PARABOLIC BGG CATEGORIES AND THEIR BLOCK DECOMPOSITION FOR LIE SUPERALGEBRAS OF CARTAN TYPE

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ABSTRACT. In this paper, we study the parabolic BGG categories for graded Lie superalgebras of Cartan type over the field of complex numbers. The gradation of such a Lie superalgebra $\mathfrak{g}$ naturally arises, with the zero component $\mathfrak{g}_0$ being a reductive Lie algebra. We first show that there are only two proper parabolic subalgebras containing Levi subalgebra $\mathfrak{g}_0$: the "maximal one" $\mathsf{P}_{\max}$ and the "minimal one" $\mathsf{P}_{\min}$. Furthermore, the parabolic BGG category arising from $\mathsf{P}_{\max}$ essentially turns out to be a subcategory of the one arising from $\mathsf{P}_{\min}$. Such a priority of $\mathsf{P}_{\min}$ in the sense of representation theory reduces the question to the study of the "minimal parabolic" BGG category $\mathcal{O}^{\min}$ associated with $\mathsf{P}_{\min}$. We prove the existence of projective covers of simple objects in these categories, which enables us to establish a satisfactory block theory. Most notably, our main results are as follows.

(1) We classify and obtain a precise description of the blocks of $\mathcal{O}^{\min}$.

(2) We investigate indecomposable tilting and indecomposable projective modules in $\mathcal{O}^{\min}$, and compute their character formulas.

CONTENTS

INTRODUCTION

0.1. By Kac's classification theorem ([13]), finite-dimensional simple Lie superalgebras over the field of complex numbers are either of classical type or of Cartan type, with the latter

consisting of infinite series of the four types $W(n)$, $S(n)$, $\tilde{S}(n)$ and $H(n)$. The simple Lie superalgebra $W(n)$ ($n \geq 3$) is the derivation algebra of the Grassmann superalgebra $\Lambda(n)$ on n generators. Arising from the natural $\mathbb{Z}$-grading on $\Lambda(n)$, $W(n)$ is also naturally $\mathbb{Z}$-graded. The Lie superalgebras $S(n)$ ($n \geq 4$), $\tilde{S}(n)$ ($n \geq 4$) and $H(n)$ ($n \geq 5$) are Lie subalgebras of $W(n)$. The superalgebra $\tilde{S}(n)$ is not $\mathbb{Z}$-graded, but carries a filtration induced by the filtration of $W(n)$.

Irreducible finite-dimensional representations of Lie superalgebras of Cartan type were studied earlier ([6], [19], etc.), motivated by Rudakov's work on irreducible representations of infinite-dimensional Lie algebras of Cartan type ([16] and [17]). In [18], Serganova considered the category of $\mathbb{Z}$-graded irreducible representations of graded Lie superalgebras of Cartan type, determined the character formulas of their $\mathbb{Z}$-graded irreducible highest weight modules. After that, there were a few papers on finite-dimensional representations over $W(n)$. For example, in [3] the authors computed the cohomological support varieties of irreducible $W(n)$-modules in a certain category, the objects of which are finite-dimensional and completely reducible over the zero component $W(n)_0$. In [20] Shomron studied the blocks of a certain category whose objects are finite-dimensional $W(n)$-modules by constructing extensions between irreducible modules. However, when considering categories containing infinite-dimensional objects, the situation becomes very complicated.

0.2. Let $\mathfrak{g}$ be a Lie superalgebra of Cartan type $X(n)$, where $X \in \{W, S, H\}$. Then $\mathfrak{g}$ is naturally endowed with a $\mathbb{Z}$-graded structure, i.e., $\mathfrak{g} = \sum_{i \geq -1} \mathfrak{g}_i$. In addition, $\mathfrak{g}_0$ is a reductive Lie algebra. When $X \in \{S, H\}$, it will be convenient to study, in place of $X(n)$, the representation category of the one-dimensional toral extension $\bar{X}(n)$ determined by the following exact sequence

$$X(n) \hookrightarrow \bar{X}(n) \twoheadrightarrow \mathbb{C}\mathsf{d},$$

where d is a canonical toral element measuring degrees in $W(n)$ (see §1.2 for details).

In the present paper, we introduce and study a parabolic BGG category for $X(n)$ with $X \in \{W, \bar{S}, \bar{H}\}$, in analogy to the Bernstein-Gelfand-Gelfand category of complex semisimple Lie algebras (see [5] and [12]). Our purpose is to investigate blocks in this category, develop a tilting module theory, and give character formulas of indecomposable tilting and indecomposable projective modules. Recall that a Lie superalgebra of Cartan type admits many mutually non-conjugate Borel subalgebras ([18, §4]), also many mutually non-conjugate Borel subalgebras containing the standard Borel subalgebra of the core reductive Lie subalgebra $\mathfrak{g}_0$ (see §2.2), and hence possibly admits many "parabolic" subalgebras. An important ingredient in our work is to discriminate these parabolic subalgebras, and choose a suitable "parabolic subalgebra". Surprisingly, there are only two such parabolic (proper) subalgebras, the maximal one $\mathsf{P}_{\max}$ which is actually $\sum_{i \geq 0} \mathfrak{g}_i$, and the minimal one $\mathsf{P}_{\min}$ which is equal to $\sum_{i \leq 0} \mathfrak{g}_i$ (see Proposition 2.3). Furthermore, the "parabolic BGG category" associated with $\mathsf{P}_{\max}$ turns out to be less interesting (see §2.4) since whose $U(\mathfrak{g})$-finitely-generated objects are finite-dimensional. Actually, the parabolic BGG category associated with $\mathsf{P}_{\max}$ is a subcategory of the parabolic BGG category associated with $\mathsf{P}_{\min}$ if only considering objects finitely generated over $U(\mathfrak{g})$.

Based on the above analysis, we only need to focus on $\mathsf{P}_{\min} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$. In this article, we simply write it as P which is naturally regarded as a "minimal parabolic" subalgebra of $\mathfrak{g}$ containing the reductive Lie algebra $\mathfrak{g}_0$. We then introduce the parabolic BGG category $\mathcal{O}^{\min}$

associated with P. This category is by definition a subcategory of the $\mathbb{Z}$-graded $U(\mathfrak{g})$-module category, satisfying some standard axioms (see Definition 3.1). What is completely different from [18] is that all standard modules have infinite composition factors. Nevertheless, we can prove the following fundamental result.

Theorem 0.1. *(See Theorem 4.2) Any simple object in $\mathcal{O}^{min}$ has a projective cover which admits a flag of standard modules.*

0.3. Along the direction just mentioned above, we can define blocks of $\mathcal{O}^{\min}$ via projective covers of irreducible modules, and get into the next topic—to classify and describe all blocks of $\mathcal{O}^{\min}$.

It can be proven that all simple objects in $\mathcal{O}^{\min}$ are parameterized by what we denote here by $\mathbf{E}$, that is, a combination of finite-dimensional irreducible modules over $\mathfrak{g}_0$ and their so-called "depths" associated with the $\mathbb{Z}$-gradation. We give one of our main results in the following.

Theorem 0.2. *(See Theorems 6.15, 6.17 and 6.19) Let $\mathfrak{g} = X(n)$, $X \in \{W, \bar{S}, \bar{H}\}$. For any given $L(\lambda), L(\mu) \in \mathbf{E}$, $L(\lambda)$ and $L(\mu)$ lie in the same block if and only if the following three conditions are satisfied.*

(1) $\mu \in \lambda + Q$;
(2) $\mathsf{dpt}(L(\mu)) = \mathsf{dpt}(L(\lambda)) + \ell(\lambda - \mu)$;
(3) $\mathsf{pty}(L(\mu)) = \mathsf{pty}(L(\lambda)) + \overline{\ell(\lambda - \mu)}$,

where $\mathsf{dpt}(L(\lambda))$ denotes the depth of $L(\lambda)$ associated with its $\mathbb{Z}$-graded structure; $\mathsf{pty}(L(\lambda))$ is the parity of the "maximal vector" v_λ^0 of $L(\lambda)$; and for each $\alpha \in Q$ we write $\ell(\alpha)$ for the length of α (see (6.12) for the definition) and $\overline{\ell(\alpha)}$ for the parity of $\ell(\alpha)$.

A more precise structural description of blocks can be found in Theorems 6.15, 6.17 and 6.19. Below, we will give an outline of the proof of Theorem 0.2.

While establishing the existence of the projective cover $P(\lambda)$ of an irreducible module $L(\lambda)$ in $\mathcal{O}^{\min}$, we consider an "enveloping" projective module $I(\lambda)$, which is induced from irreducible modules $L^0(\lambda)$ over the graded-zero component $\mathfrak{g}_0$, endowed with a flag of standard modules. Then we can prove that $I(\lambda)$ lies in the same block $\mathbf{B}(\lambda)$ as $P(\lambda)$. Based on the construction of $I(\lambda)$, we use various strategies to read off information about $\mathbf{B}(\lambda)$. In particular, we examine maximal vectors. Along this way, the block decomposition becomes easy for $W(n)$ and $\bar{S}(n)$. However, it does not work well for $\bar{H}(n)$. The solution is to establish the relations between the standard modules of $\overline{CH}(n)$ and the standard modules of $\bar{H}(n)$. (Here $\overline{CH}(n)$ is a Lie subalgebra of $W(n)$ while $\bar{H}(n)$ is the derived subalgebra of $\overline{CH}(n)$ with codimension one in $\overline{CH}(n)$). The most important step in this approach is the non-trivial observation that all standard modules for $\overline{CH}(n)$ are indecomposable over $\bar{H}(n)$ (see Corollary 6.10). Another thing to notice is that the behavior of $\bar{H}(2r+1)$ at the root lattice is critically different from that of $\bar{H}(2r)$, which is ultimately a consequence of the difference of orthogonal classical Lie algebras of types B_r and D_r. So proving the final results on blocks for $\bar{H}(2r+1)$ and $\bar{H}(2r)$ will require separate arguments (see Theorems 6.17 and 6.19).

The above block theorem actually reveals a somewhat degenerate behavior of blocks for algebraic models of Cartan series, in comparison with the classical theory of complex semi-simple Lie algebras and basic classical Lie superalgebras (see [12] and [10]). The intrinsic mechanism should be further investigated.

0.4. Another important ingredient in our arguments is to prove that each of $W(n)$, $\bar{S}(n)$ and $\bar{H}(n)$ admits a semi-infinite character. The notation of semi-infinite character put forward by Soergel was derived from the work on semi-cohomology by Feigin, Voronov and Arkhipov (cf. [2], [11] and [22]). For $\mathbb{Z}$-graded Lie algebras admitting semi-infinite characters, Soergel established in [21] a framework for some $\mathbb{Z}$-graded representation category. Following Soergel's work [21], Brundan investigated some general theory of category $\mathcal{O}$ for a general $\mathbb{Z}$-graded Lie superalgebra in [7], which can be used to study representations of classical Lie superalgebras, and especially to deal with $\mathfrak{gl}(m,n)$ and $\mathfrak{q}(n)$. Fortunately, the general theory of Brundan's work is available to the case of $\mathbb{Z}$-graded Lie superalgebras of Cartan type, so we have the category $\mathcal{O}^{\min}$ in the present paper. Especially, a BGG reciprocity for truncated categories in [7] is true for $\mathcal{O}^{\min}$. Furthermore, we can investigate tilting modules in $\mathcal{O}^{\min}$ on the basis of Soergel's and Brundan's work. Most notably, we establish Soergel's reciprocity for tilting modules in our $\mathcal{O}^{\min}$.

Recall that our category $\mathcal{O}^{\min}$ is associated with the "minimal parabolic" subalgebra P, which enables us to obtain a realization of co-standard modules in $\mathcal{O}^{\min}$ via Kac modules. This is very important for us to go further, and in particular it leads to the following reciprocities.

Theorem 0.3. *(See Theorem 5.6, Propositions 7.5) Let $P(\lambda)$(resp. $T(\lambda)$) be the indecomposable projective (resp. tilting) module in $\mathcal{O}^{min}$ corresponding to the simple object $L(\lambda) \in \mathbf{E}$, and $K(\lambda)$ be the corresponding Kac module. Let $[P(\lambda) : \Delta(\mu)]$ (resp. $[T(\lambda) : \Delta(\mu)]$) denote the multiplicity of the standard module $\Delta(\mu)$ in $P(\lambda)$ (resp. $T(\lambda)$). Then the following statements hold.*

(1) *If $\mathfrak{g} = W(n)$, then*

$$\begin{aligned} [P(\mu) : \Delta(\lambda)] &= (K(\lambda + \Xi) : L(\mu)); \\ [T(\mu) : \Delta(\lambda)] &= (K(-w_0\lambda + 2\Xi) : L(-w_0\mu + \Xi)). \end{aligned}$$

(2) *If $\mathfrak{g} = \bar{S}(n)$, then*

$$\begin{aligned} [P(\mu) : \Delta(\lambda)] &= (K(\lambda + \Xi) : L(\mu)); \\ [T(\mu) : \Delta(\lambda)] &= (K(-w_0\lambda + \Xi) : L(-w_0\mu)). \end{aligned}$$

(3) *If $\mathfrak{g} = \bar{H}(n)$, then*

$$\begin{aligned} [P(\mu) : \Delta(\lambda)] &= (K(\lambda + n\delta) : L(\mu)); \\ [T(\mu) : \Delta(\lambda)] &= (K(-w_0\lambda + n\delta) : L(-w_0\mu)). \end{aligned}$$

Here w_0 is the longest element of the Weyl group of $\mathfrak{g}_0$, δ is the linear dual of extended toral element d, $\Xi = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ and $(K(\cdot) : L(\cdot))$ denotes the multiplicity of a composition factor in certain Kac module.

From the above theorem, Serganova's character formulas on Kac modules in [18] allow us finally to obtain the character formulas of both indecomposable projective and indecomposable tilting modules in $\mathcal{O}^{\min}$.

0.5. This paper is organized as follows. In Section 1, we introduce some basic notions and notations for Lie superalgebras of Cartan type. Most notably, we show in §1.3 that the $\mathbb{Z}$-graded Cartan type Lie superalgebras admit semi-infinite characters. In Section 2, we make a precisely construction of adjacent Borel subalgebras, and then show the surprising result that any parabolic subalgebra containing $\mathfrak{g}_0$ is either the maximal one or the minimal one. Then the BGG category arising from any parabolic subalgebra is essentially the subcategory of the one arising from the minimal. In Section 3, we introduce the category $\mathcal{O}^{\min}$, investigate some natural representations and list some properties of their weights. These arguments are very important to the study of blocks. In Section 4, we consider the projective modules in $\mathcal{O}^{\min}$. In particular, we establish that all simple objects in $\mathcal{O}^{\min}$ have projective covers, and every indecomposable projective module admits a flag of standard modules (Theorem 4.2). In Section 5, we obtain a degenerate BGG reciprocity (Theorem 5.3). In Section 6, we investigate and describe the blocks in $\mathcal{O}^{\min}$, see Theorems 6.15, 6.17 and 6.19. In Section 7, we obtain a version of Soergel's reciprocity for indecomposable tilting modules via a realization of co-standard modules in terms of Kac modules. Then we apply the degenerate BGG reciprocity and Soergel's reciprocity to give character formulas of the indecomposable projective and the indecomposable tilting modules. The last two sections are appendixes, where we give a detailed computation for semi-infinite characters (Appendix A) and character formulas of tilting modules (Appendix B).

In the same spirit, it is also possible to extend parts of the theory of the category $\mathcal{O}^{\min}$ (including tilting modules and their character theory) to infinite-dimensional Lie algebras of Cartan type (see [8]).

1. PRELIMINARIES

In this paper, we always assume that the base field is the complex field $\mathbb{C}$. All vector superspaces (resp. supermodules, superalgebras) are over $\mathbb{C}$, and will be simply called spaces (resp. modules, algebras).

1.1. **The Lie superalgebras of Cartan type.** In this subsection, we recall the definitions of finite-dimensional Lie superalgebras of Cartan type (see [13] for details).

Let $\Lambda(n)$ be the Grassmann superalgebra on n odd generators $\xi_1, \ldots, \xi_n$ ($n \geq 2$). Let $\deg(\xi_i) = 1$ for $1 \leq i \leq n$. Then $\Lambda(n)$ has a natural $\mathbb{Z}$-grading with $\Lambda(n)_j = \operatorname{span}_{\mathbb{C}}\{\xi_{k_1} \wedge \cdots \wedge \xi_{k_j} \mid 1 \leq k_1 < \cdots < k_j \leq n\}$. The Witt type Lie superalgebra $W(n)$ is defined to be the set of all superderivations of $\Lambda(n)$. Then

$$W(n) = \left\{ \sum_{i=1}^{n} f_i D_i \mid f_i \in \Lambda(n) \right\},$$

where D_i is the superderivation of $\Lambda(n)$ defined through $D_i(\xi_j) = \delta_{ij}$ for $1 \le i, j \le n$. The Witt type Lie superalgebra $W(n)$ has a natural $\mathbb{Z}$-grading with

$$W(n)_j = \left\{ \sum_{i=1}^{n} f_i D_i \mid f_i \in \Lambda(n)_{j+1} \right\}. \tag{1.1}$$

Let **div** be the divergence mapping from the Witt type Lie superalgebra $W(n)$ to the Grassmann superalgebra $\Lambda(n)$ defined as:

$$\begin{aligned}
\mathbf{div} : W(n) &\to \Lambda(n) \\
\sum_{i=1}^{n} f_i D_i &\mapsto \sum_{i=1}^{n} D_i(f_i).
\end{aligned}$$

The special Lie superalgebra $S(n)$ is defined as the Lie subalgebra of $W(n)$, consisting of all elements $x \in W(n)$ such that $\mathbf{div}(x) = 0$. Since the divergence mapping is a homogeneous operator of degree 0, the special Lie superalgebra $S(n)$ inherits the $\mathbb{Z}$-gradation of $W(n)$, i.e., $S(n) = \bigoplus_{i=-1}^{n-2} S(n)_i$, where $S(n)_i = W(n)_i \cap S(n)$. Now we introduce the mapping D_{ij} :

$$\begin{aligned}
D_{ij} : \Lambda(n) &\to W(n) \\
f &\mapsto D_i(f)D_j + D_j(f)D_i.
\end{aligned}$$

We can check that $S(n)$ is the $\mathbb{C}$-linear span of the elements belonging to $\{D_{ij}(f) \mid f \in \Lambda(n), 1 \le i, j \le n\}$.

Up to isomorphism, there is a different class of simple Lie superalgebras of another special type $\tilde{S}(n)$. The Lie superalgebra $\tilde{S}(n)$ is defined only for even n, and it consists of all $x \in W(n)$ such that

$$(1 + \xi_1 \cdots \xi_n)\mathbf{div}(x) + x(\xi_1 \cdots \xi_n) = 0.$$

It is not a $\mathbb{Z}$-graded subalgebra of $W(n)$ as the defining condition is not homogeneous. Hence we ignore $\tilde{S}(n)$ in this paper.

Next, we introduce the Hamiltonian Lie superalgebra $H(n)$ with $n \ge 5$ (Note that $H(4) \cong A(1,1)$. We do not care about this case in the present paper. So we assume $n \ge 5$ for type H). Assume that $n = 2r$ or $n = 2r + 1$, set

$$i' = \begin{cases} i + r, & \text{if } 1 \le i \le r; \\ i - r, & \text{if } r + 1 \le i \le 2r; \\ i, & \text{if } i = 2r + 1. \end{cases}$$

The Hamiltonian operator D_H from the Grassmann superalgebra $\Lambda(n)$ to the Witt Lie superalgebra $W(n)$ is defined as:

$$\begin{aligned}
D_H : \Lambda(n) &\to W(n) \\
f &\mapsto D_H(f) = \sum_{i=1}^{n} (-1)^{\bar{f}} D_i(f)D_{i'},
\end{aligned}$$

where f is a homogeneous element in $\Lambda(n)$ and $\bar{f}$ denotes the parity of f. Set $CH(n) = \{D_H(f) \mid f \in \Lambda(n)\}$. Then the Hamiltonian Lie superalgebra $H(n)$ is by definition, the derived algebra of $CH(n)$, i.e.,

$$H(n) = [CH(n), CH(n))]; \quad CH(n) = H(n) + \mathbb{C}D_H(\xi_1 \cdots \xi_n), \tag{1.2}$$

which can be further described as follows

$$H(n) = \left\{ D_H(f) \mid f \in \sum_{i=0}^{n-1} \Lambda(n)_i \right\}.$$

Moreover, $H(n)$ is a $\mathbb{Z}$-graded subalgebra of $W(n)$ with $H(n) = \bigoplus_{i=-1}^{n-3} H(n)_i$, where $H(n)_i = W(n)_i \cap H(n)$. Generally, for a graded subalgebra $L = \sum_{i=-1}^{\infty} L_i$ of $W(n)$, we set $L_{\geq j} := \sum_{i \geq j} L_i$. Especially, we have the following structure

$$L_{-1} = \sum_{i=1}^{n} \mathbb{C} D_i \quad \text{for } L = X(n), X \in \{W, S, H\}. \tag{1.3}$$

Let $L = W(n), S(n)$ or $H(n)$. By the following canonical map

$$\begin{array}{ccc} L_0 & \to & \mathfrak{gl}(n) \\ \sum_{1 \leq i,j \leq n} k_{ij} \xi_i D_j & \mapsto & \sum_{1 \leq i,j \leq n} k_{ij} E_{ij}, \end{array}$$

we get

$$L_0 \cong \begin{cases} \mathfrak{gl}(n), & \text{if } \mathfrak{g} = W(n); \\ \mathfrak{sl}(n), & \text{if } \mathfrak{g} = S(n); \\ \mathfrak{so}(n), & \text{if } \mathfrak{g} = H(n), \end{cases} \tag{1.4}$$

and correspondingly have the standard triangular decomposition $L_0 = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

1.2. **Toral extension $\bar{S}(n)$, $\bar{H}(n)$ and $\overline{CH}(n)$.** Set $\mathsf{d} = \sum_{i=1}^{n} \xi_i D_i$. Then d is a canonical toral element of $W(n)$. The element d measures the degrees of homogenous spaces of $W(n)$, thereby it normalizes any graded subalgebra $\mathfrak{s}$ of $W(n)$, i.e., $[\mathsf{d}, \mathfrak{s}] \subseteq \mathfrak{s}$. Set

$$\bar{S}(n) = S(n) \oplus \mathbb{C}\mathsf{d},$$
$$\bar{H}(n) = H(n) \oplus \mathbb{C}\mathsf{d},$$
$$\overline{CH}(n) = CH(n) \oplus \mathbb{C}\mathsf{d},$$

and $\bar{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}\mathsf{d}$ for $\mathfrak{g} = \bar{S}(n)$, $\bar{H}(n)$ or $\overline{CH}(n)$, $\bar{\mathfrak{h}} := \mathfrak{h}$ for $\mathfrak{g} = W(n)$. We then have the following standard basis of $\bar{\mathfrak{h}}$:

$$\begin{cases} \{\xi_i D_i \mid 1 \leq i \leq n\}, & \text{if } \mathfrak{g} = W(n), \bar{S}(n); \\ \{\xi_i D_i - \xi_{i+r} D_{i+r}, \mathsf{d} \mid 1 \leq i \leq r\}, & \text{if } \mathfrak{g} = \bar{H}(2r), \overline{CH}(2r), \bar{H}(2r+1), \overline{CH}(2r+1), \end{cases} \tag{1.5}$$

whose dual basis can be described as follows:

$$\begin{cases} \{\epsilon_i \mid 1 \leq i \leq n\}, & \text{if } \mathfrak{g} = W(n), \bar{S}(n); \\ \{\epsilon_i, \delta \mid 1 \leq i \leq r\}, & \text{if } \mathfrak{g} = \bar{H}(2r), \overline{CH}(2r), \bar{H}(2r+1), \overline{CH}(2r+1). \end{cases}$$

This means $\epsilon_i(\xi_j D_j) = \delta_{ij}$ when $\mathfrak{g} = W(n)$ or $\bar{S}(n)$; and $\epsilon_i(\xi_j D_j - \xi_{j+r} D_{j+r}) = \delta_{ij}$, $\epsilon_i(\mathsf{d}) = 0$, $\delta(\xi_j D_j - \xi_{j+r} D_{j+r}) = 0$ for $1 \leq i, j \leq r$, and $\delta(\mathsf{d}) = 1$ when $\mathfrak{g} = \bar{H}(2r), \overline{CH}(2r), \bar{H}(2r+1), \overline{CH}(2r+1)$.

Set $V = \sum_{i=1}^{n} \mathbb{C}\xi_i$. We can further regard

$$\mathfrak{g}_0 = \begin{cases} \mathfrak{gl}(V), & \text{if } \mathfrak{g} = W(n), \bar{S}(n); \\ \mathfrak{so}(V) + \mathbb{C}\mathsf{d}, & \text{if } \mathfrak{g} = \bar{H}(n), \overline{CH}(n). \end{cases} \tag{1.6}$$

Convention 1.1. *In the sequel, whenever the context is clear, we don't distinguish ϵ_i and $\epsilon_i|_{\mathfrak{h}}$ for $1 \le i \le m$. Here, $m = n$ when $\mathfrak{g} = W(n)$ or $\bar{S}(n)$; $m = r$ when $\mathfrak{g} = \bar{H}(n), \overline{CH}(n)$.*

1.3. **Root systems and closed subalgebras.** Note that the Cartan subalgebras of $\mathfrak{g}$ coincide with the Cartan subalgebras of $\mathfrak{g}_0$. Associated with the Cartan subalgebra $\bar{\mathfrak{h}}$, there is a root system $\Phi(\mathfrak{g})$ and the corresponding root space decomposition $\mathfrak{g} = \bar{\mathfrak{h}} + \sum_{\alpha \in \Phi(\mathfrak{g})} \mathfrak{g}_\alpha$ for $\mathfrak{g} = X(n)$ ($X \in \{W, \bar{S}, \bar{H}, \overline{CH}\}$). The root system $\Phi(\mathfrak{g})$ can be described as below.

(1) For $\mathfrak{g} = W(n)$, $\Phi(\mathfrak{g}) = \{\epsilon_{i_1} + \cdots + \epsilon_{i_k} - \epsilon_j \mid 1 \le i_1 < \cdots < i_k \le n; k = 0, 1, ..., n; 1 \le j \le n\}$.

(2) For $\mathfrak{g} = \bar{S}(n)$, $\Phi(\mathfrak{g}) = \Phi(W(n)) \backslash \{(\sum_{i=1}^{n} \epsilon_i) - \epsilon_j \mid j = 1, ..., n\}$.

(3) For $\mathfrak{g} = \bar{H}(2r)$,

$$\Phi(\mathfrak{g}) = \{\pm\epsilon_{i_1} \pm \cdots \pm \epsilon_{i_k} + l\delta \mid 1 \le i_1 < i_2 < \cdots < i_k \le r;$$
$$k - 2 \le l < n - 2, l - k \in 2\mathbb{Z}\}.$$

(4) For $\mathfrak{g} = \bar{H}(2r + 1)$,

$$\Phi(\mathfrak{g}) = \{\pm\epsilon_{i_1} \pm \cdots \pm \epsilon_{i_k} + l\delta \mid 1 \le i_1 < i_2 < \cdots < i_k \le r;$$
$$k - 2 \le l < n - 2\}.$$

(5) For $\mathfrak{g} = \overline{CH}(n)$, $\Phi(\mathfrak{g}) = \Phi(\bar{H}(n)) \cup \{(n - 2)\delta\}$.

In particular, Φ_0 (resp. Φ_0^+) will denote the root system of $\mathfrak{g}_0$ (resp. $\mathfrak{n}^+$). Correspondingly, we have the Borel subalgebra $\mathfrak{b} = \bar{\mathfrak{h}} \oplus \mathfrak{n}^+$.

A subset Ψ of Φ is called a closed one if for any $\alpha, \beta \in \Psi$, we always have $\alpha + \beta \in \Psi$ provided that $\alpha + \beta \in \Phi$. We say a subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ to be closed if there is a closed subset Ψ of Φ such that $\mathfrak{q} = \mathfrak{h} + \sum_{\alpha \in \Psi} \mathfrak{g}_\alpha$.

We also need a convention $\Xi \in \mathfrak{h}^*$ for $\mathfrak{g} = W(n)$ or $\bar{S}(n)$ which means $\sum_{i=1}^{n} \epsilon_i$.

1.4. **Semi-infinite characters.**

Definition 1.2. *Let $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie superalgebra with $\dim \mathfrak{g}_i < \infty$ for all $i \in \mathbb{Z}$. A character $\gamma : \mathfrak{g}_0 \to \mathbb{C}$ is called a semi-infinite character for $\mathfrak{g}$ if the following items are satisfied.*

(SI-1) *As a Lie superalgebra, $\mathfrak{g}$ is generated by $\mathfrak{g}_1, \mathfrak{g}_0$ and $\mathfrak{g}_{-1}$;*

(SI-2) $\gamma([x, y]) = str((adx \circ ady)|_{\mathfrak{g}_0}), \forall x \in \mathfrak{g}_1$ *and* $y \in \mathfrak{g}_{-1}$.

Now we turn to $\mathfrak{g} = X(n)$ for $X \in \{W, \bar{S}, \bar{H}, \overline{CH}\}$. We define $\mathcal{E}_W : \mathfrak{g}_0 \longrightarrow \mathbb{F}$ to be a linear map with $\mathcal{E}_W(\xi_i D_j) = -\delta_{ij}$ for $1 \le i, j \le n$. Set $\mathcal{E}_{\bar{S}} = \mathcal{E}_{\bar{H}} = \mathcal{E}_{\overline{CH}} = 0$. By a direct computation, it is not hard (but tedious) to verify the following fact.

Lemma 1.3. *The linear map $\mathcal{E}_X$ is a semi-infinite character for $X(n)$, where $X \in \{W, \bar{S}, \bar{H}, \overline{CH}\}$.*

Proof. The proof is left in Appendix A. □

2. Borel subalgebras and parabolic subalgebras

In the following we denote $\Phi := \Phi(\mathfrak{g})$ if the context is clear. Following Serganova ([18]), we call a root $\alpha \in \Phi$ nonessential if $-\alpha \notin \Phi(\mathfrak{g})$, and essential if $-\alpha \in \Phi(\mathfrak{g})$. By (1.5), we have $\bar{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}$-span{standard basis}. Define

$$\bar{\mathfrak{h}}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}\text{-span}\{\text{standard basis}\}.$$

Call $h \in \bar{\mathfrak{h}}_{\mathbb{R}}$ regular if $\alpha(h) \neq 0$ for all $\alpha \in \Phi(\mathfrak{g})$. According to [18], any regular h deduces a subdivision $\Phi = \Phi_h^+ \cup \Phi_h^-$, where $\Phi_h^{\pm} = \{\alpha \in \Phi \mid \alpha(h) \in \mathbb{R}^{\pm}\}$. That defines a triangular decomposition $\mathfrak{g} = \mathfrak{N}_h^+ \oplus \bar{\mathfrak{h}} \oplus \mathfrak{N}_h^-$ for $\mathfrak{N}_h^{\pm} = \sum_{\alpha \in \Phi_h^{\pm}} \mathfrak{g}_{\alpha}$, where $\mathfrak{g}_{\alpha}$ is the corresponding root space. A Borel subalgebra $\mathfrak{B} := \mathfrak{B}_h$ is defined as $\bar{\mathfrak{h}} \oplus \mathfrak{N}_h^+$. Sometimes, we write Φ_h^+ as $\Phi(\mathfrak{B})$ if without any confusion. There are only finitely many Borel subalgebras (containing the given $\bar{\mathfrak{h}}$). An $\alpha \in \Phi_h^+$ is called simple for the Borel subalgebra $\mathfrak{B}$ if after removing α from Φ_h^+ and adding $-\alpha$ (if it does exist) we obtain a set of positive roots for some other Borel subalgebra $\mathfrak{B}'$. In this case, we call $\mathfrak{B}$ and $\mathfrak{B}'$ are adjacent, and related by even reflection if α is even essential, by odd reflection if α is odd essential, by nonessential reflection if α is nonessential. Denote $\mathfrak{B}' = r_{\alpha}(\mathfrak{B})$. For any two Borel subalgebras (containing $\bar{\mathfrak{h}}$), one is linked to the other one by a chain of reflections (see [18]).

