Characterization of the Two-Dimensional Fivefold and Sixfold Lattice Tiles

Chuanming Zong

Abstract. In 1885, Fedorov discovered that a convex domain can form a lattice tiling of the Euclidean plane if and only if it is a parallelogram or a centrally symmetric hexagon. It is known that there is no other convex domain which can form a two, three or fourfold lattice tiling in the Euclidean plane, but there are centrally symmetric convex octagons and decagons which can form fivefold lattice tilings. This paper characterizes all the convex domains which can form five or sixfold lattice tilings of the Euclidean plane. They are parallelograms, centrally symmetric hexagons, three types of centrally symmetric octagons and three types of centrally symmetric decagons.

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1. Introduction

Planar tilings is an ancient subject in our civilization. It has been considered in the arts by craftsmen since antiquity. Up to now, it is still an active research field in mathematics and some basic problems remain unsolved. In 1885, Fedorov [6] discovered that there are only two types of two-dimensional lattice tiles: *parallelograms and centrally symmetric hexagons.* In 1917, for the purpose to verify the second part of Hilbert's 18th problem in $\mathbb{E}^2$, Bieberbach suggested Reinhardt (see [23]) to determine all the two-dimensional congruent tiles. However, to complete the list turns out to be challenging and dramatic. Over the years, the list has been successively extended by Reinhardt, Kershner, James, Rice, Stein, Mann, McLoud-Mann and Von Derau (see [19]), its completeness has been mistakenly announced several times! In 2017, Rao [22] announced a completeness proof based on computer checks. For an updated survey on this topic, we refer to Zong [33].

The three-dimensional case was also studied in the ancient time. More than 2,300 years ago, Aristotle claimed that both identical regular tetrahedra and identical cubes can fill the whole space without gap. The cube case is obvious! However, the tetrahedron case is wrong and such a tiling is impossible (see [16]).

Let K be a convex body with (relative) interior $\mathrm{int}(K)$, (relative) boundary $\partial(K)$ and volume $\mathrm{vol}(K)$, and let X be a discrete set, both in $\mathbb{E}^n$. We call $K+X$ a *translative tiling* of $\mathbb{E}^n$ and call K a *translative tile* if $K+X=\mathbb{E}^n$ and the translates $\mathrm{int}(K)+\mathbf{x}_i$ are pairwise disjoint, where $\mathbf{x}_i\in X$. In other words, if $K+X$ is both a packing and a covering in $\mathbb{E}^n$ (see [7, 32]). In particular, we call $K+\Lambda$ a *lattice tiling* of $\mathbb{E}^n$ and call K a *lattice tile* if Λ is an n-dimensional lattice. Apparently, a translative tile must be a convex polytope. Usually, a lattice tile is called a *parallelohedron.*

In 1885, Fedorov [6] also characterized the three-dimensional lattice tiles: *A three-dimensional lattice tile must be a parallelotope, a hexagonal prism, a rhombic dodecahedron, an elongated dodecahedron, or a truncated octahedron.* The situations in higher dimensions turn out to be very complicated. Through the works of Delone [3], Štogrin [25] and Engel [5], we know that there are exactly 52 combinatorially different types of parallelohedra in $\mathbb{E}^4$. A computer classification for the five-dimensional parallelohedra was announced by Dutour Sikirić, Garber, Schürmann and Waldmann [4] only in 2015.

Let Λ be an n-dimensional lattice. The *Dirichlet-Voronoi cell* of Λ is defined by

$$C=\left\{\mathbf{x}:\mathbf{x}\in\mathbb{E}^n,\ |\mathbf{x},\mathbf{o}|\leq|\mathbf{x},\Lambda|\right\},$$

where $|X,Y|$ denotes the Euclidean distance between X and Y. Clearly, $C+\Lambda$ is a lattice tiling and the Dirichlet-Voronoi cell C is a parallelohedron. In 1908, Voronoi [27] made a conjecture that *every parallelohedron is a linear transformation image of the Dirichlet-Voronoi cell of a suitable* *lattice.* In $\mathbb{E}^2$, $\mathbb{E}^3$ and $\mathbb{E}^4$, this conjecture was confirmed by Delone [3] in 1929. In higher dimensions, it is still open.

To characterize the translative tiles is another fascinating problem. At the first glance, translative tilings should be more complicated than lattice tilings. However, the dramatic story had a happy end! It was shown by Minkowski [21] in 1897 that *every translative tile must be centrally symmetric.* In 1954, Venkov [26] proved that *every translative tile must be a lattice tile* (*parallelohedron*) (see [1] for generalizations). Later, a new proof for this beautiful result was independently discovered by McMullen [20].

Let X be a discrete multiset in $\mathbb{E}^n$ and let k be a positive integer. We call $K+X$ a *k-fold translative tiling* of $\mathbb{E}^n$ and call K a *k-fold translative tile* if every point $\mathbf{x}\in\mathbb{E}^n$ belongs to at least k translates of K in $K+X$ and every point $\mathbf{x}\in\mathbb{E}^n$ belongs to at most k translates of $\mathrm{int}(K)$ in $\mathrm{int}(K)+X$. In other words, $K+X$ is both a k-fold packing and a k-fold covering in $\mathbb{E}^n$ (see [7, 32]). In particular, we call $K+\Lambda$ a k-fold lattice tiling of $\mathbb{E}^n$ and call K a *k-fold lattice tile* if Λ is an n-dimensional lattice. Apparently, a k-fold translative tile must be a convex polytope. In fact, similar to Minkowski's characterization, it was shown by Gravin, Robins and Shiryaev [9] that *a k-fold translative tile must be a centrally symmetric polytope with centrally symmetric facets.* Let $\det(\Lambda)$ denote the *determinant* of a lattice Λ. One can easily deduce that $\mathrm{vol}(K)=k\cdot\det(\Lambda)$ if $K+\Lambda$ is a k-fold lattice tiling of $\mathbb{E}^n$.

