well defined for any Lipschitz function ζ when $w(t)$ is asymptotically contractive. Moreover, the following properties hold.

Lemma 3.4. *Let $\mathcal{L}$ denote a differential operator on $C^2(\mathbb{R}^n)$ of the form*

$$\mathcal{L} = \sum_{i=1}^n h_i(w)\frac{\partial}{\partial w_i} + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2}{\partial w_i \partial w_j}.$$

Assume that $w(t)$ is asymptotically contractive and $\zeta : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz. Then $\xi(w)$ defined in (3.10) *satisfies*

$$\mathcal{L}\xi(w) = \zeta(w) - \langle \zeta, \pi \rangle, \tag{3.11}$$

and the gradient of $\xi(w)$ with respect to w is bounded, more precisely,

$$\|\nabla_w \xi(w)\|_{l^2} \leq \frac{C_\gamma}{\gamma}\|\zeta\|_{Lip}. \tag{3.12}$$

Proof. Note that

$$\xi(w) = -\int_0^\infty \int_{\mathbb{R}^n} (\zeta(z) - \langle \zeta, \pi \rangle) p_t^w(z) dz dt,$$

where p_t^w is the density of $\mathcal{P}_t^w$. By the Kolmogorov backward equation

$$\frac{\partial}{\partial t} p_t^w = \mathcal{L} p_t^w,$$

we obtain $\xi \in C^2(\mathbb{R}^n)$. Thus the operator $\mathcal{L}$ can be applied to ξ, and the remaining results follow from a combination of Lemma A.2 and Lemma 2.7 in [34]. □

Theorem 3.5. *Let $q > \frac{2}{2\alpha-1}$ with $\alpha \in (\frac{1}{2}, 1)$. Consider $\zeta \in Lip_u(\mathbb{R}^n, \mathbb{R})$ and $G \in Lip_u(L^q, L_q(\mathcal{U}, L^q))$ with $G(t, 0) = 0$ for all $t \geq 0$. In addition to the assumptions*

($\mathbf{A}_1$) *$w(t)$ is asymptotically contractive,*

($\mathbf{A}_2$) *$w(t)$ is ergodic and π is its equilibrium distribution,*

$(\mathbf{A}_3)$ *h dissipates the energy, that is for some constants λ and M_λ,*

$$(h(w), w) \le -\lambda \|w\|_{l^2}^2 + M_\lambda, \tag{3.13}$$

we assume that

$$2\langle \zeta, \pi \rangle > (q-1)\|G\|_{Lip}^2 + q\frac{C_\gamma^2}{\gamma^2}\|\zeta\|_{Lip}^2. \tag{3.14}$$

Then there exist some constants $a_1 > 0$ and $M_1 = M_1(\theta(0), w(0)) > 0$ such that

$$\mathbb{E}\|\theta(t)\|_{L^q}^{q^-} \le M_1 e^{-a_1 t}, \quad t \ge 0. \tag{3.15}$$

Proof. Applying the Itô formula to the function $F_q(\theta) = \|\theta\|_{L^q}^q$ gives

$$\begin{aligned} dF_q(\theta(t)) = &-q\kappa \int_{\mathbb{T}^2} |\theta(t)|^{q-2}\theta(t)(-\Delta)^\alpha \theta(t) dx dt - q\zeta(w(t))\|\theta(t)\|_{L^q}^q dt \\ &+ \frac{q(q-1)}{2}\int_{\mathbb{T}^2} |\theta(t)|^{q-2}\sum_{k=1}^\infty |G(t,\theta(t))e_k|^2 dx dt \\ &+ q\int_{\mathbb{T}^2} |\theta(t)|^{q-2}\theta(t) G(t,\theta(t)) dx dW(t), \end{aligned} \tag{3.16}$$

where we have used (3.7). With the aid of the Itô formula for $\xi(w) \in C^2(\mathbb{R}^n)$ defined in (3.10) and Lemma 3.4, we find that

$$\begin{aligned} d\xi(w(t)) &= \sum_{i=1}^n h_i(w(t))\frac{\partial \xi}{\partial w_i}(w(t))dt + \sum_{i=1}^n \frac{\partial \xi}{\partial w_i}(w(t))dB_i(t) + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 \xi}{\partial w_i \partial w_j}(w(t))dt \\ &= (\zeta(w(t)) - \langle \zeta, \pi \rangle)dt + (\nabla_w \xi(w(t)), dB(t)). \end{aligned} \tag{3.17}$$

Applying the Itô formula to $g(w) = e^{q\xi(w)}$ again yields

$$\begin{aligned} dg(w(t)) &= qg(w(t))d\xi(w(t)) + \frac{1}{2}q^2 g(w(t))d\xi(w(t))d\xi(w(t)) \\ &= \left(q(\zeta(w(t)) - \langle \zeta, \pi \rangle) + \frac{1}{2}q^2\|\nabla_w \xi(w(t))\|_{l^2}^2\right) g(w(t))dt \\ &\quad + qg(w(t))(\nabla_w \xi(w(t)), dB(t)). \end{aligned} \tag{3.18}$$

Take a constant $\sigma > 0$ and let $U_1(t,\theta,w) = e^{\sigma t}F_q(\theta)g(w)$. Since $W(t)$ and $B(t)$ are mutually independent, in view of Lemma 2.6, it follows from (3.16) and (3.18) that

$$\begin{aligned} dU_1(t,\theta(t),w(t)) &= \sigma U_1(t,\theta(t),w(t))dt + e^{\sigma t}g(w(t))dF_q(\theta(t)) \\ &\quad + e^{\sigma t}F_q(\theta(t))dg(w(t)) + e^{\sigma t}dF_q(\theta(t))dg(w(t)) \\ &= \sigma U_1(t,\theta(t),w(t))dt - q\kappa e^{\sigma t}g(w(t))\int_{\mathbb{T}^2} |\theta(t)|^{q-2}\theta(t)(-\Delta)^\alpha \theta(t) dx dt \end{aligned}$$

$$
\begin{aligned}
&+\frac{q(q-1)}{2}e^{\sigma t}g(w(t))\int_{\mathbb{T}^2}|\theta(t)|^{q-2}\sum_{k=1}^{\infty}|G(t,\theta(t))e_k|^2dxdt\\
&-q\langle\zeta,\pi\rangle U_1(t,\theta(t),w(t))dt+\frac{1}{2}q^2\|\nabla_w\xi(w(t))\|_{l^2}^2U_1(t,\theta(t),w(t))dt\\
&+qe^{\sigma t}g(w(t))\int_{\mathbb{T}^2}|\theta(t)|^{q-2}\theta(t)G(t,\theta(t))dxdW(t)\\
&+qU_1(t,\theta(t),w(t))(\nabla_w\xi(w(t)),dB(t)).
\end{aligned}
\tag{3.19}
$$

Similar to the arguments of (3.6), the second term on the right-hand side of the above inequality is bounded by

$$
-q\kappa e^{\sigma t}g(w(t))\int_{\mathbb{T}^2}|\theta(t)|^{q-2}\theta(t)(-\Delta)^{\alpha}\theta(t)dx\le 0. \tag{3.20}
$$

For the third term on the right-hand side of (3.19), we make use of the Hölder inequality and the conditions imposed on G to obtain

$$
\begin{aligned}
&\frac{q(q-1)}{2}e^{\sigma t}g(w(t))\int_{\mathbb{T}^2}|\theta(t)|^{q-2}\sum_{k=1}^{\infty}|G(t,\theta(t))e_k|^2dx\\
&\le\frac{1}{2}q(q-1)e^{\sigma t}g(w(t))\|\theta(t)\|_{L^q}^{q-2}\|G(t,\theta(t))\|_{L_q(\mathcal{U},L^q)}^2\\
&\le\frac{1}{2}q(q-1)\|G\|_{Lip}^2U_1(t,\theta(t),w(t)).
\end{aligned}
\tag{3.21}
$$

Then integrating the equality (3.19) over $[0,t]$ and taking expectation result in

$$
\begin{aligned}
\mathbb{E}U_1(t,\theta(t),w(t))\le\mathbb{E}U_1(0,\theta(0),w(0))+\Big(&\sigma+\frac{1}{2}q(q-1)\|G\|_{Lip}^2\\
&-q\langle\zeta,\pi\rangle+\frac{1}{2}q^2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|_{Lip}^2\Big)\mathbb{E}\int_0^tU_1(r,\theta(r),w(r))dr,
\end{aligned}
\tag{3.22}
$$

where we have used (3.12), (3.20) and (3.21). By the condition (3.14), we choose σ sufficiently small such that

$$
\sigma+\frac{1}{2}q(q-1)\|G\|_{Lip}^2-q\langle\zeta,\pi\rangle+\frac{1}{2}q^2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|_{Lip}^2<0.
$$

Therefore,

$$
\mathbb{E}\left(\|\theta(t)\|_{L^q}^qe^{q\xi(w(t))}\right)\le e^{-\sigma t}\mathbb{E}\left(\|\theta(0)\|_{L^q}^qe^{q\xi(w(0))}\right). \tag{3.23}
$$

In order to remove $g(w(t))$ inside the expectation, we need an estimate of $g(w(t))$. Applying the Itô formula to $e^{\beta\|w\|_{l^2}^2}$ with $\beta<\lambda$, we obtain

$$
e^{\beta\|w(t)\|_{l^2}^2}=e^{\beta\|w(0)\|_{l^2}^2}+2\int_0^t\beta(w(r),h(w(r)))e^{\beta\|w(r)\|_{l^2}^2}dr
$$

$$+\int_0^t \left(n\beta + 2\beta^2\|w(r)\|_{l^2}^2\right) e^{\beta\|w(r)\|_{l^2}^2} dr$$
$$+ 2\beta\int_0^t e^{\beta\|w(r)\|_{l^2}^2}(w(r), dB(r)).$$