2.1. **Borel subalgebras containing $\mathfrak{b}$ and strongly regular toral elements.** In the following, what we are interested in are Borel subalgebras $\mathfrak{B}$ containing $\mathfrak{b}$. Among such Borel subalgebras we distinguish $\mathfrak{B}_{\max} = \mathfrak{b} + \sum_{i>0} \mathfrak{g}_i$ and $\mathfrak{B}_{\min} = \mathfrak{b} + \mathfrak{g}_{-1}$, whose dimensions are of maximal and minimal respectively.

A defining toral element $h \in \mathfrak{h}_{\mathbb{R}}$ of a Borel subalgebra containing $\mathfrak{b}$ is said to be strongly regular. We will denote by $\mathrm{diag}(a_1, \ldots, a_n)$ the diagonal matrix of size $n \times n$ with the entries a_i on the ith diagonal positions. The toral element h can be identified with $\mathrm{diag}(a_1, \ldots, a_n)$. The following facts are clear.

Lemma 2.1. *Let $\mathfrak{g} = W(n)$ or $\bar{S}(n)$. Suppose that $h = \mathrm{diag}(a_1, \ldots, a_n) \in \bar{\mathfrak{h}}_{\mathbb{R}}$ is strongly regular. The following statements hold.*

(1) *The toral element h satisfies $a_i > a_j$ for $1 \leq i < j \leq n$.*
(2) *If $\mathfrak{B}_h = \mathfrak{B}_{max}$, then $a_i + a_j > a_k$ for any different $i, j, k \in \{1, 2, \ldots n\}$.*
(3) *If $\mathfrak{B}_h \not\supseteq \mathfrak{B}_{min}$, then $\epsilon_1 \in \Phi_h^+$.*

Proof. (1) It follows from the fact $\Phi_0^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}$.

(2) This is due to the fact that $\epsilon_i + \epsilon_j - \epsilon_k \in \Phi^+$.

(3) Suppose $\epsilon_1 \notin \Phi_h^+$. Then $a_1 < 0$. By (1), we have all $a_i < 0, 1 \leq i \leq n$. Correspondingly, $\mathfrak{B}_h$ contains $\mathfrak{g}_{-1}$. Hence $\mathfrak{B}_h \supset \mathfrak{B}_{\min}$. $\qquad\square$

Before the arguments on $\mathfrak{g} = \bar{H}(n)$, we need the following information on the root set $\Phi(\mathfrak{g}_{-1})$ of $\mathfrak{g}_{-1}$.

$$\Phi(\mathfrak{g}_{-1}) = \begin{cases} \{\pm\epsilon_i - \delta \mid i = 1, \ldots, r\}, & \text{for } \bar{H}(2r); \\ \{\pm\epsilon_i - \delta \mid i = 1, \ldots, r\} \cup \{-\delta\}, & \text{for } \bar{H}(2r+1). \end{cases} \tag{2.1}$$

Lemma 2.2. *Let $\mathfrak{g} = \bar{H}(n)$ with $n = 2r$ or $n = 2r + 1$. Suppose that h is strongly regular with $h = \mathrm{diag}(a + a_1, \ldots, a + a_r; a - a_1, \ldots, a - a_r) \in \bar{\mathfrak{h}}_{\mathbb{R}}$ when $n = 2r$, or $h = \mathrm{diag}(a + a_1, \ldots, a + a_r; a - a_1, \ldots, a - a_r, a) \in \bar{\mathfrak{h}}_{\mathbb{R}}$ when $n = 2r + 1$. The following statements hold.*

(1) *The toral element h satisfies $a_i > a_j$, $a_i + a_j > 0$ for $1 \le i < j \le r$. Additionally, $a_r > 0$ for $\mathfrak{g} = \bar{H}(2r+1)$.*
(2) *If $\mathfrak{B}_h = \mathfrak{B}_{max}$, $(n-3)a > a_1$.*
(3) *If $n = 2r$ and $\mathfrak{B}_h$ does not contain $\mathfrak{g}_{-1}$, then $\epsilon_1 + \delta \in \Phi_h^+$, or $-\epsilon_r + \delta \in \Phi_h^+$.*
(4) *If $n = 2r+1$ and $\mathfrak{B}_h$ does not contain $\mathfrak{g}_{-1}$, then one of the following items occurs.*
    (i) *$\delta \in \Phi_h^+$.*
    (ii) *$-\delta \in \Phi_h^+$, and either $\epsilon_1 + \delta \in \Phi_h^+$ or $-\epsilon_r + \delta \in \Phi_h^+$.*

Proof. (1), (2) By the same reason as in the proof of Lemma 2.1, the first two statements are clear.

(3) Suppose $\epsilon_1 + \delta \notin \Phi_h^+$ and $-\epsilon_r + \delta \notin \Phi_h^+$, then $-\epsilon_1 - \delta \in \Phi_h^+$ and $\epsilon_r - \delta \in \Phi_h^+$. This implies that $-a_1 - a = (-\epsilon_1 - \delta)(h) > 0$ and $a_r - a = (\epsilon_r - \delta)(h) > 0$. Hence, $a_1 - a > a_2 - a > \cdots > a_r - a > 0$ and $-a_r - a > -a_{r-1} - a > \cdots > -a_1 - a > 0$ by (1). Consequently, $\Phi(\mathfrak{g}_{-1}) \subseteq \Phi_h^+$ by (2.1), and $\mathfrak{g}_{-1} \subseteq \mathfrak{B}_h$, a contradiction.

(4) Suppose $\delta \notin \Phi_h^+$, then $-\delta \in \Phi_h^+$. Assume in contrary that $\epsilon_1 + \delta \notin \Phi_h^+$ and $-\epsilon_r + \delta \notin \Phi_h^+$. Similar arguments as in (3) yield that $\{\pm\epsilon_i - \delta \mid i = 1, \ldots, r\} \subseteq \Phi_h^+$. Hence $\Phi(\mathfrak{g}_{-1}) \subseteq \Phi_h^+$ by (2.1), and $\mathfrak{g}_{-1} \subseteq \mathfrak{B}_h$, a contradiction. $\square$

2.2. Variation of Borel subalgebras from $\mathfrak{B}_{\mathbf{max}}$ to $\mathfrak{B}_{\mathbf{min}}$ for $\mathfrak{g} = W(n)$ or $\bar{S}(n)$.
Certainly, it is interesting and nontrivial to construct an adjacent chain of Borel subalgebras. It is a good way to do that via strongly regular toral elements. Here, we list them for $\mathfrak{g} = W(n)$ and $\tilde{S}(n)$.

2.2.1. Set

$$h_{\mathrm{max}} = \mathrm{diag}(\frac{n}{n+1}, \frac{n-1}{n}, \frac{n-2}{n-1}, \ldots, \frac{2}{3}, \frac{1}{2}),$$
$$h_{\mathrm{min}} = \mathrm{diag}(-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, \ldots, -\frac{n-1}{n}, -\frac{n}{n+1}).$$

By a straightforward computation, h_{max} and h_{min} are exactly the defining strongly regular toral elements in $\bar{\mathfrak{h}}_{\mathbb{R}}$ for $\mathfrak{B}_{\mathrm{max}}$ and $\mathfrak{B}_{\mathrm{min}}$ respectively. Now we can show that there are a sequence of reflections $r_\alpha, \ldots, r_\gamma$ such that $\mathfrak{B}_{\mathrm{min}} = r_\alpha(\cdots(r_\gamma(\mathfrak{B}_{\mathrm{max}})))$. Actually, we take a sequence of regular toral elements in $\mathfrak{h}_{\mathbb{R}}$ as below. Set $h_0 = h_{\mathrm{max}}$, and consider

$$h_r = \mathrm{diag}(\frac{n}{n+1}, \frac{n-1}{n}, \frac{n-2}{n-1}, \ldots, \frac{r+1}{r+2}; -\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, \ldots, -\frac{r}{r+1}),$$

$r = 1, \ldots, n$. Naturally $h_n = h_{\mathrm{min}}$. Readers can verify that each h_r is strongly regular and the corresponding positive root set is

$$\Phi_{h_r}^+ = \Phi_0^+ \cup \Phi^+ \cup (-\Pi_r)\backslash \mathbb{X}_r \tag{2.2}$$

where

$$\Pi_r = \{\epsilon_{n-r+j} \mid j = 1, \ldots, r\},$$
$$-\Pi_r = \{-\epsilon_{n-r+j} \mid j = 1, \ldots, r\},$$
$$\mathbb{X}_r = \{\sum_{q=1}^{k} \epsilon_{i_q} + \sum_{q=1}^{l} \epsilon_{n-r+j_q} - \epsilon_d \mid (*)_r < 0 \text{ for } 2 \leq l + k \leq n,$$
$$1 \leq i_1 < \cdots < i_k \leq n - r; 1 \leq j_1 < \cdots < j_l \leq r; \ 1 \leq d \leq n\}.$$

with

$$(*)_r = \begin{cases} \sum_{q=1}^{k} \frac{n-i_q+1}{n-i_q+2} - \sum_{q=1}^{l} \frac{j_q}{j_q+1} - \frac{d}{d+1} & \text{if } 1 \leq d \leq n - r; \\ \sum_{q=1}^{k} \frac{n-i_q+1}{n-i_q+2} - \sum_{q=1}^{l} \frac{j_q}{j_q+1} + \frac{d'}{d'+1} & \text{if } d = n - r + d' \text{ with } 1 \leq d' \leq r. \end{cases}$$

Obviously, $\mathbb{X}_r \supset \Pi_r$.

2.2.2. Now we continue to refine the above process. Recall $h_1 = \text{diag}(\frac{n}{n+1}, \frac{n-1}{n}, \frac{n-1}{n-3}, \ldots, \frac{2}{3}; -\frac{1}{2})$. Set $h_1^{(1)} := h_1$ and for $q = 2, \ldots, n$,

$$h_1^{(q)} := \text{diag}(\frac{n}{n+1}, \ldots, \frac{q}{q+1}, \frac{qn-1}{2qn}, \ldots, \frac{n-1}{2n}; -\frac{1}{2}).$$

Note that from the beginning, we have set an appointment $n > 2$. So it is easily known that $h_1^{(q)}$ is strongly regular. Inductively, for a given $r > 1$ set $h_r^{(r)} := h_r$ and

$$h_r^{(q)} := \text{diag}(\frac{n}{n+1}, \ldots, \frac{q}{q+1}, \frac{rqn-1}{(r+1)qn}, \ldots, \frac{r(r+1)n-1}{(r+1)(r+2)n}; -\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, \ldots, -\frac{r}{r+1})$$

for $q = r + 1, \ldots, n$. All of $h_r^{(q)}$'s are strongly regular. The corresponding Borel subalgebra of $h_r^{(q)}$ is denoted by $\mathfrak{B}_r^{(q)}$, $r = 0, 1, \ldots, n$, $q = r, r+1, \ldots, n$. In particular, $\mathfrak{B}_0 = \mathfrak{B}_{\max}$ and $\mathfrak{B}_n = \mathfrak{B}_{\min}$ (here set $\mathfrak{B}_r = \mathfrak{B}_r^{(r)}$).

2.3. **Parabolic subalgebras containing $\mathfrak{g}_0$.** For a given strongly regular element h, we have $\Phi = \Phi_h^+ \cup \Phi_h^-$ and the corresponding Borel subalgebra

$$\mathfrak{B}_h = \mathfrak{b} \oplus \sum_{\alpha \in \Phi_h^+ \setminus \Phi_0^+} \mathfrak{g}_\alpha.$$

We define a parabolic subalgebra P_h associated with h (and then with $\mathfrak{B}_h$) as the closed subalgebra generated by $\mathfrak{g}_0$ and $\mathfrak{B}_h$.

Associated with the maximal Borel subalgebra $\mathfrak{B}_{\max} = \mathfrak{b} + \sum_{i>0} \mathfrak{g}_i$, and the minimal Borel subalgebra $\mathfrak{B}_{\min} = \mathfrak{b} + \mathfrak{g}_{-1}$, it is readily known that the corresponding parabolic subalgebras are, respectively,

$$\mathsf{P}_{\max} = \mathfrak{g}_0 + \sum_{i>0} \mathfrak{g}_i \text{ and } \mathsf{P}_{\min} = \mathfrak{g}_0 + \mathfrak{g}_{-1}.$$

The minimal parabolic subalgebra will be the most interesting, playing a crucial role in the theory of parabolic BGG categories. The following basic observation preliminarily reveals its importance.

Proposition 2.3. *Let* $\mathfrak{g} = X(n)$, $X \in \{W, \bar{S}, \bar{H}, \overline{CH}\}$. *Then any proper parabolic subalgebra coincides with either* $\mathsf{P}_{\max}$ *or* $\mathsf{P}_{\min}$.

Proof. Let P be an arbitrarily given parabolic subalgebra generated by $\mathfrak{g}_0$ and $\mathfrak{B}_h$, where

$$h = \begin{cases} \mathrm{diag}(a_1, \ldots, a_n), & \text{if } \mathfrak{g} = W(n), \bar{S}(n); \\ \mathrm{diag}(a + a_1, \ldots, a + a_r; a - a_1, \ldots, a - a_r), & \text{if } \mathfrak{g} = H(2r); \\ \mathrm{diag}(a + a_1, \ldots, a + a_r; a - a_1, \ldots, a - a_r, a), & \text{if } \mathfrak{g} = H(2r+1) \end{cases}$$

is a defining strongly regular toral element of $\mathfrak{B}_h$. Then $a_1 > a_2 > \cdots > a_n$ for $\mathfrak{g} = W(n), \bar{S}(n)$, and $a_1 > a_2 > \cdots > a_r$ for $\mathfrak{g} = H(n)$ with $n = 2r, 2r+1$. Denote by Ψ_h the root set of P. Then Ψ_h is a closed root subsystem of Φ.

Firstly we assume that P does not contain $\mathsf{P}_{\min}$. In this situation, we will show $\mathsf{P} = \mathsf{P}_{\max}$. We proceed it case by case.

Case 1: $\mathfrak{g} = W(n), \bar{S}(n)$.

By Lemma 2.1(3), Ψ_h contains the root ϵ_1. Note that by definition, $\Phi_0^- \subseteq \Psi_h$. Hence, Ψ_h contains all $\epsilon_k = (\epsilon_k - \epsilon_1) + \epsilon_1$ for $k = 2, \ldots, n$. Correspondingly, Ψ_h contains all ϵ_k for $k = 1, \ldots, n$. Hence Ψ_h contains all Φ^+ because Ψ_h is a closed root subsystem, containing Φ_0 and all ϵ_k, $k = 1, \ldots, n$. This means that the parabolic subalgebra associated with $\mathfrak{B}_h$ contains $\mathsf{P}_{\max}$. On the other hand, a parabolic subalgebra containing $\mathsf{P}_{\max}$ is either $\mathfrak{g}$ itself or equal to $\mathsf{P}_{\max}$. We are done.

Case 2: $\mathfrak{g} = H(2r)$.

By Lemma 2.2(3), Ψ_h contains the root $-\epsilon_r + \delta$ or the root $\epsilon_1 + \delta$. If $\epsilon_1 + \delta \in \Psi_h$, then $-\epsilon_r + \delta = (-\epsilon_r - \epsilon_1) + (\epsilon_1 + \delta) \in \Psi_h$, because $-\epsilon_r - \epsilon_1 \in \Phi_0 \subseteq \Psi_h$ and Ψ_h is closed. Consequently, we see that Ψ_h always contains the root $-\epsilon_r + \delta$. Since $\Phi_0 = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq n\} \subseteq \Psi_h$ and Ψ_h is closed, we have $\pm\epsilon_i + \delta = (\pm\epsilon_i + \epsilon_r) + (-\epsilon_r + \delta) \in \Psi_h$ for $1 \leq i \leq r-1$. In addition, $\epsilon_r + \delta = (\epsilon_r + \epsilon_1) + (-\epsilon_1 + \delta) \in \Psi_h$. Hence, $\pm\epsilon_j + \delta \in \Psi_h$ for any $1 \leq j \leq n$. In particular, $2\delta = (\epsilon_1 + \delta) + (-\epsilon_1 + \delta) \in \Psi_h$, so that $2m\delta \in \Psi_h$ for any $m \in \mathbb{Z}^+$. Now let $\alpha = \pm\epsilon_{i_1} \pm \cdots \pm \epsilon_{i_k} + l\delta$ be an arbitrary root in $\mathfrak{g}_{\geq 1}$, where $k - 2 \leq l \leq n - 2$ and $l - k$ is even. Since α can be written as

$$\alpha = (\pm\epsilon_{i_1} \pm \epsilon_{i_2}) + (\pm\epsilon_{i_3} + \delta) + \cdots + (\pm\epsilon_{i_k} + \delta) + (l - k + 2)\delta,$$

we get that $\alpha \in \Psi_h$ by induction on k. This implies $\mathsf{P}_{\max} \subseteq \mathsf{P}$. On the other hand, a parabolic subalgebra containing $\mathsf{P}_{\max}$ is either $\mathfrak{g}$ itself or equal to $\mathsf{P}_{\max}$. We are done.

Case 3: $\mathfrak{g} = H(2r+1)$.

By Lemma 2.2(4), if $-\delta \in \Psi_h$, Ψ_h contains the root $-\epsilon_r + \delta$ or the root $\epsilon_1 + \delta$. While if $\delta \in \Psi_h$, then $\epsilon_1 + \delta \in \Psi_h$, because $\epsilon_1 \in \Psi_h$ and Ψ_h is closed. Similar arguments as in Case 2 yield the desired assertion in this case.

Secondly we assume that P contains $\mathsf{P}_{\min}$. In this case, it suffices to show that P must coincide with $\mathfrak{g}$ itself as long as P properly contains $\mathsf{P}_{\min}$. We also proceed it case by case.

Case 1: $\mathfrak{g} = W(n), \bar{S}(n)$.

In this case, under the assumption $\mathsf{P} \supsetneq \mathsf{P}_{\min}$, the root set Ψ_h of P contains $\Phi(\mathsf{P}_{\min}) \cup \{\epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_t} - \epsilon_k\}$ for some sequence $(1 \leq i_1 < i_2 < \cdots < i_t \leq n)$ and $k \in \{1, 2, \ldots, n\}$ with $t > 1$. By definition, Ψ_h is closed. Note that $\Phi(\mathsf{P}_{\min}) = \{-\epsilon_i \mid i = 1, \ldots, n\} \cup \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$ is already contained in Ψ_h. So it is easily deduced that all $\epsilon_i, i = 1, \ldots, n$, are contained in $\Psi(\mathsf{P})$. Consequently, it is further deduced that $\Psi(\mathsf{P}) = \Phi$ and then $\mathsf{P} = \mathfrak{g}$.

Case 2: $\mathfrak{g} = H(n)$.

In this case, under the assumption $\mathsf{P} \supsetneq \mathsf{P}_{\min}$, the root set Ψ_h of P contains $\Phi(\mathsf{P}_{\min}) \cup \{\pm\epsilon_{i_1} + \cdots + \pm\epsilon_{i_k} + l\delta\}$ for some $k \geq 1$ and $k - 2 \leq l \leq n - 2$. By definition, Ψ_h is closed.

Note that

$$\Phi(\mathsf{P}_{\min}) = \begin{cases} \{\pm\epsilon_i - \delta \mid i = 1, \ldots, n\} \cup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i \ne j \le n\}, & \text{if } n = 2r; \\ \{\pm\epsilon_i - \delta, -\delta \mid i = 1, \ldots, n\} \cup \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i \mid 1 \le i \ne j \le n\}, & \text{if } n = 2r + 1 \end{cases}$$

is already contained in Ψ_h. So it is easily deduced that all roots in $\mathfrak{g}_{\ge 1}$ are contained in Ψ_h. Consequently, $\Psi_h = \Phi$, and then $\mathsf{P} = \mathfrak{g}$.

The proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 2.4. *There is a natural question when it is true that the subalgebra generated by $\mathfrak{g}_0$ and a Borel subalgebra $\mathfrak{B}_h$ is closed, i.e. it coincides with P_h. This question can be positively answered for $\mathfrak{g} = \bar{S}_n$ because in this case, all $\mathfrak{g}_i$ $(i \ne 0)$ are irreducible $\mathfrak{g}_0$-modules (see [13, Proposition 3.3.1]).*

2.4. **Parabolic categories.** In general, we can consider a parabolic BGG category $\mathcal{O}_h$ of $\mathfrak{g}$ associated with P_h, whose objects are super $\mathfrak{g}$-modules endowed with an admissible $\mathbb{Z}$-graded structure, locally finite over P_h and semisimple over $\bar{\mathfrak{h}}$. The morphisms in $\mathcal{O}_h$ are even homomorphisms of $\mathbb{Z}$-graded $\mathfrak{g}$-modules.

By Proposition 2.3, there are only two possibilities for a proper parabolic subalgebra P_h, that is, it coincides with either $\mathsf{P}_{\max}$ or $\mathsf{P}_{\min}$. If the objects of $\mathcal{O}_h$ are additionally required to be finitely generated over $U(\mathfrak{g})$, then it is readily seen that any objects in the BGG category arising from $\mathsf{P}_{\max}$ is finite-dimensional. All such objects belong to the other BGG category $\mathcal{O}^{\min}$ arising from $\mathsf{P}_{\min}$.

3. THE CATEGORY $\mathcal{O}^{\mathrm{MIN}}$

3.1.   From now on we always assume that $\mathfrak{g} = X(n)$ with $X \in \{W, \bar{S}, \bar{H}, \overline{CH}\}$. Keeping in mind, we have $\mathfrak{g}_0 = \mathfrak{n}^+ \oplus \bar{\mathfrak{h}} \oplus \mathfrak{n}^-$ with $\bar{\mathfrak{h}}$ defined in §1.2, and the minimal parabolic subalgebra $\mathsf{P}_{\min}$ defined in §2. From now on, we simply write $\mathsf{P} = \mathsf{P}_{\min}$.

Definition 3.1. *We define a category $\mathcal{O}^{min}$ whose objects are $\mathbb{Z}_2$-graded vector spaces $M = M_{\bar{0}} \oplus M_{\bar{1}}$ satisfying the following axioms:*
   (1) M is an admissible $\mathbb{Z}$-graded $\mathfrak{g}$-module, i.e., $M = \bigoplus_{i \in \mathbb{Z}} M_i$ with $M_i = (M_i \cap M_{\bar{0}}) \oplus (M_i \cap M_{\bar{1}})$, $\dim M_i < \infty$, and $\mathfrak{g}_i M_j \subseteq M_{i+j}, \forall i, j \in \mathbb{Z}$.
   (2) M is locally finite as a P-module.
   (3) M is semisimple over $\bar{\mathfrak{h}}$.
The morphisms in $\mathcal{O}^{min}$ are always assumed to be even (see Remark 3.2(4) below), and they are $\mathfrak{g}$-module morphisms compatible with the $\mathbb{Z}$-gradation, i.e., for any $M, N \in \mathcal{O}^{min}$,

$$\mathrm{Hom}_{\mathcal{O}^{min}}(M, N) = \{f \in \mathrm{Hom}_{U(\mathfrak{g})}(M, N) \mid f(M_i) \subseteq N_i, \forall i \in \mathbb{Z}\}.$$

Remark 3.2. (1) *Since $U(\mathsf{P}) \cong \bigwedge(\mathfrak{g}_{-1}) \otimes U(\mathfrak{g}_0)$ and $\dim \bigwedge(\mathfrak{g}_{-1}) = 2^n$. The condition being locally finite-dimensional over P is equivalent to being locally finite-dimensional over $\mathfrak{g}_0$.*
   (2) *The isomorphism classes of irreducible finite-dimensional $\mathfrak{g}_0$-modules are parameterized by Λ^+, the set of the weights whose restriction to $[\mathfrak{g}_0, \mathfrak{g}_0]$ are dominant and integral. Denote by $L^0(\lambda)$ the finite-dimensional irreducible $\mathfrak{g}_0$-module corresponding to $\lambda \in \Lambda^+$, which is a highest weight module associated with the Borel subalgebra $\mathfrak{b} = \bar{\mathfrak{h}} + \mathfrak{n}^+$.*
   (3) *The $\mathbb{Z}$-graded module category of $\bar{X}(n)$ can be naturally identified with the $\mathbb{Z}$-graded module category of $X(n)$ $(X \in \{S, H, CH\})$.*

(4) *Recall that the Lie superalgebra* $\mathfrak{g}$ *is equal to* $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *with* $\mathfrak{g}_{\bar{0}} = \sum_{all\ even\ i} \mathfrak{g}_i$, *and* $\mathfrak{g}_{\bar{1}} = \sum_{all\ odd\ i} \mathfrak{g}_i$. *For any two* $\mathfrak{g}$-*modules* M, N, *we say a homomorphism* $\varphi : M \to N$ *is of parity* $|\varphi|$ *if* $\varphi(xm) = (-1)^{|\varphi||x|} x\varphi(m)$ *for any* $\mathbb{Z}_2$-*homogeneous element* $x \in \mathfrak{g}_{|x|}$, *and* $m \in M$. *In this paper we always assume that the homomorphism* φ *in* $\mathcal{O}^{min}$ *is of even parity, i.e.,* $\varphi(xm) = x\varphi(m)$ *for any* $x \in \mathfrak{g}$ *and* $m \in M$. *So* $\mathcal{O}^{min}$ *is an abelian category.*

(5) *If forgetting the* $\underline{\mathbb{Z}\text{-graded}}$ *structure of* $\mathcal{O}^{min}$, *then we have the category* $\overline{\mathcal{O}^{min}}$. *A* $U(\mathfrak{g})$-*module* M *belongs to* $\overline{\mathcal{O}^{min}}$ *if and only if it is a* $\underline{weight\ module}$ *and locally* P-*finite. Denote by* $\mathcal{F}$ *the natural forgetful functor from* $\mathcal{O}^{min}$ *to* $\overline{\mathcal{O}^{min}}$.

Let $\mathbf{E}$ be a complete set of pairwise non-isomorphic irreducible $\mathbb{Z}$-graded modules of $\mathfrak{g}_0$. Each $E \in \mathbf{E}$ is necessarily concentrated in a single degree $\lfloor E \rfloor \in \mathbb{Z}$, which means that $E = E_{\lfloor E \rfloor}$. So, $\mathbf{E}$ can be parameterized by $\Lambda^+ \times \mathbb{Z}$.

Denote by $\mathcal{O}^{min}_{\geq d}$ the full subcategory of $\mathcal{O}^{min}$ consisting of all objects that are zero in degrees less than d (called a truncated subcategory by d).

3.2. **Standard and co-standard modules.** For a given $(\lambda, d) \in \mathbf{E} = \Lambda^+ \times \mathbb{Z}$, we have a $\mathbb{Z}$-graded (finite-dimensional) irreducible $\mathfrak{g}_0$-module $L^0(\lambda)$ whose degree $\lfloor L^0(\lambda) \rfloor$ is equal to d. Let us introduce the standard modules $\Delta(\lambda)$ and co-standard modules $\nabla(\lambda)$ in $\mathcal{O}^{min}$ as below:

$$\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathsf{P})} L^0(\lambda) \tag{3.1}$$

and

$$\nabla(\lambda) = \mathrm{Hom}_{\mathfrak{g}_{\geq 0}}(U(\mathfrak{g}), L^0(\lambda)). \tag{3.2}$$

with trivial $\mathfrak{g}_{-1}$(resp. $\mathfrak{g}_{\geq 1}$)-action on $L^0(\lambda)$ in $\Delta(\lambda)$ (resp. $\nabla(\lambda)$). Once the parity $|v^0_\lambda|$ of a maximal vector v^0_λ in $L^0(\lambda)$ is given[1], say $\epsilon \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, the super-structure of $\Delta(\lambda)$ is determined by the super structure of $U(\mathfrak{g}_{\geq 1}) = U(\mathfrak{g}_{\geq 1})_{\bar{0}} \oplus U(\mathfrak{g}_{\geq 1})_{\bar{1}}$ together with ϵ as follows

$$\Delta(\lambda) = \Delta(\lambda)_{\bar{0}} \oplus \Delta(\lambda)_{\bar{1}}, \text{ where } \Delta(\lambda)_{\delta+\epsilon} = U(\mathfrak{g}_{\geq 1})_\delta \otimes L^0(\lambda) \text{ for } \delta \in \mathbb{Z}_2.$$

Obviously, $U(\mathfrak{g})$ has a $\mathbb{Z}$-grading induced by the $\mathbb{Z}$-grading of $\mathfrak{g}$. So for $\Delta(\lambda)$, we have the following decomposition as a $\mathfrak{g}_0$-module

$$\Delta(\lambda) \cong \bigoplus_{i \geq 1} U(\mathfrak{g}_{\geq 1})_i \otimes_{\mathbb{C}} L^0(\lambda)$$

where $U(\mathfrak{g}_{\geq 1})_i$ denotes the ith homogeneous part of $U(\mathfrak{g}_{\geq 1})$. Because $U(\mathfrak{g}_{\geq 1})_i, i \geq 0$, is finite-dimensional, $U(\mathfrak{g}_{\geq 1})_i \otimes_{\mathbb{C}} L^0(\lambda)$ is a finite-dimensional $\mathfrak{g}_0$-module. Hence, $\Delta(\lambda)$ is locally finite over $\mathfrak{g}_0$. Consequently, $\Delta(\lambda)$ is an object in $\mathcal{O}^{min}$. As to the co-standard module, we have the following isomorphisms over $\mathfrak{g}_{\leq 0}$:

$$\begin{aligned}
\nabla(\lambda) &\cong \mathrm{Hom}_{\mathfrak{g}_0}(U(\mathfrak{g}_{\leq 0}), L^0(\lambda)) \\
&\cong \mathrm{Hom}_{\mathfrak{g}_0}(U(\mathfrak{g}_{\leq 0}), \mathbb{C}) \otimes_{\mathbb{C}} L^0(\lambda) \\
&\cong \mathrm{Hom}_{\mathbb{C}}(\bigwedge(\mathfrak{g}_{-1}), \mathbb{C}) \otimes_{\mathbb{C}} L^0(\lambda),
\end{aligned}$$

where $\bigwedge(\mathfrak{g}_{-1})$ denotes the exterior product space on the abelian Lie (super)algebra $\mathfrak{g}_{-1}$, and the last isomorphism above is due to the fact that by definition $U(\mathfrak{g}_{-1}) = \bigwedge(\mathfrak{g}_{-1})$. Hence $\dim \nabla(\lambda) = 2^n \dim L^0(\lambda)$, and $\nabla(\lambda)$ is an object in $\mathcal{O}^{min}$. Especially, $\nabla(\lambda)$ admits a simple

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[1]For $\mathcal{O}^{min}$, one can give parities for weight spaces similar to [9, §6].

socle $L^0(\lambda)$ over $\mathfrak{g}_{\leq 0}$. For the detailed description of $\nabla(\lambda)$, readers can refer to Proposition 5.5 later.