Multiple tilings were first investigated by Furtwängler [8] in 1936 as a generalization of Minkowski's conjecture on cube tilings. Let C denote the n-dimensional unit cube. Furtwängler made a conjecture that *every k-fold lattice tiling $C+\Lambda$ has twin cubes. In other words, every multiple lattice tiling $C+\Lambda$ has two cubes sharing a whole facet.* In the same paper, he proved the two- and three-dimensional cases. Unfortunately, when $n\geq 4$, this beautiful conjecture was disproved by Hajós [13] in 1941. In 1979, Robinson [24] determined all the integer pairs (n,k) for which Furtwängler's conjecture is false. We refer to Zong [30, 31] for detailed accounts on this fascinating problem, and to pages 82-84 of Gruber and Lekkerkerker [12] for some generalizations.

Let P be an n-dimensional centrally symmetric convex polytope, let $\tau(P)$ denote the smallest integer k such that P is a k-fold translative tile, and let $\tau^*(P)$ denote the smallest integer k such that P is a k-fold lattice tile. For convenience, we define $\tau(P)=\infty$ if P cannot form translative tiling of any multiplicity. Clearly, for every convex polytope we have

$$\tau(P)\leq\tau^*(P).$$

It is a basic problem (see [28]) to determine if $\tau(P)=\tau^*(P)$ holds for every polytope. Up to now, this problem is open even in the plane.

If σ is a non-singular affine linear transformation from $\mathbb{E}^n$ to $\mathbb{E}^n$, it can be easily verified that $P+X$ is a k-fold tiling of $\mathbb{E}^n$ if and only if $\sigma(P)+\sigma(X)$ is a k-fold tiling of $\mathbb{E}^n$. Thus, both $\tau(\sigma(P))=\tau(P)$ and $\tau^*(\sigma(P))=\tau^*(P)$ hold for all convex polytopes P and all non-singular affine linear transformations σ.

In 1994, Bolle [2] proved that *every centrally symmetric lattice polygon is a multiple lattice tile*, where a lattice polygon means a polygon with lattice point vertices. However, little is known about the multiplicity. Let Λ denote the two-dimensional integer lattice $\mathbb{Z}^2$, and let D_8 denote the octagon with vertices $(1,0)$, $(2,0)$, $(3,1)$, $(3,2)$, $(2,3)$, $(1,3)$, $(0,2)$ and $(0,1)$. As a particular example of Bolle's theorem, it was discovered by Gravin, Robins and Shiryaev [9] that $D_8+\Lambda$ *is a sevenfold lattice tiling of* $\mathbb{E}^2$. Consequently, we have

$$\tau^*(D_8)\leq 7.$$

In 2000, Kolountzakis [14] proved that, if D is a two-dimensional convex domain which is not a parallelogram and $D+X$ is a multiple tiling in $\mathbb{E}^2$, then X must be a finite union of translated two-dimensional lattices. In 2013, a similar result in $\mathbb{E}^3$ was discovered by Gravin, Kolountzakis, Robins and Shiryaev [10]. Afterwards, Lev and Liu [17], Liu [18] and Kolountzakis [15] made important progress on this topic.

Recently, Yang and Zong [28] proved the following results: *Besides parallelograms and centrally symmetric hexagons, there is no other convex domain which can form a two, three or fourfold lattice tiling in the Euclidean plane. However, there are convex octagons and decagons which can* *form fivefold lattice tilings. Consequently, whenever $n \geq 3$, there are non-parallelohedral polytopes which can form fivefold lattice tilings in the n-dimensional Euclidean space.*

This paper characterizes all the two-dimensional five and sixfold lattice tiles by proving the following results.

Theorem 1. *A convex domain can form a fivefold lattice tiling of the Euclidean plane if and only if it is a parallelogram, a centrally symmetric hexagon, a centrally symmetric octagon (under a suitable affine linear transformation) with vertices* $\mathbf{v}_1 = (-\alpha, -\frac{3}{2})$, $\mathbf{v}_2 = (1-\alpha, -\frac{3}{2})$, $\mathbf{v}_3 = (1+\alpha, -\frac{1}{2})$, $\mathbf{v}_4 = (1-\alpha, \frac{1}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \alpha < \frac{1}{4}$, *or with vertices* $\mathbf{v}_1 = (\beta, -2)$, $\mathbf{v}_2 = (1+\beta, -2)$, $\mathbf{v}_3 = (1-\beta, 0)$, $\mathbf{v}_4 = (\beta, 1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $\frac{1}{4} < \beta < \frac{1}{3}$, *or a centrally symmetric decagon (under a suitable affine linear transformation) with* $\mathbf{u}_1 = (0, 1)$, $\mathbf{u}_2 = (1, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (\frac{3}{2}, 0)$, $\mathbf{u}_5 = (1, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ *and* $\mathbf{u}_{10} = -\mathbf{u}_5$ *as the middle points of its edges.*

Theorem 2. *Let W denote the quadrilateral with vertices* $\mathbf{w}_1 = (-\frac{1}{2}, 1)$, $\mathbf{w}_2 = (-\frac{1}{2}, \frac{3}{4})$, $\mathbf{w}_3 = (-\frac{2}{3}, \frac{2}{3})$ *and* $\mathbf{w}_4 = (-\frac{3}{4}, \frac{3}{4})$. *A centrally symmetric convex decagon can take* $\mathbf{u}_1 = (0, 1)$, $\mathbf{u}_2 = (1, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (\frac{3}{2}, 0)$, $\mathbf{u}_5 = (1, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ *and* $\mathbf{u}_{10} = -\mathbf{u}_5$ *as the middle points of its edges if and only if one of its vertices is an interior point of W.*