Set $\mathcal{H}(t) = \mathbb{E}e^{\beta\|w(t)\|_{l^2}^2}$. Then

$$\mathcal{H}'(t) \le \beta\left(-2(\lambda-\beta)\|w(t)\|_{l^2}^2 + n + 2M_\lambda\right)\mathcal{H}(t), \tag{3.24}$$

thanks to the condition (3.13). When $(\lambda-\beta)\|w(t)\|_{l^2}^2 \le n + 2M_\lambda$,

$$\mathcal{H}(t) \le e^{\frac{\beta(n+2M_\lambda)}{\lambda-\beta}}, \tag{3.25}$$

otherwise,

$$\mathcal{H}'(t) \le -\beta(n+2M_\lambda)\mathcal{H}(t). \tag{3.26}$$

Combining (3.24)-(3.26) gives

$$\mathcal{H}'(t) \le -\beta(n+2M_\lambda)\mathcal{H}(t) + \beta(n+2M_\lambda)e^{\frac{\beta(n+2M_\lambda)}{\lambda-\beta}}.$$

Applying the Gronwall lemma results in

$$\mathcal{H}(t) \le e^{-\beta(n+2M_\lambda)t}\mathcal{H}(0) + e^{\frac{\beta(n+2M_\lambda)}{\lambda-\beta}},$$

which implies that $\mathbb{E}e^{\beta\|w(t)\|_{l^2}^2}$ is uniformly bounded in time t. By the Young inequality, for any constant $k \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}g(w(t))^k = \mathbb{E}e^{kq\xi(w(t))} &\le \mathbb{E}e^{|kq|(\|\xi\|_{Lip}\|w(t)\|_{l^2}+|\xi(0)|)} \\ &\le \mathbb{E}e^{\beta\|w(t)\|_{l^2}^2+C} \le C\mathbb{E}\mathcal{H}(w(t)) < \infty,\end{aligned} \tag{3.27}$$

where we have used the fact that ξ is Lipschitz from (3.12).

Finally, using the Hölder inequality, it follows from (3.23) and (3.27) that there exists a real number $M_1 = M_1(\theta(0), w(0))$ such that

$$\mathbb{E}\|\theta(t)\|_{L^q}^{q^-} \le \left(\mathbb{E}\left(\|\theta(t)\|_{L^q}^q e^{q\xi(w(t))}\right)\right)^{\frac{q^-}{q}} \left(\mathbb{E}g(w(t))^{-\frac{q^-}{q-q^-}}\right)^{\frac{q-q^-}{q}} \le M_1 e^{-\frac{q^-}{q}\sigma t},$$

which completes the proof. □

The above theorem indicates that wether $\zeta(w(t))$ is positive or not, the null solution to Eq. (1.1) is stable in the sense of moments of order q^- as long as the average damping $\langle\zeta,\pi\rangle > 0$ is sufficiently large. Here we use some simple example to explain the intuition.

Example 3.6. *Consider the affine function $\zeta(w(t)) = aw(t) + b$ and the one-dimensional stochastic process*

$$dw(t) = -\gamma w(t)dt + dB(t),$$

where $a \in \mathbb{R}$, $b, \gamma > 0$ are constants and the initial datum w follows the equilibrium distribution π which is a normal distribution with zero mean.

By the Duhamel formula, its solution is given by

$$w(t) = e^{-\gamma t}w + \int_0^t e^{-(t-r)\gamma}dB(r).$$

Therefore, $\xi(w)$ can be written explicitly as below

$$\xi(w) = -a\int_0^\infty \mathbb{E}^w w(t)dt = -aw\int_0^\infty e^{-\gamma t}dt = -\frac{aw}{\gamma}.$$

Thanks to Lemma 2.8, we obtain that $w(t)$ is asymptotically contractive with $C_\gamma = 1$. Moreover, it is easy to see that $w(t)$ is geometrically ergodic (see more details in [34, Theorem 2.3] and [36, Theorem 2.5]) and dissipative. Applying the Itô formula to $g(w) = e^{q\xi(w)} = e^{-\frac{qaw}{\gamma}}$ results in

$$dg(w(t)) = \left(qaw(t) + \frac{q^2a^2}{2\gamma^2}\right)g(w(t))dt - \frac{qa}{\gamma}g(w(t))dB(t).$$

Let $U_0(t,\theta,w) = e^{\sigma t}\|\theta\|_{L^q}^q e^{-\frac{qaw}{\gamma}}$. Similar to the arguments of (3.19) and (3.22), we have

$$\begin{aligned}\mathbb{E}U_0(t,\theta(t),w(t)) &\le \mathbb{E}U_0(0,\theta(0),w(0))\\ &\quad + \left(\sigma + \frac{1}{2}q(q-1)\|G\|_{Lip}^2 - qb + \frac{q^2a^2}{2\gamma^2}\right)\int_0^t \mathbb{E}U_0(r,\theta(r),w(r))dr.\end{aligned}$$

If we take $b > 0$ sufficiently large such that

$$b > \frac{1}{2}(q-1)\|G\|_{Lip}^2 + \frac{qa^2}{2\gamma^2},$$

then by a similar method to the one in the proof of Theorem 3.5, we can obtain that the null solution to Eq. (1.1) is stable in the sense of moments of order q^-.

Based on the result in Theorem 3.5, we further show that the sample paths of solutions converge to the null solution almost surely as time goes to infinity.

Theorem 3.7. *Let the assumptions in Theorem 3.5 be satisfied. Then any pathwise solution $\theta(t)$ to Eq. (1.1) converges to the null solution almost surely exponentially in L^q, i.e., there exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$, such that for $\omega \notin \Omega_0$ there exists a random variable $T(\omega) > 0$ such that*

$$\|\theta(t)\|_{L^q} \le M_2 e^{-a_2 t}, \quad \forall t \ge T(\omega),$$

for some positive constants $M_2 = M_2(\theta(0), w(0)) > 0$ and $a_2 > 0$.

Proof. Applying the Itô formula to $U_2(\theta,w)=F_q(\theta)g(w)$ and arguing as in (3.19), we obtain that for any natural number N and $t\geq N$,

$$\begin{aligned}
U_2(\theta(t),w(t)) = U_2(\theta(N),w(N)) &- q\kappa\int_N^t g(w(r))\int_{\mathbb{T}^2}|\theta(r)|^{q-2}\theta(r)(-\Delta)^{\alpha}\theta(r)dxdr\\
&+\frac{q(q-1)}{2}\int_N^t g(w(r))\int_{\mathbb{T}^2}|\theta(r)|^{q-2}\sum_{k=1}^{\infty}|G(r,\theta(r))e_k|^2dxdr\\
&-q\int_N^t\langle\zeta,\pi\rangle U_2(\theta(r),w(r))dr+\frac{1}{2}q^2\int_N^t\|\nabla_w\xi(w(r))\|_{l^2}^2U_2(\theta(r),w(r))dr\\
&+q\int_N^t g(w(r))\int_{\mathbb{T}^2}|\theta(r)|^{q-2}\theta(r)G(r,\theta(r))dxdW(r)\\
&+q\int_N^t U_2(\theta(r),w(r))(\nabla_w\xi(w(r)),dB(r)).
\end{aligned}\tag{3.28}$$

By the Burkholder-Davis-Gundy inequality, the Hölder inequality and the Young inequality, in view of the conditions on G, we deduce that

$$\begin{aligned}
&q\mathbb{E}\sup_{N\leq t\leq N+1}\left|\int_N^t g(w(r))\int_{\mathbb{T}^2}|\theta(r)|^{q-2}\theta(r)G(r,\theta(r))dxdW(r)\right|\\
&\leq C_q\mathbb{E}\left(\int_N^{N+1}\left(g(w(r))\int_{\mathbb{T}^2}|\theta(r)|^{q-1}\left(\sum_{k=1}^{\infty}|G(r,\theta(r))e_k|^2\right)^{\frac{1}{2}}dx\right)^2dr\right)^{\frac{1}{2}}\\
&\leq C_q\mathbb{E}\left(\int_N^{N+1}g(w(r))^2\|\theta(r)\|_{L^q}^{2(q-1)}\|G(r,\theta(r))\|^2_{L_q(\mathcal{U},L^q)}dr\right)^{\frac{1}{2}}\\
&\leq C_q\mathbb{E}\left(\sup_{N\leq r\leq N+1}\left(g(w(r))\|\theta(r)\|_{L^q}^q\right)\int_N^{N+1}g(w(r))\|\theta(r)\|_{L^q}^q dr\right)^{\frac{1}{2}}\\
&\leq\frac{1}{4}\mathbb{E}\left(\sup_{N\leq r\leq N+1}U_2(\theta(r),w(r))\right)+C_q\mathbb{E}\int_N^{N+1}U_2(\theta(r),w(r))dr,
\end{aligned}\tag{3.29}$$

and

$$\begin{aligned}
&q\mathbb{E}\sup_{N\leq t\leq N+1}\left|\int_N^t U_2(\theta(r),w(r))(\nabla_w\xi(w(r)),dB(r))\right|\\
&\leq C_q\mathbb{E}\left(\int_N^{N+1}U_2(\theta(r),w(r))^2\|\nabla_w\xi(w(r))\|_{l^2}^2dr\right)^{\frac{1}{2}}\\
&\leq C_q\mathbb{E}\left(\sup_{N\leq r\leq N+1}U_2(\theta(r),w(r))\int_N^{N+1}U_2(\theta(r),w(r))dr\right)^{\frac{1}{2}}\\
&\leq\frac{1}{4}\mathbb{E}\left(\sup_{N\leq r\leq N+1}U_2(\theta(r),w(r))\right)+C_q\mathbb{E}\int_N^{N+1}U_2(\theta(r),w(r))dr,
\end{aligned}\tag{3.30}$$

where we have used the estimate (3.12) in (3.30). Combining (3.28)-(3.30) and arguing as in (3.22), we have

$$\begin{aligned}\mathbb{E}\bigg(\sup_{N\le r\le N+1} U_2(\theta(r),w(r))\bigg) &\le \mathbb{E}U_2(\theta(N),w(N))\\ &+\bigg(\frac{1}{2}q(q-1)\|G\|^2_{Lip}-q\langle\zeta,\pi\rangle+\frac{1}{2}q^2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|^2_{Lip}\bigg)\mathbb{E}\int_N^{N+1}U_2(\theta(r),w(r))dr\\ &+\frac{1}{2}\mathbb{E}\bigg(\sup_{N\le r\le N+1} U_2(\theta(r),w(r))\bigg)+C_q\mathbb{E}\int_N^{N+1}U_2(\theta(r),w(r))dr.\end{aligned}$$