Furthermore, both $\Delta(\lambda)$ and $\nabla(\lambda)$ belong to $\mathcal{O}_{\geq d'}^{\min}$ as long as $\lfloor L^0(\lambda) \rfloor = d \geq d'$. In this case, we say that *both of them have depth d*. Generally, for $M \in \mathcal{O}_{\geq d}^{\min}$, define the depth of M to be the least number t with $M_t \neq 0$ for the gradation $M = \sum_{i=d}^{\infty} M_i$. Denote by $\mathsf{dpt}(M)$ the depth of M. By definition, $\mathsf{dpt}(M) \geq d$ for $M \in \mathcal{O}_{\geq d}^{\min}$. The following basic observation is clear.

Lemma 3.3. *Both $\Delta(\lambda)$ and $\nabla(\lambda)$ are indecomposable.*

Actually, it is readily known that $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) has a unique maximal submodule. Hence, $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) has a unique simple quotient, which is denoted by $L(\lambda)$ (resp. $\tilde{L}(\lambda)$).

Lemma 3.4. *Maintain the notations as above. Then $\{L(\lambda)\}_{(\lambda,d) \in \mathbf{E}}$ and $\{\tilde{L}(\lambda)\}_{(\lambda,d) \in \mathbf{E}}$ are two complete sets of pairwise non-isomorphic irreducibles in $\mathcal{O}^{min}$ respectively. Hence every simple object in $\mathcal{O}^{min}$ is finite-dimensional.*

Proof. Let E be any simple object of $\mathcal{O}^{\min}$ and v be a non-zero weight vector belonging to E. Consider the finite-dimensional $U(\mathsf{P})$-module $U(\mathsf{P}).v$. Obviously, $U(\mathsf{P}).v$ has a non-zero $U(\mathsf{P})$-irreducible submodule E_0. Assume that E_0 is isomorphic to $L^0(\lambda)$ for some $\lambda \in \Lambda^+$, then we have a non-zero homomorphism from $\Delta(\lambda)$ to E. Hence E is isomorphic to $L(\lambda)$, with the depth of E equal to $\lfloor E_0 \rfloor$.

On the other hand, assume that $L(\lambda)$ and $L(\mu)$ are two irreducible modules with depths d_λ and d_μ respectively. By the construction, $L^0(\lambda)$ is the unique simple socle of $L(\lambda)$ over $\mathfrak{g}_{\leq 0}$. If $L(\lambda)$ and $L(\mu)$ are isomorphic, then $L^0(\lambda)$ and $L^0(\mu)$ must be isomorphic as $\mathfrak{g}_{\leq 0}$-modules. Hence $\lambda = \mu$. Naturally, $d_\lambda = d_\mu$. Thus, we already prove that the set $\{L(\lambda)\}_{(\lambda,d) \in \mathbf{E}}$ forms a complete set of pairwise non-isomorphic simple objects in $\mathcal{O}^{\min}$.

By the same arguments, one can similarly prove the statement for $\{\tilde{L}(\lambda)\}_{(\lambda,d) \in \mathbf{E}}$. Since $\nabla(\lambda)$ is finite-dimensional, any simple object of $\mathcal{O}^{\min}$ is finite-dimensional. $\qquad\square$

Remark 3.5. *(1) By the above lemma, we can see that for any $\lambda \in \Lambda^+$ (modulo the depths), there is a unique $\tilde{\lambda} \in \Lambda^+$ such that $L(\lambda) \cong \tilde{L}(\tilde{\lambda})$. Thus, the correspondence sending λ to $\tilde{\lambda}$ gives rise to a permutation on Λ^+. The precise description can be given in §9 with aid of Proposition 5.5.*

(2) For $M \in \mathcal{O}^{min}$, we write $(M : L(\lambda))$ for the multiplicity of the simple object $L(\lambda)$ in M, i.e., the supremum of $\#\{i \mid M^i/M^{i-1} \cong L(\lambda)\}$ over all finite filtration $\{M = M^k \supset \ldots \supset M^i \supset M^{i-1} \supset \ldots \supset M^1 \supset M^0 = 0 \mid i \in \mathbb{Z}_{>0}\}$. Suppose $M = \oplus_{k \in \mathbb{Z}} M_k$. Note that $\dim M_d < \infty$ for $d = \lfloor L(\lambda) \rfloor$. So $(M : L(\lambda))$ is finite. Especially, we will call $L(\lambda)$ a composition factor of M if $(M : L(\lambda))$ is nonzero.

3.3. **Some natural representations and related notations.** We collect some basic facts on natural representations of $\mathfrak{g} = X(n)$ for $X \in \{W, \bar{S}, \bar{H}, \overline{CH}\}$, which will be used later for the study of blocks of $\mathcal{O}^{\min}$.

Recall that for $\mathfrak{g} = \sum_{i \geq -1} \mathfrak{g}_i$, the graded subspaces $\mathfrak{g}_{-1} = \sum_{i=1}^{n} \mathbb{C} D_i$ and

$$\mathfrak{g}_0 = \begin{cases} \mathfrak{gl}(V), & \text{if } \mathfrak{g} = W(n), \bar{S}(n); \\ \mathfrak{so}(V) + \mathbb{C}\mathsf{d}, & \text{if } \mathfrak{g} = \bar{H}(n), \overline{CH}(n) \end{cases} \tag{3.3}$$

for $V = \sum_{i=1}^{n} \mathbb{C}\xi_i$. Especially, $\mathfrak{g}_{-1}$ becomes the contragredient module V^* of V over $\mathfrak{g}_0$ with the weight set

$$\mathsf{Wt}(\mathfrak{g}_{-1}) = \begin{cases} \{-\epsilon_1, ..., -\epsilon_n\}, & \text{for } X(n), X \in \{W, \bar{S}\}; \\ \{-\epsilon_1 - \delta, ..., -\epsilon_r - \delta, \epsilon_1 - \delta, ..., \epsilon_r - \delta\}, & \text{for } \bar{H}(2r) \text{ or } \overline{CH}(2r); \\ \{-\epsilon_1 - \delta, ..., -\epsilon_r - \delta, \epsilon_1 - \delta, ..., \epsilon_r - \delta, -\delta\}, & \text{for } \bar{H}(2r+1) \text{ or } \overline{CH}(2r+1). \end{cases}$$

Furthermore, $\bigwedge^n(\mathfrak{g}_{-1})$ is a one-dimensional $\mathfrak{g}_0$-module generated by $D_1 \wedge \cdots \wedge D_n$, of weight $-\sum_{i=1}^{n} \epsilon_i$ when $\mathfrak{g} = X(n)$ for $X \in \{W, \bar{S}\}$, or of weight $-n\delta$ when $\mathfrak{g} = \bar{H}(n)$. Furthermore, $\bigwedge(\mathfrak{g}_{-1}) = \sum_{i=0}^{n} \bigwedge^i(\mathfrak{g}_{-1})$ admits a weight set

$$\mathsf{Wt}\left(\bigwedge(\mathfrak{g}_{-1})\right) = \{-(\epsilon_{i_1} + \cdots + \epsilon_{i_k}) \mid 1 \le i_1 < \cdots < i_k \le n, k = 0, 1, ..., n\} \qquad (3.4)$$

for $X(n)$, $X \in \{W, \bar{S}\}$.

From now on, we set

$$\epsilon_{r+k} := -\epsilon_k \quad (k = 1, ..., r) \text{ for } \mathfrak{g} = \bar{H}(n) \text{ or } \overline{CH}(n). \qquad (3.5)$$

Let $\aleph$ be 0 or 1 in the following. Then we can write

$$\mathsf{Wt}\left(\bigwedge(\mathfrak{g}_{-1})\right) \qquad (3.6)$$

$$= \begin{cases} \{-(\epsilon_{i_1} + \cdots + \epsilon_{i_k}) - k\delta \mid 1 \le i_1 < \cdots < i_k \le n, k = 0, 1, ..., n\}, & \text{for } \bar{H}(2r); \\ \{-(\epsilon_{i_1} + \cdots + \epsilon_{i_k}) - (k + \aleph)\delta \mid 1 \le i_1 < \cdots < i_k \le n - 1, k = 0, 1, ..., n\}, & \text{for } \bar{H}(2r+1). \end{cases}$$
$$\qquad (3.7)$$

The above is also true for $\overline{CH}(n)$.

We always set $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$ throughout the paper. Then $\mathfrak{g}_0'$ is a semisimple Lie algebra.

Lemma 3.6. *Let $\mathfrak{g} = W(n)$. The following statements hold.*

(1) *Set $M := \sum_{i=1}^{n} \mathbb{C}m_i \in \mathfrak{g}_1$ with $m_i = \xi_i\mathsf{d} \in \mathfrak{g}_1$ for $\mathsf{d} = \sum_{j=1}^{n} \xi_j D_j$. Then both $\mathfrak{g}_{-1}$ and M are not only abelian subalgebras but also $\mathfrak{g}_0$-modules. Especially $U(\mathfrak{g}_{-1}) = \bigwedge \mathfrak{g}_{-1}$ and $U(M) = \bigwedge M$. Here and after, $\bigwedge L$ denotes the exterior-product space of a vector space L.*

(2) *Under the identification between $\mathfrak{g}_0$ and $\mathfrak{gl}(V)$ for $V = \sum_{i=1}^{n} \mathbb{C}\xi_i$, the $\mathfrak{g}_0$-module M is isomorphic to V while $\mathfrak{g}_{-1}$ is isomorphic to its linear dual V^* (as $\mathfrak{g}_0$-modules).*

(3) *Consider the following tensor products*

$$M^-(\lambda) := \bigwedge \mathfrak{g}_{-1} \otimes_{\mathbb{C}} L^0(\lambda)$$

and

$$M^+(\mu) := \bigwedge M \otimes_{\mathbb{C}} L^0(\mu)$$

in the category of $\mathfrak{g}_0$-modules, where $\lambda, \mu \in \Lambda^+$. If $L^0(\mu)$ is a composition factor of the $\mathfrak{g}_0$-module $M^-(\lambda)$, then $L^0(\lambda)$ must be a composition factor of the $\mathfrak{g}_0$-module $M^+(\mu)$.

Proof. By a straightforward computation, the statements in (1) and (2) can be easily verified. (3) Note that $\mathfrak{g}_0 \cong \mathfrak{gl}(V)$. The statement follows from (2) and the following isomorphism

$$\mathrm{Hom}_{\mathfrak{g}_0}(L^0(\mu), \bigwedge V^* \otimes_{\mathbb{C}} L^0(\lambda)) \cong \mathrm{Hom}_{\mathfrak{g}_0}(L^0(\mu) \otimes_{\mathbb{C}} \bigwedge V, L^0(\lambda)). \qquad (3.8)$$

$\square$

In the following, we will generalize Lemma 3.6(3) to the situation when $\mathfrak{g} = \overline{CH}(n)$. Set $L := \mathfrak{g}_{\geq 1} \subseteq U(\mathfrak{g})$. Consider the $\mathfrak{g}_0$-modules $M^+(\mu) := L \otimes_{\mathbb{C}} L^0(\mu)$ and $M^-(\lambda) := \sum_{i=0}^n M^-(\lambda)_{-i}$ with $M^-(\lambda)_{-i} = \bigwedge^i(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} L^0(\lambda)$. The following lemma is somewhat a bridge to understand the block structure of $\mathcal{O}^{\min}$ for the case $\overline{CH}(n)$ (see Proposition 6.11).

Lemma 3.7. *Let* $\mathfrak{g} = \overline{CH}(n)$ *and* $L^0(\mu)$ *be an irreducible composition factor of the* $\mathfrak{g}_0$-*module* $M^-(\lambda)_i$, $\lambda, \mu \in \Lambda^+$. *The following statements hold.*

(1) *If* $i \leq -3$, *then* $L^0(\lambda - 2\delta)$ *is a composition factor of the* $\mathfrak{g}_0$-*module* $M^+(\mu)$.
(2) *If* $i = -1$, *then* $L^0(\lambda - 2\delta)$ *is a composition factor of the* $\mathfrak{g}_0$-*module* $M^+(\mu - (n-2)\delta)$.
(3) *If* $i = -2$, *then* $L^0(\lambda - 2\delta)$ *is a composition factor of the* $\mathfrak{g}_0$-*module* $M^+(\mu - (n-4)\delta)$.

Proof. (1) Recall that $\mathfrak{g} = \sum_{i=-1}^{n-2} \mathfrak{g}_i$ with $\mathfrak{g}_i = W(n)_i \cap \mathfrak{g}$ for $\mathfrak{g} = \overline{CH}(n)$. We still set $V = \sum_{i=1}^n \mathbb{C}\xi_i$. Then we can identify $\mathfrak{g}_0'$ with $\mathfrak{so}(V)$, which admits a natural representation on V. The $\mathfrak{g}_0'$-module $\mathfrak{g}_{-1} = \sum_{i=1}^n \mathbb{C}D_i$ is isomorphic to the contragredient $\mathfrak{g}_0'$-module V^* of V. Furthermore, for $i \in \{1, 2, \cdots, n-2\}$, $\mathfrak{g}_i$ is isomorphic to $\bigwedge^{i+2}(V)$ and admits eigenvalue i for the action of d. Actually, we can identify $\mathfrak{g}_i$ with the space spanned by $D_H(\xi_{j_1} \cdots \xi_{j_{i+2}})$ for all $(j_1, ..., j_{i+2})$ satisfying $1 \leq j_1 < \cdots < j_{i+2} \leq n$, the latter of which is isomorphic to $\bigwedge^{i+2} V$ as vector spaces. We can further say that $\mathfrak{g}_i$ is isomorphic to $\bigwedge^{i+2} V$ as $\mathfrak{so}(V)$-modules. This is ensured by the definition of D_H and the fact that for the basis elements $X = D_H(\xi_s\xi_t) \in \mathfrak{g}_0'$ ($1 \leq s < t \leq n$), the following identity holds.

$$\mathsf{ad}X.D_H(\xi_{j_1} \cdots \xi_{j_{i+2}}) = \sum_{k=1}^{i+2} D_H(\xi_{j_1} \cdots \xi_{j_{k-1}} \cdot X(\xi_{j_k}) \cdot \xi_{j_{k+1}} \cdots \xi_{j_{i+2}}).$$

We continue to apply the isomorphism presented in (3.8) for $\mathfrak{g}_0'$-modules in the current case. For $i \in \{1, ..., n-2\}$, we further have the following identity

$$\mathrm{Hom}_{\mathfrak{g}_0'}(L^0(\mu), \bigwedge^{i+2} V^* \otimes_{\mathbb{C}} L^0(\lambda)) \cong \mathrm{Hom}_{\mathfrak{g}_0'}(L^0(\mu) \otimes_{\mathbb{C}} \mathfrak{g}_i, L^0(\lambda)).$$

Or to say, for $i \in \{1, ..., n-2\}$,

$$\mathrm{Hom}_{\mathfrak{g}_0'}(L^0(\mu), M^-(\lambda)_{-(i+2)}) \cong \mathrm{Hom}_{\mathfrak{g}_0'}(L^0(\mu) \otimes_{\mathbb{C}} \mathfrak{g}_i, L^0(\lambda)).$$

Taking the eigenvalues of d into account, we get the first statement.

(2) Recall that as $\mathfrak{g}_0'$-modules, $\bigwedge^{n-1} V^* \cong V^*$ and $\bigwedge^{n-2} V^* \cong \bigwedge^2 V^*$. Taking the eigenvalues of d into account, the second and the third statements follow from the first one. $\square$

Remark 3.8. *The* $\mathfrak{g}_0$-*module* $M^+(\mu)$ *in the arguments of Lemmas 3.6 and 3.7 can be regarded as a* $U(\mathfrak{g}_0)$-*submodule in* $U(\mathfrak{g}_{\geq 0}) \otimes_{U(\mathfrak{g}_0)} L^0(\mu)$. *In general, for a* $\mathfrak{g}_0$-*submodule* L *of* $U(\mathfrak{g}_{\geq 0})$ *by adjoint action, the tensor product module* $M^+(\mu) = L \otimes_{\mathbb{C}} L^0(\mu)$ *can be regarded as* $LU(\mathfrak{g}_0) \otimes_{U(\mathfrak{g}_0)} L^0(\mu)$, *the latter of which is a* $\mathfrak{g}_0$-*submodule of the induced module* $U(\mathfrak{g}_{\geq 0}) \otimes_{U(\mathfrak{g}_0)} L^0(\mu)$. *Similarly,* $M^-(\lambda)$ *can be regarded as a* $\mathfrak{g}_0$-*submodule of the induced module* $U(\mathsf{P}) \otimes_{U(\mathfrak{g}_0)} L^0(\lambda)$.

4. PROJECTIVE COVERS

Keep the notations as the previous sections.

4.1. **Projective covers in $\mathcal{O}^{\mathbf{min}}$.** By Lemma 3.4, $\{L(\lambda)\}_{(\lambda,d)\in\mathbf{E}}$ form a complete set of pairwise non-isomorphic simple objects in $\mathcal{O}^{\min}$. By abuse of notations, we don't distinguish $\mathbf{E}$ and the set of iso-classes of irreducible modules in $\mathcal{O}^{\min}$ from now on. Especially, we make an appointment that the simple object $L(\lambda)$ with depth d will be written as $L(\lambda) = L(\lambda)_d$. We first have the following basic observations.

Lemma 4.1.   (1) *Suppose that M is a $\bar{\mathfrak{h}}$-semisimple and locally finite $U(\mathfrak{g}_0)$-module. Then M is semisimple over $\mathfrak{g}_0$.*
  (2) *Suppose that M is a finite-dimensional $U(\mathsf{P})$-module generated by a maximal λ-weighted vector v. Then M admits a unique irreducible quotient module, which is isomorphic to $L^0(\lambda)$ as a $\mathfrak{g}_0$-module, endowed with trivial $\mathfrak{g}_{-1}$-action.*
  (3) *Denote by $\mathcal{O}^0_{fin}$ the category of $\bar{\mathfrak{h}}$-semisimple and locally finite $\mathfrak{g}_0$-modules. If $V \in \mathcal{O}^0_{fin}$ is a highest weight module, i.e., generated by a maximal vector of weight λ, then $V \cong L^0(\lambda)$.*
  (4) *Any finite-dimensional irreducible $\mathfrak{g}_0$-module $L^0(\lambda)$ for $\lambda \in \Lambda^+$ is projective in $\mathcal{O}^0_{fin}$.*

Proof. (1) For any nonzero $v \in M$, $V := U(\mathfrak{g}_0)v$ is finite-dimensional. As M is $\bar{\mathfrak{h}}$-semisimple, so is V. We write $V = \sum_{\lambda\in\bar{\mathfrak{h}}^*} V_\lambda$. The finite-dimensionality of V entails, by some routine arguments, that V can be decomposed into a direct sum of irreducible $\mathfrak{g}_0$-modules which are generated by maximal (weighted-) vectors in $V = \sum V_\lambda$. Therefore, M is semisimple over $\mathfrak{g}_0$.

(2) Recall for $\mu, \tau \in \bar{\mathfrak{h}}^*$, $\mu \succeq \tau$ means that $\mu - \tau$ lies in $\mathbb{Z}_{\geq 0}$-span of $\Phi_{\geq 1} \cup \Phi_0^+$. Clearly M admits one-dimensional weight space M'_λ of the highest weight λ. Furthermore, any proper submodule of M admits weight spaces less than λ. Hence M admits a unique maximal submodule, thereby M as a $U(\mathsf{P})$-module, has a quotient isomorphic to $L^0(\lambda)$, which can be viewed as an irreducible $\mathfrak{g}_0$-module, endowed with a trivial $\mathfrak{g}_{-1}$-action.

(3) This is a direct consequence of (1). Otherwise, $V = V_1 \oplus V_2 \oplus \cdots \oplus V_s, s \geq 2$, and $V_i's$ are all finite-dimensional simple $\mathfrak{g}_0$-modules. Then V can not be generated by a single maximal vector of weight λ.

(4) It follows from the statements (1). $\square$

The following result asserts the existence of projective covers of simple modules in $\mathcal{O}^{\min}$.

Theorem 4.2. *Each simple object $L(\lambda)$ in $\mathcal{O}^{min}$ has a projective cover $P(\lambda)$. Furthermore, $P(\lambda)$ admits a flag of standard modules, i.e., there is a sequence of submodules of $P(\lambda)$*

$$P(\lambda) = P_0 \supset P_1 \supset \cdots \supset P_l \supset P_{l+1} = 0$$

such that $P_i/P_{i+1} \cong \Delta(\lambda_i)$ for some λ_i, $i = 0, 1, \cdots, l$.

Proof. Set $I(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} L^0(\lambda)$. Then $I(\lambda)$ lies in $\mathcal{O}^{\min}$ (see Definition 3.1). Our arguments are divided into different steps.

(i) We first claim that $I(\lambda)$ is a projective object in $\mathcal{O}^{\min}$.

Indeed, thanks to Lemma 4.1, $L^0(\lambda)$ is a projective $\mathfrak{g}_0$-module in $\mathcal{O}^0_{\mathrm{fin}}$. Note that the induction functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} -$ is left adjoint to the restriction functor. The claim follows.

(ii) We next show that $I(\lambda)$ has a finite filtration such that each sub-quotient is isomorphic to a standard module.

Note that $I(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathsf{P})} (U(\mathsf{P}) \otimes_{U(\mathfrak{g}_0)} L^0(\lambda))$. Now we consider the $U(\mathsf{P})$-module $U(\mathsf{P}) \otimes_{U(\mathfrak{g}_0)} L^0(\lambda)$. As a vector space, $U(\mathsf{P}) \otimes_{U(\mathfrak{g}_0)} L^0(\lambda) \cong \bigwedge(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} L^0(\lambda)$. Denote $\mathcal{L}^j(\lambda) :=$

$\bigwedge^j \mathfrak{g}_{-1} \otimes_\mathbb{C} L^0(\lambda)$ and $\mathscr{L}^{\geq j}(\lambda) := \bigoplus_{i=j}^n \mathscr{L}^i(\lambda), 0 \leq j \leq n$. By a simple calculation, we can check that each $\mathscr{L}^{\geq j}(\lambda), 0 \leq j \leq n$, is a $U(\mathsf{P})$-submodule of $U(\mathsf{P}) \otimes_{U(\mathfrak{g}_0)} L^0(\lambda)$. In particular, $\mathscr{L}^{\geq 0}(\lambda) = U(\mathsf{P}) \otimes_{U(\mathfrak{g}_0)} L^0(\lambda)$. Then we have the following subsequence of $U(\mathsf{P})$-modules.

$$U(\mathsf{P}) \otimes_{U(\mathfrak{g}_0)} L^0(\lambda) = \mathscr{L}^{\geq 0}(\lambda) \supseteq \mathscr{L}^{\geq 1}(\lambda) \supseteq \cdots \supseteq \mathscr{L}^{\geq(n-1)}(\lambda) \supseteq \mathscr{L}^{\geq n}(\lambda) \supseteq 0, \qquad (4.1)$$

which satisfies that $\mathscr{L}^{\geq i}(\lambda)/\mathscr{L}^{\geq(i+1)}(\lambda) \cong \mathscr{L}^i(\lambda), 0 \leq i \leq n-1$. Here the subquotient $\mathscr{L}^i(\lambda)$ has trivial $\mathfrak{g}_{-1}$-action and is finite-dimensional.

Since $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(n)$ (resp. $\mathfrak{sl}(n)+\mathbb{C}\mathsf{d}$ or $\mathfrak{so}(n)+\mathbb{C}\mathsf{d}$) for $\mathfrak{g}$ being of type W (resp. $\bar{S}$ or $\bar{H}$) and d acts on $\mathscr{L}^i(\lambda)$ as a scalar $\lambda(\mathsf{d}) - i$, Weyl's completely reducible theorem is available to $\mathscr{L}^j(\lambda)$, which means that $\mathscr{L}^j(\lambda)$ can be certainly decomposed into the following sum of irreducible $\mathfrak{g}_0$-modules:

$$\mathscr{L}^j(\lambda) = \bigoplus_{k=1}^{n_j} L^0\left(\eta_k^{(j)}\right), \qquad (4.2)$$

where $\eta_k^{(j)} \in \Lambda^+$ satisfies $\mathrm{Hom}_{\mathfrak{g}_0}\left(L^0\left(\eta_k^{(j)}\right), \bigwedge^j \mathfrak{g}_{-1} \otimes L^0(\lambda)\right) \neq 0$.

So as a $U(\mathsf{P})$-module, there is a filtration of $\mathscr{L}^{\geq j}(\lambda)$

$$\mathscr{L}^{\geq j}(\lambda) =: \mathscr{L}_1^{\geq j}(\lambda) \supseteq \mathscr{L}_2^{\geq j}(\lambda) \supseteq \cdots \supseteq \mathscr{L}_{n_j}^{\geq j}(\lambda) \supseteq \mathscr{L}^{\geq j+1}(\lambda) = \mathscr{L}_1^{\geq j+1}(\lambda) \qquad (4.3)$$

such that $\mathscr{L}_k^{\geq j}(\lambda)/\mathscr{L}_{k+1}^{\geq j}(\lambda) \cong L^0\left(\eta_k^{(j)}\right)$ and $\mathscr{L}_{n_j}^{\geq j}(\lambda)/\mathscr{L}_1^{\geq j+1}(\lambda) \cong L^0\left(\eta_{n_j}^{(j)}\right)$. From (4.1) and (4.3), we then get the following $U(\mathsf{P})$-module filtration,

$$\mathscr{L}^{\geq 0}(\lambda) \supseteq \mathscr{L}^{\geq 1}(\lambda) \supseteq \cdots \supseteq \mathscr{L}_1^{\geq j}(\lambda) \supseteq \mathscr{L}_2^{\geq j}(\lambda) \supseteq \cdots \supseteq \mathscr{L}_{n_j}^{\geq j}(\lambda) \supseteq$$
$$\supseteq \mathscr{L}_1^{\geq j+1}(\lambda) \supseteq \mathscr{L}_2^{\geq j+1}(\lambda) \supseteq \cdots \supseteq \mathscr{L}_{n_{j+1}}^{\geq j+1}(\lambda) \supseteq \cdots$$
$$\cdots \supseteq \mathscr{L}^{\geq n}(\lambda) \supseteq 0, \qquad (4.4)$$

such that $\mathscr{L}_k^{\geq j}(\lambda)/\mathscr{L}_{k+1}^{\geq j}(\lambda) \cong L^0\left(\eta_k^{(j)}\right)$ and $\mathscr{L}_{n_j}^{\geq j}(\lambda)/\mathscr{L}_1^{\geq j+1}(\lambda) \cong L^0\left(\eta_{n_j}^{(j)}\right)$ for $j = 0, 1, \cdots, n$. Now set $I_k^{\geq j}(\lambda) = U(\mathfrak{g}) \otimes_{U(P)} \mathscr{L}_k^{\geq j}(\lambda)$. Then we have the following $U(\mathfrak{g})$-module filtration,

$$I^{\geq 0}(\lambda) \supseteq I_1^{\geq 1}(\lambda) \supseteq \cdots \supseteq I_1^{\geq j}(\lambda) \supseteq I_2^{\geq j}(\lambda) \supseteq \cdots \supseteq I_{n_j}^{\geq j}(\lambda) \supseteq I_1^{\geq j+1}(\lambda) \supseteq \cdots \supseteq I^{\geq n}(\lambda) \supseteq 0. \qquad (4.5)$$

By the construction, $I_k^{\geq j}(\lambda)/I_{k+1}^{\geq j}(\lambda)$ is isomorphic to $\Delta\left(\eta_k^{(j)}\right)$ for $1 \leq k < n_j$, and $I_{n_j}^{\geq j}/I_1^{\geq j+1}$ is isomorphic to $\Delta\left(\eta_{n_j}^{(j)}\right)$.

(iii) Thirdly, we prove that any direct summand of $I(\lambda)$ admits a Δ-flag.

By the construction in (ii), we have got that $I(\lambda)$ admits a Δ-flag of finite length, in which the bottom one is a submodule $\Delta(\gamma)$ with $\gamma = \lambda - \sum_{i=1}^n \epsilon_i$ for $\mathfrak{g} = W(n)$ or $\bar{S}(n)$, and $\gamma = \lambda - n\delta$ for $\mathfrak{g} = \bar{H}(n)$ or $\overline{CH}(n)$. This means that $\gamma \in \Lambda^+$ is the minimal one in $\mathsf{Wt}(I(\lambda)) \cap \Lambda^+$ (the set of the dominant and integral weights of $I(\lambda)$ is in the same sense as in the proof of the above lemma). Actually, one can prove the general result that if $V \in \mathcal{O}^{\min}$ admits a Δ-flag of finite length with the bottom standard module factor $\Delta(\gamma)$ satisfying that γ is minimal in $\mathsf{Wt}(V) \cap \Lambda^+$, then any direct summand of V admits a Δ-flag. This can be done by some standard inductive arguments on the lengths of Δ-flags (see [12, §3.7]).

(iv) Fourthly, we prove that there exists an indecomposable projective module J_0 such that $J_0 \to L(\lambda)$ is an epimorphism as $U(\mathfrak{g})$-modules.

From the arguments in (ii), we know that as $U(\mathfrak{g})$-modules,

$$I(\lambda)/I_1^{\geq 1}(\lambda) \cong \Delta(\lambda).$$

So there are natural surjective morphisms $I(\lambda) \xrightarrow{\pi_2} \Delta(\lambda) \xrightarrow{\pi_1} L(\lambda)$. Denote $\pi := \pi_1 \circ \pi_2$. So we have

$$\pi : I(\lambda) \twoheadrightarrow L(\lambda). \tag{4.6}$$

Assume $I(\lambda) = \bigoplus_{i=0}^k J_i$ (the finiteness of k is ensured by (ii) and (iii)). Then there is a summand of $I(\lambda)$, written as J_0 without loss of generality, such that $\pi|_{J_0}$ is non-zero. We denote $\pi|_{J_0}$ by π_0. Because $\Delta(\lambda) \xrightarrow{\pi_1} L(\lambda)$ is surjective, the projective property of J_0 entails that π_0 can be lifted to a morphism $\bar{\pi}_0 : J_0 \longrightarrow \Delta(\lambda)$.

(v) We claim that J_0 is the projective cover of both $\Delta(\lambda)$ and $L(\lambda)$.

By the above argument, we already have the following commutative diagram:

$$
\begin{array}{ccc}
& & J_0 \\
& \overset{\bar{\pi}_0}{\swarrow} & \downarrow \pi_0 \\
\Delta(\lambda) & \xrightarrow{\pi_1} & L(\lambda).
\end{array}
$$

In fact, $\bar{\pi}_0$ is surjective. Otherwise, the image of $\bar{\pi}_0$ will be contained in the maximal submodule of $\Delta(\lambda)$, so $\pi_1 \circ \bar{\pi}_0(J_0) = 0 \neq \pi_0(J_0)$, which contradicts to the above commutative diagram.

What remains is to prove that π_0 is essential. Consider $A := \mathrm{Hom}_{\mathcal{O}^{\min}}(I(\lambda), I(\lambda))$. Then we have an isomorphism of vector spaces: $A \cong \mathrm{Hom}_{U(\mathfrak{g}_0)}(L^0(\lambda), I(\lambda)|_{U(\mathfrak{g}_0)})$. Because $L^0(\lambda)$ is generated by v_λ and $I(\lambda)_\lambda$ is finite-dimensional, $\dim A < \infty$. Hence, as a subalgebra of A, $A_0 := \mathrm{Hom}_{\mathcal{O}^{\min}}(J_0, J_0)$ is finite-dimensional. Then, by some standard arguments on Fitting decomposition we can prove that π_0 is indeed essential.

We can further have that J_0 is also the projective cover of $\Delta(\lambda)$. This is because the essential property of $\bar{\pi}_0$ can be ensured by that of π_0.