Theorem 3. *A convex domain can form a sixfold lattice tiling of the Euclidean plane if and only if it is a parallelogram, a centrally symmetric hexagon, a centrally symmetric octagon (under suitable affine linear transformations) with vertices* $\mathbf{v}_1 = (-\alpha, -2)$, $\mathbf{v}_2 = (1-\alpha, -2)$, $\mathbf{v}_3 = (1+\alpha, -1)$, $\mathbf{v}_4 = (1-\alpha, 0)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \alpha < \frac{1}{6}$, *a centrally symmetric decagon (under suitable affine linear transformations) with* $\mathbf{u}_1 = (-1, \frac{1}{2})$, $\mathbf{u}_2 = (\frac{1}{2}, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, 1)$, $\mathbf{u}_4 = (2, \frac{1}{2})$, $\mathbf{u}_5 = (2, 0)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ *and* $\mathbf{u}_{10} = -\mathbf{u}_5$ *as the middle points of its edges, or with* $\mathbf{u}_1 = (-\frac{1}{2}, 1)$, $\mathbf{u}_2 = (\frac{1}{2}, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (2, 0)$, $\mathbf{u}_5 = (\frac{3}{2}, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ *and* $\mathbf{u}_{10} = -\mathbf{u}_5$ *as the middle points of its edges.*

Theorem 4. *Let Q denote the quadrilateral with vertices* $\mathbf{q}_1 = (0, 1)$, $\mathbf{q}_2 = (0, \frac{5}{6})$, $\mathbf{q}_3 = (-\frac{1}{4}, \frac{3}{4})$ *and* $\mathbf{q}_4 = (-\frac{1}{3}, \frac{5}{6})$. *A centrally symmetric convex decagon P_{10} can take* $\mathbf{u}_1 = (-1, \frac{1}{2})$, $\mathbf{u}_2 = (\frac{1}{2}, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, 1)$, $\mathbf{u}_4 = (2, \frac{1}{2})$, $\mathbf{u}_5 = (2, 0)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ *and* $\mathbf{u}_{10} = -\mathbf{u}_5$ *as the middle points of its edges if and only if one of its vertices is an interior point of Q.*

Let Q^* denote the quadrilateral with vertices $\mathbf{q}_1 = (0, \frac{5}{4})$, $\mathbf{q}_2 = (\frac{1}{6}, \frac{7}{6})$, $\mathbf{q}_3 = (0, 1)$ *and* $\mathbf{q}_4 = (-\frac{1}{6}, \frac{7}{6})$. *A centrally symmetric convex decagon P_{10}^* can take* $\mathbf{u}_1 = (\frac{1}{2}, -1)$, $\mathbf{u}_2 = (\frac{3}{2}, -\frac{1}{2})$, $\mathbf{u}_3 = (2, 0)$, $\mathbf{u}_4 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_5 = (\frac{1}{2}, 1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ *and* $\mathbf{u}_{10} = -\mathbf{u}_5$ *as the middle points of its edges if and only if one of its vertices is an interior point of Q^*.*

Remark 1. In principle, our method can characterize all k-fold lattice tiles for any given k. Of course, the complexity increases along with the multiplicity k.

2. Basic Results

Let P_{2m} denote a centrally symmetric convex $2m$-gon centered at the origin, let $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_{2m}$ be the $2m$ vertices of P_{2m} enumerated clock-wise, and let G_1, G_2, $\ldots$, G_{2m} be the $2m$ edges of P_{2m}, where G_i has two vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$. For convenience, we write $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2m}\}$ and $\Gamma = \{G_1, G_2, \ldots, G_{2m}\}$.

Assume that $P_{2m} + X$ is a $\tau(P_{2m})$-fold translative tiling of $\mathbb{E}^2$, where $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$ is a discrete multiset with $\mathbf{x}_1 = \mathbf{o}$. Now, let us observe the local structures of $P_{2m} + X$ at the vertices $\mathbf{v} \in V + X$.

Let $X^{\mathbf{v}}$ denote the subset of X consisting of all points $\mathbf{x}_i$ such that

$$\mathbf{v} \in \partial(P_{2m}) + \mathbf{x}_i.$$

Since $P_{2m}+X$ is a multiple tiling, the set $X^{\mathbf{v}}$ can be divided into disjoint subsets $X_1^{\mathbf{v}}$, $X_2^{\mathbf{v}}$, $\ldots$, $X_t^{\mathbf{v}}$ such that the translates in $P_{2m}+X_j^{\mathbf{v}}$ can be re-enumerated as $P_{2m}+\mathbf{x}_1^j$, $P_{2m}+\mathbf{x}_2^j$, …, $P_{2m}+\mathbf{x}_{s_j}^j$ satisfying the following conditions:

1. $\mathbf{v}\in\partial(P_{2m})+\mathbf{x}_i^j$ *holds for all* $i=1,2,\ldots,s_j$.

2. *Let* $\angle_i^j$ *denote the inner angle of* $P_{2m}+\mathbf{x}_i^j$ *at* $\mathbf{v}$ *with two half-line edges* $L_{i,1}^j$ *and* $L_{i,2}^j$*, where* $L_{i,1}^j$, $\mathbf{x}_i^j-\mathbf{v}$ *and* $L_{i,2}^j$ *are in clock-wise order. Then, the inner angles join properly as*

$$L_{i,2}^j=L_{i+1,1}^j$$

holds for all $i=1,\ 2,\ \ldots,\ s_j$*, where* $L_{s_j+1,1}^j=L_{1,1}^j$.