Thanks to the condition (3.14), it can be deduced that

$$\frac{1}{2}\mathbb{E}\bigg(\sup_{N\le r\le N+1} U_2(\theta(r),w(r))\bigg)\le \mathbb{E}U_2(\theta(N),w(N))+C_q\mathbb{E}\int_N^{N+1}U_2(\theta(r),w(r))dr.$$

Then by the Markov inequality, the Hölder inequality and the estimate (3.27), we infer that for any $\varepsilon_N>0$,

$$\begin{aligned}&P\bigg(\sup_{N\le r\le N+1}\|\theta(r)\|^{q^-}_{L^q}\ge\varepsilon_N\bigg)\le\varepsilon_N^{-1}\mathbb{E}\bigg(\sup_{N\le r\le N+1}\|\theta(r)\|^{q^-}_{L^q}\bigg)\\ &\le\varepsilon_N^{-1}\Bigg(\mathbb{E}\bigg(\sup_{N\le r\le N+1}\|\theta(r)\|^{q^-}_{L^q}e^{q^-\xi(w(r))}\bigg)^{\frac{q}{q^-}}\Bigg)^{\frac{q^-}{q}}\Bigg(\mathbb{E}\bigg(\sup_{N\le r\le N+1}e^{-q^-\xi(w(r))}\bigg)^{\frac{q}{q-q^-}}\Bigg)^{\frac{q-q^-}{q}}\\ &\le C\varepsilon_N^{-1}\Bigg(\mathbb{E}\bigg(\sup_{N\le r\le N+1}U_2(\theta(r),w(r))\bigg)\Bigg)^{\frac{q^-}{q}}\\ &\le C_q\varepsilon_N^{-1}\bigg(\mathbb{E}U_2(\theta(N),w(N))+\mathbb{E}\int_N^{N+1}U_2(\theta(r),w(r))dr\bigg)^{\frac{q^-}{q}}.\end{aligned}$$

Therefore, it follows from (3.23) that there exists a constant $M_2=M_2(\theta(0),w(0))>0$ such that

$$P\bigg(\sup_{N\le r\le N+1}\|\theta(r)\|^{q^-}_{L^q}\ge\varepsilon_N\bigg)\le M_2\varepsilon_N^{-1}e^{-\frac{q^-}{q}\sigma N}.$$

Let $\varepsilon_N=M_2e^{-\frac{q^-}{2q}\sigma N}$. We find that

$$P\bigg(\sup_{N\le r\le N+1}\|\theta(r)\|^{q^-}_{L^q}\ge M_2e^{-\frac{q^-}{2q}\sigma N}\bigg)\le e^{-\frac{q^-}{2q}\sigma N}.$$

Using the Borel-Cantelli lemma, we conclude that there exists $\Omega_0\subset\Omega$ with $P(\Omega_0)=0$, such that for $\omega\notin\Omega_0$ there exists a random variable $N_0(\omega)>0$ such that if $N\ge N_0(\omega)$,

$$\sup_{N\le r\le N+1}\|\theta(r)\|^{q^-}_{L^q}\le M_2e^{-\frac{q^-}{2q}\sigma N}.$$

Thus the proof is complete. □

The Lipschitz requirement of G in Theorem 3.5 can be relaxed to include all the functions satisfying (3.31). The more general result is stated in the following theorem.

Theorem 3.8. *Fix $q > \frac{2}{2\alpha-1}$ with $\alpha \in (\frac{1}{2}, 1)$. Let the conditions $(\mathbf{A}_1)$-$(\mathbf{A}_3)$ be satisfied and $\zeta \in Lip_u(\mathbb{R}^n, \mathbb{R})$. Assume that G satisfies*

$$\|G(t,\theta)\|^2_{L_q(\mathcal{U},L^q)} \leq \ell_1(t) + (\ell_0 + \ell_2(t))\|\theta\|^2_{L^q}, \tag{3.31}$$

where $\ell_0 > 0$ is a constant and $\ell_1(t)$, $\ell_2(t)$ are nonnegative integrable functions such that there exist real number $\delta > 0$ such that

$$\int_0^\infty \ell_1(t)e^{\delta t}dt < \infty, \quad \int_0^\infty \ell_2(t)dt < \infty. \tag{3.32}$$

If it holds that

$$q\langle\zeta,\pi\rangle \geq \delta + \frac{1}{2}q(q-1)\ell_0 + \frac{1}{2}q^2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|^2_{Lip}, \tag{3.33}$$

there exist some positive constants a_3 and $M_3 = M_3(\theta(0), w(0))$ such that

$$\mathbb{E}\|\theta(t)\|^{q^-}_{L^q} \leq M_3 e^{-a_3 t}, \quad t \geq 0.$$

Furthermore, almost sure exponential stability of the null solution also holds true.

Proof. Let $U_3(t,\theta,w) = e^{\delta t}F_q(\theta)g(w)$. Similar to the arguments of (3.19), then taking expectation yields

$$\begin{aligned}
\mathbb{E}U_3(t,\theta(t),w(t)) = \mathbb{E}U_3(0,\theta(0),w(0)) &+ \delta\mathbb{E}\int_0^t U_3(r,\theta(r),w(r))dr \\
&- q\kappa\mathbb{E}\int_0^t e^{\delta r}e^{q\xi(w(r))}\int_{\mathbb{T}^2}|\theta(r)|^{q-2}\theta(r)(-\Delta)^\alpha\theta(r)dxdr \\
&+ \frac{q(q-1)}{2}\mathbb{E}\int_0^t e^{\delta r}e^{q\xi(w(r))}\int_{\mathbb{T}^2}|\theta(r)|^{q-2}\sum_{k=1}^\infty|G(r,\theta(r))e_k|^2dxdr \\
&- q\langle\zeta,\pi\rangle\mathbb{E}\int_0^t U_3(r,\theta(r),w(r))dr \\
&+ \frac{1}{2}q^2\mathbb{E}\int_0^t \|\nabla_w\xi(w(r))\|^2_{l^2}U_3(r,\theta(r),w(r))dr.
\end{aligned} \tag{3.34}$$

By the Hölder inequality, the Young inequality and the condition (3.31), we have

$$\begin{aligned}
&\frac{q(q-1)}{2}\mathbb{E}\int_0^t e^{\delta r}e^{q\xi(w(r))}\int_{\mathbb{T}^2}|\theta(r)|^{q-2}\sum_{k=1}^\infty|G(r,\theta(r))e_k|^2dxdr \\
&\leq \frac{1}{2}q(q-1)\mathbb{E}\int_0^t e^{\delta r}e^{q\xi(w(r))}\|\theta(r)\|^{q-2}_{L^q}\|G(r,\theta(r))\|^2_{L_q(\mathcal{U},L^q)}dr
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}q(q-1)\mathbb{E}\int_0^t e^{\delta r} e^{q\xi(w(r))}\|\theta(r)\|_{L^q}^{q-2}\left(\ell_1(r) + (\ell_0 + \ell_2(r))\|\theta(r)\|_{L^q}^2\right)dr \\
&\le \frac{1}{2}q(q-1)\ell_0\mathbb{E}\int_0^t U_3(r,\theta(r),w(r))dr + C_q\mathbb{E}\int_0^t \ell_1(r)e^{\delta r}e^{q\xi(w(r))}dr \\
&\quad + C_q\mathbb{E}\int_0^t (\ell_1(r)+\ell_2(r))\,U_3(r,\theta(r),w(r))dr. \qquad (3.35)
\end{aligned}
$$

Inserting (3.35) into (3.34) results in

$$
\begin{aligned}
&\mathbb{E}U_3(t,\theta(t),w(t)) \le \mathbb{E}U_3(0,\theta(0),w(0)) \\
&\qquad + \left(\delta + \frac{1}{2}q(q-1)\ell_0 - q\langle\zeta,\pi\rangle + \frac{1}{2}q^2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|_{Lip}^2\right)\mathbb{E}\int_0^t U_3(r,\theta(r),w(r))dr \\
&\qquad + C_q\mathbb{E}\int_0^t \ell_1(r)e^{\delta r}e^{q\xi(w(r))}dr + C_q\mathbb{E}\int_0^t (\ell_1(r)+\ell_2(r))\,U_3(r,\theta(r),w(r))dr, \qquad (3.36)
\end{aligned}
$$

where we have used (3.6) and (3.12). In view of the condition (3.33), the second term on the right-hand side of (3.36) is negative. Then the inequality (3.36), combined with (3.27), can be transformed as follows

$$
\begin{aligned}
\mathbb{E}U_3(t,\theta(t),w(t)) \le \mathbb{E}U_3(0,\theta(0),w(0)) + C_q\int_0^t \ell_1(r)e^{\delta r}dr \\
+ C_q\mathbb{E}\int_0^t (\ell_1(r)+\ell_2(r))\,U_3(r,\theta(r),w(r))dr.
\end{aligned}
$$