As $I(\lambda)$ admits a unique factor $\Delta(\lambda)$ in its Δ-flag, it is easy to deduce that the choice of J_0 is unique among all indecomposable direct summands of $I(\lambda)$. $\square$

Remark 4.3. (1) *We can precisely construct such a $P(\lambda)$ (=J_0) as below. From $\pi_0 = \pi|_{J_0}$ and the definition of the category $\mathcal{O}^{min}$, it follows that J_0 contains a vector v_0 of the form like*

$$1 \otimes v_\lambda^0 + \sum_i u_i \otimes v_i$$

where $u_i \in U(\mathfrak{g}_{\geq 1})\mathfrak{g}_{\geq 1}U(\mathfrak{g}_{-1})\mathfrak{g}_{-1}$ with the weight of all $u_i \otimes v_i$ being λ. Set

$$\tilde{J}_0 := U(\mathfrak{g})v_0.$$

By the arguments as above, we actually have the following commutative diagram

$$
\begin{array}{ccc}
 & \tilde{J}_0 & \\
{\scriptstyle \bar{\pi}_0|_{\tilde{J}_0}} \swarrow & \downarrow {\scriptstyle \pi_0|_{\tilde{J}_0}} & \\
\Delta(\lambda) & \xrightarrow{\ \pi_1\ } & L(\lambda).
\end{array}
$$

The essential property of π_0 entails that $J_0 = \tilde{J}_0$.

(2) From the proof (v) of Theorem 4.2, we know that J_0 admits a unique maximal submodule, which is exactly $\ker(\pi_0)$. So an irreducible module in $\mathcal{O}^{min}$ is naturally the unique irreducible quotient of its projective cover.

(3) Let $\lambda \in \Lambda^+$. Set

$$\Upsilon(\lambda) := \{\mu \in \Lambda^+ \mid (\bigwedge \mathfrak{g}_{-1} \otimes L^0(\lambda) : L^0(\mu))_{\mathfrak{g}_0} \neq 0\}, \tag{4.7}$$

where $(L : L^0(\mu))_{\mathfrak{g}_0}$ denotes the multiplicity of $L^0(\mu)$ in the composition series of the finite-dimensional $\mathfrak{g}_0$-module L. As in the proof (ii) of Theorem 4.2, we have the following decomposition of as $\mathfrak{g}_0'$-modules:

$$\bigwedge \mathfrak{g}_{-1} \otimes_{\mathbb{C}} L^0(\lambda) = \bigoplus_{\mu \in \Upsilon(\lambda)} n_{\lambda,\mu} L^0(\mu).$$

Moreover, the following statements hold.

  (1°) $n_{\lambda,\mu} = 0$ for any $\mu \notin \Upsilon(\lambda)$.
  (2°) For a projective object $Q \in \mathcal{O}^{min}$, denote by $[Q : \Delta(\mu)]$ the multiplicity of $\Delta(\mu)$ in its Δ-flag. Then $[I(\lambda) : \Delta(\mu)] = n_{\lambda,\mu}$ for any $\mu \in \Lambda^+$. In particular, $[I(\lambda) : \Delta(\lambda)] = 1$.
  (3°) Suppose $\lambda - \sum_{i=k}^n \epsilon_i \in \Lambda^+$. Then $[I(\lambda) : \Delta(\lambda - \sum_{i=k}^n \epsilon_i)] \neq 0, 1 \leq k \leq n$, for $\mathfrak{g} = W(n), \bar{S}(n)$, In particular, $[I(\lambda) : \Delta(\lambda - \sum_{i=1}^n \epsilon_i)] \neq 0$ for $\mathfrak{g} = W(n), \bar{S}(n)$.
  (4°) $[I(\lambda) : \Delta(\lambda + \sum_{i=1}^k \epsilon_i - (n-k)\delta)] \neq 0$ $(k = 0, 1, ..., r)$ for $\mathfrak{g} = \bar{H}(n), \overline{CH}(n)$. In particular, $[I(\lambda) : \Delta(\lambda - n\delta)] \neq 0$ for $\mathfrak{g} = \bar{H}(n), \overline{CH}(n)$.

These statements (1°)-(4°) will be used in the sequel. The statements (1°)-(2°) are direct consequences of the theorem. For (3°), we remind that $D_k \wedge D_{k+1} \wedge \cdots \wedge D_n \otimes v_\lambda^0$ with $1 \leq k \leq n$ is a maximal weight vector of $\bigwedge^{n-k+1} \mathfrak{g}_{-1} \otimes_{\mathbb{C}} L^0(\lambda)$ for $\mathfrak{g} = W(n), \bar{S}(n)$. As for (4°), we can check that $D_{k+1} \wedge \cdots \wedge D_r \wedge D_{r+1} \wedge \cdots \wedge D_n \otimes v_\lambda^0$ is a $\mathfrak{g}_0$-maximal weight vector. Now the results in (3°) and (4°) hold by (3.4), (3.6) and the formula (4.2) in the proof of Theorem 4.2.

4.2. The category $\mathcal{O}_{\mathfrak{f}}^{\mathbf{min}}$.

Denote by $\mathcal{O}_{\mathfrak{f}}^{min}$ the full subcategory of $\mathcal{O}^{min}$ whose objects are finitely-generated $U(\mathfrak{g})$-modules in $\mathcal{O}^{min}$. Then we have the following consequence based on Theorem 4.2.

Theorem 4.4. *The category $\mathcal{O}_{\mathfrak{f}}^{min}$ has enough projective objects, this is to say, for each $M \in \mathcal{O}_{\mathfrak{f}}^{min}$, there is a projective object $P \in \mathcal{O}_{\mathfrak{f}}^{min}$ and an epimorphism $P \twoheadrightarrow M$.*

Proof. Note that $P(\lambda)$, $\Delta(\lambda)$, $\nabla(\lambda)$ and $L(\lambda)$ are all in $\mathcal{O}_{\mathfrak{f}}^{min}$. And it is still true that $P(\lambda)$ is a projective cover of $L(\lambda)$ in $\mathcal{O}_{\mathfrak{f}}^{min}$. For any nonzero object $M \in \mathcal{O}_{\mathfrak{f}}^{min}$, M admits a filtration of finite length

$$M = M^0 \supset M^1 \supset M^2 \supset \cdots \supset M^{t-1} \supset M^t = 0 \tag{4.8}$$

such that M^{i-1}/M^i is isomorphic to a non-zero quotient of $\Delta(\lambda_i)$ for some $L(\lambda_i) \in \mathbf{E}$, $i = 1, \cdots, t$. The least number t in all possible filtrations as in (4.8) is called the standard length of M, denoted by $l(M)$.

Set $P = \bigoplus_{i=1}^t P(\lambda_i)$. Then by induction on t, there will be a covering morphism from P onto M. The proof is completed. $\qquad\square$

Proposition 4.5. *In $\mathcal{O}_{\mathfrak{f}}^{min}$, every indecomposable projective module is isomorphic to the projective cover $P(\lambda)$ of some irreducible module $L(\lambda)$.*

Proof. Suppose P is an indecomposable projective module in $\mathcal{O}_{\mathfrak{f}}^{\min}$. By the definition of $\mathcal{O}_{\mathfrak{f}}^{\min}$, P has an irreducible quotient $L(\lambda)$, which defines an epimorphism $\phi : P \to L(\lambda)$. The projective property of P yields the following commutative diagram

$$
\begin{array}{ccc}
 & & P \\
 & \overset{\bar\phi}{\swarrow} & \downarrow \phi \\
P(\lambda) & \overset{\pi_0}{\longrightarrow} & L(\lambda).
\end{array}
$$

Because π_0 is essential, $\bar\phi$ must be surjective. Hence $P(\lambda)$ is isomorphic to a direct summand of P. The assumption of indecomposability of P entails that P is isomorphic to $P(\lambda)$. $\qquad\square$

5. Degenerate BGG reciprocity and typical functor

Maintain the previous notations and assumptions.

5.1. Thanks to Lemma 4.1, Brundan's arguments in [7] are available to $\mathcal{O}^{\min}$.

Theorem 5.1. *([7, Theorem 4.4] and [21, Theorem 3.2]) Every simple object $L(\lambda) = L(\lambda)_d$ in $\mathcal{O}_{\geq d'}^{min}$ admits a projective cover $P_{\geq d'}(\lambda)$ with $d \geq d'$, the head of $P_{\geq d'}(\lambda)$ is isomorphic to $L(\lambda) = L(\lambda)_d$. Moreover,*

(1) *$P_{\geq d'}(\lambda)$ admits a finite Δ-flag with $\Delta(\lambda)$ at the top.*
(2) *For $m < l$, the kernel of any surjection $P_{\geq m}(\lambda) \to P_{\geq l}(\lambda)$ admits a finite Δ-flag with subquotients of the form $\Delta(\mu)$ for $m \leq \lfloor \bar{L}^0(\mu) \rfloor < l$.*
(3) *$L(\lambda)$ admits a projective cover in $\mathcal{O}^{min}$ if and only if there exists $l \ll 0$ with $P_{\geq l}(\lambda) = P_{\geq l-1}(\lambda) = P_{\geq l-2}(\lambda) = \cdots$, in which case $P(\lambda) = P_{\geq l}(\lambda)$.*

In our case we have a stronger result (Theorem 4.2). This is to say, the projective covers of $L(\lambda)$ in $\mathcal{O}_{\geq d}^{\min}$ and $\mathcal{O}^{\min}$ exist. But the above theorem can help us to give more information on $P(\lambda)$ in the next subsection.

5.2. By Theorem 5.1, every simple object $L(\lambda) = L(\lambda)_{d'}$ in $\mathcal{O}_{\geq d}^{\min}$ admits a projective cover $P_{\geq d}(\lambda)$ with $d' \geq d$, the head of $P_{\geq d}(\lambda)$ is isomorphic to $L(\lambda) = L(\lambda)_{d'}$. Theorem 5.1 along with Theorem 4.2 implies that there exists $l \ll 0$ with $P_{\geq l}(\lambda) = P_{\geq l-1}(\lambda) = P_{\geq l-2}(\lambda) = \cdots$, and $P(\lambda) = P_{\geq l}(\lambda)$. Furthermore, by Theorem 5.1, any $P_{\geq l}(\lambda)$ admits a Δ-flag. Denote by $[P_{\geq l}(\lambda) : \Delta(\mu)]$ the multiplicity of $\Delta(\mu)$ in the Δ-flag of $P_{\geq l}(\lambda)$. By [21, §4] or [7, Lemma 4.5], we have the following result.

Lemma 5.2. $[P_{\geq l}(\lambda) : \Delta(\mu)] = (\nabla(\mu) : L(\lambda))$ *for all $L(\lambda)$ and $L(\mu) \in \mathbf{E}$ as long as $\mathsf{dpt}(L(\lambda)) \geq l$ and $\mathsf{dpt}(L(\mu)) \geq l$.*

5.3. **Degenerate BGG reciprocity.**

Theorem 5.3. *In the category $\mathcal{O}^{min}$, the following statement holds*

$$[P(\lambda) : \Delta(\mu)] = (\nabla(\mu) : L(\lambda))$$

for all $L(\lambda), L(\mu) \in \mathbf{E}$.

Proof. For any given $L(\lambda) \in \mathbf{E}$, assume $\mathsf{dpt}(L(\lambda)) = d$. By Theorem 5.1(3), there exists some $l \ll 0$ such that $P(\lambda)$ in $\mathcal{O}^{min}$ and $P(\lambda) = P_{\geq l}(\lambda) = P_{\geq l-i}(\lambda)$ for all $i \in \mathbb{Z}_{\geq 0}$ (certainly, $l < d$). For any $L(\mu) \in \mathbf{E}$, there exists some $i_0 \in \mathbb{Z}_{\geq 0}$ such that $\mathsf{dpt}(L(\mu)) \geq \bar{l} - i_0$. Since $l < d$ and $i_0 \in \mathbb{Z}_{\geq 0}$, we have $P(\lambda) = P_{\geq l-i_0}(\lambda)$ and $\mathsf{dpt}(L(\lambda)) \geq l - i_0$. Now applying Lemma 5.2 to $P_{\geq l-i_0}(\lambda)$, we have $[P(\lambda) : \Delta(\mu)] = (\nabla(\mu) : L(\lambda))$. $\qquad\square$

5.4. **The Kac-module realization of co-standard modules.** Set $\mathfrak{g}^+ := \oplus_{i \geq 0} \mathfrak{g}_i$. The following module is the so-called Kac-module

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} L^0(\lambda),$$

where $L^0(\lambda)$ is regarded as a $\mathfrak{g}^+$-module with trivial $\mathfrak{g}_{\geq 1}$-action. One can check that $K(\lambda)$ has a simple head, denoted by $\overline{L}(\lambda)$.

Following [18], we introduce the set Ω of the so-called Serganova atypical weights as follows. If $\mathfrak{g} = W(n)$, set

$$\Omega = \{a\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n \mid a \in \mathbb{C}, 1 \leq i \leq n\}.$$

If $\mathfrak{g} = \bar{S}(n)$, set

$$\Omega = \{a\epsilon_1 + \cdots + a\epsilon_{i-1} + b\epsilon_i + (a+1)\epsilon_{i+1} + \cdots + (a+1)\epsilon_n \mid a, b \in \mathbb{C}, 1 \leq i \leq n\}.$$

If $\mathfrak{g} = \bar{H}(n)$, set

$$\Omega = \{-\epsilon_1 - \cdots - \epsilon_{i-1} + b\epsilon_i + a\delta \mid a, b \in \mathbb{C}, 1 \leq i \leq r\}.$$

Definition 5.4. *A weight λ is called Serganova atypical if λ belongs to Ω. Otherwise, λ is called Serganova typical.*

Keep it in mind that the notation $\Xi = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ for $\mathfrak{g} = W(n)$ or $\bar{S}(n)$.

Proposition 5.5. *Let $\mathfrak{g} = X(n), X \in \{W, \bar{S}, \bar{H}\}$.*
  (1) *If $\mathfrak{g} = W(n), \bar{S}(n)$, then $\nabla(\lambda) \cong K(\lambda + \Xi)$.*
  (2) *If $\mathfrak{g} = \bar{H}(n)$, then $\nabla(\lambda) \cong K(\lambda + n\delta)$.*
  (3) *The Kac-module $K(\lambda)$ is irreducible if and only if λ is Serganova typical.*

Proof. The third statement follows from [18, Theorem 6.3]. We proceed to prove the first two statements.

Note that $\mathfrak{g}^+$ is a subalgebra of $\mathfrak{g}$ with codimension n and $\mathfrak{g}_{\bar{0}} \subseteq \mathfrak{g}^+$. Let $f : \mathfrak{g}^+ \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{g}^+) = \mathfrak{gl}(\mathfrak{g}_{-1})$ be the map defined by $f(a)(b + \mathfrak{g}^+) = [a, b] + \mathfrak{g}^+$. Then it follows from [4, Theorem 2.2] that $U(\mathfrak{g}) : U(\mathfrak{g}^+)$ is a free θ-Frobenius extension, where θ is the unique automorphism of $U(\mathfrak{g}^+)$ defined by

$$\theta(a) = \begin{cases} a + \mu(a) \cdot 1, & \text{if } a \in \mathfrak{g}_{\bar{0}}^+, \\ (-1)^n a, & \text{if } a \in \mathfrak{g}_{\bar{1}}^+, \end{cases}$$

and $\mu : \mathfrak{g}^+ \to \mathbb{C}$ is defined by $\mu(a) = \mathrm{tr} f(a)$. Thus by [15, §3], we have

$$\mathrm{Hom}_{\mathfrak{g}^+}\left(U(\mathfrak{g}),\ {}_\theta L^0(\lambda)\right) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} L^0(\lambda) = K(\lambda), \tag{5.1}$$

where ${}_\theta L^0(\lambda)$ is a $\mathfrak{g}^+$-module with action twisted by θ, i.e., $s*v := \theta(s)v$ for any $s \in \mathfrak{g}^+$ and $v \in L^0(\lambda)$.

Now let v_λ^0 be a maximal vector of $L^0(\lambda)$ corresponding to the standard Borel subalgebra $\mathfrak{b}_0^+$. Since

$$\theta(x) = \begin{cases} x, & \text{if } x \in \mathfrak{g}_{\geq 1} \cap \mathfrak{g}_{\bar{0}}, \\ (-1)^n x, & \text{if } x \in \mathfrak{g}_{\geq 1} \cap \mathfrak{g}_{\bar{1}}, \end{cases}$$

${}_\theta L^0(\lambda)$ is still an irreducible $\mathfrak{g}^+$-module with trivial $\mathfrak{g}_{\geq 1}$-action. Because $\mu(x) = 0$ for $x \in \mathfrak{n}^+$, v_λ^0 is still a maximal vector of ${}_\theta L^0(\lambda)$. Let $h \in \bar{\mathfrak{h}}$.

Case (i): $\mathfrak{g} = W(n)$ or $\bar{S}(n)$.

Since $f(h)(D_i + \mathfrak{g}^+) = [h, D_i] + \mathfrak{g}^+ = -\epsilon_i(h)(D_i) + \mathfrak{g}^+$, it follows that $\mu(h) = \mathrm{tr} f(h) = -\epsilon_1(h) - \epsilon_2(h) - \cdots - \epsilon_n(h) = -\Xi(h)$. Consequently,

$$h*v_\lambda = \theta(h)v_\lambda = (h + \mu(h).1)v_\lambda = (\lambda - \Xi)(h)v_\lambda.$$

Hence by (5.1) we get that

$$\nabla(\lambda - \Xi) \cong K(\lambda) \text{ for any } \lambda \in \Lambda^+.$$

Equivalently,

$$\nabla(\lambda) \cong K(\lambda + \Xi) \text{ for any } \lambda \in \Lambda^+. \tag{5.2}$$

Case (ii): $\mathfrak{g} = \bar{H}(n)$.

Subcase (ii-1): $n = 2r$.

In this subcase,

$$f(h)(D_i + \mathfrak{g}^+) = (-\epsilon_i - \delta)(h)(D_i + \mathfrak{g}^+) \text{ for } 1 \leq i \leq r,$$
$$f(h)(D_i + \mathfrak{g}^+) = (\epsilon_{i'} - \delta)(h)(D_i + \mathfrak{g}^+) \text{ for } r+1 \leq i \leq 2r.$$

Subcase (ii-2): $n = 2r + 1$.

In this subcase,

$$f(h)(D_i + \mathfrak{g}^+) = (-\epsilon_i - \delta)(h)(D_i + \mathfrak{g}^+) \text{ for } 1 \leq i \leq r,$$
$$f(h)(D_i + \mathfrak{g}^+) = (\epsilon_{i'} - \delta)(h)(D_i + \mathfrak{g}^+) \text{ for } r+1 \leq i \leq 2r,$$
$$f(h)(D_n + \mathfrak{g}^+) = -\delta(h)(D_n + \mathfrak{g}^+).$$

It follows that ${}_\theta L^0(\lambda) \cong L^0(\lambda - n\delta)$. Hence, by (5.1), we get

$$\nabla(\lambda - n\delta) \cong K(\lambda),\ \forall\, \lambda \in \Lambda^+.$$

Equivalently,

$$\nabla(\lambda) \cong K(\lambda + n\delta),\ \forall\, \lambda \in \Lambda^+.$$

$\square$

Theorem 5.6. *Let* $\lambda, \mu \in \mathbf{E}$. *Then the following statements hold.*

(1) *If* $\mathfrak{g} = W(n)$ *or* $\mathfrak{g} = \bar{S}(n)$, *then*

$$[P(\lambda) : \Delta(\mu)] = (K(\mu + \Xi) : L(\lambda)).$$

(2) *If* $\mathfrak{g} = \bar{H}(n)$, *then*

$$[P(\lambda) : \Delta(\mu)] = (K(\mu + n\delta) : L(\lambda)).$$

Proof. Theorem 5.3 and Proposition 5.5 can be applied to get these results.   □

5.5. **Typical blocks and the typical functor.** We begin this subsection with the following consequence of indecomposable projective modules in $\mathcal{O}^{\min}$, which is well known for Noetherian categories.

Lemma 5.7. *Suppose $M \in \mathcal{O}^{min}$. Then the following statements hold.*
 (1) *For any $L(\lambda) \in \mathbf{E}$,*
$$(M : L(\lambda)) = \dim \operatorname{Hom}_{\mathcal{O}^{min}}(P(\lambda), M).$$
 (2) *If there exists a nonzero vector $v \in M$ of weight λ, which is annihilated by $\mathfrak{g}_{-1} + \mathfrak{n}^+$, then $(M : L(\lambda)) \neq 0$.*

Proof. (1) Suppose $\mathsf{dpt}(L(\lambda)) = t$. If $(M : L(\lambda)) \neq 0$, then the multiplicity is less than the dimension of M_t. By the definition of $\mathcal{O}^{\min}$, $\dim M_t < \infty$. Thus, it is a routine way to prove the lemma by induction on $(M : L(\lambda)) < \infty$.

(2) Consider the submodule N generated by v in M. Note that by assumption the module $U(\mathfrak{g}_0)v$ is a finite-dimensional highest weight module over $U(\mathfrak{g}_0)$, generated by the maximal vector v. Therefore $U(\mathfrak{g}_0)v$ is isomorphic to $L^0(\lambda)$. Furthermore, the assumption of $\mathfrak{g}_{-1}$-annihilation of v implies that as a $U(\mathfrak{g})$-module, N is a homomorphism image of $\Delta(\lambda)$. Hence $(M : L(\lambda)) \neq 0$.   □

In general, define $A_\lambda := \operatorname{Hom}_{\mathcal{O}^{\min}}(P(\lambda), P(\lambda))$. Then A_λ is a finite-dimensional $\mathbb{C}$-algebra, whose dimension is exactly $(P(\lambda) : L(\lambda))$ by Lemma 5.7(1). Let $\mathcal{O}_\lambda^{\min}$ stand for the block in which $L(\lambda)$ lies. In general, we can define a functor
$$\mathsf{S}_\lambda := \operatorname{Hom}_{\mathcal{O}^{\min}}(P(\lambda), -).$$
By Lemma 5.7(1) again, S_λ gives rise to a functor from $\mathcal{O}_\lambda^{\min}$ to the category of finite-dimensional A_λ-modules, the latter of which is denoted by $A_\lambda\text{-}\mathbf{mod}^f$.

Denote by Λ_{st} the set of all Serganova typical weights. All dominant Serganova typical weights can be clearly described. For example, if $\mathfrak{g} = W(n)$, then $\Lambda_{st}^+ = \{\lambda = \sum_{i=1}^n a_i\epsilon_i \mid a_i - a_{i+1} \in \mathbb{Z}_{\geq 0}\}\backslash\Omega^+$ with $\Omega^+ := \Omega \cap \Lambda^+$.

Set $\Lambda_{\mathsf{t}} := \{\lambda - \Xi \mid \lambda \in \Lambda_{st}\}$ if $\mathfrak{g} = W(n), \bar{S}(n)$, and $\Lambda_{\mathsf{t}} := \{\lambda - n\delta \mid \lambda \in \Lambda_{st}\}$ if $\mathfrak{g} = \bar{H}(n)$. All weights lying in $\Lambda_{\mathsf{t}}^+ := \Lambda^+ \bigcap \Lambda_{\mathsf{t}}$ are called typical. According to Theorem 5.3 and Proposition 5.5 (or Theorem 7.8), we have that for $\lambda \in \Lambda_{\mathsf{t}}^+$,
$$P(\lambda) = \Delta(\lambda). \tag{5.3}$$
So when λ is typical, $A_\lambda = \operatorname{End}_{\mathcal{O}^{\min}}(\Delta(\lambda))$, which is one-dimensional. The functor S_λ is degenerated.

Proposition 5.8. *Let λ be a typical weight and M be an object of $\mathcal{O}_\lambda^{min}$. Then the functor S_λ measures the multiplicity of $L(\lambda)$ in M. This is to say, if $(M : L(\lambda)) = m$, then $\mathsf{S}_\lambda(M) = \mathbb{C}^m$, the unique m-dimensional A_λ-module up to isomorphisms.*

Proof. Note that A_λ is a one-dimensional algebra over $\mathbb{C}$, which is isomorphic to $\mathbb{C}$. The isomorphism class of an object in $A_\lambda\text{-}\mathbf{mod}^f$ is only dependent on the dimension. So the statement is a direct consequence of Lemma 5.7(1).   □

6. Blocks of $\mathcal{O}^{\mathrm{MIN}}$

6.1. **Definition.** Due to Theorem 4.2, we define an equivalent relation $\sim$ in $\mathbf{E}$. For any simple objects $L(\lambda_1), L(\lambda_2)$ in $\mathbf{E}$, we say that $L(\lambda_1)$ and $L(\lambda_2)$ are linked (or λ_1 and λ_2 are linked) if there exists $L(\mu) \in \mathbf{E}$ such that $(P(\mu) : L(\lambda_i)) \neq 0$ for $i = 1, 2$. We say that $L(\lambda) \sim L(\mu)$ (or $\lambda \sim \mu$) if there exist a sequence $L(\lambda) = L(\lambda_1), L(\lambda_2),......,L(\lambda_k) = L(\mu)$ in $\mathbf{E}$ such that $L(\lambda_i)$ and $L(\lambda_{i+1})$ are linked (or λ_i and λ_{i+1} are linked) for every $i = 1, ..., k-1$.

For a given element $\theta \in \mathbf{E}/\sim$, we define a full subcategory $\mathcal{O}_\theta^{\mathrm{min}}$ of $\mathcal{O}^{\mathrm{min}}$ whose objects are those modules M only admitting composition factors from θ. We call $\mathcal{O}_\theta^{\mathrm{min}}$ a block corresponding to θ.

Lemma 6.1. *Any indecomposable object in $\mathcal{O}_{\mathfrak{f}}^{min}$ must belong to a certain $\mathcal{O}_\theta^{min}$.*

Proof. Suppose that M is a nonzero indecomposable module belonging to $\mathcal{O}_{\mathfrak{f}}^{\mathrm{min}}$. As in the proof of Theorem 4.4, there is a projective module $P := \oplus_{i=1}^t P(\lambda_i)$ and an epimorphism $\pi : P \twoheadrightarrow M$. So

$$M = \pi(P(\lambda_1)) + \pi(P(\lambda_2)) + \cdots + \pi(P(\lambda_t)). \tag{6.1}$$

This ensures that we can define a non-zero submodule M_θ of M, which is a sum of submodules belonging to $\mathcal{O}_\theta^{\mathrm{min}}$.

If M_θ coincides with M, then we are done. Otherwise, we have a non-zero submodule M_θ' of M, which is the sum of all submodules belonging to the blocks outside $\mathcal{O}_\theta^{\mathrm{min}}$. Then $M = M_\theta + M_\theta'$ by (6.1). Furthermore, $M_\theta + M_\theta'$ is a direct sum through the definition of blocks. This contradicts to the indecomposability of M. The proof is completed. $\square$

Recall that all standard modules $\Delta(\lambda)$ and costandard modules $\nabla(\lambda)$ are indecomposable and finitely generated. In addition, we have the following stronger results.

Lemma 6.2. *Let $(\lambda, d) \in \mathbf{E} = \Lambda^+ \times \mathbb{Z}$. Then $\Delta(\lambda)$ and $\nabla(\lambda)$ are in the same block.*

Proof. One can give a proof following [7, Lemma 3.5]. Here we give another one. By the arguments in the proof of Proposition 5.5, as a vector space $\nabla(\lambda)$ can be identified with $\bigwedge \mathfrak{g}_{-1} \otimes_{-\theta} L^0(\lambda)$. Take a maximal vector v_0 of $L^0(\lambda)$, and set $v = \bigwedge_{i=1}^n D_i \otimes v_0$. By definition, v has weight λ. Furthermore, v is annihilated by $\mathfrak{g}_{-1} + \mathfrak{n}^+$. Hence by Lemma 5.7(2), $\nabla(\lambda)$ shares the same composition factor $L(\lambda)$ with $\Delta(\lambda)$. So this lemma is a direct consequence of Lemmas 3.3 and 6.1. $\square$

Remark 6.3. *In [7], the definition of blocks was introduced via standard modules and co-standard modules because of the loss of projective covers of simple objects. Lemma 6.2 shows that our definition of blocks is compatible with the one introduced therein.*

6.2. In the following, we discuss some block properties through investigating standard modules. Recall that $\mathfrak{g}$ admits a $\mathbb{Z}$-gradation which gives rise to the $\mathbb{Z}$-gradation $U(\mathfrak{g}) = \sum_{i \in \mathbb{Z}} U(\mathfrak{g})_i$. Similarly, we can talk about the gradation of $U(\mathfrak{g}_{\geq 1}) = \sum_{i \geq 0} U(\mathfrak{g}_{\geq 1})_i$.

Consider $\Delta(\mu) = \sum_{i \geq 0} \Delta(\mu)_i$ for $\Delta(\mu)_i = U(\mathfrak{g}_{\geq 1})_i \otimes L^0(\lambda)$. Set $\Delta(\mu)^{(j)} = \sum_{i \geq j} \Delta(\mu)_i$ for $j \in \mathbb{N}$. Then as a $\mathfrak{g}_{\geq 0}$-module, $\Delta(\mu)$ has the natural descending filtration $\{\Delta(\mu)^{(j)}\}_{j \in \mathbb{N}}$.

Lemma 6.4. *Let $\lambda, \mu \in \Lambda^+$. If v_λ is a nonzero λ-weighted vector of $\Delta(\mu)$ annihilated by $\mathfrak{n}^+$, then $\lambda \sim \mu$.*

Proof. By definition, $\Delta(\mu) = U(\mathfrak{g}_{\geq 1}) \otimes L^0(\mu)$ as a vector space. For any $\mathfrak{n}^+$-annihilating vector $v_\lambda \in \Delta(\mu)$ of weight λ, if v_λ lies in $1 \otimes L^0(\mu)$, then λ coincides with μ, and the statement of the lemma is obvious. In the following, we suppose $v_\lambda \in \Delta(\mu)^j \backslash \Delta(\mu)^{j-1}$ for some $j > 0$. Still set $\mathfrak{g}^+ = \mathfrak{g}_{\geq 0}$. Consider the $U(\mathfrak{g}^+)$-submodule generated by v_λ in $\Delta(\mu)$, denoted by $\mathcal{M}$. Clearly, $\mathcal{M}$ has a proper submodule $\mathcal{N} := U(\mathfrak{g}_{\geq 1})\mathfrak{g}_{\geq 1}U(\mathfrak{g}_0)v_\lambda$. So we have a $U(\mathfrak{g}^+)$-module $\overline{\mathcal{M}} := \mathcal{M}/\mathcal{N}$. This $\overline{\mathcal{M}}$ is generated by the image of v_λ in $\overline{\mathcal{M}}$, denoted by $\bar{v}_\lambda$ which has weight λ and is annihilated by $\mathfrak{n}^+ \oplus \mathfrak{g}_{\geq 1}$. So we have surjective morphisms

$$\mathcal{M} \twoheadrightarrow \overline{\mathcal{M}} \twoheadrightarrow L^0(\lambda),$$

where $L^0(\lambda)$ is an irreducible $\mathfrak{g}^+$-module with highest weigh λ and trivial $\mathfrak{g}_{\geq 1}$-action. Consider the functor $\Gamma = \mathrm{Hom}_{\mathfrak{g}_{\geq 0}}(U(\mathfrak{g}), -)$ from the category of $U(\mathfrak{g}^+)$-modules to the one of $U(\mathfrak{g})$-modules. Then $\Gamma(L^0(\lambda)) = \nabla(\lambda)$.