For convenience, we call such a sequence $P_{2m}+\mathbf{x}_1^j$, $P_{2m}+\mathbf{x}_2^j$, …, $P_{2m}+\mathbf{x}_{s_j}^j$ an *adjacent wheel* at $\mathbf{v}$. It is easy to see that

$$\sum_{i=1}^{s_j}\angle_i^j=2w_j\cdot\pi$$

hold for positive integers w_j. Then we define

$$\varpi(\mathbf{v})=\sum_{j=1}^{t}w_j=\frac{1}{2\pi}\sum_{j=1}^{t}\sum_{i=1}^{s_j}\angle_i^j$$

and

$$\varphi(\mathbf{v})=\sharp\left\{\mathbf{x}_i:\ \mathbf{x}_i\in X,\ \mathbf{v}\in \mathrm{int}(P_{2m})+\mathbf{x}_i\right\}.$$

Clearly, if $P_{2m}+X$ is a $\tau(P_{2m})$-fold translative tiling of $\mathbb{E}^2$, then

$$\tau(P_{2m})=\varphi(\mathbf{v})+\varpi(\mathbf{v}) \tag{1}$$

holds for all $\mathbf{v}\in V+X$.

First, let us introduce some basic results which will be useful in this paper.

Lemma 1 (Bolle [2]**).** *A convex polygon is a k-fold lattice tile for a lattice* Λ *and some positive integer* k *if and only if the following conditions are satisfied:*
1. *It is centrally symmetric.*
2. *When it is centered at the origin, in the relative interior of each edge* G *there is a point of* $\frac{1}{2}\Lambda$.
3. *If the midpoint of an edge* G *is not in* $\frac{1}{2}\Lambda$ *then* G *is a lattice vector of* Λ.

Lemma 2 (Yang and Zong [28]**).** *If* D *is a two-dimensional convex domain which is neither a parallelogram nor a centrally symmetric hexagon, then we have*

$$\tau^*(D)\geq 5.$$

Lemma 3. *If* m *is even and* $P_{2m}+\Lambda$ *is a multiple lattice tiling, then* P_{2m} *has an edge* G *which is a lattice vector of* Λ.

Proof. We assume that $\Lambda=\mathbb{Z}^2$. Let $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_{2m}$ be the $2m$ vertices of P_{2m} arranged in clock-wise. Let G_i denote the edge of P_{2m} with vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$, where $\mathbf{v}_{2m+1}=\mathbf{v}_1$.

If the midpoint of one of the $2m$ edges, say G_1, is not in $\frac{1}{2}\Lambda$, then it follows from Lemma 1 that G_1 is a lattice vector of Λ.

Let $\mathbf{u}_i$ denote the midpoint of G_i. If $\mathbf{u}_i\in\frac{1}{2}\Lambda$ hold for all $i=1,2,\ldots,2m$, then we have

$$\begin{cases}\mathbf{v}_2-\mathbf{u}_1=\mathbf{u}_1-\mathbf{v}_1,\\ \mathbf{v}_3-\mathbf{u}_2=\mathbf{u}_2-\mathbf{v}_2,\\ \qquad\cdots\\ \mathbf{v}_{m+1}-\mathbf{u}_m=\mathbf{u}_m-\mathbf{v}_m,\end{cases}$$

which implies that

$$\mathbf{v}_{m+1}=(-1)^m\mathbf{v}_1+2\sum_{i=1}^{m}(-1)^{m-i}\mathbf{u}_i. \tag{2}$$

Since m is even and $\mathbf{v}_{m+1} = -\mathbf{v}_1$, it can be deduced by (2) that

$$\mathbf{v}_1 = \sum_{i=1}^{m} (-1)^{i+1}\mathbf{u}_i \in \tfrac{1}{2}\Lambda.$$

If fact, in this case all the vertices belong to $\frac{1}{2}\Lambda$. Then, we get

$$\mathbf{v}_2 - \mathbf{v}_1 = 2\,(\mathbf{u}_1 - \mathbf{v}_1) \in \Lambda.$$

The lemma is proved. □

Lemma 4. *Let $\mathbf{u}_i$ be the middle point of G_i. If m is an odd positive integer, $P_{2m} + \Lambda$ is a k-fold lattice tiling of $\mathbb{E}^2$, and all $\mathbf{u}_i$ belong to $\frac{1}{2}\Lambda$, then we have*

$$\sum_{i=1}^{m} (-1)^{i}\mathbf{u}_i = \mathbf{o},$$

where $\mathbf{o} = (0,0)$ is the origin of $\mathbb{E}^2$.

Proof. Since $\mathbf{u}_i$ is the middle point of G_i, we have

$$\begin{cases} \mathbf{v}_2 &= 2\mathbf{u}_1 - \mathbf{v}_1, \\ \mathbf{v}_3 &= 2\mathbf{u}_2 - \mathbf{v}_2, \\ & \cdots \\ \mathbf{v}_{m+1} &= 2\mathbf{u}_m - \mathbf{v}_m, \end{cases}$$

which implies

$$-\mathbf{v}_1 = \mathbf{v}_{m+1} = -\mathbf{v}_1 - 2\sum_{i=1}^{m} (-1)^{i}\mathbf{u}_i$$

and finally

$$\sum_{i=1}^{m} (-1)^{i}\mathbf{u}_i = \mathbf{o}.$$

The lemma is proved. □

Lemma 5 (Yang and Zong [29]). *Assume that P_{2m} is a centrally symmetric convex $2m$-gon centered at the origin and $P_{2m} + X$ is a $\tau(P_{2m})$-fold translative tiling of the plane, where $m \geq 4$. If $\mathbf{v} \in V + X$ is a vertex and $G \in \Gamma + X$ is an edge with $\mathbf{v}$ as one of its two vertices, then there are at least $\lceil (m-3)/2 \rceil$ different translates $P_{2m} + \mathbf{x}_i$ satisfying both*

$$\mathbf{v} \in \partial(P_{2m}) + \mathbf{x}_i$$

and

$$G \setminus \{\mathbf{v}\} \subset \mathrm{int}(P_{2m}) + \mathbf{x}_i.$$

Lemma 6 (Yang and Zong [29]). *Let P_{2m} be a centrally symmetric convex $2m$-gon, then*

$$\tau^*(P_{2m}) \geq \tau(P_{2m}) \geq \begin{cases} m-1, & \text{if } m \text{ is even}, \\ m-2, & \text{if } m \text{ is odd}. \end{cases}$$

Lemma 7 (Yang and Zong [29]). *Assume that P_{2m} is a centrally symmetric convex $2m$-gon centered at the origin, $P_{2m} + X$ is a translative multiple tiling of the plane, and $\mathbf{v} \in V + X$. Then we have*

$$\varpi(\mathbf{v}) = \kappa \cdot \frac{m-1}{2} + \ell \cdot \frac{1}{2},$$

where κ is a positive integer and ℓ is the number of the edges in $\Gamma + X$ which take $\mathbf{v}$ as an interior point.