By the Gronwall lemma, we arrive at

$$
\begin{aligned}
\mathbb{E}U_3(t,\theta(t),w(t)) \le \mathbb{E}U_3(0,\theta(0),w(0))\exp\left(C_q\int_0^t \ell_1(r)+\ell_2(r)dr\right) \\
+ C_q\int_0^t \exp\left(C_q\int_r^t \ell_1(\tau)+\ell_2(\tau)d\tau\right)\ell_1(r)e^{\sigma r}dr.
\end{aligned}
$$

This together with the condition (3.32) implies that there exists a positive constant $\widetilde{M} = \widetilde{M}(\theta(0),w(0))$ such that

$$
\mathbb{E}\left(\|\theta(t)\|_{L^q}^q e^{q\xi(w(t))}\right) \le \widetilde{M}e^{-\delta t}.
$$

Using the Hölder inequality and (3.27), we deduce that there exists a real number $M_3 = M_3(\theta(0),w(0))$ such that

$$
\mathbb{E}\|\theta(t)\|_{L^q}^{q^-} \le \left(\mathbb{E}\left(\|\theta(t)\|_{L^q}^q e^{q\xi(w(t))}\right)\right)^{\frac{q^-}{q}}\left(\mathbb{E}e^{-\frac{qq^-}{q-q^-}\xi(w(t))}\right)^{\frac{q-q^-}{q}} \le M_3 e^{-\frac{q^-}{q}\delta t}.
$$

This completes the proof of the first part of the theorem. The rest of the theorem can be proved by the same way as in the proof of Theorem 3.7, and thus we omit it here. □

Remark 3.9. It is easy to find two functions such that (3.32) holds true. For example, if

$$\ell_1(t) = M_{\ell_1} e^{-\hat{\delta} t}, \quad \ell_2(t) = M_{\ell_2} e^{-\hat{\delta} t},$$

with $\hat{\delta} > \delta$, then the condition (3.32) will be satisfied.

4 Feller properties and invariant measures

We have shown that the null solution to Eq. (1.1) is stable in L^q. However, it is impossible to obtain the stability of the null solution in H^s because of the quadratic nonlinear term. In this section, we shall establish the uniform boundedness of pathwise solutions in H^s with $s \geq 2 - 2\alpha$, which implies the existence of non-trivial invariant measures. From now on, we assume that G is independent of time t. Hence the solution to Eq. (1.1) is a time-homogeneous Markov process with the state space H^s.

4.1 Uniform boundedness of solutions

In this subsection, we prove that the solution to Eq. (1.1) is uniformly bounded in H^s provided that the average damping $\langle \zeta, \pi \rangle$ is sufficiently large.

Lemma 4.1. *Let $G \in Bnd_u(L^p, L_p(\mathcal{U}, L^p))$ and $\zeta \in Lip_u(\mathbb{R}^n, \mathbb{R})$, with $2 \leq p < \infty$. Assume that the assumptions $(\mathbf{A}_1)$-$(\mathbf{A}_3)$ and*

$$\langle \zeta, \pi \rangle > \frac{1}{2}(p-1)\|G\|_{Bnd}^2 + \frac{1}{2} p \frac{C_\gamma^2}{\gamma^2} \|\zeta\|_{Lip}^2 \tag{4.1}$$

hold true. Then for any initial valve $\theta(0) \in L^p$, there exists a positive constant $C = C(\theta(0), w(0))$ such that

$$\mathbb{E}\left(\|\theta(t)\|_{L^p}^p e^{p\xi(w(t))} \right) \leq C, \quad t \geq 0, \tag{4.2}$$

where θ is the solution to Eq. (1.1) *and $\xi(w)$ is defined as in* (3.10).

Proof. The same arguments as in (3.19) leads to

$$\begin{aligned} \|\theta(t)\|_{L^p}^p e^{p\xi(w(t))} = & \|\theta(0)\|_{L^p}^p e^{p\xi(w(0))} - p\kappa \int_0^t e^{p\xi(w(r))} \int_{\mathbb{T}^2} |\theta(r)|^{p-2} \theta(r) (-\Delta)^\alpha \theta(r) dx dr \\ & + \frac{p(p-1)}{2} \int_0^t e^{p\xi(w(r))} \int_{\mathbb{T}^2} |\theta(r)|^{p-2} \sum_{k=1}^\infty |G(\theta(r)) e_k|^2 dx dr \\ & - p \langle \zeta, \pi \rangle \int_0^t \|\theta(r)\|_{L^p}^p e^{p\xi(w(r))} dr \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}p^2\int_0^t \|\nabla_w\xi(w(r))\|_{l^2}^2\|\theta(r)\|_{L^p}^p e^{p\xi(w(r))}dr\\
&+p\int_0^t e^{p\xi(w(r))}\int_{\mathbb{T}^2}|\theta(r)|^{p-2}\theta(r)G(\theta(r))dxdW(r)\\
&+p\int_0^t \|\theta(r)\|_{L^p}^p e^{p\xi(w(r))}(\nabla_w\xi(w(r)),dB(r)).
\end{aligned}
$$

Let $\mathcal{Z}(t)=\mathbb{E}\left(\|\theta(t)\|_{L^p}^p e^{p\xi(w(t))}\right)$. Then the above inequality can be transformed as follows

$$
\begin{aligned}
\mathcal{Z}'(t)=&-p\kappa\mathbb{E}\left(e^{p\xi(w(t))}\int_{\mathbb{T}^2}|\theta(t)|^{p-2}\theta(t)(-\Delta)^\alpha\theta(t)dx\right)\\
&+\frac{p(p-1)}{2}\mathbb{E}\left(e^{p\xi(w(t))}\int_{\mathbb{T}^2}|\theta(t)|^{p-2}\sum_{k=1}^\infty|G(\theta(t))e_k|^2dx\right)\\
&-p\langle\zeta,\pi\rangle\mathcal{Z}(t)+\frac{1}{2}p^2\mathbb{E}\left(\|\nabla_w\xi(w(t))\|_{l^2}^2\|\theta(t)\|_{L^p}^p e^{p\xi(w(t))}\right).
\end{aligned}
\tag{4.3}
$$

By the condition (4.1), there exists a constant $\varepsilon_0>0$ sufficiently small such that

$$
2\varepsilon_0+\frac{1}{2}p(p-1)\|G\|_{Bnd}^2-p\langle\zeta,\pi\rangle+\frac{1}{2}p^2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|_{Lip}^2\le 0.
\tag{4.4}
$$

For this $\varepsilon_0>0$, taking into account that $G\in Bnd_u(L^p,L_p(\mathcal{U},L^p))$, we make use of the Hölder inequality and the Young inequality to obtain that

$$
\begin{aligned}
&\frac{p(p-1)}{2}\mathbb{E}\left(e^{p\xi(w(t))}\int_{\mathbb{T}^2}|\theta(t)|^{p-2}\sum_{k=1}^\infty|G(\theta(t))e_k|^2dx\right)\\
&\le\frac{1}{2}p(p-1)\mathbb{E}\left(e^{p\xi(w(t))}\|\theta(t)\|_{L^p}^{p-2}\|G(\theta(t))\|_{L_p(\mathcal{U},L^p)}^2\right)\\
&\le\frac{1}{2}p(p-1)\|G\|_{Bnd}^2\mathbb{E}\left(e^{p\xi(w(t))}\|\theta(t)\|_{L^p}^{p-2}(1+\|\theta(t)\|_{L^p})^2\right)\\
&\le\frac{1}{2}p(p-1)\|G\|_{Bnd}^2\mathcal{Z}(t)+\varepsilon_0\mathcal{Z}(t)+C_p\mathbb{E}e^{p\xi(w(t))}.
\end{aligned}
\tag{4.5}
$$

Using (3.6) and (3.12), we conclude from (4.3) and (4.5) that

$$
\mathcal{Z}'(t)+\varepsilon_0\mathcal{Z}(t)\le\left(2\varepsilon_0+\frac{1}{2}p(p-1)\|G\|_{Bnd}^2-p\langle\zeta,\pi\rangle+\frac{1}{2}p^2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|_{Lip}^2\right)\mathcal{Z}(t)+C_p\mathbb{E}e^{p\xi(w(t))}.
$$

Furthermore, it follows from (4.4) that

$$
\mathcal{Z}'(t)+\varepsilon_0\mathcal{Z}(t)\le C_p\mathbb{E}e^{p\xi(w(t))}.
$$

Applying the Gronwall lemma gives

$$
\mathcal{Z}(t)\le e^{-\varepsilon_0 t}\mathcal{Z}(0)+C_p\int_0^t e^{-\varepsilon_0(t-r)}\mathbb{E}e^{p\xi(w(r))}dr,
$$

which, together with the estimate (3.27), completes the proof. $\square$

Lemma 4.2. *Let $\alpha \in (\frac{1}{2}, 1)$ and $s \geq 2 - 2\alpha$. Assume that, in addition to the assumptions imposed in Lemma 4.1 for $p = 2\eta^* := 2\frac{2(2-\eta_0)}{1-\eta_0}$, $\zeta \in Lip_u(\mathbb{R}^n, \mathbb{R})$ and $G \in Lip_u(H^s, L_2(\mathcal{U}, H^s))$ with $G(0) = 0$, where $\eta_0 = \frac{s+2-\alpha-\frac{2}{q_0}}{s+\alpha}$, $q_0 = \frac{1}{1-\alpha^-}$ and $\alpha^- \in (\frac{1}{2}, \alpha)$. If the assumptions $(\mathbf{A}_1)$-$(\mathbf{A}_3)$ and*

$$2\kappa\lambda_1^\alpha + 2\langle\zeta, \pi\rangle > \|G\|_{Lip}^2 + 2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|_{Lip}^2 \tag{4.6}$$

hold true, then for any initial valve $\theta(0) \in L^p \cap H^s$, the solution to Eq. (1.1) is uniformly bounded in the sense of 1-th moment of $\|\cdot\|_{H^s}$.