In the following we focus on the subcategory $\mathcal{C}_{\mathfrak{g}^+}$ of $U(\mathfrak{g}^+)$-module category which consists of objects C satisfying: (i) it has $\mathbb{Z}$-gradation, and finitely generated over $U(\mathfrak{g}^+)$, (ii) C is locally finite over $\mathfrak{g}_0$, i.e. for any $v \in C$ the $U(\mathfrak{g}_0)$-submodule generated by v is finite-dimensional. Then all irreducible objects in $\mathcal{C}_{\mathfrak{g}^+}$ are finite-dimensional, and the isomorphism classes of irreducible objects in $\mathcal{C}_{\mathfrak{g}^+}$ coincide with $\{L^0(\lambda) \mid \lambda \in \Lambda^+\}$ (see the forthcoming Lemma 6.5). The functor Γ is regarded as a functor from $\mathcal{C}_{\mathfrak{g}^+}$ to the $\mathbb{Z}$-graded $U(\mathfrak{g})$-module category. Furthermore, by the same arguments as in the proof of Proposition 5.5, $\Gamma(M)$ for any $M \in \mathcal{C}_{\mathfrak{g}^+}$, can be identified with $\bigwedge \mathfrak{g}_{-1} \otimes {}_{-\theta}M$ where the meaning of ${}_{-\theta}M$ are the same as in the paragraph around (5.1).

Note that $\Delta(\mu)$ belongs to $\mathcal{C}_{\mathfrak{g}^+}$, and is still an indecomposable $U(\mathfrak{g}^+)$-module. The irreducible $U(\mathfrak{g}^+)$-module $L^0(\lambda)$ is already known as a composition factor of $\Delta(\mu)$. Hence, there is a series of irreducible $U(\mathfrak{g}^+)$-modules $L^0(\lambda_i)$, $i = 0, 1, \ldots, s$ for $\lambda_i \in \Lambda^+$ such that $\lambda_0 = \lambda$ and $\lambda_s = \mu$ with $\mathrm{Ext}^1_{\mathcal{C}_{\mathfrak{g}^+}}(L^0(\lambda_{i-1}), L^0(\lambda_i)) \neq 0$ or $\mathrm{Ext}^1_{\mathcal{C}_{\mathfrak{g}^+}}(L^0(\lambda_i), L^0(\lambda_{i-1})) \neq 0$ for $i = 1, \ldots, s$ (see the forthcoming Lemma 6.6). Note that Γ is an exact functor. Under the former situation, for example, we claim that

$$\mathrm{Ext}^1_{U(\mathfrak{g})}(\nabla(\lambda_{i-1}), \nabla(\lambda_i)) \neq 0. \tag{6.2}$$

Actually, taking in $\mathcal{C}_{\mathfrak{g}^+}$ a non-split extension

$$0 \longrightarrow L^0(\lambda_i) \overset{\varphi}{\longrightarrow} N \overset{\psi}{\longrightarrow} L^0(\lambda_{i-1}) \longrightarrow 0, \tag{6.3}$$

one has a short exact sequence over $U(\mathfrak{g})$:

$$0 \longrightarrow \nabla(\lambda_i) \overset{\Gamma(\varphi)}{\longrightarrow} \Gamma(N) \overset{\Gamma(\psi)}{\longrightarrow} \nabla(\lambda_{i-1}) \longrightarrow 0.$$

If this one is split, i.e. there exists a $U(\mathfrak{g})$-module homomorphism $\pi : \nabla(\lambda_{i-1}) \longrightarrow \Gamma(N)$ such that $\Gamma(\psi) \circ \pi = \mathrm{id}_{\nabla(\lambda_{i-1})}$, then one in particular has $\Gamma(\psi) \circ \pi|_{1 \otimes L^0(\lambda_{i-1})} = \mathrm{id}_{1 \otimes L^0(\lambda_{i-1})}$. Notice that $\Gamma(\psi)^{-1}(1 \otimes L^0(\lambda_{i-1})) = 1 \otimes N$. Hence π maps $1 \otimes L^0(\lambda_{i-1})$ to $1 \otimes N$. This implies that the extension (6.3) is split, which contradicts to the assumption. The claim (6.2) is proven.

Hence as $U(\mathfrak{g})$-modules, the indecomposable module $\nabla(\lambda)$ must lie in the same block as the indecomposable module $\nabla(\mu)$ does.

Thanks to Lemma 6.1, it follows that $\lambda \sim \mu$. The proof is completed. $\square$

Lemma 6.5. *Let M be an irreducible module in the category $\mathcal{C}_{\mathfrak{g}^+}$ which is defined in the proof above. Then M is finite-dimensional, which is actually an irreducible $\mathfrak{g}_0$-module annihilated by $\mathfrak{g}_{\geq 1}$.*

Proof. At first, $U(\mathfrak{g}^+)$ has the $\mathbb{Z}$-graded structure defined by the $\mathbb{Z}$-gradation of $\mathfrak{g}^+$. That is, $U(\mathfrak{g}^+) = \bigoplus_{i \in \mathbb{Z}} U(\mathfrak{g}^+)_i$ with all $U(\mathfrak{g}^+)_i$ being $\mathfrak{g}_0$-modules. Furthermore, $U(\mathfrak{g}^+)_{\geq k} := \bigoplus_{i \geq k} U(\mathfrak{g}^+)_i$ is a regular $U(\mathfrak{g}^+)$-module.

For $M = \sum_{i \in \mathbb{Z}} M_i$, we suppose that M_0 is nonzero without loss of generality. Take a nonzero vector $v \in M_0$. By assumption, $V_0 = U(\mathfrak{g}_0)v$ is a finite-dimensional subspace in M_0. By the irreducibility of M, we have $M = \sum_{i \geq 0} V_i$ where $V_i = U(\mathfrak{g}^+)_i V_0$. Furthermore, set

$$M^{(k)} := \sum_{i \geq k} V_i.$$

Then all $M^{(k)}$ are $U(\mathfrak{g}^+)$-submodules of M and $M/M^{(1)}$ is finite-dimensional. The irreducibility of M yields that for any k, $M^{(k)}$ either coincides with M itself or equals to zero. Combining with the filtration $M = M^{(0)} \supset M^{(1)} \supset M^{(2)} \supset \cdots$ along with the fact that $\mathfrak{g}_{\geq 1}$ is nilpotent, we have that if $M = M^{(1)}$, by Nakayama Lemma $M = 0$. It's a contradiction. So it must happen that $M^{(1)} = 0$. Note that $M = M/M^{(1)}$ is finite-dimensional, which actually coincides with V_0. Consequently, M is irreducible over $\mathfrak{g}_0$, annihilated by $\mathfrak{g}_{\geq 1}$. $\square$

Lemma 6.6. *In $\mathcal{C}_{\mathfrak{g}^+}$, any two composition factors of $\Delta(\mu)$ lie in a connected Ext-quiver. This is to say, if $L^0(\lambda), L^0(\lambda')$ are two composition factors of $\Delta(\mu)$, then there are a series of different λ_i, $i = 0, 1, \ldots, s$ with $\lambda_0 = \lambda$ and $\lambda_s = \lambda'$ such that $Ext^1_{\mathcal{C}_{\mathfrak{g}^+}}(\lambda_{i-1}, \lambda_i) \neq 0$ or $Ext^1_{\mathcal{C}_{\mathfrak{g}^+}}(\lambda_i, \lambda_{i-1}) \neq 0$ for all $i = 1, \ldots, s$.*

Proof. We only need to show the lemma for the fixed $\lambda' = \mu$ because $\Delta(\mu)$ has a simple head isomorphic to $L^0(\mu)$ in $\mathcal{C}_{\mathfrak{g}^+}$. For this we write $\Delta(\mu) = \bigoplus_{i \geq 0} \Delta(\mu)_i$ which has a natural $\mathbb{Z}$-grading arising from the gradation of $\mathfrak{g}^+ = \sum_{i \geq 0} \mathfrak{g}_i$, furthermore $\Delta(\mu)$ admits a $U(\mathfrak{g}^+)$-module filtration $\{\Delta(\mu)^{(k)} := \bigoplus_{i \geq k} \Delta(\mu)_i \mid k \in \mathbb{Z}_{\geq 0}\}$.

By construction, there is $k \geq 1$ such that $L^0(\lambda)$ is a subquotient of $\Delta(\mu)^{(k)}$. We further suppose without loss of generality, that $L^0(\lambda) \cong M/N$ for $M, N \in \mathcal{C}_{\mathfrak{g}^+}$ satisfying $M, N \subseteq \Delta(\mu)^{(k)}$.

Consider $\overline{\Delta(\mu)} := \Delta(\mu)/\Delta(\mu)^{(k+1)}$ which is a finite-dimensional and indecomposable object in $\mathcal{C}_{\mathfrak{g}^+}$. Clearly $\overline{\Delta(\mu)}$ also has a head isomorphic to $L^0(\mu)$. Set $\phi : \Delta(\mu) \to \overline{\Delta(\mu)}$ to be the canonical surjective homomorphism in $\mathcal{C}_{\mathfrak{g}^+}$. Then $L^0(\lambda) \cong \overline{\phi^{-1}(\phi(M))}/\overline{\phi^{-1}(\phi(N))}$ which is still a composition factor of $\overline{\Delta(\mu)}$. Here $\phi^{-1}(\bullet)$ stands for the preimage in $\Delta(\mu)$ of $\bullet$.

By the same arguments as in the finite-dimensional module category, it can be shown that $L^0(\lambda)$ and $L^0(\mu)$ lie in a connected Ext-quiver in $\mathcal{C}_{\mathfrak{g}^+}$.

The proof is completed. $\square$

6.3. Recall $\Xi = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$. We have the following elementary observation, the proof of which follows directly from the forthcoming Lemma 9.7 in Appendix B.

Lemma 6.7. *Let $\mathfrak{g} = \bar{S}(n)$. Then the following statements hold.*

(1) *For $l \in \mathbb{C}$, we have $l\Xi \sim (l + \mathbb{Z})\Xi$.*

(2) *For $\lambda = b\Xi + c\epsilon_n, b \in \mathbb{C}, c \in \mathbb{Z}_{\leq 0}$, we have $\lambda \sim b\Xi$.*
(3) *For $\lambda = a\epsilon_1 + b\Xi, a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{C}$, we have $\lambda \sim b\Xi$.*

The following result is crucial for determining the blocks for the Lie superalgebra $\bar{S}(n)$ of special type.

Proposition 6.8. *Let $\mathfrak{g} = \bar{S}(n)$ and $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_n\epsilon_n \in \Lambda^+$. Then $\lambda \sim \lambda_1\Xi$, i.e., $L(\lambda)$ belongs to the same block as that $L(\lambda_1\Xi)$ lies in.*

Proof. We begin with the following Claim.

CLAIM: if there exists some i with $2 \leq i \leq n - 2$ such that $\lambda_i > \lambda_{i+1}$ and $\lambda_{n-1} \geq \lambda_n + 1$, then

$$\lambda \sim \lambda + \epsilon_{i+1}. \tag{6.4}$$

Indeed, for any j with $2 \leq j \leq n-1$, $\xi_1\xi_2\cdots\xi_j D_n \otimes v_\lambda^0$ is an $\mathfrak{n}^+$-maximal weight vector of weight $\lambda+\epsilon_1+\epsilon_2+\cdots+\epsilon_j-\epsilon_n$. By Lemma 6.4, we know that $L(\lambda)$ and $L(\lambda+\epsilon_1+\epsilon_2+\cdots+\epsilon_j-\epsilon_n)$ lie in the same block. i.e., $\lambda \sim \lambda + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_j - \epsilon_n$. In particular,

$$\lambda \sim \lambda + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} - \epsilon_n. \tag{6.5}$$

By the condition of the claim, $\lambda - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_i + \epsilon_n \in \Lambda^+$, so $\lambda \sim \lambda - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_i + \epsilon_n$ and $\lambda - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_i + \epsilon_n \sim \lambda + \epsilon_{i+1}$, it follows that $\lambda \sim \lambda + \epsilon_{i+1}$. The claim follows.

With the above claim, we carry on the proof by taking all possibilities of λ_1 into the arguments.

CASE 1: $\lambda_1 = \lambda_2$.

In this case, set $\mu = \lambda + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} - \epsilon_n = (\lambda_1 + 1)\epsilon_1 + (\lambda_2 + 1)\epsilon_2 + \cdots + (\lambda_{n-1} + 1)\epsilon_{n-1} + (\lambda_n - 1)\epsilon_n$. Then $\lambda \sim \mu$ by (6.5). Moreover, we can use (6.4) successively to obtain $\mu \sim (\lambda_1 + 1)(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1}) + (\lambda_n - 1)\epsilon_n$. Hence $\lambda \sim \mu \sim (\lambda_1 + 1)\Xi \sim \lambda_1\Xi$ by Lemma 6.7(1) and (2), as desired.

CASE 2: $\lambda_1 \neq \lambda_2$.

By using similar arguments as in CASE 1, without loss of generality, we can assume

$$\lambda_1 > \lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} \gg \lambda_n.$$

SUBCASE (I): $\lambda_1 - \lambda_2$ is even.

Recall that $\Delta(\lambda)$ contains a $\mathfrak{g}_0$-submodule $\bar{S}(n)_1 \otimes_{\mathbb{C}} L^0(\lambda)$, and $\bar{S}(n)_1 \cong L^0(\epsilon_1 + \epsilon_2 - \epsilon_n)$ as $\mathfrak{g}_0$-modules. Take $w_1 = (1n) \in \mathfrak{S}_n$ (the symmetric group on n letters), which is the Weyl group of $\mathfrak{g}_0$. Set

$$\nu_1 := \lambda + \frac{1}{2}(\lambda_1 - \lambda_2)w_1(\epsilon_1 + \epsilon_2 - \epsilon_n)$$

$$= \frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_1 + \frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_2 + \lambda_3\epsilon_3 + \cdots + \lambda_{n-1}\epsilon_{n-1} + \left(\lambda_n + \frac{1}{2}(\lambda_1 - \lambda_2)\right)\epsilon_n \in \Lambda^+.$$

It follows from [14, Theorem 2.10] and Lemma 6.4 that $\lambda \sim \nu_1$. Furthermore, $\nu_1 \sim \frac{1}{2}(\lambda_1 + \lambda_2)\Xi \sim \lambda_1\Xi$ by the claim in CASE 1 and Lemma 6.7(1). Consequently, $\lambda \sim \lambda_1\Xi$.

SUBCASE (II): $\lambda_1 - \lambda_2$ is odd.

In this case, take $w_2 = (13)(2n) \in \mathfrak{S}_n$. Set

$$\nu_2 := (\lambda + 2(\epsilon_1 + \epsilon_2 - \epsilon_n)) + w_2(\epsilon_1 + \epsilon_2 - \epsilon_n)$$

$$= (\lambda_1 + 2)\epsilon_1 + (\lambda_2 + 1)\epsilon_2 + (\lambda_3 + 1)\epsilon_3 + \lambda_4\epsilon_4 \cdots + \lambda_{n-1}\epsilon_{n-1} + (\lambda_n - 1)\epsilon_n \in \Lambda^+.$$

It follows from [14, Theorem 2.10] and Lemma 6.4 that $\lambda \sim \nu_2$. Now, $(\lambda_1 + 2) - (\lambda_2 + 1)$ is even in ν_2. The claim in SUBCASE (I) implies that $\nu_2 \sim (\lambda_1 + 2)\Xi \sim \lambda_1\Xi$. Hence, we also have $\lambda \sim \lambda_1\Xi$, as desired. We complete the proof. $\qquad\square$

6.4. Continue to investigate the standard modules. Denote by $\Delta(\lambda)_X$ the standard $X(n)$-module for $X \in \{W, \bar{S}, \bar{H}, \overline{CH}\}$. i.e. $\Delta(\lambda)_X = U(X(n)) \otimes_{U(P)} L^0(\lambda)$. Similarly, we can define $\mathcal{O}_X^{\min}$, $I(\lambda)_X$, $L(\lambda)_X$, $\mathbf{E}_X$ and $\Upsilon(\lambda)_X$. In this subsection, we establish some relation between standard modules for $\overline{CH}(n)$ and $\bar{H}(n)$. The following preliminary result is important for us.

Lemma 6.9. *Let* $\phi \in \mathrm{Hom}_{\bar{H}(n)}(\Delta(\lambda)_{\overline{CH}(n)}, \Delta(\lambda)_{\overline{CH}(n)})$ *with* $\phi^2 = \phi$. *If* $\phi|_{\Delta(\lambda)_{\bar{H}(n)}} = 0$, *then* $\phi = 0$.

Proof. Recall that $\overline{CH}(n) = \bar{H}(n) \oplus \mathbb{C}D_H(\xi_1 \cdots \xi_n)$. It suffices to show that

$$\phi((D_H(\xi_1 \cdots \xi_n))^k \otimes v) = 0, \ \forall\, k \in \mathbb{N}^+, v \in L^0(\lambda). \tag{6.6}$$

We use induction on k to show (6.6).

Since ϕ keeps the grading and weight spaces invariant, we can assume

$$\phi(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0) = cD_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0 + \sum_{i=1}^s u_i \otimes v_i,$$

where $c \in \mathbb{C}$, $u_i \in U(\bar{H}(n)_{\geq 1}), v_i \in L^0(\lambda), 1 \leq i \leq s$, and all $v_i's$ are linearly independent. On one hand,

$$\begin{aligned}
\phi^2(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0) &= \phi(cD_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0 + \sum_{i=1}^s u_i \otimes v_i) \\
&= c\phi(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0) \\
&= c^2 D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0 + \sum_{i=1}^s cu_i \otimes v_i.
\end{aligned}$$

On the other hand, we have

$$\phi^2(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0) = \phi(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0) = cD_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0 + \sum_{i=1}^s u_i \otimes v_i.$$

Hence, $c = 1$, or $c = 0$ and $\sum_{i=1}^s u_i \otimes v_i = 0$. We claim that the latter happens. Indeed, if $c = 1$, then for any $1 \leq j \leq n$, we have

$$\phi(D_j(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0)) = D_j\phi(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0) = (-1)^{j-1}D_H(\xi_1 \cdots \hat{\xi}_j \cdots \xi_n) \otimes v_\lambda^0 + \sum_{i=1}^s [D_j, u_i] \otimes v_i.$$

However,

$$\phi(D_j(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0)) = (-1)^{j-1}\phi(D_H(\xi_1 \cdots \hat{\xi}_j \cdots \xi_n) \otimes v_\lambda^0) = 0.$$

We get a contradiction. Hence, $c = 0$ and $\sum_{i=1}^{s} u_i \otimes v_i = 0$, i.e., $\phi(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0) = 0$. Since $L^0(\lambda) = U(\mathfrak{n}^-)v_\lambda^0$, it follows that

$$
\begin{aligned}
&\phi(D_H(\xi_1 \cdots \xi_n) \otimes L^0(\lambda)) \\
=&\phi(D_H(\xi_1 \cdots \xi_n)U(\mathfrak{n}^-) \otimes v_\lambda^0) \\
=&\phi(U(\mathfrak{n}^-)D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0) \\
=&U(\mathfrak{n}^-)\phi(D_H(\xi_1 \cdots \xi_n) \otimes v_\lambda^0) \\
=&0,
\end{aligned}
$$

i.e., (6.6) holds for $k = 1$.

Now suppose $\phi((D_H(\xi_1 \cdots \xi_n)^l \otimes L^0(\lambda)) = 0$ for $l < k$. We need to show that

$$
\phi((D_H(\xi_1 \cdots \xi_n)^k \otimes L^0(\lambda)) = 0.
$$

Since ϕ keeps the grading and weight spaces invariant, we can assume

$$
\phi((D_H(\xi_1 \cdots \xi_n))^k \otimes v_\lambda^0) = a(D_H(\xi_1 \cdots \xi_n))^k \otimes v_\lambda^0 + \sum_{i=1}^{t} w_i \otimes \nu_i,
$$

where $a \in \mathbb{C}$, $w_i \in U(\bar{H}(n)_{\geq 1})\mathcal{B}, \nu_i \in L^0(\lambda), 1 \leq i \leq t$, $\mathcal{B} = \mathrm{span}_{\mathbb{C}}\{(D_H(\xi_1 \cdots \xi_n))^i \mid 0 \leq i \leq k-1\}$, and all $\nu_i's, 1 \leq i \leq t$, are linearly independent. On one hand,

$$
\begin{aligned}
\phi^2((D_H(\xi_1 \cdots \xi_n))^k \otimes v_\lambda^0) &= \phi(a(D_H(\xi_1 \cdots \xi_n))^k \otimes v_\lambda^0 + \sum_{i=1}^{t} w_i \otimes \nu_i) \\
&= a\phi((D_H(\xi_1 \cdots \xi_n))^k \otimes v_\lambda^0) \\
&= a^2(D_H(\xi_1 \cdots \xi_n))^k \otimes v_\lambda^0 + \sum_{i=1}^{t} a w_i \otimes \nu_i.
\end{aligned}
$$

On the other hand, we have

$$
\phi^2((D_H(\xi_1 \cdots \xi_n))^k \otimes v_\lambda^0) = \phi((D_H(\xi_1 \cdots \xi_n))^k \otimes v_\lambda^0) = a(D_H(\xi_1 \cdots \xi_n))^k \otimes v_\lambda^0 + \sum_{i=1}^{t} w_i \otimes \nu_i.
$$

Similar arguments as in the case $k = 1$ yield that $a = 0$ and $\sum_{i=1}^{t} w_i \otimes \nu_i = 0$, and furthermore $\phi((D_H(\xi_1 \cdots \xi_n))^k \otimes v) = 0$, i.e., (6.6) holds for k. Consequently, $\phi = 0$, as desired. $\qquad\square$

As a consequence of Lemma 6.9, we have the following result.

Corollary 6.10. *As an $\bar{H}(n)$-module, $\Delta(\lambda)_{\overline{CH}(n)}$ is indecomposable.*

Proof. Let f be an element of $\mathrm{Hom}_{\bar{H}(n)}(\Delta(\lambda)_{\overline{CH}(n)}, \Delta(\lambda)_{\overline{CH}(n)})$ with $f^2 = f$. Then $f|_{\Delta(\lambda)_{\bar{H}(n)}} = 0$ or 1 due to the indecomposability of $\Delta(\lambda)_{\bar{H}(n)}$ as an $\bar{H}(n)$-module. If $f|_{\Delta(\lambda)_{\bar{H}(n)}} = 0$, then $f = 0$ by Lemma 6.9. If $f|_{\Delta(\lambda)_{\bar{H}(n)}} = 1$, then $(\mathbf{id} - f)|_{\Delta(\lambda)_{\bar{H}(n)}} = 0$ and $(\mathbf{id} - f)^2 = \mathbf{id} - 2f + f^2 = (\mathbf{id} - f)$. It also follows from Lemma 6.9 that $(\mathbf{id} - f)|_{\Delta(\lambda)_{\overline{CH}(n)}} = 0$, i.e., $f = \mathbf{id}$. This implies

that 0 and **id** are the only two idempotents in $\mathrm{Hom}_{\bar{H}(n)}(\Delta(\lambda)_{\overline{CH}(n)}, \Delta(\lambda)_{\overline{CH}(n)})$. Then it follows from [1, Proposition 5.10] that $\Delta(\lambda)_{\overline{CH}(n)}$ is an indecomposable $\bar{H}(n)$-module. $\square$

6.5. Revisit to $I(\lambda)$.

Proposition 6.11. *Let* $\mathfrak{g} = W(n), \bar{S}(n), \bar{H}(n), \overline{CH}(n)$. *Then all composition factors in* $I(\lambda)$ *lie in the same block.*

Proof. Note that $I(\lambda) = \bigoplus_{\mu \in \Upsilon(\lambda)} P(\mu)^{\oplus a_{\lambda\mu}}$, here $a_{\lambda\mu} \in \mathbb{Z}_{>0}$. Each $P(\mu)$ $(\mu \in \Upsilon(\lambda))$ is actually the projective cover of both $L(\mu)$ and $\Delta(\mu)$. In order to prove the proposition, it suffices by the definition of blocks to prove

$$\mu \sim \lambda, \ \forall \mu \in \Upsilon(\lambda). \tag{6.7}$$

In the following we will prove the proposition for the case of $W(n)$ by verifying the formula (6.7) (consequently, the case of $\bar{S}(n)$ is easily solved). For the case $\overline{CH}(n)$, we will prove the proposition by partially verifing the formula (6.7) and accomplishing the remaining cases by using Corollary 6.10. So the arguments will be divided into cases.

(i) Assume $\mathfrak{g} = W(n)$. Take $\mu \in \Upsilon(\lambda)$. For $\Delta(\mu) = U(\mathfrak{g}) \otimes_{U(\mathsf{P})} L^0(\mu)$, keeping the notations in Lemma 3.6, we see that $\Delta(\mu)$ contains a $\mathfrak{g}_0$-submodule $M^+(\mu)$ (see Remark 3.8). From Lemma 3.6(3), there is a $\mathfrak{g}_0$-maximal vector m_λ in $M^+(\mu)$, i.e. $\mathfrak{n}^+ m_\lambda = 0$, $H m_\lambda = \lambda(H) m_\lambda$ for any $H \in \bar{\mathfrak{h}}$. By Lemma 6.4 we know $\mu \sim \lambda$, as desired.

(ii) Assume $\mathfrak{g} = \bar{S}(n)$. For any $\mu \in \Upsilon(\lambda)$, it follows from Lemma 6.7, Proposition 6.8 and Remar 4.3(3) that $\mu \sim \lambda$. Hence, the assertion for $\mathfrak{g} = \bar{S}(n)$ is proven.

(iii) Assume $\mathfrak{g} = \overline{CH}(n)$. By the definition of $\Upsilon(\lambda)$ (see (4.7)), we can write $\Upsilon(\lambda) = \bigcup_{i=0}^n \Upsilon_i(\lambda)$ with

$$\Upsilon_i(\lambda) = \{\mu \in \Upsilon(\lambda) \mid (\textstyle\bigwedge^i \mathfrak{g}_{-1} \otimes_{\mathbb{C}} L^0(\lambda) : L^0(\mu))_{\mathfrak{g}_0} \neq 0\}. \tag{6.8}$$

By the same arguments as (i), it follows from Lemmas 3.7 and 6.4 that for $\mu \in \Upsilon(\lambda)$:

$$\mu \sim \lambda - 2\delta \text{ if } \mu \in \bigcup_{i \geq 3} \Upsilon_i(\lambda); \tag{6.9}$$

$$\mu - (n-2)\delta \sim \lambda - 2\delta \text{ if } \mu \in \Upsilon_1(\lambda); \tag{6.10}$$

$$\mu - (n-4)\delta \sim \lambda - 2\delta \text{ if } \mu \in \Upsilon_2(\lambda). \tag{6.11}$$

As all standard modules are indecomposable, the above formula (6.9) implies that all $\Delta(\mu)$ for $\mu \in \bigcup_{i \geq 3} \Upsilon_i(\lambda)$ lie in the same block as $L(\lambda - 2\delta)$ does. Especially, by Remark 4.3(3)(4°), We have the following result:

$$L(\lambda + \epsilon_1 + \cdots + \epsilon_k - (n-k)\delta) \text{ and } L(\lambda - 2\delta) \text{ lie in the same block.}$$

Furthermore, we divide the following arguments into two different cases.

(Case 1) For $\mathfrak{g} = \overline{CH}(n)$ with $n = 2r$. In this case, $r \geq 3$ by the assumption that $n \geq 5$.

Claim 1: $L(\lambda)$ and $L(\lambda + \sum_{i=1}^r \epsilon_i + r\delta)$ lie in the same block. Let $k = r$ and $k = 0$, by the above result we know that both $L(\lambda + \sum_{i=1}^r \epsilon_i - r\delta)$ and $L(\lambda - n\delta)$ lie in the same block. Due to the arbitrariness of λ, one can change λ to $\lambda - n\delta$, then the claim follows.

Claim 2: $L(\lambda)$ and $L(\lambda + \sum_{i=1}^r \epsilon_i + (r-2)\delta)$ share one block. This claim can be checked by the fact that $D_H(\Pi_{i=1}^r \xi_i) \otimes v_\lambda^0$ is a $\mathfrak{g}_0$-maximal vector and Lemma 6.4.

By the arbitrariness of λ (or by translating λ to $\lambda - (\sum_{i=1}^{r} \epsilon_i + (r-2)\delta)$ in the previous claims), we have that $L(\lambda)$ and $L(\lambda \pm 2\delta)$ lie in the same block. Furthermore, we see that $L(\lambda)$, $L(\lambda \pm 2\delta)$ and $L(\lambda \pm n\delta)$ lie in the same block.

(Case 2) For $\mathfrak{g} = \overline{CH}(n)$ with $n = 2r + 1$, by a direct verification, the standard module $\Delta(\lambda)$ admits $\mathfrak{g}_0$-maximal vectors $D_H(\prod_{i=1}^{r} \xi_i) \otimes v_\lambda^0$ and $D_H((\prod_{i=1}^{r} \xi_i)\xi_{2r+1}) \otimes v_\lambda^0$. Hence by Lemma 6.4 we get $L(\lambda)$, $L(\lambda + \sum_{i=1}^{r} \epsilon_i + (r-2)\delta)$ and $L(\lambda + \sum_{i=1}^{r} \epsilon_i + (r-1)\delta)$ share the same block. By the arbitrariness of λ again (or by translating λ to $\lambda - (\sum_{i=1}^{r} \epsilon_i) + (2-r)\delta$ in the above), we have that in this case, $L(\lambda)$ and $L(\lambda \pm \delta)$ lie in the same block. Consequently, $L(\lambda)$, $L(\lambda \pm 2\delta)$ and $L(\lambda \pm n\delta)$ share the same block.

With the above arguments, we can directly deduce that not only for $\mu \in \Upsilon_{\geq 3}(\lambda)$ but also for $\mu \in \Upsilon_1(\lambda) \cup \Upsilon_2(\lambda)$, all $L(\mu)$ lie in the same block as $L(\lambda)$ does. Hence we indeed prove that all composition factors in $\Delta(\mu)$ for $\mu \in \Upsilon(\lambda)$, thereby all composition factors in $I(\lambda)$, lie in the same block. We have proven the proposition in this case.

(iv) Assume $\mathfrak{g} = \bar{H}(n)$. Recall that $I(\lambda)_{\bar{H}(n)}$ has a $\Delta(\mu)_{\bar{H}(n)}$-filtration and $L(\mu)_{\bar{H}(n)}$ is the head of $\Delta(\mu)_{\bar{H}(n)}$. By Lemma 6.1 and the indecomposability of $\Delta(\mu)_{\bar{H}(n)}$, we need to show that all $L(\mu)_{\bar{H}(n)}, \mu \in \Upsilon(\lambda)_{\bar{H}(n)}$, belong to the same block.

Recall that both $\bar{H}(n)$ and $\overline{CH}(n)$ have the same 0-graded spaces $\mathfrak{so}(n) \oplus \mathbb{C}\mathsf{d}$. The parameters of isomorphism classes of irreducible modules for $\mathcal{O}^{\min}{}_{\overline{CH}(n)}$ and for $\mathcal{O}^{\min}{}_{\bar{H}(n)}$ are the same, arising from $\{L^0(\lambda) \mid \lambda \in \Lambda^+\}$ for $\mathfrak{g}_0$. Since $\bar{H}(n)$ and $\overline{CH}(n)$ has the same -1-graded spaces, $\Upsilon(\lambda)_{\bar{H}(n)} = \Upsilon(\lambda)_{\overline{CH}(n)}$.