3. Technical Lemmas

Lemma 8. *Let P_{14} be a centrally symmetric convex tetradecagon, then*

$$\tau^*(P_{14}) \geq \tau(P_{14}) \geq 7.$$

Proof. We take $\mathbf{v} \in V + X$ and assume that $P_{14} + \mathbf{x}_1$, $P_{14} + \mathbf{x}_2$, $\ldots$, $P_{14} + \mathbf{x}_s$ is an adjacent wheel at $\mathbf{v}$. First, it follows from Lemma 5 and Lemma 7 that

$$\varphi(\mathbf{v}) \geq 2 \tag{3}$$

and

$$\varpi(\mathbf{v}) \geq 3.$$

Now, we consider three cases.

Case 1. *$\varpi(\mathbf{v}) \geq 5$ holds for a vertex $\mathbf{v} \in V + X$.* Then, by (1) and (3) we get

$$\tau(P_{14}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 7.$$

Case 2. *$\varpi(\mathbf{v}) = 4$ holds for a vertex $\mathbf{v} \in V + X$.* Then, by Lemma 7 we get $\ell \neq 0$. If $\mathbf{v} \in \mathrm{int}(G)$ holds for a suitable edge G, applying Lemma 5 to G and its two vertices we get

$$\varphi(\mathbf{v}) \geq 4.$$

Then it follows by (1) that

$$\tau(P_{14}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 8.$$

Case 3. *$\varpi(\mathbf{v}) = 3$ holds for every vertex $\mathbf{v} \in V + X$.* Then, the adjacent wheels at all $\mathbf{v} \in V$ are essentially unique, as shown by Figure 1. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_{14}$ be the fourteen vertices of P_{14}. It follows that there are five point $\mathbf{y}_i \in X$ such that $P_{14} + \mathbf{x}_1$, $P_{14} + \mathbf{x}_7$, $P_{14} + \mathbf{y}_1$, $\ldots$, $P_{14} + \mathbf{y}_5$ is the adjacent wheel at $\mathbf{v}_1^*$. Then we have $\mathbf{v}_{10} + \mathbf{y}_2 = \mathbf{v}_1^*$, $\mathbf{v}_8 + \mathbf{y}_4 = \mathbf{v}_1^*$ and

$$\mathbf{v} \in \mathrm{int}(P_{14}) + \mathbf{y}_i \quad i = 2,\ 4.$$

By convexity, it can be easily deduced that

$$\mathbf{v}_4^* \in \mathrm{int}(P_{14}) + \mathbf{y}_i, \quad i = 2,\ 4.$$

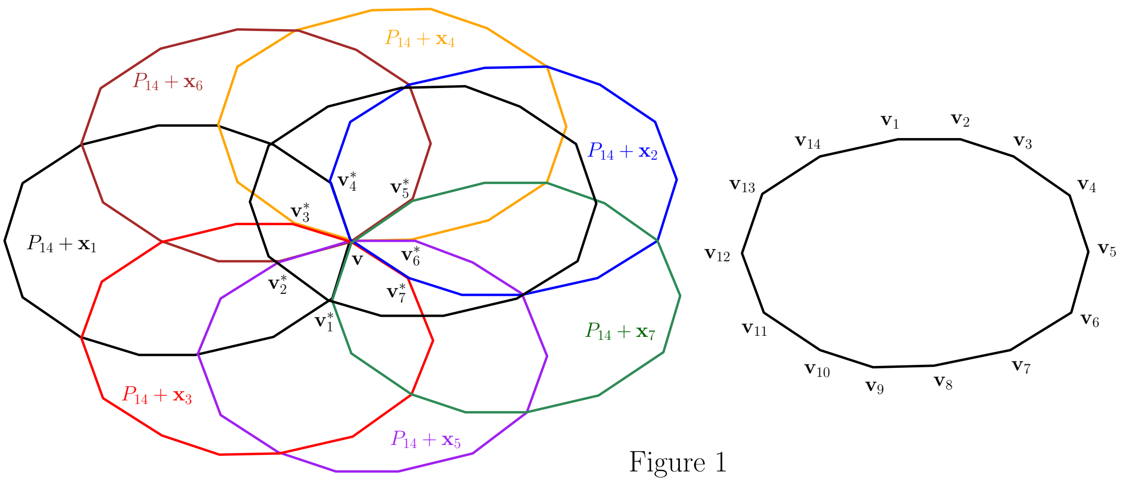

Figure 1

On the other hand, the adjacent wheel at $\mathbf{v}_4^*$ has two different translates taking $\mathbf{v}$ as an interior point as well. Thus, we have

$$\varphi(\mathbf{v}) \geq 4$$

and

$$\tau(P_{14}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 7. \tag{4}$$

The lemma is proved. $\square$

Lemma 9. *Let P_{12} be a centrally symmetric convex dodecagon, then*

$$\tau^*(P_{12}) \geq 7.$$

Proof. Since $\tau^*(P_{2m})$ is invariant under linear transformations on P_{2m}, we assume that $\Lambda = \mathbb{Z}^2$ and $P_{12} + \Lambda$ is a $\tau^*(P_{12})$-fold lattice tiling. Let $\mathbf{u}_i$ denote the middle point of G_i and write $\mathbf{v}_i = (x_i, y_i)$ and $\mathbf{u}_i = (x_i', y_i')$. By Lemma 3 and a uni-modular transformation, we may assume that $\mathbf{v}_2-\mathbf{v}_1=(k,0)$ and $y_1'>0$, where k is a positive integer. By reduction (as shown by Figure 8), we may assume further that $\mathbf{v}_2-\mathbf{v}_1=(1,0)$. For convenience, let P denote the parallelogram with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_7=-\mathbf{v}_1$ and $\mathbf{v}_8=-\mathbf{v}_2$.