Proof. Applying Λ^s to Eq. (1.1) and using the Itô formula to the function $\|\Lambda^s\theta\|_{L^2}^2$, we deduce that

$$\begin{aligned} d\|\theta(t)\|_{H^s}^2 = &-2\kappa\|\theta(t)\|_{H^{s+\alpha}}^2 dt - 2\zeta(w(t))\|\theta(t)\|_{H^s}^2 dt - 2\langle\Lambda^s\theta(t), \Lambda^s(u(t)\cdot\nabla\theta(t))\rangle dt \\ &+ \operatorname{tr}\Big(\Lambda^s G(\theta(t))(\Lambda^s G(\theta(t)))^*\Big)dt + 2\langle\Lambda^s\theta(t), \Lambda^s G(\theta(t))dW(t)\rangle. \end{aligned}$$

Since $B(t)$ is independent of $W(t)$, by Lemma 2.6 and (3.18) for the case $q = 2$, we have

$$\begin{aligned} &\|\theta(t)\|_{H^s}^2 e^{2\xi(w(t))} + 2\kappa\int_0^t e^{2\xi(w(r))}\|\theta(r)\|_{H^{s+\alpha}}^2 dr \\ &= \|\theta(0)\|_{H^s}^2 e^{2\xi(w(0))} - 2\int_0^t e^{2\xi(w(r))}\langle\Lambda^s\theta(r), \Lambda^s(u(r)\cdot\nabla\theta(r))\rangle dr \\ &\quad + \int_0^t e^{2\xi(w(r))}\operatorname{tr}\Big(\Lambda^s G(\theta(r))(\Lambda^s G(\theta(r)))^*\Big)dr \\ &\quad - 2\langle\zeta, \pi\rangle\int_0^t e^{2\xi(w(r))}\|\theta(r)\|_{H^s}^2 dr + 2\int_0^t \|\nabla_w\xi(w(r))\|_{l^2}^2\|\theta(r)\|_{H^s}^2 e^{2\xi(w(r))}dr \\ &\quad + 2\int_0^t e^{2\xi(w(r))}\langle\Lambda^s\theta(r), \Lambda^s G(\theta(r))dW(r)\rangle \\ &\quad + 2\int_0^t \|\theta(r)\|_{H^s}^2 e^{2\xi(w(r))}(\nabla_w\xi(w(r)), dB(r)). \end{aligned}$$

Let $\mathcal{V}(t) = \mathbb{E}\left(\|\theta(t)\|_{H^s}^2 e^{2\xi(w(t))}\right)$. Then it follows that

$$\begin{aligned} &\mathcal{V}'(t) + 2\kappa\mathbb{E}\left(e^{2\xi(w(t))}\|\theta(t)\|_{H^{s+\alpha}}^2\right) \\ &= -2\mathbb{E}\left(e^{2\xi(w(t))}\langle\Lambda^s\theta(t), \Lambda^s(u(t)\cdot\nabla\theta(t))\rangle\right) \\ &\quad + \mathbb{E}\left(e^{2\xi(w(t))}\operatorname{tr}\Big(\Lambda^s G(\theta(t))(\Lambda^s G(\theta(t)))^*\Big)\right) \\ &\quad - 2\langle\zeta, \pi\rangle\mathcal{V}(t) + p\mathbb{E}\left(\|\nabla_w\xi(w(t))\|_{l^2}^2\|\theta(t)\|_{H^s}^2 e^{2\xi(w(t))}\right) \end{aligned}$$

$$:= \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4. \tag{4.7}$$

By the condition (4.6), we choose $\varepsilon_0 > 0$ sufficient small such that

$$2\varepsilon_0 - 2\kappa\lambda_1^\alpha + \|G\|_{Lip}^2 - 2\langle\zeta,\pi\rangle > +2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|_{Lip}^2 \leq 0. \tag{4.8}$$

Note that $\nabla \cdot u = 0$. Making use of the Schwarz inequality and Lemmas 2.1-2.2 yields

$$\begin{aligned}
\mathscr{I}_1 &\leq 2\mathbb{E}\left(e^{2\xi(w(t))}\|\Lambda^{s+1-\alpha}(u(t)\theta(t))\|_{L^2}\|\Lambda^{s+\alpha}\theta(t)\|_{L^2}\right)\\
&\leq C\mathbb{E}\left(e^{2\xi(w(t))}(\|u(t)\|_{L^{p_0}}\|\theta(t)\|_{H^{s+1-\alpha,q_0}} + \|u(t)\|_{H^{s+1-\alpha,q_0}}\|\theta(t)\|_{L^{p_0}})\|\theta(t)\|_{H^{s+\alpha}}\right)\\
&\leq C\mathbb{E}\left(e^{2\xi(w(t))}\|\theta(t)\|_{L^{p_0}}\|\theta(t)\|_{H^{s+1-\alpha,q_0}}\|\theta(t)\|_{H^{s+\alpha}}\right),
\end{aligned} \tag{4.9}$$

where $p_0 = \frac{2}{2\alpha^- - 1}$, $q_0 = \frac{1}{1-\alpha^-}$ and α^- denotes a number strictly less than α but close to it. With the Sobolev embedding $H^{s+2-\alpha-\frac{2}{q_0}} \subset H^{s+1-\alpha,q_0}$ and the following Nirenberg-Gagliardo inequality (cf. [5]):

$$\|\theta\|_{H^{s+2-\alpha-\frac{2}{q_0}}} \leq C\|\theta\|_{H^{s+\alpha}}^{\eta_0}\|\theta\|_{L^2}^{1-\eta_0},$$

where $0 < \eta_0 = \frac{s+2-\alpha-\frac{2}{q_0}}{s+\alpha} < 1$, in view of the Young inequality and the Sobolev embedding $L^{p_0} \subset L^2$, the first term on the right-hand side of (4.7) can be further bounded by

$$\begin{aligned}
\mathscr{I}_1 &\leq C\mathbb{E}\left(e^{2\xi(w(t))}\|\theta(t)\|_{H^{s+\alpha}}^{1+\eta_0}\|\theta(t)\|_{L^{p_0}}^{2-\eta_0}\right)\\
&\leq \frac{\varepsilon_0}{2\lambda_1^\alpha}\mathbb{E}\left(e^{2\xi(w(t))}\|\theta(t)\|_{H^{s+\alpha}}^2\right) + C\mathbb{E}\left(e^{2\xi(w(t))}\|\theta(t)\|_{L^{p_0}}^{\frac{2(2-\eta_0)}{1-\eta_0}}\right)\\
&\leq \frac{\varepsilon_0}{2\lambda_1^\alpha}\mathbb{E}\left(e^{2\xi(w(t))}\|\theta(t)\|_{H^{s+\alpha}}^2\right) + \frac{\varepsilon_0}{2}\mathcal{V}(t) + C\mathbb{E}\left(e^{2\xi(w(t))}\|\theta(t)\|_{L^{p_0}}^{\frac{2(2-\eta_0)}{1-\eta_0}}\right).
\end{aligned} \tag{4.10}$$

Recall that $G \in Lip_u(H^s, L_2(\mathcal{U}, H^s))$ with $G(0) = 0$. We estimate the second term on the right-hand side of (4.7) as follows

$$\mathscr{I}_2 \leq \mathbb{E}\left(e^{2\xi(w(t))}\|G(\theta(t))\|_{L_2(\mathcal{U},H^s)}^2\right) \leq \|G\|_{Lip}^2\mathcal{V}(t). \tag{4.11}$$

Inserting (4.10)-(4.11) into (4.7) results in

$$\begin{aligned}
\mathcal{V}'(t) + \varepsilon_0\mathcal{V}(t) \leq{}& C\mathbb{E}\left(e^{2\xi(w(t))}\|\theta(t)\|_{L^{p_0}}^{\frac{2(2-\eta_0)}{1-\eta_0}}\right)\\
&+ \left(2\varepsilon_0 - 2\kappa\lambda_1^\alpha + \|G\|_{Lip}^2 - 2\langle\zeta,\pi\rangle + 2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|_{Lip}^2\right)\mathcal{V}(t),
\end{aligned} \tag{4.12}$$

where we have used (3.12) and the following inequality:

$$\lambda_1^\alpha\|\theta\|_{H^s}^2 \leq \|\theta\|_{H^{s+\alpha}}^2. \tag{4.13}$$

Due to (4.8), we leave out the last term on the right-hand side of (4.12) and apply the Gronwall lemma to obtain that

$$\mathcal{V}(t) \leq e^{-\varepsilon_0 t}\mathcal{V}(0) + C\int_0^t e^{-\varepsilon_0(t-r)}\mathbb{E}\left(e^{2\xi(w(r))}\|\theta(r)\|_{L^{p_0}}^{\frac{2(2-\eta_0)}{1-\eta_0}}\right)dr. \tag{4.14}$$

Note that

$$\mathbb{E}\left(e^{2\xi(w(r))}\|\theta(r)\|_{L^{p_0}}^{\eta^*}\right) \leq \frac{1}{2}\mathbb{E}\left(\|\theta(r)\|_{L^{2\eta^*}}^{2\eta^*}e^{2\eta^*\xi(w(r))}\right) + \frac{1}{2}\mathbb{E}e^{-2(\eta^*-2)\xi(w(r))} < \infty, \tag{4.15}$$

where $\eta^* := \frac{2(2-\eta_0)}{1-\eta_0}$ and we have used (4.2), (3.27) and the embedding $L^{2\eta^*} \subset L^{p_0}$. Then we conclude from (4.14) and (4.15) that

$$\mathcal{V}(t) \text{ is uniformly bounded.} \tag{4.16}$$

Using the Hölder inequality and (3.27), we find that

$$\mathbb{E}\|\theta(t)\|_{H^s} \leq \left(\mathbb{E}\left(\|\theta(t)\|_{H^s}^2 e^{2\xi(w(t))}\right)\right)^{\frac{1}{2}}\left(\mathbb{E}e^{-2\xi(w(t))}\right)^{\frac{1}{2}} < \infty,$$

which completes the proof. □

4.2 Feller property of solutions

To mark the dependence of the solution $\theta(t)$ to Eq. (1.1) on each fixed initial value $\theta_0 = x \in H^s$ with $s \geq 2-2\alpha$ and $\alpha \in (\frac{1}{2}, 1)$, we denote it by $\theta(t;x)$ (whose existence is guaranteed by Theorem 3.1). Next we present a continuous dependence estimate.