By the arguments in (iii), we have known that all $\Delta(\mu)_{\overline{CH}(n)}, \mu \in \Upsilon(\lambda)_{\bar{H}(n)}$, lie in the same block in $\mathcal{O}^{\min}{}_{\overline{CH}(n)}$ as $L(\lambda)_{\overline{CH}(n)}$ does. Hence all $\bar{H}(n)$-modules $\Delta(\mu)_{\overline{CH}(n)}, \mu \in \Upsilon(\lambda)_{\overline{CH}(n)}$, lie in the same block of $\mathcal{O}^{\min}{}_{\bar{H}(n)}$. Because $\Delta(\mu)_{\overline{CH}(n)}$ admits an $\bar{H}(n)$-irreducible quotient $L(\mu)_{\bar{H}(n)}$, By Lemma 6.1 and Corollary 6.10, all $L(\mu)_{\bar{H}(n)}, \mu \in \Upsilon(\lambda)_{\bar{H}(n)}$, belong to the same block. So the desired result follows.

Summing up, we finish the proof. $\qquad\square$

When $\mathfrak{g} = \bar{H}(2r)$, set $\aleph_r \in \{0, 1\}$ satisfying $\aleph_r \equiv r \mod 2$ for $\mathfrak{g} = \bar{H}(2r)$. Then we put forward some additional new notations

$$\begin{cases} \Theta_B := \epsilon_1 + \cdots + \epsilon_{r-1} + \epsilon_r, & \text{for } H(2r+1). \\ \Theta_{D,\aleph_r} := \epsilon_1 + \cdots + \epsilon_{r-1} + \epsilon_r + \aleph_r \delta, & \text{for } H(2r). \end{cases}$$

(along with already appointing $\Xi := \sum_{i=1}^{n} \epsilon_i$, for $\mathfrak{g} = X(n)$, with $X \in \{W, \bar{S}\}$).

Set

$$\tilde{\delta} := \begin{cases} \delta, & \text{for } H(2r+1); \\ 2\delta, & \text{for } H(2r). \end{cases}$$

Then we have the following corollary.

Corollary 6.12. *For any $L(\lambda) \in \mathbf{E}$, the following statements hold.*

(1) *When $\mathfrak{g} = X(n)$ with $X \in \{W, \bar{S}, \bar{H}\}$. If there exists $-\beta \in \mathsf{Wt}(\bigwedge(\mathfrak{g}_{-1}))$ such that $\lambda - \beta \in \Upsilon(\lambda)$, then $L(\lambda)$ and $L(\lambda - \beta)$ share the same block.*

(2) *When $\mathfrak{g} = X(n)$ with $X \in \{W, \bar{S}\}$, then $L(\lambda)$ and $L(\lambda + \sum_{i=k}^{n} \epsilon_i)$, $1 \leq k \leq n$, lie in the same block. In particular, $L(\lambda)$ and $L(\lambda - \Xi)$ lie in the same block.*

(3) *When $\mathfrak{g} = \bar{H}(n)$, then $L(\lambda)$ and $L(\lambda - \tilde{\delta})$ lie in the same block.*

(4) *When $\mathfrak{g} = \bar{H}(2r+1)$, then $L(\lambda)$ and $L(\lambda + \sum_{i=1}^{k} \epsilon_i - (n-k)\delta)$ lie in the same block. In particular, $L(\lambda)$ and $L(\lambda + \sum_{i=1}^{r} \epsilon_i)$ lie in the same block.*

(5) *When $\mathfrak{g} = \bar{H}(2r)$, then $L(\lambda)$ and $L(\lambda - \Theta_{D,\aleph_r})$ lie in the same block ($\aleph_r \in \{0,1\}$ with $\aleph_r \equiv r \mod 2$).*

Proof. (1) This is a direct consequence of the proof for Proposition 6.11.

(2) This is the consequence of Remark 4.3(3)(3°) and the result in (1).

(3) When $n = 2r$ (*resp.* $n = 2r + 1$), one can check it by the same arguments as the process for proof of Proposition 6.11(iii) for case 1(resp. case 2).

(4)-(5) By Remark 4.3 (3)(4°), $I(\lambda)$ admits the composition factor $L(\lambda + \sum_{i=1}^{k} \epsilon_i - (n-k)\delta)$. So $L(\lambda)$ and $L(\lambda + \sum_{i=1}^{k} \epsilon_i - (n-k)\delta)$ lie in the same block due to Proposition 6.11. Thanks to (3), we further have that $L(\lambda)$ and $L(\lambda + \sum_{i=1}^{r} \epsilon_i)$ lie in the same block for $H(2r + 1)$ and $H(2r)$ with even r. Similarly, for $H(2r)$ with odd r, we can check that $L(\lambda)$ and $L(\lambda + \sum_{i=1}^{r} \epsilon_i + \delta)$ lie in the same block. $\square$

6.6. **Depth Lemma and parity Lemma.** We will analyse the relation of depths for simple objects in a block. Suppose that $L(\lambda)$ is given, and $\mathsf{dpt}(L(\lambda)) = d$. Then by the construction of $P(\lambda)$ (see Remark 4.3(1)), the depth of each composition factor is consequently determined. Conversely, for any given composition factor $L(\mu') = L(\mu')_{d'}$ in $P(\lambda')$, the depth of $P(\lambda')$ (thereby the depth of $L(\lambda')$) is definitely determined by the predefined depth of $L(\mu')$. From this fact and the definition of blocks we can easily have the following depth lemma. We firstly introduce some new notations before the following lemma. Let $\mu = \mu_1\epsilon_1 + \mu_2\epsilon_2 + \cdots + \mu_n\epsilon_n$ be an element of $\bar{\mathfrak{h}}^*$ for $\mathfrak{g} = X(n)$ with $X \in \{W, \bar{S}\}$, and $\mu = \mu_1\epsilon_1 + \mu_2\epsilon_2 + \cdots + \mu_r\epsilon_r + c\delta$ for $\mathfrak{g} = \bar{H}(n)$. We define the length of μ, which is denoted by $\ell(\mu)$, as below

$$\ell(\mu) = \begin{cases} \sum_{i=1}^{n} \mu_i, & \text{for } \mathfrak{g} = X(n), \text{ with } X \in \{W, \bar{S}\}; \\ c & \text{for } \mathfrak{g} = \bar{H}(n). \end{cases} \tag{6.12}$$

Obviously,

$$\ell(\lambda \pm \mu) = \ell(\lambda) \pm \ell(\mu).$$

Lemma 6.13. (DEPTH LEMMA)

(1) *If $L(\mu)$ and $L(\nu)$ are in the same block, then $\mathsf{dpt}(L(\mu)) - \mathsf{dpt}(L(\nu)) = \ell(\mu - \nu)$.*

(2) *For any $\lambda \in \Lambda^+$, and different $d_1, d_2 \in \mathbb{Z}$, $L(\lambda)_{d_1}$ and $L(\lambda)_{d_2}$ do not lie in the same block.*

Proof. (1) The proof is divided into the following steps.

CLAIM I: If $(\Delta(\lambda) : L(\mu)) \neq 0$ and $(\Delta(\lambda) : L(\nu)) \neq 0$, then

$$\mathsf{dpt}(L(\mu)) - \mathsf{dpt}(L(\nu)) = \ell(\mu - \nu). \tag{6.13}$$

Set $\lfloor L^0(\lambda) \rfloor = d$. So $\Delta(\lambda), L(\lambda)$ are all of depth d. Recall that $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(P)} L^0(\lambda) \cong U(\mathfrak{g}_{\geq 1}) \otimes_{\mathbb{C}} L^0(\lambda)$ as a vector space. So if $v \in U(\mathfrak{g}_{\geq 1})_i \otimes_{\mathbb{C}} L^0(\lambda)$ is a homogeneous element of $\Delta(\lambda)$, then $\text{degree}(v) = d + i$.

Now let $L(\mu)$ be an irreducible $U(\mathfrak{g})$-module with $(\Delta(\lambda) : L(\mu)) \neq 0$. Then there exists an inclusion of submodule $\Delta(\lambda) \supseteq M \supseteq N \supseteq 0$ such that $M/N \cong L(\mu)$. Let $v_\mu \in M/N$ be a maximal vector of $L(\mu)$. If $v_\mu \in U(\mathfrak{g}_{\geq 1})_i \otimes_{\mathbb{C}} L^0(\lambda)$, then

$$\mathsf{dpt}(L(\mu)) = i + d = \mathsf{dpt}(L(\lambda)) + \ell(\mu - \lambda). \tag{6.14}$$

Similarly, we have

$$\mathsf{dpt}(L(\nu)) = \mathsf{dpt}(L(\lambda)) + \ell(\nu - \lambda). \tag{6.15}$$

Consequently, the equality (6.13) holds due to $\ell(\mu - \lambda) - \ell(\nu - \lambda) = \ell(\mu - \nu)$. The first claim is proven.

CLAIM II: If $(P(\lambda) : L(\mu)) \neq 0$ and $(P(\lambda) : L(\nu)) \neq 0$, then $\mathsf{dpt}(L(\mu)) - \mathsf{dpt}(L(\nu)) = \ell(\mu - \nu)$.

Since $P(\lambda)$ is a direct summand of $I(\lambda)$ (see Theorem 4.2), it suffices to prove this claim for $I(\lambda)$, i.e. If $(I(\lambda) : L(\mu)) \neq 0$ and $(I(\lambda) : L(\nu)) \neq 0$, then $\mathsf{dpt}(L(\mu)) - \mathsf{dpt}(L(\nu)) = \ell(\mu - \nu)$. Set $\mathsf{dpt}(L(\lambda)) = d$. Assume that

$$I(\lambda) = M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots \supseteq M_l \supseteq 0$$

is the descending sequence such that $M_i/M_{i+1} \cong \Delta(\lambda_i)$ shown in Theorem 4.2. Then we have that $\ell(\Delta(\lambda_i)) = d + \ell(\lambda_i - \lambda)$. Denote by $s = \max\{i \mid L(\mu) \text{ is a subquotient of } M_i\}$, $t = \max\{j \mid L(\nu) \text{ is a subquotient of } M_j\}$.

If $s = t$, then there exists the following down sequence

$$M_s \supseteq N_1 \supseteq N_2 \supseteq M_{s+1}$$

such that $N_1/N_2 \cong L(\mu)$. because

$$N_1/N_2 \hookrightarrow M_s/N_2 \cong M_s/M_{s+1}\big/N_2/M_{s+1} \cong \Delta(\lambda_s)\big/N_2/M_{s+1},$$

$L(\mu)$ can be realized as a sub-quotient of $\Delta(\lambda_s)_{d+\ell(\lambda_s-\lambda)}$. Meanwhile, $L(\nu)$ can be also realized as a sub-quotient of $\Delta(\lambda_s)_{d+\ell(\lambda_s-\lambda)}$. Thus $L(\mu)$ and $L(\nu)$ are two sub-quotients of $\Delta(\lambda_s)_{d+\ell(\lambda_s-\lambda)}$. Then CLAIM I implies CLAIM II.

If $s \neq t$, assume $s < t$ without loss of generality. Then by the above discuss, $L(\mu)$ (resp. $L(\nu)$) is a sub-quotient of $\Delta(\lambda_s)_{d+\ell(\lambda_s-\lambda)}$ (resp. $\Delta(\lambda_t)_{d+\ell(\lambda_t-\lambda)}$). So by the equality (6.14) we have

$$\mathsf{dpt}(L(\mu)) = d + \ell(\lambda_s - \lambda) + \ell(\mu - \lambda_s) \tag{6.16}$$

$$\mathsf{dpt}(L(\nu)) = d + \ell(\lambda_t - \lambda) + \ell(\nu - \lambda_t) \tag{6.17}$$

Then the desired assertion follows from (6.16)-(6.17).

Now the statement (1) of the theorem holds due to the definition of blocks in Subsection 6.1.

For (2), this is a direct consequence of (1). $\qquad\square$

Because $L(\lambda)$ (resp. $\Delta(\lambda)$, $P(\lambda)$) is generated by v_λ^0, which is a maximal vector of $L^0(\lambda)$, the super structure of $L(\lambda)$ (resp. $\Delta(\lambda)$, $P(\lambda)$) is completely determined by the predefined parity $|v_\lambda^0|$ of v_λ^0. By abuse of the notions and notations with the context being clear, we say that $L(\lambda)$ is of parity $|v_\lambda^0|$, denote $\mathsf{pty}(L(\lambda)) := |v_\lambda^0|$, or write $L(\lambda) = L(\lambda)^\iota$ for $\iota = |v_\lambda^0|$. Meanwhile, we have the following parity Lemma.

Lemma 6.14. *(PARITY LEMMA) Keep the notations as above. The following statements hold.*

(1) *If $L(\mu)$ and $L(\nu)$ are in the same block, then $|v_\mu^0| - |v_\nu^0| = \overline{\ell(\mu - \nu)}$ where $\overline{\ell(\mu - \nu)} \in \mathbb{Z}_2$ denotes the parity of $\ell(\mu - \nu)$.*

(2) *For any $\lambda \in \Lambda^+$, and different parities $\iota_1, \iota_2 \in \mathbb{Z}_2$, $L(\lambda)^{\iota_1}$ and $L(\lambda)^{\iota_2}$ do not lie in the same block.*

Proof. By arguments similar to the proof of Lemma 6.13, the lemma is readily justified. □

6.7. **Blocks of $\mathcal{O}^{\mathbf{min}}$ for $\mathfrak{g} = W(n)$ or $\bar{S}(n)$.** In this subsection, we focus our concern on $W(n)$ and $\bar{S}(n)$. Recall the notation $\Xi = \sum_{i=1}^{n} \epsilon_i$. Let $\lambda = \lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \cdots + \lambda_n \epsilon_n$ be an element of Λ^+. Write λ in the following form

$$\lambda = \lambda_n \Xi + (\lambda_1 - \lambda_n)\epsilon_1 + \cdots + (\lambda_{n-1} - \lambda_n)\epsilon_{n-1}$$

$$= \lambda_n(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n) + \alpha, \text{ with } \lambda_n \in \mathbb{C}, \ \alpha \in Q^+ := (\sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\epsilon_i) \cap \Lambda^+.$$

(6.18)

Denote by Q the root lattice of $\mathfrak{g}$ with respect to the root system $\Phi(\mathfrak{g})$ (see §1.3). Then set

$$\mathcal{O}^{\min}(c, i) = \{L(\lambda) \in \mathbf{E} \mid \lambda \in (c + \mathbb{Z})\Xi + Q, \ \mathsf{dpt}(L(c\Xi)) = i;$$

$$\mathsf{dpt}(L(\lambda)) = i + \ell(\lambda - c\Xi)\}.$$

It further splits into

$$\mathcal{O}^{\min}(c, i) = \mathcal{O}^{\min}(c, \bar{0}, i) \cup \mathcal{O}^{\min}(c, \bar{1}, i)$$

where

$$\mathcal{O}^{\min}(c, \iota, i) = \{L(\lambda) \in \mathcal{O}^{\min}(c, i) \mid \mathsf{pty}(L(c\Xi)) = \iota;$$

$$\mathsf{pty}(L(\lambda)) = \iota + \overline{\ell(\lambda - c\Xi)}\}$$

for $\iota \in \mathbb{Z}_2$. Here $\overline{\ell(\lambda - c\Xi)} \in \mathbb{Z}_2$ denotes the parity of $\ell(\lambda - c\Xi)$.

Let $\mu = \mu_1\epsilon_1 + \mu_2\epsilon_2 + \cdots + \mu_n\epsilon_n$ be an element of $\bar{\mathfrak{h}}^*$ for $\mathfrak{g} = X(n)$ with $X \in \{W, \bar{S}\}$. We define the height of μ, which is denoted by $\mathrm{ht}(\mu)$, as $\mathrm{ht}(\mu) = \sum_{i=1}^{n} \mu_i$.

Theorem 6.15. *Assume that $\mathfrak{g} = X(n)$ with $X \in \{W, \bar{S}\}$. The complete set of all different blocks in $\mathcal{O}^{min}$ is described as follows*

$$\{\mathcal{O}^{min}(c, \iota, i) \mid (c, \iota, i) \in \mathbb{C}/\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}\}.$$

Proof. Firstly, we will prove that simple objects belonging to $\mathcal{O}^{\min}(c, \iota, i)$ are indeed in the same block.

For any given $L(\lambda) \in \mathcal{O}^{\min}(c, \iota, i)$, naturally $\lambda \in \Lambda^+$. By (6.18) and Corollary 6.12(2), we can write $\lambda = c\Xi + \alpha$ for some $\alpha \in Q^+$ without loss of generality. We will prove that $L(\lambda)$ lies in the block where $L(c\Xi)$ lies by induction on $\mathrm{ht}(\alpha)$.

When $\mathrm{ht}(\alpha) = 0$, then $\alpha = 0$ because $\alpha \in Q^+$. So the conclusion is true.

When $\mathrm{ht}(\alpha) > 0$, suppose that the conclusion has been true for the situation of being less than $\mathrm{ht}(\alpha)$. Assume $\alpha = \sum_{i=1}^{n-1} a_i\epsilon_i$ with $a_i \in \mathbb{Z}_{\geq 0}$. Then there exists $1 \leq t \leq n-1$ such that $a_t > a_{t+1}$, where we make convention that $a_n = 0$. Take $\beta = \sum_{k=t+1}^{n} \epsilon_k$. Consider $\lambda' := \lambda + \beta$ which lies in Λ^+. By Corollary 6.12(2), $L(\lambda')$ and $L(\lambda' - \beta) = L(\lambda)$ lie in the same block. Note that $\alpha + \beta = \sum_{i=1}^{t} a_i\epsilon_i + \sum_{k=t+1}^{n}(a_k + 1)\epsilon_i = \Xi + \gamma$, where $\gamma = \alpha - \sum_{i=1}^{t} \epsilon_i \in Q^+$. So we have $\lambda' = (c+1)\Xi + \gamma$. By Corollary 6.12(2), $L(\lambda')$ and $L(c\Xi + \gamma)$ lie in the same block. Furthermore, $\mathrm{ht}(\gamma) < \mathrm{ht}(\alpha)$. Thus, by inductive hypothesis, $L(c\Xi + \gamma)$ and $L(c\Xi)$ already lie in the same block. Hence, $L(\lambda)$ and $L(c\Xi)$ finally turn out to lie in the same block.

Secondly, for any $L(\lambda) \in \mathbf{E}$, we see that $L(\lambda) \in \mathcal{O}^{\min}(c, \iota, i)$ for some $c \in \mathbb{C}, \iota \in \mathbb{Z}_2, i \in \mathbb{Z}$ by (6.18). Moreover, we will prove that if a simple object $L(\mu)$ lies in the block where $L(c\Xi)_i^\gamma$ lies, then $L(\mu)$ must lie in $\mathcal{O}^{\min}(c, \iota, i)$. For this, we only need to note the following two facts:

(i) For any indecomposable projective module $P(\lambda)$ with $\lambda = c\Xi + \alpha$ for $c \in \mathbb{C}$ and $\alpha \in Q$, and its composition factor $L(\mu)$, by Remark 4.3 we have $\mu - \lambda \in \sum_{i=1}^n \mathbb{Z}\epsilon_i$, thereby $\mu \in c\Xi + Q$.

(ii) If $L(\mu')$ is a composition factor of $P(\lambda')$ and $\mu' \in c\Xi + Q$. By Theorem 4.2, $P(\lambda')$ is a direct summand of $I(\lambda')$ and $L(\mu')$ is a composition factor of $I(\lambda')$. Hence $\lambda' \in c\Xi + Q$.

Thus, by the definition of blocks and taking Depth Lemma and Parity Lemma into account, we have proven that if a simple object $L(\mu)$ lies in the block where $L(c\Xi)_i^\iota$ lies, then $L(\mu)$ must lie in $\mathcal{O}^{\min}(c, \iota, i)$. The proof is completed. $\qquad\square$

6.8. **Blocks of $\mathcal{O}^{\min}$ for $\mathfrak{g} = \bar{H}(n)$.** In this case, $n = 2r$ or $n = 2r + 1$. Recall that the notation $\aleph_r \in \{0, 1\}$ satisfies $\aleph_r \equiv r \mod 2$ for $H(2r)$. And recall that there is a standard dual δ of d in $\bar{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}\mathsf{d}$. Let $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_r\epsilon_r + c\delta$ be an element of $\bar{\mathfrak{h}}^*$, We define the height of λ, which is denoted by $\mathrm{ht}(\lambda)$, as $\mathrm{ht}(\lambda) = \sum_{i=1}^r \lambda_i$.

Recall the notations $\Theta_{D,\aleph_r} = \epsilon_1 + \cdots + \epsilon_{r-1} + \epsilon_r + \aleph_r\delta$ for $\bar{H}(2r)$, and $\Theta_B = \epsilon_1 + \cdots + \epsilon_{r-1} + \epsilon_r$ for $\bar{H}(2r + 1)$. For $\lambda \in \Lambda^+ \subseteq \bar{\mathfrak{h}}^*$, it can be further presented as

$$
\lambda = \begin{cases} \lambda_r\Theta_{D,\aleph_r} + c\delta + \sum_{i=1}^{r-1}(\lambda_i - \lambda_r)\epsilon_i, & \text{if } n = 2r; \\ \lambda_r\Theta_B + c\delta + \sum_{i=1}^{r-1}(\lambda_i - \lambda_r)\epsilon_i, & \text{if } n = 2r + 1 \end{cases} \tag{6.19}
$$

satisfying that $\sum_{i=1}^{r-1}(\lambda_i - \lambda_r)\epsilon_i \in \sum_{i=1}^{r-1}\mathbb{Z}_{\geq 0}\epsilon_i \cap \Lambda^+$ for both $\bar{H}(2r)$ and $\bar{H}(2r + 1)$. So for $\lambda \in \Lambda^+$, by (6.19) we can write

$$
\lambda = c\delta + d\Theta + \alpha \text{ with } c, d \in \mathbb{C}, \ \alpha = \gamma + \mathrm{ht}(\gamma)\delta \tag{6.20}
$$

where $\gamma \in Q^+ := \sum_{i=1}^{r-1}\mathbb{Z}_{\geq 0}\epsilon_i \cap \Lambda^+$ and $\gamma + \mathrm{ht}(\gamma)\delta \in Q := \mathbb{Z}\Phi(\mathfrak{g})$, here $\mathbb{Z}\Phi(\mathfrak{g})$ denotes the root lattice of $\mathfrak{g}$.

In the following, *we will simply write $\Theta = \Theta_B$ or $\Theta_{D,\aleph_r}$ according to the situation $n = 2r+1$ or $n = 2r$ respectively.*

Lemma 6.16. (INDEPENDENCE LEMMA) *Let $\mathfrak{g} = \bar{H}(n)$ and $\lambda \in \Lambda^+$. Then the expression of λ in (6.20) is unique.*

Proof. Suppose $\lambda = c_i\delta + d_i\Theta + \alpha_i$, $i = 1, 2$. We need to prove that $c_1 = c_2$, $d_1 = d_2$ and $\alpha_1 = \alpha_2$. We know $d_1 = d_2 = \lambda_r$. So we have $\lambda - \lambda_r\Theta = c_1\delta + \alpha_1 = c_2\delta + \alpha_2$, Hence $(c_1 - c_2)\delta + (\alpha_1 - \alpha_2) = 0$. According to (6.20), assume that $\alpha_i = \gamma_i + \mathrm{ht}(\gamma_i)\delta$, $i = 1, 2$ with $\gamma_i \in Q^+$. Then $(c_1 + \mathrm{ht}(\gamma_1) - c_2 - \mathrm{ht}(\gamma_2))\delta = \gamma_2 - \gamma_1$. Since $\gamma_1, \gamma_2 \in Q^+$, $\gamma_2 - \gamma_1 = 0$, we have $\gamma_1 = \gamma_2$. Consequently, $\alpha_1 = \alpha_2$ and $c_1 = c_2$. $\qquad\square$

6.8.1. *Case $H(2r + 1)$.* In this case $\tilde{\delta} = \delta$ and $\Theta = \Theta_B = \sum_{i=1}^r \epsilon_i$. Recall that $Q = \mathbb{Z}\Phi(\mathfrak{g})$ is the root lattice of $\mathfrak{g}$. By Lemma 6.16 it does make sense to set

$$
\mathcal{O}^{\min}(c, d, i) = \{L(\lambda) \mid \lambda \in (c + \mathbb{Z})\delta + (d + \mathbb{Z})\Theta + Q, \ \mathsf{dpt}(L(c\delta + d\Theta)) = i;
$$
$$
\mathsf{dpt}(L(\lambda)) = i + \ell(\lambda - c\delta - d\Theta)\}.
$$

It further splits into

$$\mathcal{O}^{\min}(c, d, i) = \mathcal{O}^{\min}(c, d, \bar{0}, i) \cup \mathcal{O}^{\min}(c, d, \bar{1}, i)$$

where

$$\mathcal{O}^{\min}(c, d, \iota, i) = \{L(\lambda) \in \mathcal{O}^{\min}(c, d, i) \mid \mathsf{pty}(L(c\delta + d\Theta)) = \iota;$$
$$\mathsf{pty}(L(\lambda)) = \iota + \overline{\ell(\lambda - c\delta - d\Theta)}\}$$

for $\iota \in \mathbb{Z}_2$. Here $\overline{\ell(\lambda - c\delta - d\Theta)} \in \mathbb{Z}_2$ denotes the parity of $\ell(\lambda - c\delta - d\Theta)$.

Theorem 6.17. *Assume* $\mathfrak{g} = \bar{H}(2r + 1)$. *The complete set of all different blocks in* $\mathcal{O}^{min}$ *is listed as follows*

$$\{\mathcal{O}^{min}(c, d, \iota, i) \mid (c, d, \iota, i) \in (\mathbb{C}/\mathbb{Z})^2 \times \mathbb{Z}_2 \times \mathbb{Z}\}.$$

Proof. We will take the same strategy as the proof of Theorem 6.15. For any given $L(\lambda) \in \mathcal{O}^{\min}(c, d, \iota, i)$, we first prove that $L(\lambda)$ lies in the block where $L(c\delta + d\Theta)$ lies. By Corollary 6.12 and Lemma 6.16, we can write $\lambda = c\delta + d\Theta + \alpha$ for some $\alpha = \gamma + \mathrm{ht}(\gamma)\delta \in Q$ with $\gamma = \sum_{i=1}^{r-1} a_i \epsilon_i \in Q^+$. By definition, we know $\mathrm{ht}(\alpha) = \mathrm{ht}(\gamma) \geq 0$. Thus, we will accomplish the proof by taking induction on $\mathrm{ht}(\alpha)$. When $\mathrm{ht}(\alpha) = 0$, then $\alpha = 0$ because $\gamma = \sum_{i=1}^{r-1} a_i \epsilon_i \in Q^+$. So the statement holds.

Suppose $\mathrm{ht}(\alpha) > 0$, and suppose that the conclusion has been true for the situation of being less than $\mathrm{ht}(\alpha)$. In this case, we can write $\gamma = \sum_{i=1}^{r-1} a_i \epsilon_i$ with $a_i \in \mathbb{Z}_{\geq 0}$ such that $a_1 \geq a_2 \geq \cdots \geq a_{r-1} \geq a_r = 0$. Because $\mathrm{ht}(\alpha) > 0$ and $a_r = 0$, there exists at least one $t \in \{1, \ldots, r-1\}$ satisfying $a_t > a_{t+1}$. Take $\beta = \sum_{i=1}^{t} \epsilon_i - (n-t)\delta$ and $\lambda' = \lambda - \beta$. Because $a_t > a_{t+1}$, $\lambda' \in \Lambda^+$. By Corollary 6.12(4), $L(\lambda')$ and $L(\lambda' + \beta) = L(\lambda)$ share the same block. On the other hand, $\lambda' = c\delta + d\Theta + (\alpha - \beta)$ with $\alpha - \beta = (\gamma - \sum_{i=1}^{t} \epsilon_i) + (\mathrm{ht}(\gamma) + n - t)\delta$. Obviously, $\gamma - \sum_{i=1}^{t} \epsilon_i \in Q^+$ and $\mathrm{ht}(\alpha - \beta) = \mathrm{ht}(\gamma) - t < \mathrm{ht}(\alpha)$. Thus, $L(\lambda')$ and $L(c\delta + d\Theta)$ lie in the same block by inductive hypothesis. Hence, $L(\lambda)$ and $L(c\delta + d\Theta)$ finally lie in the same block.

Conversely, we have the following clear observation.

(i) Let $P(\lambda)$ be any indecomposable projective module where $\lambda = c\delta + d\Theta + \alpha$ with $c, d \in \mathbb{C}$ and $\alpha \in Q$. By the construction of $P(\lambda)$ (Remark 4.3(1)), all weights of $P(\lambda)$ are in $\lambda + \mathbb{Z}\delta + Q$. So if $L(\mu)$ is a composition factor of $P(\lambda)$, then $\mu \in (c + \mathbb{Z})\delta + (d + \mathbb{Z})\Theta + Q$.

(ii) If $L(\mu')$ is a composition factor of $P(\lambda')$ and $\mu' \in (c + \mathbb{Z})\delta + (d + \mathbb{Z})\Theta + Q$, then by Remark 4.3 again, we have $\lambda' \in (c + \mathbb{Z})\delta + (d + \mathbb{Z})\Theta + Q$.

Thus, by the definition of blocks and taking Depth Lemma and Parity Lemma into account, we have proven that if a simple object $L(\mu)$ lies in the block where $L(c\delta + d\Theta)_i^\iota$ lies, then $L(\mu)$ must lie in $\mathcal{O}^{\min}(c, d, \iota, i)$. The proof is completed. □

6.8.2. *Case* $\bar{H}(2r)$. In this case, $\Theta = \Theta_{D,\aleph_r}$. Recall that $\mathfrak{g}$ admits the root lattice Q (see §6.7). In contrast with the block structure of $H(2r + 1)$, there is a crucial difference in the case of $H(2r)$, that is, $L(\lambda)$ and $L(\lambda + \delta)$ do not lie in the same block. The following lemma is a clue to it.

Lemma 6.18. *Let* $\mathfrak{g} = \bar{H}(2r)$. *Then the following statements hold.*

(1) *The root lattice* Q *contains* $\pm 2\delta$, *but does not contain* $\pm \delta$.

(2) *If $L(\mu)$ and $L(\nu)$ are in the same block, then $(\mu - \nu) \in Q$. In particular, $L(\lambda)$ and $L(\lambda \pm \delta)$ can not belong to the same block.*

(3) *Let $\beta = \beta_1 \epsilon_1 + \cdots + \beta_r \epsilon_r$ be an element of $Q \cap \Lambda^+$. Then there exist $m \in \mathbb{Z}$ and $\gamma \in Q^+$ such that*

$$\beta = 2m\delta + \beta_r \Theta + \gamma + ht(\gamma)\delta.$$

Proof. (1) Recall that the root system is

$$\Phi = \{\pm \epsilon_{i_1} \pm \cdots \pm \epsilon_{i_k} + l\delta \mid 1 \le i_1 < i_2 < \cdots < i_k \le r;$$
$$k - 2 < l < n - 2, l - k \in 2\mathbb{Z}\}.$$

It is easily seen that $\pm \delta$ does not appear in the $\mathbb{Z}$-linear combinations of roots.

(2) Consider $I(\lambda)$. Any of its weights is of the form $\lambda + \alpha$ for some $\alpha \in Q$. Because $P(\lambda)$ is a direct summand of $I(\lambda)$, if $L(\mu)$ and $L(\nu)$ are two composition factors of $P(\lambda)$, then $(\mu - \nu) \in Q$. The statement (2) follows due to the definition of blocks and the statement (1).