By Lemma 1 it follows that all y_2-y_3, y_3-y_4, y_4-y_5, y_5-y_6 and y_6-y_7 are positive integers. Thus, we have

$$y_1=y_1'=y_2\geq \frac{5}{2}.$$

If $y_1=y_1'=y_2\geq 3$, then we have

$$\tau^*(P_{12})=\mathrm{vol}(P_{12})>\mathrm{vol}(P)\geq 6.$$

If $y_1=y_1'=y_2=\frac{5}{2}$, then all $\mathbf{u}_i$ belong to $\frac{1}{2}\Lambda$. Let T_i denote the triangle with vertices $\mathbf{u}_i$, $\mathbf{u}_{i+1}$ and $\mathbf{u}_6$, where $i=2,3$ and 4. Clearly, all $y_i'-y_6'$ are positive integers. Thus, we have

$$\mathrm{vol}(T_i)=\frac{1}{2}\begin{vmatrix} x_i'-x_6' & y_i'-y_6' \\ x_{i+1}'-x_6' & y_{i+1}'-y_6' \end{vmatrix}\geq\frac{1}{4}$$

and

$$\tau^*(P_{12})=\mathrm{vol}(P_{12})>\mathrm{vol}(P)+2(\mathrm{vol}(T_2)+\mathrm{vol}(T_3)+\mathrm{vol}(T_4))\geq 5+6\cdot\frac{1}{4}>6.$$

The lemma is proved. □

Let P be a lattice polygon with vertices in $\mathbb{Z}^2$. Let $\alpha(P)$ denote the area of P, let $\ell(P)$ denote the number of the points in $P\cap\mathbb{Z}^2$, and let $\ell^*(P)$ denote the number of the points in $\partial(P)\cap\mathbb{Z}^2$. Then we have the following result (see page 316 of [11]):

Pick's Theorem.

$$\alpha(P)=\ell(P)-\tfrac{1}{2}\ell^*(P)-1.$$

Lemma 10. *For every centrally symmetric convex decagon P_{10} we have*

$$\tau^*(P_{10})\geq 5,$$

where the equality holds if and only if, under a suitable affine linear transformation, it takes $\mathbf{u}_1=(0,1)$, $\mathbf{u}_2=(1,1)$, $\mathbf{u}_3=(\frac{3}{2},\frac{1}{2})$, $\mathbf{u}_4=(\frac{3}{2},0)$, $\mathbf{u}_5=(1,-\frac{1}{2})$, $\mathbf{u}_6=-\mathbf{u}_1$, $\mathbf{u}_7=-\mathbf{u}_2$, $\mathbf{u}_8=-\mathbf{u}_3$, $\mathbf{u}_9=-\mathbf{u}_4$ and $\mathbf{u}_{10}=-\mathbf{u}_5$ as the middle points of its edges. Furthermore

$$\tau^*(P_{10})=6$$

holds if and only if, under a suitable affine linear transformation, it takes $\mathbf{u}_1=(-1,\frac{1}{2})$, $\mathbf{u}_2=(\frac{1}{2},1)$, $\mathbf{u}_3=(\frac{3}{2},1)$, $\mathbf{u}_4=(2,\frac{1}{2})$, $\mathbf{u}_5=(2,0)$, $\mathbf{u}_6=-\mathbf{u}_1$, $\mathbf{u}_7=-\mathbf{u}_2$, $\mathbf{u}_8=-\mathbf{u}_3$, $\mathbf{u}_9=-\mathbf{u}_4$ and $\mathbf{u}_{10}=-\mathbf{u}_5$ as the middle points of its edges, or takes $\mathbf{u}_1=(-\frac{1}{2},1)$, $\mathbf{u}_2=(\frac{1}{2},1)$, $\mathbf{u}_3=(\frac{3}{2},\frac{1}{2})$, $\mathbf{u}_4=(2,0)$, $\mathbf{u}_5=(\frac{3}{2},-\frac{1}{2})$, $\mathbf{u}_6=-\mathbf{u}_1$, $\mathbf{u}_7=-\mathbf{u}_2$, $\mathbf{u}_8=-\mathbf{u}_3$, $\mathbf{u}_9=-\mathbf{u}_4$ and $\mathbf{u}_{10}=-\mathbf{u}_5$ as the middle points of its edges.

Proof. Let $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_{10}$ be the ten vertices of P_{10} enumerated clock-wise, let G_i denote the edge of P_{10} with vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$, where $\mathbf{v}_{11}=\mathbf{v}_1$, and let $\mathbf{u}_i$ denote the middle point of G_i. For convenience, we write $\mathbf{v}_i=(x_i,y_i)$ and $\mathbf{u}_i=(x_i',y_i')$.

It is known that $\sigma(D)+\sigma(\Lambda)$ is a k-fold lattice tiling of $\mathbb{E}^2$ whenever $D+\Lambda$ is such a tiling and σ is a non-singular linear transformation from $\mathbb{E}^2$ to $\mathbb{E}^2$. Therefore, without loss of generality, by Lemma 2 we may assume that $\Lambda=\mathbb{Z}^2$ and $P_{10}+\Lambda$ is a five or sixfold lattice tiling of $\mathbb{E}^2$.