Lemma 4.3. *Let the conditions of Lemma 4.2 be satisfied. Then for any fixed $T \geq 0$ and $x_1, x_2 \in H^s$,*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|\theta(t;x_1) - \theta(t;x_2)\|_{H^s}^2\right) \leq C_T\|x_1 - x_2\|_{H^s}^2. \tag{4.17}$$

Proof. Let $\theta_1(t) := \theta(t;x_1)$ and $\theta_2(t) := \theta(t;x_2)$ be global solutions to Eq. (1.1) with initial values x_1 and x_2, respectively. Define the stopping time

$$\tau_n = \inf_{t\geq 0}\left\{\int_0^t \|\theta_1(r)\|_{H^{s+\alpha}}^2 + \|\theta_2(r)\|_{H^{s+\alpha}}^2 + 1dr \geq n\right\}.$$

Clearly this is an increasing sequence. Furthermore, since θ_1 and θ_2 are global solutions, we may infer from (3.1) that $\lim_{n\to\infty}\tau_n = \infty$ a.s. Set $\varrho(t) = \theta_1(t) - \theta_2(t)$. Applying the Itô formula to the function $\|\varrho\|_{H^s}^2$ yields

$$d\|\varrho(t)\|_{H^s}^2 + 2\kappa\|\varrho(t)\|_{H^{s+\alpha}}^2 dt$$

$$
\begin{aligned}
&= -2\langle \Lambda^s(u_1(t)\cdot\nabla\theta_1(t) - u_2(t)\cdot\nabla\theta_2(t)), \Lambda^s\varrho(t)\rangle dt \\
&\quad - 2\zeta(w(t))\|\varrho(t)\|_{H^s}^2 dt + \|G(\theta_1(t)) - G(\theta_2(t))\|_{L_2(\mathcal{U},H^s)}^2 dt \\
&\quad + 2\langle \Lambda^s(G(\theta_1(t)) - G(\theta_2(t)))dW(t), \Lambda^s\varrho(t)\rangle. \qquad (4.18)
\end{aligned}
$$

Fix n and stopping times τ_a, τ_b such that $0 \le \tau_a \le \tau_b \le \tau_n \wedge T$. Integrating (4.18) in time and taking supremum, finally taking expectation we arrive at

$$
\begin{aligned}
&\mathbb{E}\bigg(\sup_{t\in[\tau_a,\tau_b]} \|\varrho(t)\|_{H^s}^2 + 2\kappa\int_{\tau_a}^{\tau_b} \|\varrho\|_{H^{s+\alpha}}^2 dt\bigg) \\
&\le \mathbb{E}\|\varrho(\tau_a)\|_{H^s}^2 + 2\mathbb{E}\int_{\tau_a}^{\tau_b} |\langle \Lambda^s(u_1(t)\cdot\nabla\varrho(t)), \Lambda^s\varrho(t)\rangle| dt \\
&\quad + 2\mathbb{E}\int_{\tau_a}^{\tau_b} |\langle \Lambda^s((u_1(t)-u_2(t))\cdot\nabla\theta_2(t)), \Lambda^s\varrho(t)\rangle| dt \\
&\quad + 2\mathbb{E}\int_{\tau_a}^{\tau_b} \zeta(w(t))\|\varrho(t)\|_{H^s}^2 dt + \mathbb{E}\int_{\tau_a}^{\tau_b} \|G(\theta_1(t)) - G(\theta_2(t))\|_{L_2(\mathcal{U},H^s)}^2 dt \\
&\quad + 2\mathbb{E}\sup_{t\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^{t} \langle \Lambda^s\left(G(\theta_1(t)) - G(\theta_2(t))\right)dW(t), \Lambda^s\varrho(t)\rangle\right| \\
&:= \mathbb{E}\|\varrho(\tau_a)\|_{H^s}^2 + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5, \qquad (4.19)
\end{aligned}
$$

where $u_1 = \mathcal{R}^\perp\theta_1$, $u_2 = \mathcal{R}^\perp\theta_2$ and we have used the bilinearity of the nonlinear term. Note that

$$
\langle u_1\cdot\nabla(\Lambda^s\varrho), \Lambda^s\varrho\rangle = 0.
$$

Since ∇ and Λ^s are commutable [22], we make use of Lemmas 2.1 and 2.2 to obtain

$$
\begin{aligned}
\mathcal{I}_1 &= 2\mathbb{E}\int_{\tau_a}^{\tau_b} |\langle \Lambda^s(u_1(t)\cdot\nabla\varrho(t)) - u_1(t)\cdot\nabla(\Lambda^s\varrho(t)), \Lambda^s\varrho(t)\rangle| dt \\
&\le 2\mathbb{E}\int_{\tau_a}^{\tau_b} \|\Lambda^s(u_1(t)\cdot\nabla\varrho(t)) - u_1(t)\cdot\Lambda^s(\nabla\varrho(t))\|_{L^2}\|\Lambda^s\varrho(t)\|_{L^2} dt \\
&\le C\mathbb{E}\int_{\tau_a}^{\tau_b} (\|u_1(t)\|_{H^{1,p_3}}\|\varrho(t)\|_{H^{s,p_4}} + \|u_1(t)\|_{H^{s,p_4}}\|\varrho(t)\|_{H^{1,p_3}})\|\varrho(t)\|_{H^s} dt \\
&\le C\mathbb{E}\int_{\tau_a}^{\tau_b} \|\theta_1(t)\|_{H^{s+\alpha}}\|\varrho(t)\|_{H^{s+\alpha}}\|\varrho(t)\|_{H^s} dt, \qquad (4.20)
\end{aligned}
$$

where $p_3 = \frac{2}{\alpha}$, $p_4 = \frac{2}{1-\alpha}$ and we have used the Sobolev embeddings $H^{s+\alpha} \subset H^{1,p_3}$ and $H^{s+\alpha} \subset H^{s,p_4}$. We estimate the term $\mathcal{I}_2$ as follows

$$
\begin{aligned}
\mathcal{I}_2 &\le 2\mathbb{E}\int_{\tau_a}^{\tau_b} |\langle \Lambda^s((u_1(t)-u_2(t))\cdot\nabla\theta_2(t)) - (u_1(t)-u_2(t))\cdot\nabla(\Lambda^s\theta_2(t)), \Lambda^s\varrho(t)\rangle| dt \\
&\quad + 2\mathbb{E}\int_{\tau_a}^{\tau_b} |\langle (u_1(t)-u_2(t))\cdot\nabla(\Lambda^s\theta_2(t)), \Lambda^s\varrho(t)\rangle| dt
\end{aligned}
$$

$$:= \mathcal{I}_{2,1} + \mathcal{I}_{2,2}. \tag{4.21}$$

For $\mathcal{I}_{2,1}$, arguing as in (4.20), we deduce that

$$\mathcal{I}_{2,1} \le C\mathbb{E}\int_{\tau_a}^{\tau_b} \|\theta_2(t)\|_{H^{s+\alpha}}\|\varrho(t)\|_{H^{s+\alpha}}\|\varrho(t)\|_{H^s}dt. \tag{4.22}$$

By the Hölder inequality and Lemmas 2.1 and 2.2, in view of the fact that $\nabla\cdot u_1 = \nabla\cdot u_2 = 0$, $\mathcal{I}_{2,2}$ is bounded by

$$\begin{aligned}
\mathcal{I}_{2,2} &\le 2\mathbb{E}\int_{\tau_a}^{\tau_b} \|\Lambda^{1-\alpha}((u_1(t)-u_2(t))\Lambda^s\theta_2(t))\|_{L^2}\|\Lambda^{s+\alpha}\varrho(t)\|_{L^2}dt\\
&\le C\mathbb{E}\int_{\tau_a}^{\tau_b} (\|u_1(t)-u_2(t)\|_{L^{p_1}}\|\theta_2(t)\|_{H^{s+1-\alpha,p_2}}\\
&\quad + \|u_1(t)-u_2(t)\|_{H^{1-\alpha,p_3}}\|\theta_2(t)\|_{H^{s,p_4}})\|\varrho(t)\|_{H^{s+\alpha}}dt\\
&\le C\mathbb{E}\int_{\tau_a}^{\tau_b} \|\varrho(t)\|_{H^s}\|\theta_2(t)\|_{H^{s+\alpha}}\|\varrho(t)\|_{H^{s+\alpha}}dt,
\end{aligned} \tag{4.23}$$

where $p_1 = \frac{2}{2\alpha-1}$, $p_2 = \frac{1}{1-\alpha}$, $p_3 = \frac{2}{\alpha}$, $p_4 = \frac{2}{1-\alpha}$, and we have used the Sobolev embeddings $H^s \subset L^{p_1}$, $H^{s+\alpha} \subset H^{s+1-\alpha,p_1}$, $H^s \subset H^{1-\alpha,p_3}$, $H^{s+\alpha} \subset H^{s,p_4}$. Inserting (4.22) and (4.23) into (4.21) gives

$$\mathcal{I}_2 \le C\mathbb{E}\int_{\tau_a}^{\tau_b} \|\theta_2(t)\|_{H^{s+\alpha}}\|\varrho(t)\|_{H^{s+\alpha}}\|\varrho(t)\|_{H^s}dt. \tag{4.24}$$