(3) Since $\epsilon_1 - \delta, \epsilon_2 - \delta, \cdots, \epsilon_r - \delta, 2\delta$ belong to Q, we can check that $\epsilon_1 + \cdots + \epsilon_{r-1} + \epsilon_r + \aleph_r \delta \in Q$. Hence,

$$\beta - \beta_r \Theta$$
$$= (\beta_1 - \beta_r)\epsilon_1 + (\beta_2 - \beta_r)\epsilon_2 + \cdots + (\beta_{r-1} - \beta_r)\epsilon_{r-1} + (\beta_1 + \cdots + \beta_{r-1} - (r-1)\beta_r)\delta$$
$$+ (-\beta_1 - \cdots - \beta_{r-1} + (r-1)\beta_r - \beta_r \aleph_r)\delta \in Q \cap \Lambda^+. \tag{6.21}$$

Write $\gamma := (\beta_1 - \beta_r)\epsilon_1 + (\beta_2 - \beta_r)\epsilon_2 + \cdots + (\beta_{r-1} - \beta_r)\epsilon_{r-1} \in Q^+$ and $\gamma_i := \beta_i - \beta_r$. Since

$$\beta = \beta_1 \epsilon_1 + \cdots + \beta_r \epsilon_r \in Q,$$

by (1) we see that $\beta_1 + \cdots + \beta_r$ is even. Then there exists $m \in \mathbb{Z}$ such that

$$-\beta_1 - \cdots - \beta_{r-1} + (r-1)\beta_r - \beta_r \aleph_r = -\beta_1 - \cdots - \beta_r + \beta_r(r - \aleph_r) = 2m.$$

By (6.21), we have

$$\beta - \beta_r \Theta = \gamma + ht(\gamma)\delta + 2m\delta$$

The statement (3) follows. $\square$

By Lemma 6.16 it does make sense to set

$$\mathcal{O}^{\min}(c, d, i) = \{L(\lambda) \mid \lambda \in (c + 2\mathbb{Z})\delta + (d + \mathbb{Z})\Theta + Q,$$
$$\mathsf{dpt}(L(c\delta + d\Theta)) = i;$$
$$\mathsf{dpt}(L(\lambda)) = i + \ell(\lambda - c\delta - d\Theta)\}.$$

It further splits into

$$\mathcal{O}^{\min}(c, d, i) = \mathcal{O}^{\min}(c, d, \bar{0}, i) \cup \mathcal{O}^{\min}(c, d, \bar{1}, i)$$

where

$$\mathcal{O}^{\min}(c, d, \iota, i) = \{L(\lambda) \in \mathcal{O}^{\min}(c, d, i) \mid \mathsf{pty}(L(c\delta + d\Theta)) = \iota;$$
$$\mathsf{pty}(L(\lambda)) = \iota + \overline{\ell(\lambda - c\delta - d\Theta)}\}$$

for $\iota \in \mathbb{Z}_2$. Here $\overline{\ell(\lambda - c\delta - d\Theta)} \in \mathbb{Z}_2$ denotes the parity of $\ell(\lambda - c\delta - d\Theta)$.

Theorem 6.19. *Assume* $\mathfrak{g} = \bar{H}(2r)$. *The complete set of all different blocks in* $\mathcal{O}^{min}$ *is listed as follows*

$$\{\mathcal{O}^{min}(c, d, \iota, i) \mid (c, d, \iota, i) \in \mathbb{C}/2\mathbb{Z} \times \mathbb{C}/\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}\}.$$

Proof. For any given $L(\lambda) \in \mathcal{O}^{min}(c, d, \iota, i)$, we first prove that $L(\lambda)$ lies in the block where $L(c\delta + d\Theta)$ lies. Assume that $\lambda = (c + 2m_1)\delta + (d + m_2)\Theta + \beta, \beta \in Q$, is an element of Λ^+. By the expression of Θ, we can deduce $\beta \in \Lambda^+ \cap Q$. Assume $\beta = \sum_{i=1}^{r} \beta_i \epsilon_i$, then $\beta_i \in \mathbb{Z}$. By Lemma 6.18(3), there exist $m \in \mathbb{Z}$ and $\gamma \in Q^+$ such that $\beta = 2m\delta + \beta_r\Theta + \gamma + \text{ht}(\gamma)\delta$. So $\lambda = (c + 2m_1 + 2m)\delta + (d + m_2 + \beta_r)\Theta + \gamma + \text{ht}(\gamma)\delta$. By Corollary 6.12(3) and (5), we can write $\lambda = c\delta + d\Theta + \alpha$ directly for some $\alpha = \gamma + \text{ht}(\gamma)\delta \in Q$ with $\gamma = \sum_{i=1}^{r-1} \gamma_i \epsilon_i \in Q^+$ without loss of generality. Thus, we can accomplish the proof similarly by taking induction on $\text{ht}(\alpha)$. By taking the same arguments as in the proof of Theorem 6.17 (here we omit the details) we can prove that $L(\lambda)$ and $L(c\delta + d\Theta)$ lie in the same block. Readers need only to notice that $n = 2r$ is even now.

What remains is to prove conversely that if a simple object $L(\mu)$ lies in the block where $L(c\delta + d\Theta)^{\iota}_i$ lies, then $L(\mu)$ must lie in $\mathcal{O}^{min}(c, d, \iota, i)$. For this, it suffices to observe the following facts.

(i) Let $P(\lambda)$ be any indecomposable projective module where $\lambda = c\delta + d\Theta + \alpha$ with $c, d \in \mathbb{C}$ and $\alpha \in Q$. We claim that any composition factor of $I(\lambda)$, say $L(\mu)$, must belong to the set $\lambda + 2\mathbb{Z}\delta + \mathbb{Z}\Theta + Q$. Recall that $I(\lambda)$ admits a Δ-flag with subquotients $\Delta(\tau)$ for $\tau \in \Upsilon(\lambda)$. So $L(\mu)$ must be a composition factor of some $\Delta(\tau)$. By the definition of $\Upsilon(\lambda)$ we can assume $\tau = \lambda - \gamma$ with $\gamma = \sum_{j=1}^{k} \pm\epsilon_{i_j} + m\delta$, where $i_j \in \{1, 2, ..., r\}$ satisfying $m \geq k$ and $m - k \in 2\mathbb{Z}$. So $\tau = \lambda - (\sum_{j=1}^{k} \pm\epsilon_{i_j} + m\delta) = \lambda - (\sum_{j=1}^{k} \pm\epsilon_{i_j} + k\delta + (m - k)\delta)$. Thus, $\tau \in \lambda + 2\mathbb{Z}\delta + Q$.

Next we investigate $L(\mu)$ from $\Delta(\tau)$. Note that by the definition of standard modules, all weights of $\Delta(\tau)$ must lie in $\tau + \mathbb{Z}_{\geq 0}\Phi(\mathfrak{g}_{\geq 1})$ where $\Phi(\mathfrak{g}_{\geq 1})$ meas the root system of $\mathfrak{g}_{\geq 1}$. So μ lies in $\tau + 2\mathbb{Z}\delta + \mathbb{Z}\Theta + Q$. The claim is true. So the claim is naturally true for $P(\lambda)$.

(ii) If $L(\mu')$ is a composition factor of $P(\lambda')$ and $\mu' \in c\delta + d\Theta + Q$, then $L(\mu')$ is naturally a composition factor of $I(\lambda')$. By the same reason as in (i) we have $\lambda' \in (c+2\mathbb{Z})\delta + (d+\mathbb{Z})\Theta + Q$. Thus, by the definition of blocks and taking Depth Lemma and Parity Lemma into account, we have that $L(\mu)$ indeed lies in $\mathcal{O}^{min}(c, d, \iota, i)$. Summing up, we finish the proof. $\square$

Remark 6.20. (1) *According to the proof, it is not hard to see that any irreducible module sharing the same block as* $L(c\delta + d\Theta + \alpha)$ *must be of the form* $L(\mu)$ *with* $\mu \in c\delta + d\Theta + Q$.

(2) *As a direct consequence of the above theorem, we know that* $L(\lambda)$ *and* $L(\lambda \pm \delta)$ *do not lie in the same block as mentioned at the beginning of the sub-subsection* §*6.8.2.*

(3) *On the basis of Proposition 6.11, one easily knows that Theorems 6.17 and 6.19 are valid in the case when* $\mathfrak{g} = \overline{CH}(n)$ *(*$n = 2r$ *or* $n = 2r + 1$*).*

6.9. **Application to the category of finite-generated modules over** $\mathfrak{g}$. We are going to consider blocks of the category of finite-generated modules over $\mathfrak{g}$. Denote this category by $\mathfrak{g}$-**mod**^{f}, whose objects are by definition, finite-generated modules, and whose morphisms are required to be even.

Recall that the forgetful functor $\mathcal{F}$ (see Remark 3.2(5)) makes $\mathcal{O}^{min}$ into the $U(\mathfrak{g})$-module category $\mathcal{F}(\mathcal{O}^{min})$ whose objects are only subjected to weighted-structure, and locally-finiteness

over $U(\mathsf{P})$. This is to say, all objects in $\mathcal{F}(\mathcal{O}^{\min})$ inherit all structures in $\mathcal{O}^{\min}$ except $\mathbb{Z}$-gradation. Then the isomorphism classes of simple objects both in $\mathcal{F}(\mathcal{O}^{\min})$ and in $\mathcal{F}(\mathcal{O}_{\bar{\mathfrak{f}}}^{\min})$ are parameterized by Λ^+ respectively, still denoted by $\{L(\lambda) \mid \lambda \in \Lambda^+\}$.

Lemma 6.21.   (1) *Any object of $\mathcal{F}(\mathcal{O}_{\bar{\mathfrak{f}}}^{min})$ can be naturally regarded as an object in $\mathcal{O}^{min}$. Any morphism in $\mathcal{F}(\mathcal{O}_{\bar{\mathfrak{f}}}^{min})$ can be lifted to $\mathcal{O}^{min}$.*
  (2) *For any $P(\lambda)$ in $\mathcal{O}^{min}$, $\mathcal{F}(P(\lambda))$ is still indecomposable and projective in $\mathcal{F}(\mathcal{O}_{\bar{\mathfrak{f}}}^{min})$.*

Proof. (1) For any given object M in $\mathcal{F}(\mathcal{O}_{\bar{\mathfrak{f}}}^{\min})$ and any given integer d, we will show that M can be endowed with a $\mathbb{Z}$-gradation related to d. By the same arguments as (4.8) in Theorem 4.4, we have that M admits a filtration of finite length

$$M = M^{(0)} \supset M^{(1)} \supset M^{(2)} \supset \cdots \supset M^{(t-1)} \supset M^{(t)} = 0 \qquad (6.22)$$

such that $M^{(i-1)}/M^{(i)}$ is isomorphic to a non-zero quotient of $\mathcal{F}(\Delta(\lambda_i))$ associated with some irreducible $U(\mathsf{P})$-module $L^0(\lambda_i) = U(\mathfrak{n}^-)v_{\lambda_i}^0$ with $\lambda_i \in \Lambda^+$, $i = 1, \cdots, t$, with t being the standard length $l(M)$. If $l(M) = 1$, $M = U(\mathfrak{g})v_{\lambda_1}^0$ which is easily endowed with a $\mathbb{Z}$-gradation, provided that $L^0(\lambda_1)$ is predefined to be of grading d. In general, we can define such a gradation on M by induction on $l(M)$. Suppose that $t = l(M) > 1$, and the gradation is defined already for less than t. Especially, $M^{(1)}$ is supposed to be already endowed with a $\mathbb{Z}$-gradation associated with d, hence all gradations of $v_{\lambda_i}^0$ $(i = 2, ..., t)$ are actually predefined, denoted by g_i. For any $m \in M$, $m \equiv m_1 \mod M^{(1)}$ for $m_1 \in U(\mathfrak{g})m_{\lambda_1}$ with m_{λ_1} being a pre-image of $v_{\lambda_1}^0$. Then we can define the gradation of m_{λ_1} to be g_1 such that g_1 is compatible with λ_i for $i = 2, ..., t$, this is to say, if $\lambda_1 - \lambda_i \in Q$, then $g_1 = g_i + \ell(\lambda_1 - \lambda_i)$. Thus, m_1, thereby M is endowed with a $\mathbb{Z}$-gradation. We have proven the first part of (1).

  Suppose that $\phi : M \to N$ is a homomorphism in $\mathcal{F}(\mathcal{O}_{\bar{\mathfrak{f}}}^{\min})$. In the way just mentioned above, M can be endowed with a $\mathbb{Z}$-gradation, thereby we can naturally endow a $\mathbb{Z}$-gradation on $\phi(M)$ such that ϕ is lifted to be a morphism in $\mathcal{O}^{\min}$. Hence we have proven the second part of (1).

  (2) Let $P(\lambda)$ be the projective cover of $L(\lambda) \in \mathbf{E}$ with $\mathsf{dpt}(L(\lambda)) = d$. Due to Remark 4.3(1), we can assume that $P(\lambda) = \sum_{g \in \mathbb{Z}} P(\lambda)_g$ is generated by some λ-weighted vector v_0 and the grading of v_0 is d. For any given surjective morphism $\phi : M \to N$ in $\mathcal{F}(\mathcal{O}_{\bar{\mathfrak{f}}}^{\min})$, and a nonzero morphism $\psi : \mathcal{F}(P(\lambda)) \to N$ in $\mathcal{O}_{\bar{\mathfrak{f}}}^{\min}$, we want to prove that there is a lift $\bar{\psi} : \mathcal{F}(P(\lambda)) \to M$. We begin with the definition of grading shift functor. Let L be a $\mathbb{Z}$-graded module belonging to $\mathcal{O}^{\min}$ and $d \in \mathbb{Z}$. Define a grading shift functor $[d] : L \mapsto L[d]$, such that as a vector space, $L[d] = L$, but the $\mathbb{Z}$-grading of $L[d]$ is changed through $L[d]_i = L_{i-d}$. We can check that $P(\lambda)[d_0]$ is the projective cover of $L(\lambda)[d_0] \in \mathbf{E}$. By (1), the surjective morphism $\phi : M \to N$ can be lifted to a surjective morphism in $\mathcal{O}^{\min}$ which becomes $\dot{\phi} : \dot{M} \to \dot{N}$. We suppose that $\psi(v_0)$ has a gradation d_0 in $\dot{N}$. Then by a suitable shift, we can re-endow a $\mathbb{Z}$-gradation on $\mathcal{F}(P(\lambda))$ such that v_0 is of gradation d_0, getting a new object $\dot{P}(\lambda)$ in $\mathcal{O}^{\min}$. By the arguments in the previous paragraph, we see that $\dot{P}(\lambda)$ is still indecomposable and projective in $\mathcal{O}^{\min}$. So we really have a morphism $\dot{\psi} : \dot{P}(\lambda) \to \dot{N}$ in $\mathcal{O}^{\min}$. The projectiveness of $\dot{P}(\lambda)$ entails that there exists a lift $\bar{\dot{\psi}} : \dot{P}(\lambda) \to \dot{M}$ of $\dot{\psi}$. After applying the forgetful functor $\mathcal{F}$, we get the desired lift $\bar{\psi}$ of ψ. The proof is completed. $\square$

  By the above lemma, we can similarly define blocks in $\mathcal{F}(\mathcal{O}^{\min})$ as below.

Set $\mathcal{F}(\mathcal{O}^{\min}(c)) := \{\mathcal{F}(L(\lambda)) \mid \lambda \in c\Xi + Q\}$ when $\mathfrak{g} = W(n)$ or $\bar{S}(n)$. Then we have in the same sense as in §6.7, that

$$\mathcal{F}(\mathcal{O}^{\min}(c)) = \mathcal{F}(\mathcal{O}^{\min}(c, \bar{0})) \cup \mathcal{F}(\mathcal{O}^{\min}(c, \bar{1})).$$

Similarly, set $\mathcal{F}(\mathcal{O}^{\min}(c, d)) := \{\mathcal{F}(L(\lambda)) \mid \lambda \in c\delta + d\Theta + Q\}$ when $\mathfrak{g} = \bar{H}(n)$. We have in the same sense as in §6.8, that

$$\mathcal{F}(\mathcal{O}^{\min}(c, d)) = \mathcal{F}(\mathcal{O}^{\min}(c, d, \bar{0})) \cup \mathcal{F}(\mathcal{O}^{\min}(c, d, \bar{1})).$$

Then we have the following direct consequence by Theorems 6.15, 6.17 and 6.19.

Corollary 6.22. *The complete classification of all different blocks in $\mathcal{F}(\mathcal{O}^{min})$ is listed as follows:*
  (1) *If $\mathfrak{g} = W(n)$, or $\bar{S}(n)$, then it is*

$$\{\mathcal{F}(\mathcal{O}^{min}(c, \gamma)) \mid (c, \gamma) \in (\mathbb{C}/\mathbb{Z}) \times \mathbb{Z}_2\}.$$

  (2) *If $\mathfrak{g} = \bar{H}(n)$, then it is*

$$\{\mathcal{F}(\mathcal{O}^{min}(c, d, \gamma)) \mid (c, d, \gamma) \in (\mathbb{C}/\mathbb{Z})^2 \times \mathbb{Z}_2\} \text{ if } n = 2r + 1; \text{ and}$$
$$\{\mathcal{F}(\mathcal{O}^{min}(c, d, \gamma)) \mid (c, d, \gamma) \in \mathbb{C}/2\mathbb{Z} \times \mathbb{C}/\mathbb{Z} \times \mathbb{Z}_2\} \text{ if } n = 2r.$$

Obviously, $\mathfrak{g}\text{-}\mathbf{mod}^f$ is a full subcategory of $\mathcal{F}(\mathcal{O}^{\min})$. We can introduce blocks of $\mathfrak{g}\text{-}\mathbf{mod}^f$ as follows.

Definition 6.23. *A block $\mathbf{B}$ of $\mathfrak{g}\text{-}\mathbf{mod}^f$ is a subcategory of $\mathfrak{g}\text{-}\mathbf{mod}^f$, satisfying that for any $B \in \mathbf{B}$, all its composition factors lie in the same block of $\mathcal{F}(\mathcal{O}^{min})$.*

We finally obtain the block theorem for $\mathfrak{g}\text{-}\mathbf{mod}^f$ as follows.

Theorem 6.24. *The following statements hold.*
  (1) *For $\mathfrak{g} = W(n)$ or $\bar{S}(n)$,*

$$\mathfrak{g}\text{-}\mathbf{mod}^f = \bigoplus_{(c, \gamma) \in \mathbb{C}/\mathbb{Z} \times \mathbb{Z}_2} \mathbf{B}(c, \gamma)$$

  (2) *For $\mathfrak{g} = \bar{H}(n)$,*

$$\mathfrak{g}\text{-}\mathbf{mod}^f = \begin{cases} \bigoplus_{(c,d,\gamma) \in (\mathbb{C}/\mathbb{Z})^2 \times \mathbb{Z}_2} \mathbf{B}(c, d, \gamma) & \text{when } n = 2r + 1; \\ \bigoplus_{(c,d,\gamma) \in \mathbb{C}/2\mathbb{Z} \times \mathbb{C}/\mathbb{Z} \times \mathbb{Z}_2} \mathbf{B}(c, d, \gamma) & \text{when } n = 2r. \end{cases}$$

Remark 6.25. (1) *In our setup, Theorem 6.24 essentially covers the main result of [20] on blocks of the category of finite-dimensional modules over $W(n)$.*
  (2) *By the same arguments as in [20], one can show that all blocks of $\mathcal{O}^{min}$ are wild.*

7. TILTING MODULES AND CHARACTER FORMULAS

Keep the same notations as in Sections 1 and 3. In particular, δ is the linear dual of d in $\bar{\mathfrak{h}}^*$ when $\mathfrak{g} = \bar{H}(n)$ (See §1.2).

7.1.   Thanks to Lemma 4.1, we can apply the arguments in [7] to our category $\mathcal{O}^{\min}$. We first recall some properties for standard and co-standard modules.

Lemma 7.1. *Keep the assumption as above. The following results hold in the category $\mathcal{O}^{min}$:*

(1) *The category $\mathcal{O}^{min}$ has enough injective objects.*
(2) *Assume that $\Delta(\lambda)$ has depth d. Then $\Delta(\lambda)$ is the projective cover of $L(\lambda)$ in $\mathcal{O}^{min}_{\geq d}$.*
(3) $\dim \operatorname{Hom}_{\mathcal{O}^{min}}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}$ *for $\lambda, \mu \in \Lambda^+$.*
(4) $\operatorname{Ext}^1_{\mathcal{O}^{min}}(\Delta(\lambda), \nabla(\mu)) = 0$ *for $\lambda, \mu \in \Lambda^+$.*

Proof. For (1), readers can refer to [7, Lemma 2.1]. For (2),(3),(4), readers can refer to [7, Lemma 3.6].   □

7.2. **Tilting modules.** Thanks to Lemma 1.3, the category $\mathcal{O}^{\min}$ is associated with a semi-infinite character of $\mathfrak{g}$. So we can apply Soergel's tilting module theory to our category $\mathcal{O}^{\min}$. The following lemma asserts the existence of the so-called indecomposable tilting modules $T(\lambda)$ for $\lambda \in \mathbf{E}$.

Lemma 7.2. ([21, Theorem 5.2] *and* [7, Theorem 5.1]) *For any given $L^0(\lambda) = L^0(\lambda)_d$ $((\lambda, d) \in \mathbf{E} = \Lambda^+ \times \mathbb{Z})$, there exists a unique up to isomorphism indecomposable object $T(\lambda) \in \mathcal{O}^{min}$ such that*

(1) $\operatorname{Ext}^1_{\mathcal{O}^{min}}(\Delta(\mu), T(\lambda)) = 0$ *for any $\mu \in \mathbf{E}$.*
(2) $T(\lambda)$ *admits a Δ-flag starting with $\Delta(\lambda)$ at the bottom.*

Definition 7.3. *An object T in $\mathcal{O}^{min}$ is called a tilting module if it satisfies (1) and (2) in Lemma 7.2 as $T(\lambda)$ does. In particular, the indecomposable tilting object $T(\lambda)$ is called the indecomposable tilting module associated with $\lambda \in \mathbf{E}$.*

In the following, we will investigate the flags of standard modules for indecomposable tilting modules, by means of Soergel reciprocity and the Kac-module realizations of co-standard modules.

7.3. **Soergel reciprocity.** By [7, Corollary 5.8], we have the following reciprocity for inde-composable tilting modules.

Proposition 7.4. *Let $\lambda, \mu \in \mathbf{E}$, and w_0 be the longest element of the Weyl group of $\mathfrak{g}_0$. Denote by $[T : \Delta(\lambda)]$ the multiplicity of $\Delta(\lambda)$ in the Δ-flag of a given tilting module T. The following statements hold.*

(1) *If $\mathfrak{g} = W(n)$, then*
$$[T(\mu) : \Delta(\lambda)] = (\nabla(-w_0\lambda + \Xi) : L(-w_0\mu + \Xi)).$$
(2) *If $\mathfrak{g} = \bar{S}(n)$ or $\bar{H}(n)$, then*
$$[T(\mu) : \Delta(\lambda)] = (\nabla(-w_0\lambda) : L(-w_0\mu)).$$

Proof. Note that the character $\mathcal{E}_X$ gives rise to a one-dimensional $\mathfrak{g}_0$-module $\mathbb{C}_{-\mathcal{E}_X}$, and we have the following $\mathfrak{g}_0$-module isomorphism
$$\mathbb{C}_{-\mathcal{E}_X} \cong \begin{cases} L^0(\Xi), & \text{if } \mathfrak{g} = W(n), \\ L^0(0), & \text{if } \mathfrak{g} = \bar{S}(n), \bar{H}(n). \end{cases}$$

Then the statements are consequences of [7, Corollary 5.8].   □

With the aid of Proposition 5.5, the above Soergel reciprocity can be rewritten below.

Proposition 7.5. *Let* $\lambda, \mu \in \Lambda^+$. *The following statements hold.*

(1) *If* $\mathfrak{g} = W(n)$, *then*

$$[T(\mu) : \Delta(\lambda)] = (K(-w_0\lambda + 2\Xi) : L(-w_0\mu + \Xi)).$$

(2) *If* $\mathfrak{g} = \bar{S}(n)$, *then*

$$[T(\mu) : \Delta(\lambda)] = (K(-w_0\lambda + \Xi) : L(-w_0\mu)).$$

(3) *If* $\mathfrak{g} = \bar{H}(n)$, *then*

$$[T(\mu) : \Delta(\lambda)] = (K(-w_0\lambda + n\delta) : L(-w_0\mu)).$$

Proof. This is a direct consequence of Propositions 7.4 and 5.5. $\qquad\square$

7.4. **Definition of character formulas for** $\mathcal{O}_{\bar{\mathfrak{f}}}^{\min}$. By Theorem 5.6 and Proposition 7.5 we have seen that the multiplicities of $\Delta(\lambda)$ in $P(\mu)$ or $T(\mu)$ can be attributed to the Cartan invariants of some finite-dimensional Kac-module, so $P(\lambda)$ and $T(\lambda)$ belong to $\mathcal{O}_{\bar{\mathfrak{f}}}^{\min}$. In this section, we compute the character formulas for those $P(\lambda)$ and $T(\lambda)$, on the basis of degenerate BGG reciprocity (Theorem 5.3) and Soergel reciprocity (Propositions 7.4 and 7.5) respectively. In the following, we first introduce the formal characters of modules in the category $\mathcal{O}_{\bar{\mathfrak{f}}}^{\min}$.

Recall that associated with the standard triangular decomposition $\mathfrak{g}_0 = \mathfrak{n}^- \oplus \bar{\mathfrak{h}} \oplus \mathfrak{n}^+$, $\mathfrak{g}_0$ admits a positive root system Φ_0^+. Furthermore, denote by $\Phi_{\geq 1}$ the root system of $\mathfrak{g}_{\geq 1}$ relative to $\bar{\mathfrak{h}}$, i.e., $\Phi_{\geq 1} := \{\alpha \in \bar{\mathfrak{h}}^* \mid (\mathfrak{g}_{\geq 1})_\alpha \neq 0\}$ where

$$(\mathfrak{g}_{\geq 1})_\alpha = \{x \in \mathfrak{g}_{\geq 1} \mid [h, x] = \alpha(h)x, \ \forall\, h \in \bar{\mathfrak{h}}\}.$$

Then we have $\mathfrak{g}_{\geq 1} = \sum_{\alpha \in \Phi_{\geq 1}} \mathfrak{g}_\alpha$. Associated with $\lambda \in \Lambda^+$, we define a subset of $\bar{\mathfrak{h}}^*$:

$$D(\lambda) = \{\mu \in \bar{\mathfrak{h}}^* \mid \mu \succeq \lambda\},$$

where $\mu \succeq \lambda$ means that $\mu - \lambda$ lies in $\mathbb{Z}_{\geq 0}$-span of $\Phi_{\geq 1} \cup \Phi_0^+$. Now we define a $\mathbb{C}$-algebra $\mathcal{A}$, whose elements are series of the form $\sum_{\lambda \in \bar{\mathfrak{h}}^*} c_\lambda e^\lambda$ with $c_\lambda \in \mathbb{C}$ and $c_\lambda = 0$ for λ outside the union of a finite number of sets of the form $D(\mu)$. Then $\mathcal{A}$ naturally becomes a commutative associative algebra if we define $e^\lambda e^\mu = e^{\lambda+\mu}$, and identify e^0 with the identity element. All formal exponentials $\{e^\lambda\}$ are linearly independent, and then in one-to-one correspondence with $\bar{\mathfrak{h}}^*$. For a semisimple $\bar{\mathfrak{h}}$-module $W = \sum_{\lambda \in \bar{\mathfrak{h}}^*} W_\lambda$, if the weight spaces are all finite-dimensional, then we can define $\mathsf{ch}(W) = \sum_{\lambda \in \bar{\mathfrak{h}}^*} (\dim W_\lambda) e^\lambda$. In particular, if V is an object in $\mathcal{O}_{\bar{\mathfrak{f}}}^{\min}$, then $\mathsf{ch}(V) \in \mathcal{A}$. We have the following fact.

Lemma 7.6. *The following statements hold.*

(1) *Let* V_1, V_2 *and* V_3 *be three* $\mathfrak{g}$-*modules in the category* $\mathcal{O}_{\bar{\mathfrak{f}}}^{min}$. *If there is an exact sequence of* $\mathfrak{g}$-*modules* $0 \to V_1 \to V_2 \to V_3 \to 0$, *then* $\mathsf{ch}(V_2) = \mathsf{ch}(V_1) + \mathsf{ch}(V_3)$.

(2) *Suppose that* $W = \sum_{\lambda \in \bar{\mathfrak{h}}^*} W_\lambda$ *is a semisimple* $\bar{\mathfrak{h}}$-*module with finite-dimensional weight spaces, and* $U = \sum_{\lambda \in \bar{\mathfrak{h}}^*} U_\lambda$ *is a finite-dimensional* $\bar{\mathfrak{h}}$-*module. If* $\mathsf{ch}(W) = \sum_{\lambda \in \bar{\mathfrak{h}}^*} c_\lambda e^\lambda$ *falls in* $\mathcal{A}$, *then* $\mathsf{ch}(W \otimes_{\mathbb{C}} U)$ *must fall in* $\mathcal{A}$ *and* $\mathsf{ch}(W \otimes_{\mathbb{C}} U) = \mathsf{ch}(W)\mathsf{ch}(U)$.

Let us investigate the formal character of a standard module $\Delta(\lambda)$ for $\lambda \in \mathbf{E}$. Recall $\Delta(\lambda) = U(\mathfrak{g}_{\geq 1}) \otimes_{\mathbb{C}} L^0(\lambda)$. As a $U(\mathfrak{g}_{\geq 1})$-module, $\Delta(\lambda)$ is a free module of rank $\dim L^0(\lambda)$ generated by $L^0(\lambda)$. By Lemma 7.6(2), we have $\mathsf{ch}(\Delta(\lambda)) = \mathsf{ch}(U(\mathfrak{g}_{\geq 1}))\mathsf{ch}L^0(\lambda)$ for $\lambda \in \mathbf{E}$. Note that

$$\Phi_{\geq 1} = \Phi_{\bar{0}}^{\geq 1} \cup \Phi_{\bar{1}}^{\geq 1}, \text{ where } \Phi_{\bar{i}}^{\geq 1} = \Phi_{\geq 1} \cap \Phi_{\bar{i}}, i \in \mathbb{Z}_2.$$

Set

$$\Theta := \prod_{\alpha \in \Phi_{\bar{1}}^{\geq 1}} (1 + e^{\alpha}) \prod_{\alpha \in \Phi_{\bar{0}}^{\geq 1}} (1 - e^{\alpha})^{-1}.$$

Then we further have $\mathsf{ch}(\Delta(\lambda)) = \Theta \mathsf{ch}L^0(\lambda)$.

7.5. **Character formulas of $T(\lambda)$.**

As a direct consequence of the forthcoming Propositions 9.5, 9.8 and 9.12 in the Appendix B, along with Lemma 7.6, Soergel reciprocity leads to the following theorem on character formulas for indecomposable tilting modules.