By Lemma 1 we know that

$$\mathrm{int}(G_i)\cap\tfrac{1}{2}\Lambda\neq\emptyset$$

holds for all the ten edges G_i and, if $\mathbf{u}_i\notin\frac{1}{2}\Lambda$, then G_i is a lattice vector of Λ. Now, we consider two cases.

Case 1. *G_1 is a lattice vector of Λ.* Without loss of generality, by a uni-modular linear transformation, we assume that $\mathbf{v}_2-\mathbf{v}_1=(k,0)$ and $y_1'>0$, where k is a positive integer. In fact, by reduction (as shown by Figure 8), one may assume that G_1 is primitive as a lattice vector and therefore $k=1$. Then, it can be deduced that

$$y_1=y_1'=y_2\in\tfrac{1}{2}\mathbb{Z}$$

and all $y_i - y_{i+1}$ are integers. In particular, when $i = 2$, 3, 4 and 5, they are positive integers. Thus, one can deduce that

$$y_1' = 2 \text{ or } \frac{5}{2}.$$

Case 1.1. $y_1' = 2$. Then we must have

$$y_2 - y_3 = y_3 - y_4 = y_4 - y_5 = y_5 - y_6 = 1.$$

By the second term of Lemma 1, one can deduce that

$$\mathbf{u}_i \in \tfrac{1}{2}\Lambda, \quad i = 2,\ 3,\ 4,\ 5.$$

Since $\mathbf{v}_2 = (1, 0) + \mathbf{v}_1$ and

$$\mathbf{v}_{i+1} = 2\mathbf{u}_i - \mathbf{v}_i, \quad i = 2,\ 3,\ 4,\ 5,$$

it can be deduced that

$$-\mathbf{v}_1 = \mathbf{v}_6 = 2(\mathbf{u}_5 - \mathbf{u}_4 + \mathbf{u}_3 - \mathbf{u}_2) + (1, 0) + \mathbf{v}_1$$

and therefore

$$\mathbf{v}_i \in \tfrac{1}{2}\Lambda, \quad i = 1,\ 2,\ \ldots,\ 10.$$

Then all G_i are lattice vectors.

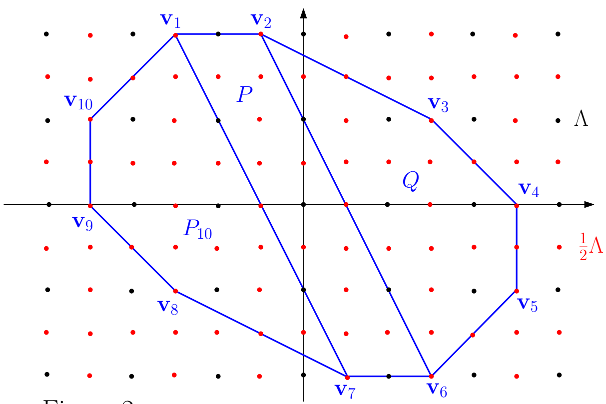

Figure 2

Let P denote the parallelogram with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_6$ and $\mathbf{v}_7$, and let Q denote the pentagon with vertices $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$ and $\mathbf{v}_6$, as shown by Figure 2. Applying Pick's theorem to Q, we get

$$\mathrm{vol}(Q) \geq \left(\frac{5}{2} - 1\right)$$

and therefore

$$\tau^*(P_{10}) = \mathrm{vol}(P_{10}) = \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(Q) \geq 4 + 2 \cdot \left(\frac{5}{2} - 1\right) = 7.$$

Case 1.2. $y_1' = \frac{5}{2}$. Then all $y_i - y_{i+1}$ are positive integers for $2 \leq i \leq 5$. If

$$\mathbf{u}_i \in \tfrac{1}{2}\Lambda$$

hold for all $i = 2$, 3, 4 and 5, similar to the previous case one can deduce

$$\tau^*(P_{10}) = \mathrm{vol}(P_{10}) \geq 7.$$

If $\mathbf{u}_i \notin \frac{1}{2}\Lambda$ holds for one of these indices, then we have $y_i - y_{i+1} = 2$. By a uni-modular transformation, we may assume that $-\frac{7}{4} \leq x_1 < \frac{3}{4}$. Then we have $\mathbf{v}_2 - \mathbf{v}_6 = (x, 5)$, where $-\frac{5}{2} \leq x < \frac{5}{2}$. If $\mathbf{v}_i - \mathbf{v}_{i+1} = (k, 2)$ with $|k| \geq 2$, let Q denote the pentagon with vertices $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$ and $\mathbf{v}_6$, then we have

$$\mathrm{vol}(Q) > \frac{1}{2} \begin{vmatrix} x & 5 \\ k & 2 \end{vmatrix} = \frac{1}{2}\,|2x - 5k| \geq \frac{5}{2}$$

and thus

$$\tau^*(P_{10}) = \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(Q) \geq 10.$$

If $\mathbf{v}_i - \mathbf{v}_{i+1} = (k, 2)$ with $k = \pm 1$, then we have $x_1 \in \frac{1}{4}\mathbb{Z}$ and therefore $x \in \frac{1}{2}\mathbb{Z}$ and $-\frac{5}{2} \leq x \leq 2$. By considering two subcases with respect to $x_1 = -\frac{7}{4}$ and $x_1 \neq -\frac{7}{4}$, we can get

$$\mathrm{vol}(Q) > \frac{1}{2}$$

and

$$\tau^*(P_{10}) = \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(Q) > 6.$$

Case 2. *All the middle points* $\mathbf{u}_i$ *belong to* $\frac{1}{2}\Lambda$. Since $P_{10} + \Lambda$ is a five or sixfold lattice tiling of $\mathbb{E}^2$, one can deduce that

$$\mathrm{vol}(2P_{10}) \leq 24$$

and all $\mathbf{u}'_i = 2\mathbf{u}_i$ belong to Λ. For convenience, we define Q_{10} to be the centrally symmetric lattice decagon with vertices $\mathbf{u}'_1$, $\mathbf{u}'_2$, ..., $\mathbf{u}'_{10}$ and write $\mathbf{u}'_i = (x'_i, y'_i)$. Since Q_{10} is a centrally symmetric lattice polygon, its area must be a positive integer. Thus, we have

$$\mathrm{vol}(Q_{10}) \leq 23. \tag{5}$$

Now, we explore Q_{10} in detail by considering the following subcases.