Taking into account that $G \in Lip_u(H^s, L_2(\mathcal{U}, H^s))$ and $|\zeta(w)| \le L$, we find that

$$\mathcal{I}_3 + \mathcal{I}_4 \le C\mathbb{E}\int_{\tau_a}^{\tau_b} \|\varrho(t)\|_{H^s}^2 dt. \tag{4.25}$$

For the last term on the right-hand side of (4.19), using the Burkholder-Davis-Gundy inequality and the Young inequality, in view of the assumptions on G, we have

$$\begin{aligned}
\mathcal{I}_5 &\le C\mathbb{E}\left(\int_{\tau_a}^{\tau_b} \|G(\theta_1(t)) - G(\theta_2(t))\|_{L_2(\mathcal{U},H^s)}^2\|\varrho(t)\|_{H^s}^2dt\right)^{\frac{1}{2}}\\
&\le C\mathbb{E}\left(\left(\sup_{t\in[\tau_a,\tau_b]}\|\varrho(t)\|_{H^s}^2\right)^{\frac{1}{2}}\left(\int_{\tau_a}^{\tau_b}\|G(\theta_1(t)) - G(\theta_2(t))\|_{L_2(\mathcal{U},H^s)}^2dt\right)^{\frac{1}{2}}\right)\\
&\le \frac{1}{2}\mathbb{E}\left(\sup_{t\in[\tau_a,\tau_b]}\|\varrho(t)\|_{H^s}^2\right) + C\mathbb{E}\int_{\tau_a}^{\tau_b}\|\varrho(t)\|_{H^s}^2dt.
\end{aligned} \tag{4.26}$$

Combining (4.20) and (4.24)-(4.26), by the Young inequality, it follows from (4.19) that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[\tau_a,\tau_b]}\|\varrho(t)\|_{H^s}^2 + \kappa\int_{\tau_a}^{\tau_b}\|\varrho(t)\|_{H^{s+\alpha}}^2dt\right)\\
&\quad\le 2\mathbb{E}\|\varrho(\tau_a)\|_{H^s}^2 + C\mathbb{E}\int_{\tau_a}^{\tau_b}(\|\theta_1(t)\|_{H^{s+\alpha}}^2 + \|\theta_2(t)\|_{H^{s+\alpha}}^2 + 1)\|\varrho(t)\|_{H^s}^2dt.
\end{aligned}$$

Then we apply the stochastic Gronwall lemma, as in [16], to complete the proof. □

Let $\mathcal{P}_t(x, A)$ be the corresponding transition function defined by

$$\mathcal{P}_t(x, A) = P(\theta(t; x) \in A), \quad t \geq 0, x \in H^s, A \in \mathscr{B}(H^s).$$

This defines the transition semigroup, also denoted by $\{\mathcal{P}_t\}_{t \geq 0}$,

$$\mathcal{P}_t \varphi(x) = \mathbb{E}(\varphi(\theta(t; x))) = \int_{H^s} \varphi(y) \mathcal{P}_t(x, dy), \quad t \geq 0, x \in H^s, \varphi \in C_b^{loc}(H^s),$$

where $C_b^{loc}(H^s)$ denotes the Banach space of all real valued, bounded, locally uniformly continuous functions, endowed with the sup norm

$$\|\varphi\|_\infty := \sup_{\theta \in H^s} |\varphi(\theta)|.$$

We will show below that $\{\mathcal{P}_t\}_{t \geq 0}$ is Feller, meaning that $\mathcal{P}_t$ maps $C_b^{loc}(H^s)$ into $C_b^{loc}(H^s)$ for every $t \geq 0$.

Theorem 4.4. *Under the assumptions of Lemma 4.2, the transition semigroup is Feller on H^s with $s \geq \frac{1}{2-\eta_0}$, where η_0 is given in Lemma 4.2, i.e.,*

$$\mathcal{P}_t : C_b^{loc}(H^s) \to C_b^{loc}(H^s), \quad \textit{for any } t \geq 0.$$

Remark 4.5. It is worth mentioning that

$$H^s \subset L^{\frac{2(2-\eta_0)}{1-\eta_0}}, \quad \text{for } s \geq \frac{1}{2-\eta_0}.$$

Recall that η_0, given in Lemma 4.2, belongs to $(0, 1)$. Hence $s \geq \frac{1}{2-\eta_0}$ follows immediately as long as we assume that $s \geq 1$ in Lemma 4.2.

Proof. Let $\varphi \in C_b^{loc}(H^s)$ be given arbitrarily. Now it suffices to prove that for any $t \geq 0$ and $m \in \mathbb{N}$,

$$\lim_{\delta \to 0} \sup_{x, x_0 \in B_m, \|x - x_0\|_{H^s} \leq \delta} |\mathcal{P}_t \varphi(x) - \mathcal{P}_t \varphi(x_0)| = 0, \tag{4.27}$$

where B_m denotes a closed ball in H^s centered at zero with radius m.

Thanks to Lemma 4.2, there exists a constant $R > m$ sufficiently large such that for any $x \in B_m$,

$$\mathbb{E}\|\theta(t; x)\|_{H^s} \leq R.$$

For this R, since φ is uniformly continuous on B_R, we choose $\eta > 0$ such that for any $\varepsilon > 0$ and any $\theta_1, \theta_2 \in B_R$ with $\|\theta_1 - \theta_2\|_{H^s} \leq \eta$,

$$|\varphi(\theta_1) - \varphi(\theta_2)| \leq \frac{\varepsilon}{2}.$$

Let

$$\delta = \frac{\sqrt{\varepsilon}\eta}{\sqrt{4\|\varphi\|_{L^\infty} C_t}},$$

where C_t is the constant appearing in (4.17). Using the Chebyshev inequality and Lemma 4.3, we obtain that for any $x, x_0 \in B_m$ with $\|x - x_0\|_{H^s} < \delta$,

$$\begin{aligned}
|\mathcal{P}_t\varphi(x) - \mathcal{P}_t\varphi(x_0)| &= |\mathbb{E}(\varphi(\theta(t;x))) - \mathbb{E}(\varphi(\theta(t;x_0)))| \\
&\le \frac{\varepsilon}{2} + 2\|\varphi\|_{L^\infty} P(\|\theta(t;x) - \theta(t;x_0)\|_{H^s} > \eta) \\
&\le \frac{\varepsilon}{2} + \frac{2\|\varphi\|_{L^\infty}}{\eta^2}\mathbb{E}\left(\sup_{r\in[0,t]} \|\theta(r;x) - \theta(r;x_0)\|^2_{H^s}\right) \le \varepsilon, \qquad (4.28)
\end{aligned}$$

which implies (4.27) as desired. Thus the proof is complete. □

4.3 Existence of invariant measures

For any Borel probability measure μ and $t \ge 0$, we define the dual semigroup $\mathcal{P}_t^*$ by

$$\mathcal{P}_t^*\mu(A) = \int_{H^s} \mathcal{P}_t(x, A)\mu(dx), \quad A \in \mathscr{B}(H^s).$$

Then μ is an invariant measure for $\{\mathcal{P}_t\}_{t\ge 0}$ if $\mathcal{P}_t^*\mu = \mu$ for all $t \ge 0$. In the following theorem, we establish the existence of non-trivial invariant measures.

Theorem 4.6. *Under the assumptions of Lemma 4.2, there exists an invariant measure associated with $\{\mathcal{P}_t\}_{t\ge 0}$ on H^s with $s \ge 2 - 2\alpha$.*

Proof. Let us return to the proof of Lemma 4.2. Putting (4.7), (4.10) and (4.11) together, in view of (3.12) and (4.13), we have

$$\begin{aligned}
&\mathcal{V}(t) + \frac{\varepsilon_0}{\lambda_1^\alpha}\mathbb{E}\int_0^t e^{2\xi(w(r))}\|\theta(r)\|^2_{H^{s+\alpha}} dt \\
&\quad \le \mathcal{V}(0) + C\mathbb{E}\int_0^t e^{2\xi(w(r))}\|\theta(r)\|_{L^{p_0}}^{\frac{2(2-\eta_0)}{1-\eta_0}} dr \\
&\qquad + \left(2\varepsilon_0 - 2\kappa\lambda_1^\alpha + \|G\|^2_{Lip} - 2\langle\zeta,\pi\rangle + 2\frac{C_\gamma^2}{\gamma^2}\|\zeta\|^2_{Lip}\right)\int_0^t \mathcal{V}(r)dr.
\end{aligned}$$

By (4.15) and (4.8), we obtain that there exists a positive constant $C_0 = C_0(\theta(0), w(0))$ such that

$$\frac{\varepsilon_0}{\lambda_1^\alpha}\mathbb{E}\int_0^t e^{2\xi(w(r))}\|\theta(r)\|^2_{H^{s+\alpha}} dt \le \mathcal{V}(0) + C_0 t.$$