Theorem 7.7. *Let $\mathfrak{g} = X(n)$ for $X \in \{W, \bar{S}, \bar{H}\}$, and $\lambda \in \Lambda^+$. The character formulas for tilting modules $T(\lambda)$ are listed as follows.*
  (1) If $\mathfrak{g} = W(n)$, then

$$\mathsf{ch}T(\lambda) = \begin{cases} \Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(2\lambda)), & \text{if } \lambda = \Xi; \\ \Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda + \epsilon_1)), & \text{if } \lambda = 2\Xi + a\epsilon_1 \text{ with } a \geq 0; \\ \Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda + \epsilon_n)), & \text{if } \lambda = \Xi + b\epsilon_n \text{ with } b \leq -1; \\ \Theta(\mathsf{ch}L^0(\lambda)), & \text{if } \lambda \notin \Omega. \end{cases}$$

  (2) If $\mathfrak{g} = \bar{S}(n)$, then

$$\mathsf{ch}T(\lambda) = \begin{cases} \Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda + \Xi) + \mathsf{ch}L^0(\lambda + \Xi - \epsilon_n) + \mathsf{ch}L^0(\lambda + \epsilon_1)), \\ \qquad \text{if } \lambda = k\Xi; \\ \Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda + \epsilon_n) + \mathsf{ch}L^0(\lambda + \epsilon_1 + \epsilon_n)), \\ \qquad \text{if } \lambda = k\Xi - \epsilon_n; \\ \Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda + \epsilon_n)), \\ \qquad \text{if } \lambda = k\Xi + b\epsilon_n \text{ with } b \in \mathbb{Z}_{\leq -2}; \\ \Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda + \epsilon_1)), \\ \qquad \text{if } \lambda = k\Xi + a\epsilon_1 \text{ with } a \in \mathbb{Z}_{\geq 1}; \\ \Theta(\mathsf{ch}L^0(\lambda)), \text{ if } \lambda \notin \Omega. \end{cases}$$

  (3) If $\mathfrak{g} = \bar{H}(n)$, then

$$\mathsf{ch}T(\lambda) = \begin{cases} \Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda + n\delta) + \mathsf{ch}L^0(\epsilon_1 + (k + n + 1)\delta) + \mathsf{ch}L^0(\epsilon_1 + (k + 3)\delta)), \\ \qquad \text{if } \lambda = k\delta; \\ \Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda + 2\delta) + \mathsf{ch}L^0(\lambda + \epsilon_1 + 3\delta) + \mathsf{ch}L^0(\lambda - \epsilon_1 - \delta)), \\ \qquad \text{if } \lambda = k\delta + a\epsilon_1 \text{ with } a \in \mathbb{Z}_{\geq 1}; \\ \Theta(\mathsf{ch}L^0(\lambda)), \text{ if } \lambda \notin \Omega. \end{cases}$$

7.6. **Character formulas of $P(\lambda)$.** According to the degenerate BGG reciprocity (Theorem 5.3), one can compute the character formulas of indecomposable projective modules precisely by the same method as Theorem 7.7. We omit the details and list the formulas as below.

Theorem 7.8. *Let $\mathfrak{g} = X(n)$ for $X \in \{W, \bar{S}, \bar{H}\}$, and $\lambda \in \Lambda^+$. The character formulas for indecomposable projective modules $P(\lambda)$ are listed as follows.*
    (1) If $\mathfrak{g} = W(n)$, then

$$
\mathsf{ch}P(\lambda) = \begin{cases}
\Theta(\mathsf{ch}L^0(0) + \mathsf{ch}L^0(-\Xi)), & \text{if } \lambda = 0; \\
\Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda - \epsilon_1)), & \text{if } \lambda = a\epsilon_1 \text{ with } a \geq 1; \\
\Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda - \epsilon_n)), & \text{if } \lambda = -\epsilon_1 - \epsilon_2 - \cdots - \epsilon_{n-1} + a\epsilon_n \text{ with } a \leq -1; \\
\Theta(\mathsf{ch}L^0(\lambda)), & \text{otherwise} .
\end{cases}
$$

    (2) If $\mathfrak{g} = \bar{S}(n)$, then

$$
\mathsf{ch}P(\lambda) = \begin{cases}
\Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda - \Xi) + \mathsf{ch}L^0(\lambda - \Xi + \epsilon_1) + \mathsf{ch}L^0(\lambda - \epsilon_n)), \\
\qquad\qquad \text{if } \lambda = k\Xi; \\
\Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda - \epsilon_1) + \mathsf{ch}L^0(\lambda - \epsilon_1 - \epsilon_n)), \\
\qquad\qquad \text{if } \lambda = \epsilon_1 + k\Xi; \\
\Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda - \epsilon_1)), \\
\qquad\qquad \text{if } \lambda = a\epsilon_1 + k\Xi \text{ with } a \in \mathbb{Z}_{\geq 2}; \\
\Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda - \epsilon_n)), \\
\qquad\qquad \text{if } \lambda = k\Xi + c\epsilon_n \text{ with } c \in \mathbb{Z}_{\leq -1}; \\
\Theta(\mathsf{ch}L^0(\lambda)), \quad \text{otherwise} .
\end{cases}
$$

    (3) If $\mathfrak{g} = \bar{H}(n)$, then

$$
\mathsf{ch}P(\lambda) = \begin{cases}
\Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda - n\delta) + \mathsf{ch}L^0(\lambda + \epsilon_1 + (1-n)\delta) + \mathsf{ch}L^0(\lambda + \epsilon_1 - \delta)), \\
\qquad\qquad \text{if } \lambda = k\delta; \\
\Theta(\mathsf{ch}L^0(\lambda) + \mathsf{ch}L^0(\lambda - 2\delta) + \mathsf{ch}L^0(\lambda + \epsilon_1 - \delta) + \mathsf{ch}L^0(\lambda - \epsilon_1 - \delta)), \\
\qquad\qquad \text{if } \lambda = a\epsilon_1 + k\delta \text{ with } a \in \mathbb{Z}_{\geq 1}; \\
\Theta(\mathsf{ch}L^0(\lambda)), \text{otherwise} .
\end{cases}
$$

7.7. **Bar-typical weights and indecomposable projective tilting modules.** Call a weight $\lambda \in \bar{\mathfrak{h}}^*$ bar-atypical if $\lambda \in \Omega^{\bar{a}}$ defined as below

$$
\Omega^{\bar{a}} = \begin{cases}
\{\pm\Xi + b\epsilon_n \mid b \in \mathbb{Z}_{\leq 0}\} \cup \{d\Xi + a\epsilon_1 \mid d = 0, 2; a \in \mathbb{Z}_{\geq 0}\}, & \text{for } W(n); \\
\{a\epsilon_1 + k\Xi \mid a \in \mathbb{Z}_{\geq 1}, k \in \mathbb{C}\} \cup \{k\Xi + c\epsilon_n \mid c \in \mathbb{Z}_{\leq 0}, k \in \mathbb{C}\}, & \text{for } \bar{S}(n); \\
\{a\epsilon_1 + k\delta \mid a \in \mathbb{Z}_{\geq 0}, k \in \mathbb{C}\}, & \text{for } \bar{H}(n).
\end{cases}
$$

Call a weight $\lambda \in \bar{\mathfrak{h}}^*$ bar-typical, if $\lambda \notin \Omega^{\bar{a}}$.

Proposition 7.9. *If $\lambda \in \Lambda^+$ is bar-typical, then $P(\lambda) = T(\lambda) = \Delta(\lambda)$. Conversely, if $P(\lambda) = T(\lambda)$, then λ must be bar-typical.*

Proof. The first part of the proposition is a direct consequence of the above theorems. As to the second part, we only need to verify that when $\lambda \in \Omega^{\bar{a}}$, $P(\lambda)$ is not a tilting module. In

this case, it is really true that $P(\lambda) = \Delta(\lambda)$ and $T(\lambda) = \Delta(\lambda)$ do not simultaneously happen. By Propositions 9.5, 9.8, 9.12 in Appendix B, Theorem 5.6 and Proposition 7.4, we can see that $[P(\lambda) : \Delta(\lambda)] = 1$ and $[T(\lambda) : \Delta(\lambda)] = 1$ in their Δ-flags. However, $\Delta(\lambda)$ is a quotient of $P(\lambda)$ and a submodule of $T(\lambda)$ (see Lemma 7.2). This implies that $P(\lambda) \not\cong T(\lambda)$ in this case. The proof is completed. $\square$

8. Appendix A: A proof for the existence of semi-infinite characters

(1) Assume $\mathfrak{g} = W(n)$. Let us first check that the linear map $\mathcal{E}_X$ is indeed a homomorphism of Lie algebras. For any basis elements $\xi_i D_j, \xi_s D_t \in \mathfrak{g}_0$,

$$\mathcal{E}_W([\xi_i D_j, \xi_s D_t]) = \mathcal{E}_W(\delta_{js}\xi_i D_t - \delta_{ti}\xi_s D_j) = 0.$$

So $\mathcal{E}_X$ is a character.

Let $\xi_{k_1}\xi_{k_2}\cdots\xi_{k_{i+1}}D_s$ be an element in $\mathfrak{g}_i, i \geq 2$. We have the following two cases.

Case (i): $s \neq k_j, \forall\, 1 \leq j \leq i+1$.

In this case,

$$\xi_{k_1}\xi_{k_2}\cdots\xi_{k_{i+1}}D_s = (-1)^i[\xi_{k_2}\cdots\xi_{k_{i+1}}D_s, \xi_s\xi_{k_1}D_{k_1}].$$

Case (ii): $s = k_j$ for some $j \in \{1, \cdots, i+1\}$.

In this case, without loss of generality, we can assume $j = i + 1$, i.e., $s = k_{i+1}$. Then we have

$$\xi_{k_1}\xi_{k_2}\cdots\xi_{k_{i+1}}D_s = [\xi_1\xi_{k_2}\cdots\xi_{k_i}D_{k_i}, \xi_{k_i}\xi_{k_{i+1}}D_s].$$

It follows that $\mathfrak{g}_i$ is included in $[\mathfrak{g}_{i-1}, \mathfrak{g}_1]$ for any $i \geq 2$. By induction on i, we see that (SI-1) holds for $W(n)$. For (SI-2), we can check it through direct calculation in the following.

Without loss of generality, we can assume $x = \xi_k\xi_i D_j, y = D_s$. We divide the proof into the following three cases.

Case (i): $s \neq k$ and $s \neq i$.

In this case, $[x, y] = 0$. And we have

$$[x, [y, z]] = \begin{cases} \xi_i D_j, & \text{if } z = \xi_s D_k; \\ -\xi_k D_j, & \text{if } z = \xi_s D_i; \\ 0, & \text{if } z \in \{\xi_u D_t \mid 1 \leq u, t \leq n\} \setminus \{\xi_s D_k, \xi_s D_i\}. \end{cases}$$

It follows that $\mathsf{str}((\mathsf{ad}x \circ \mathsf{ad}y)|_{\mathfrak{g}_0}) = 0 = \mathcal{E}_W([x, y])$.

Case (ii): $s = k$ and $i = j$.

In this case, $[x, y] = \xi_j D_j$, and we have

$$[x, [y, z]] = \begin{cases} \xi_j D_j, & \text{if } z = \xi_s D_s; \\ -\xi_s D_j, & \text{if } z = \xi_s D_j; \\ 0, & \text{if } z \in \{\xi_u D_t \mid 1 \leq u, t \leq n\} \setminus \{\xi_s D_s, \xi_s D_j\}. \end{cases}$$

It follows that $\mathsf{str}((\mathsf{ad}x \circ \mathsf{ad}y)|_{\mathfrak{g}_0}) = -1 = \mathcal{E}_W([x, y])$.

Case (iii): $s = k$ and $i \neq j$.

In this case, $[x, y] = \xi_i D_j$, and we have

$$[x, [y, z]] = \begin{cases} \xi_i D_j, & \text{if } z = \xi_s D_s; \\ -\xi_s D_j, & \text{if } z = \xi_s D_i; \\ 0, & \text{if } z \in \{\xi_u D_t \mid 1 \leq u, t \leq n\} \setminus \{\xi_s D_s, \xi_s D_i\}. \end{cases}$$

It follows that $\mathsf{str}((\mathsf{ad}x \circ \mathsf{ad}y)|_{\mathfrak{g}_0}) = 0 = \mathcal{E}_W([x, y])$.

Thus, (SI-2) holds for $W(n)$. Consequently, $\mathcal{E}_W$ is a semi-infinite character for $W(n)$.

(2) Assume $\mathfrak{g} = \bar{S}(n)$. For (SI-1), one can refer to [13, Proposition 4.1.1]. Moreover, since $\mathfrak{g}_0$ coincides with $W(n)_0$, and str is linear, it follows that $\mathcal{E}_{\bar{S}}$ is a semi-infinite character for $\bar{S}(n)$.

(3) Assume $\mathfrak{g} = \bar{H}(n)$ or $\overline{CH}(n)$. For (SI-1), one can refer to [13, Proposition 4.1.1]. For (SI-2), we can check it through direct calculation in the following.

Without loss of generality, we can assume $x = D_H(\xi_i \xi_j \xi_k)$ and $y = D_s$. We divide the proof into the following two cases.

Case (i): $s = i$.

In this case, $[x, y] = D_H(\xi_j \xi_k)$, and

$$[x, [y, z]] = \begin{cases} D_H(\xi_j \xi_k), & \text{if } z = D_H(\xi_s \xi_{s'}) \text{ or } \mathsf{d}; \\ D_H(\xi_s \xi_k), & \text{if } z = D_H(\xi_s \xi_{j'}); \\ -D_H(\xi_s \xi_j), & \text{if } z = D_H(\xi_s \xi_{k'}); \\ 0, & \text{if } z = D_H(\xi_s \xi_t) \text{ for } t \neq s', j', k'; \\ 0, & \text{if } z = D_H(\xi_l \xi_m) \text{ for } l \neq s, m \neq s. \end{cases}$$

It follows that $\mathsf{str}((\mathsf{ad}x \circ \mathsf{ad}y)|_{\mathfrak{g}_0}) = 0 = \mathcal{E}_{\bar{H}}([x, y])$.

Case (ii): $s \neq i, j, k$.

In this case, $[x, y] = 0$, and

$$[x, [y, z]] = \begin{cases} \delta_{it'} D_H(\xi_j \xi_k) - \delta_{jt'} D_H(\xi_i \xi_k) + \delta_{kt'} D_H(\xi_i \xi_j), & \text{if } z = D_H(\xi_s \xi_t); \\ 0, & \text{if } z = \mathsf{d} \text{ or } D_H(\xi_l \xi_m) \text{ for } l \neq s, m \neq s. \end{cases}$$

It follows that $\mathsf{str}((\mathsf{ad}x \circ \mathsf{ad}y)|_{\mathfrak{g}_0}) = 0 = \mathcal{E}_{\bar{H}}([x, y])$.

Thus, (SI-2) holds both for $\bar{H}(n)$ and $\overline{CH}(n)$. Hence, $\mathcal{E}_{\bar{H}}$ (resp. $\mathcal{E}_{\overline{CH}}$) is a semi-infinite character for $\bar{H}(n)$ (resp. $\overline{CH}(n)$).

9. Appendix B: Computations for character formulas

In this appendix, we list the composition factors of Kac-module which is contributed to compute the character formulas of tilting modules and indecomposable projective modules. Recall that we have introduced the set Ω of the so-called Serganova atypical weights in subsection 5.4.

9.1. The case of $W(n)$.

Lemma 9.1. *Let $\lambda \in \Lambda^+$. Then the following statements hold.*

  (1) *If $\lambda \neq a\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n$, $L(\lambda - \Xi) \cong \overline{L}(\lambda)$.*
  (2) *If $\lambda = a\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n$ and $\lambda \neq 0$, then $L(\lambda - \Xi + \epsilon_i) \cong \overline{L}(\lambda)$.*
  (3) *If $\lambda = 0$, $L(0) \cong \overline{L}(0)$.*

Based on [18, Theorem 7.6] and Lemma 9.1, the following lemma holds.

Lemma 9.2. *Let $\lambda, \mu \in \Lambda^+$. Then the following statements hold.*

  (1) *If $\lambda = 0$, then there is the following exact sequence*

$$0 \to L(-\Xi) \to K(0) \to L(0) \to 0.$$

  (2) *If $\lambda = a\epsilon_n$, $a < 0$, then there is the following exact sequence*

$$0 \to L(a\epsilon_n - \Xi) \to K(a\epsilon_n) \to L((a+1)\epsilon_n - \Xi) \to 0.$$

(3) If $\lambda = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$, then there is the following exact sequence

$$0 \to L(0) \to K(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n) \to L(\epsilon_1) \to 0.$$

(4) If $\lambda = a\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n, a \geq 2$, then there is the following exact sequence

$$0 \to L((a-1)\epsilon_1) \to K(a\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n) \to L(a\epsilon_1) \to 0.$$

(5) If $(K(\lambda) : L(\mu)) \neq 0$, then $(K(\lambda) : L(\mu)) = 1$.

By the definition of $\mathcal{O}^{\min}$ we only need to consider the weights belonging to Λ^+, i.e., the weights $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_n\epsilon_n$ such that $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{n-1} - \lambda_n$ are all non-negative integers. Obviously, the following lemma holds.

Lemma 9.3. *Let λ be a weight belonging to Λ^+ such that $-w_0\lambda + 2\Xi$ is Serganova atypical. Then λ has to be one of the following two forms*

(1) $\lambda = (2-a)\epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_n$, *for some $a \in \mathbb{Z}_{\leq n}$.*
(2) $\lambda = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} + (2-b)\epsilon_n$, *for some $b \in \mathbb{Z}_{\geq 1}$.*

In case (1), $-w_0\lambda + 2\Xi = a\epsilon_n$, while in case (2), $-w_0\lambda + 2\Xi = b\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$.

Proof. Assume

$$-w_0\lambda + 2\Xi = a\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n.$$

It follows that

$$\lambda = \epsilon_1 + \epsilon_2 + \cdots + \cdots + \epsilon_{n-i} + (2-a)\epsilon_{n-i+1} + 2\epsilon_{n-i+2} + \cdots + 2\epsilon_{n-1} + 2\epsilon_n.$$

Since λ is an element in Λ^+, λ has to be one of the following two forms:

$$\lambda = (2-a)\epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_n, a \in \mathbb{Z}_{\leq 0},$$

or

$$\lambda = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} + (2-b)\epsilon_n, b \in \mathbb{Z}_{\geq 1}.$$

Consequently, $-w_0\lambda + 2\Xi = a\epsilon_n$ or $b\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$, respectively. $\square$

Now we are in the position to determine the multiplicities of standard modules appearing in each tilting module.

Proposition 9.4. *Let $\lambda, \mu \in \Lambda^+$. Then the following statements hold.*

(1) *In the case $\lambda = 2\Xi$, $[T(\mu) : \Delta(\lambda)] \neq 0$ if and only if $\mu = \Xi$ or $\mu = \lambda$.*
(2) *In the case $\lambda = (2-a)\epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_n, a \in \mathbb{Z}_{\leq -1}$, $[T(\mu) : \Delta(\lambda)] \neq 0$ if and only if $\mu = \lambda - \epsilon_1$ or $\mu = \lambda$.*
(3) *In the case $\lambda = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} + (2-b)\epsilon_n, b \in \mathbb{Z}_{\geq 1}$, $[T(\mu) : \Delta(\lambda)] \neq 0$ if and only if $\mu = \lambda - \epsilon_n$ or $\mu = \lambda$.*
(4) *In the case that λ is not any one of the forms in Cases (i), (ii), (iii), $[T(\mu) : \Delta(\lambda)] \neq 0$ if and only if $\lambda = \mu$.*

Moreover, if $[T(\mu) : \Delta(\lambda)] \neq 0$, $[T(\mu) : \Delta(\lambda)] = 1$.

Proof. (1) Let $\lambda = (2-a)\epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_n, a \in \mathbb{Z}_{\leq 0}$. By Proposition 7.5 and Lemma 9.3, we have

$$[T(\mu) : \Delta(\lambda)] = (K(a\epsilon_n) : L(-w_0\mu + \Xi)).$$

(1-i) $a = 0$.

In this case,

$$
\begin{aligned}
&[T(\mu) : \Delta(\lambda)] \neq 0 \\
\Longleftrightarrow\quad & (K(0) : L(-w_0\mu + \Xi)) \neq 0 \\
\Longleftrightarrow\quad & L(-w_0\mu + \Xi) \cong L(0) \text{ or } L(-w_0\mu + \Xi) \cong L(-\Xi) \text{ (by Lemma 9.2)} \\
\Longleftrightarrow\quad & -w_0\mu + \Xi = 0 \text{ or } -w_0\mu + \Xi = -\Xi \\
\Longleftrightarrow\quad & \mu = \Xi \text{ or } \mu = 2\Xi \\
i.e.,\quad & \mu = \Xi \text{ or } \mu = \lambda.
\end{aligned}
$$

(1-ii) $a \leq -1$.

In this case,

$$
\begin{aligned}
&[T(\mu) : \Delta(\lambda)] \neq 0 \\
\Longleftrightarrow\quad & (K(a\epsilon_n) : L(-w_0\mu + \Xi)) \neq 0 \\
\Longleftrightarrow\quad & L(-w_0\mu + \Xi) \cong L(a\epsilon_n - \epsilon_{n-1} - \cdots - \epsilon_1) \text{ or} \\
& L(-w_0\mu + \Xi) \cong L((a-1)\epsilon_n - \epsilon_{n-1} - \cdots - \epsilon_1) \text{ (by Lemma 9.2)} \\
\Longleftrightarrow\quad & -w_0\mu + \Xi = a\epsilon_n - \epsilon_{n-1} - \cdots - \epsilon_1 \\
& \text{or } -w_0\mu + \Xi = (a-1)\epsilon_n - \epsilon_{n-1} - \cdots - \epsilon_1 \\
\Longleftrightarrow\quad & \mu = (1-a)\epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_n \text{ or } \mu = (2-a)\epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_n \\
i.e.,\quad & \mu = \lambda - \epsilon_1 \text{ or } \mu = \lambda.
\end{aligned}
$$

For the results in (2)-(4), we can calculate them similarly. $\qquad\square$

As a direct consequence, the following proposition holds.

Proposition 9.5. *Let $\mathfrak{g} = W(n)$ and $\mu \in \Lambda^+$. Then the following statements hold.*

(1) *If $\mu = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$, we have the following exact sequence:*
$$0 \to \Delta(\mu) \to T(\mu) \to \Delta(2\mu) \to 0.$$

(2) *If $\mu = a\epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_n$ with $a \geq 2$, then we have the following exact sequence:*
$$0 \to \Delta(\mu) \to T(\mu) \to \Delta(\mu + \epsilon_1) \to 0.$$

(3) *If $\mu = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} + b\epsilon_n$ with $b \leq 0$, then we have the following exact sequence:*
$$0 \to \Delta(\mu) \to T(\mu) \to \Delta(\mu + \epsilon_n) \to 0.$$

(4) *Otherwise, $T(\mu) = \Delta(\mu)$.*

9.2. **The case $\bar{S}(n)$.** Let λ be an element in Ω. Then it is easy to see that λ belongs to Λ^+ if and only if
$$\lambda = b\epsilon_1 + a\epsilon_2 + a\epsilon_3 + \cdots + a\epsilon_n \text{ with } (b-a) \in \mathbb{Z}_{\geq 0},$$
or
$$\lambda = a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + c\epsilon_n \text{ with } (a-c) \in \mathbb{Z}_{\geq 0}.$$

The following result follows directly from [18, Lemma 5.1].

Lemma 9.6. *Let $\lambda \in \Lambda^+$. Then the following statements hold.*

(1) *If $\lambda = a\Xi - \epsilon_n$, $\overline{L}(\lambda) \cong L(\lambda - \Xi + \epsilon_1 + \epsilon_n)$, i.e.,*
$$\overline{L}(a\Xi - \epsilon_n) \cong L(a\Xi - \epsilon_n - \epsilon_{n-1} - \cdots - \epsilon_2).$$

(2) *If $\lambda = a\Xi$, $\overline{L}(\lambda) \cong L(\lambda)$.*

(3) *If $\lambda = a\Xi - b\epsilon_n$ for $b \in \mathbb{Z}_{\geq 2}$, then $\overline{L}(\lambda) \cong L(\lambda - \Xi + \epsilon_n)$.*

(4) If $\lambda = a\Xi + b\epsilon_1$ for $b \in \mathbb{Z}_{\geq 1}$, then $\overline{L}(\lambda) \cong L(\lambda - \Xi + \epsilon_1)$.

(5) If $\lambda \notin \Omega$, $\overline{L}(\lambda) \cong L(\lambda - \Xi)$.

Based on the results in [18, §8] and Lemma 9.6, we get the following lemma.

Lemma 9.7. *Let* $\lambda \in \Lambda^+$. *Then the following statements hold.*

(1) *If* $\lambda = a\Xi$, *then we have the following exact sequences*
$$0 \to M \to K(a\Xi) \to L(a\Xi) \to 0,$$
$$0 \to L((a-1)\Xi) \to M \to L((a-1)\Xi + \epsilon_1) \to 0.$$

(2) *If* $\lambda = a\Xi + \epsilon_1$, *then we have the following exact sequences*
$$0 \to M \to K(a\Xi + \epsilon_1) \to L((a-1)\Xi + 2\epsilon_1) \to 0,$$
$$0 \to L((a-1)\Xi + \epsilon_1) \to M \to L(a\Xi) \to 0.$$

(3) *If* $\lambda = a\Xi - \epsilon_n$, *then we have the following exact sequences*
$$0 \to M \to K(a\Xi - \epsilon_n) \to L((a-1)\Xi + \epsilon_1) \to 0,$$
$$0 \to L((a-1)\Xi - \epsilon_n) \to M \to L((a-1)\Xi) \to 0.$$

(4) *If* $\lambda = b\epsilon_1 + a\Xi, b \in \mathbb{Z}_{\geq 2}$, *then we have the following exact sequence*
$$0 \to L((a-1)\Xi + b\epsilon_1) \to K(\lambda) \to L((a-1)\Xi + (b+1)\epsilon_1) \to 0.$$

(5) *If* $\lambda = a\Xi - c\epsilon_n, c \in \mathbb{Z}_{\geq 2}$, *then we have the following exact sequence*
$$0 \to L((a-1)\Xi - c\epsilon_n) \to K(\lambda) \to L((a-1)\Xi - (c+1)\epsilon_n) \to 0.$$

(6) *If* $(K(\lambda) : L(\mu)) \neq 0$, *then* $(K(\lambda) : L(\mu)) = 1$.

Similar arguments as in the proof of Proposition 9.4 yield the following result on the multiplicities of standard modules in each tilting module for $\bar{S}(n)$.

Proposition 9.8. *Let* $\mathfrak{g} = \bar{S}(n)$ *and* λ *be an element in* Λ^+. *Then* $[T(\lambda) : \Delta(\mu)] \neq 0$ *implies* $[T(\lambda) : \Delta(\mu)] = 1$. *Furthermore, the following statements hold.*

(1) *Assume that* λ *is Serganova atypical.*

(1-i) *If* $\lambda = k\Xi$, *then*
$$[T(\lambda) : \Delta(\mu)] \neq 0 \iff \mu \in \{\lambda, \lambda + \Xi, \lambda + \Xi - \epsilon_n, \lambda + \epsilon_1\}.$$

(1-ii) *If* $\lambda = k\Xi - \epsilon_n$, *then*
$$[T(\lambda) : \Delta(\mu)] \neq 0 \iff \mu \in \{\lambda + \epsilon_n, \lambda, \lambda + \epsilon_1 + \epsilon_n\}.$$

(1-iii) *If* $\lambda = k\Xi + b\epsilon_n$ *with* $b \in \mathbb{Z}_{\leq -2}$, *then*
$$[T(\lambda) : \Delta(\mu)] \neq 0 \iff \mu \in \{\lambda, \lambda + \epsilon_n\}.$$

(1-iv) *If* $\lambda = k\Xi + a\epsilon_1$ *with* $a \in \mathbb{Z}_{\geq 1}$, *then*
$$[T(\lambda) : \Delta(\mu)] \neq 0 \iff \mu \in \{\lambda, \lambda + \epsilon_1\}.$$

(2) *In the case that* λ *is Serganova typical,* $T(\lambda) = \Delta(\lambda)$.

9.3. **The case $\bar{H}(n)$.**

Lemma 9.9. *Let $\lambda \in \Lambda^+$ be a Serganova atypical weight. Then $\lambda = a\epsilon_1 + m\delta$ for some $a \in \mathbb{Z}_{\geq 0}$.*

Proof. With respect to our choice of positive roots, we can get that if $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_r\epsilon_r + b\delta$ is an element of Λ^+, then it must satisfy the following conditions:

   (i) when $n = 2r$, then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r|$, $\lambda_i - \lambda_j \in \mathbb{Z}$ *and* $\lambda_i \in \frac{1}{2}\mathbb{Z}$;
   (ii) when $n = 2r + 1$, then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq \lambda_r$, $\lambda_i - \lambda_j \in \mathbb{Z}$ *and* $\lambda_r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

Consequently, from the expression of Ω in §5.4 the lemma follows. $\square$

   The following result follows from [18, Lemma 5.1].

Lemma 9.10. *Let $\lambda \in \Lambda^+$. Then the following statements hold.*
   (1) *If λ is Serganova typical, then $\bar{L}(\lambda) \cong L(\lambda - n\delta)$.*
   (2) *If λ is Serganova atypical and $\lambda \neq a\delta$, then $\bar{L}(\lambda) \cong L(\lambda + (2 - n)\delta)$.*
   (3) *If $\lambda = a\delta$, $\bar{L}(\lambda) \cong L(\lambda)$.*

   The following description on composition factors of Kac modules with Serganova atypical weights follows from Lemma 9.10 and [18, Section 9].

Lemma 9.11. *Let $\lambda \in \Omega$. Then the following statements hold.*
   (1) *If $\lambda = a\delta$, then the irreducible composition factors of $K(\lambda)$ are*
$$L(a\delta), L((a - n)\delta), L(\epsilon_1 + (a + 1 - n)\delta).$$
   (2) *If $\lambda = \epsilon_1 + a\delta$, then the irreducible composition factors of $K(\lambda)$ are*
$$L((a - 1)\delta), L((a + 1 - n)\delta), L(\lambda + \epsilon_1 + (1 - n)\delta), L(\lambda + (2 - n)\delta), L(\lambda - n\delta).$$
   (3) *If $\lambda = b\epsilon_1 + a\delta, b \in \mathbb{Z}_{\geq 2}$, then the irreducible composition factors of $K(\lambda)$ are*
$$L(\lambda + (2 - n)\delta), L(\lambda - n\delta), L(\lambda + \epsilon_1 + (1 - n)\delta), L(\lambda - \epsilon_1 + (1 - n)\delta).$$
   (4) *If $(K(\lambda) : L(\mu)) \neq 0$, then $(K(\lambda) : L(\mu)) = 1$.*

   Let $\lambda = a\epsilon_1 + m\delta$ and $\mu = b\epsilon_1 + l\delta$ be elements in Λ^+, we have $-\omega_0\lambda = \lambda + (2a - 2m)\delta, -\omega_0\mu = \lambda + (2b - 2l)\delta$. So $[T(\mu) : \Delta(\lambda)] = (K(\lambda + (2a - 2m + n)\delta) : L(\mu + (2b - 2l)\delta))$ due to Proposition 7.5. Then we obtain the following result on the multiplicities of standard modules in each tilting module for $\bar{H}(n)$.

Proposition 9.12. *Let $\mathfrak{g} = \bar{H}(n)$ and $\lambda \in \Lambda^+$. Then $[T(\lambda) : \Delta(\mu)] \neq 0$ implies $[T(\lambda) : \Delta(\mu)] = 1$. Moreover, the following statements hold.*
   (1) *Assume that λ is Serganova atypical.*
      (1-i) *If $\lambda = m\delta$, then*
$$[T(\lambda) : \Delta(\mu)] \neq 0 \Longleftrightarrow \mu \in \{\lambda, \lambda + n\delta, \epsilon_1 + (m + n + 1)\delta, \epsilon_1 + (m + 3)\delta\}.$$
      (1-ii) *If $\lambda = a\epsilon_1 + m\delta, a \geq 1$, then*
$$[T(\lambda) : \Delta(\mu)] \neq 0 \Longleftrightarrow \mu \in \{\lambda, \lambda + 2\delta, \lambda + \epsilon_1 + 3\delta, \lambda - \epsilon_1 - \delta\}.$$
   (2) *If λ is Serganova typical, $T(\lambda) = \Delta(\lambda)$.*

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