Then using the Young inequality and (3.27), we deduce that

$$
\begin{aligned}
\mathbb{E}\int_0^T \|\theta(t)\|_{H^{s+\alpha}}dt &\leq \mathbb{E}\int_0^T e^{2\xi(w(t))}\|\theta(t)\|_{H^{s+\alpha}}^2 dt + \frac{1}{4}\mathbb{E}\int_0^T e^{-2\xi(w(t))}dt \\
&\leq \frac{\lambda_1^\alpha}{\varepsilon_0}\left(\mathcal{V}(0) + C_0 T\right) + CT,
\end{aligned} \tag{4.29}
$$

which, by the Markov inequality, implies that

$$
\begin{aligned}
&\lim_{R\to\infty}\liminf_{T\to\infty}\frac{1}{T}\int_0^T P\left(\|\theta(t)\|_{H^{s+\alpha}} > R\right)dt \\
&\leq \lim_{R\to\infty}\liminf_{T\to\infty}\frac{1}{RT}\mathbb{E}\int_0^T \|\theta(t)\|_{H^{s+\alpha}}dt \\
&\leq \lim_{R\to\infty}\liminf_{T\to\infty}\frac{1}{R}\left(\frac{\lambda_1^\alpha}{\varepsilon_0 T}\mathcal{V}(0) + \frac{\lambda_1^\alpha C_0}{\varepsilon_0} + C\right) = 0.
\end{aligned} \tag{4.30}
$$

Consider the sequence of time average measures

$$
\mu_n(\Gamma) := \frac{1}{T_n}\int_0^{T_n}\mathcal{P}_t(x,\Gamma)dt = \frac{1}{T_n}\int_0^{T_n} P(\theta(t)\in\Gamma)dt, \quad \Gamma\in\mathscr{B}(H^s).
$$

To obtain the existence of invariant measures, it suffices to show that μ_n is weakly compact, due to the classical Krylov-Bogoliubov method (cf. [12, Theorem 11.7]). For any $R>0$, let $B_R = \{\theta\in H^{s+\alpha} : \|\theta\|_{H^{s+\alpha}} \leq R\}$ and $B_R^{c_{H^s}} = \{\theta\in H^s : \theta\notin B_R\}$. It is clear that B_R is compact in H^s and therefore by (4.30), for any $\varepsilon>0$, there exists a compact set $B_R\subset H^s$ such that

$$
\mu_n(H^s\backslash B_R) = \mu_n(B_R^{c_{H^s}}) < \varepsilon, \qquad \text{for all } n\geq 1.
$$

By the well-known Prokhorov theorem (see, e.g. [12, Theorem 2.3]), the family $\{\mu_n\}_{n\geq1}$ is weakly compact as desired. □

References

[1] L. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. (2), 171 (2010), 1903-1930.

[2] N. Chen, A.J. Majda, X.T. Tong, Information barriers for noisy Lagrangian tracers in filtering random incompressible flows, Nonlinearity, 27 (2014), 2133-2163.

[3] N. Chen, A.J. Majda, X.T. Tong, Noisy Lagrangian tracers for filtering random rotating compressible flows, J. Nonlinear Sci., 25 (2015), 451-488.

[4] X. Cheng, H. Kwon, D. Li, Non-uniqueness of steady-state weak solutions to the surface quasi-geostrophic equations, Comm. Math. Phys., 388 (2021), 1281-1295.

[5] J.W. Cholewa, T. Dlotko, Global Attractors in Abstract Parabolic Problems, Cambridge University Press, Cambridge, 2000.

[6] P. Constantin, N. Glatt-Holtz, V. Vicol, Unique ergodicity for fractionally dissipated, stochastically forced 2D Euler equations, Comm. Math. Phys., 330 (2014), 819-857.

[7] P. Constantin, A.J. Majda, E. Tabak, Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar, Nonlinearity, 7 (1994), 1495-1533.

[8] P. Constantin, A. Tarfulea, V. Vicol, Long time dynamics of forced critical SQG, Comm. Math. Phys., 335 (2015), 93-141.

[9] A. Córdoba, D. Córdoba, A maximum principle applied to quasi-geostrophic equations, Commun. Math. Phys., 249 (2004), 511-528.

[10] M. Coti-Zelati, V. Vicol, On the global regularity for the supercritical SQG equation, Indiana Univ. Math. J., 65 (2016), 535-552.

[11] M. Dai, Existence and stability of steady-state solutions to the quasi-geostrophic equations in $\mathbb{R}^2$, Nonlinearity, 28 (2015), 4227-4248.

[12] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, 1992.

[13] B. Dong, Z. Chen, Asymptotic stability of the critical and super-critical dissipative quasi-geostrophic equation, Nonlinearity, 19 (2006), 2919-2928.

[14] R. Farwig, C. Qian, Asymptotic behavior for the quasi-geostrophic equations with fractional dissipation in $\mathbb{R}^2$, J. Differential Equations, 266 (2019), 6525-6579.

[15] M. Fujii, Long time existence and asymptotic behavior of solutions for the 2d quasi-geostrophic equation with large dispersive forcing, J. Math. Fluid Mech., 23 (2021), 1-19.

[16] N. Glatt-Holtz, M. Ziane, Strong pathwise solutions of the stochastic Navier-Stokes system, Adv. Differential Equations, 14 (2009), 567-600.

[17] S.M. Griffies, Fundamentals of Ocean Climate Models, Princeton University Press, Princeton, NJ, 2004.

[18] F. Hadadifard, A.G. Stefanov, On the forced surface quasi-geostrophic equation: existence of steady states and sharp relaxation rates, J. Math. Fluid Mech., 23 (2021), 1-27.

[19] X. Han, W. Shen, S. Zhou, Random attractors for stochastic lattice dynamical systems in weighted spaces, J. Differential Equations, 250 (2011), 1235-1266.

[20] R.Z. Has'minskiĭ, Stochastic Stability of Differential Equations, Sijthoff & Noordhoff, Alphen aan den Rijn-Germantown, Md., 1980.

[21] L. Hsiao, Quasilinear Hyperbolic Systems and Dissipative Mechanisms, World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[22] N. Ju, Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space, Comm. Math. Phys., 251 (2004), 365-376.

[23] N. Ju, The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations, Comm. Math. Phys., 255 (2005), 161-181.

[24] T. Kato, G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math., 41 (1988), 891-907.

[25] C. Kenig, G. Ponce, L. Vega, Well-posedness of the initial value problem for the Korteweg-de Vries equation, J. Amer. Math. Soc., 4 (1991), 323-347.

[26] A. Kiselev, F. Nazarov, A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, Invent. Math., 167 (2007), 445-453.

[27] N.V. Krylov, Itô's formula for the L_p-norm of stochastic W_p^1-valued processes, Probab. Theory Relat. Fields, 147 (2010), 583-605.

[28] O. Lazar, L. Xue, Regularity results for a class of generalized surface quasi-geostrophic equations, J. Math. Pures Appl., 130 (2019), 200-250.

[29] T. Liang, Y. Wang, T. Caraballo, Large time behavior of stochastic fractionally dissipative quasi-geostrophic equations, submitted.

[30] T. Liang, Y. Wang, T. Caraballo, Stability of fractionally dissipative 2D quasi-geostrophic equation with infinite delay, J. Dynam. Differential Equations, 33 (2021), 2047-2074.

[31] K. Liu, Stability of Infinite Dimensional Stochastic Differential Equations with Applications, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[32] X. Luo, Stationary solutions and nonuniqueness of weak solutions for the Navier-Stokes equations in high dimensions, Arch. Ration. Mech. Anal., 233 (2019), 701-747.

[33] A.J. Majda, B. Gershgorin, Elementary models for turbulent diffusion with complex physical features: eddy diffusivity, spectrum and intermittency, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 371 (2013), 1-36.

[34] A.J. Majda, X.T. Tong, Simple nonlinear models with rigorous extreme events and heavy tails, Nonlinearity, 32 (2019), 1641-1674.

[35] C. Martínez Carracedo, M. Sanz Alix, The Theory of Fractional Powers of Operators, Elsevier, Amsterdam, 2001.

[36] J.C. Mattingly, A.M. Stuart, D.J. Higham, Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise, Stochastic Process. Appl., 101 (2002), 185-232.

[37] B. Øksendal, Stochastic Differential Equations. An Introduction with Applications, Springer-Verlag, Berlin, 2003.

[38] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.

[39] C. Prévôt, M. Röckner, A Concise Course on Stochastic Partial Differential Equations, in: Lecture Notes in Mathematics, Springer, Berlin, 2007.

[40] M. Röckner, R. Zhu, X. Zhu, Sub and supercritical stochastic quasi-geostrophic equation, Ann. Probab., 43 (2015), 1202-1273.

[41] M.E. Schonbek, T.P. Schonbek, Asymptotic behavior to dissipative quasi-geostrophic flows, SIAM J. Math. Anal., 35 (2003), 357-375.

[42] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton, Princeton University Press, 1970.

[43] X. Wan, D. Xiu, G.E. Karniadakis, Stochastic solutions for the two-dimensional advection-diffusion equation, SIAM J. Sci. Comput., 26 (2004), 578-590.

[44] M. Yamamoto, Y. Sugiyama, Spatial-decay of solutions to the quasi-geostrophic equation with the critical and supercritical dissipation, Nonlinearity, 32 (2019), 2467-2480.

[45] S. Yang, Y. Li, Asymptotic autonomous attractors for a stochastic lattice model with random viscosity, J. Difference Equ. Appl., 26 (2020), 540-560.

[46] L. Yang, X. Pu, Ergodicity of large scale stochastic geophysical flows with degenerate Gaussian noise, Appl. Math. Lett., 64 (2017), 27-33.

[47] Z. Ye, On the global regularity for the anisotropic dissipative surface quasi-geostrophic equation, Nonlinearity, 33 (2020), 72-105.