Case 2.1. $\mathbf{u}'_1$ *is primitive in* Λ. Without loss of generality, guaranteed by uni-modular linear transformations, we take $\mathbf{u}'_1 = (0, 1)$. Then, Lemma 4 implies

$$\begin{cases} x'_4 - x'_5 = x'_3 - x'_2, \\ y'_4 - y'_5 = y'_3 - y'_2 + 1. \end{cases} \tag{6}$$

If $x'_2 \geq x'_3$ or $x'_3 = x'_4$, one can easily deduce contradiction with convexity from (6). For example, if $x'_3 = x'_4 > x'_2$, then it can be deduced by (6) that

$$\mathbf{u}'_2 - \mathbf{u}'_5 = \mathbf{u}'_{10} - \mathbf{u}'_7 = k\mathbf{u}'_1$$

with $k \geq 2$, which contradicts the assumption that Q_{10} is a centrally symmetric convex decagon. Therefore, we may assume that

$$x'_3 > x'_i$$

for all $i \neq 3$.

Let T' denote the lattice triangle with vertices $\mathbf{u}'_1$, $\mathbf{u}'_2$ and $\mathbf{u}'_3$, let Q denote the lattice quadrilateral with vertices $\mathbf{u}'_3$, $\mathbf{u}'_4$, $\mathbf{u}'_5$ and $\mathbf{u}'_6$, and let T denote the lattice triangle with vertices $\mathbf{u}'_1$, $\mathbf{u}'_3$ and $\mathbf{u}'_6$ (as shown by Figure 3). It follows from (5) and Pick's theorem that

$$\mathrm{vol}(T) \leq \frac{1}{2}\Big(23 - 2\big(\mathrm{vol}(T') + \mathrm{vol}(Q)\big)\Big) \leq 10$$

and therefore

$$x'_3 \leq 10. \tag{7}$$

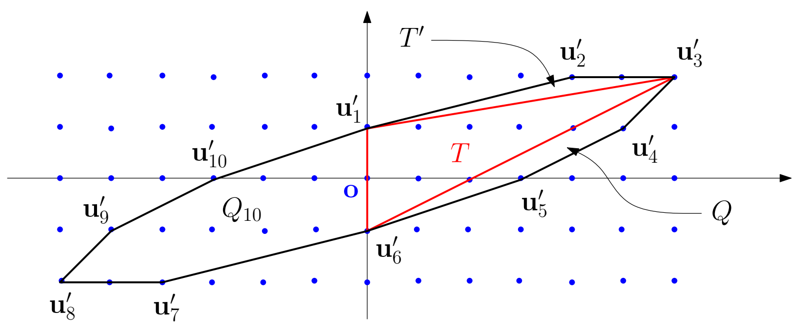

Figure 3

Let α denote the slope of G_1, that is

$$\alpha = \frac{y_2 - y_1}{x_2 - x_1}.$$

By a uni-modular linear transformation such as

$$\begin{cases} x' = x, \\ y' = y + kx, \end{cases}$$

where k is a suitable integer, we may assume that

$$0 \le \alpha < 1. \tag{8}$$

Let L_i denote the straight line containing G_i, it is obvious that P_{10} is in the strip bounded by L_1 and L_6. Furthermore, we define five slopes

$$\beta_i = \frac{y'_{i+1} - y'_i}{x'_{i+1} - x'_i}, \quad i = 1,\ 2,\ \ldots,\ 5.$$

By convexity it can be shown that there is no sixfold lattice decagon tile with $\alpha = 0$ in our setting. When $\alpha > 0$, by (6) and convexity it follows that $y'_4 - y'_5 \ge 1$ and therefore

$$y'_3 - y'_2 \ge 0.$$

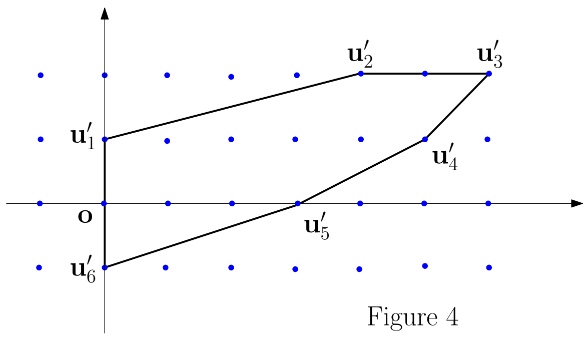

Figure 4

As shown by Figure 4, we assume that

$$\mathbf{u}'_3 - \mathbf{u}'_4 = (p_1, q_1)$$

and

$$\mathbf{u}'_5 - \mathbf{u}'_6 = (p_2, q_2),$$

where all p_i and q_i are positive integers. Then, by (7) we have

$$x'_3 - x'_2 = x'_3 - (x'_2 - x'_1) = x'_3 - (p_1 + p_2) \le 8.$$

Now, we consider in subcases with respect to the different orientations of $\mathbf{u}'_3 - \mathbf{u}'_2$.

Case 2.1.1. $y'_3 - y'_2 = 0$ *and* $x'_3 - x'_2 = 1$. By (6) and convexity we have $x'_4 - x'_5 = 1$, $y'_4 - y'_5 = 1$, $\beta_4 = 1$,

$$\beta_3 = \frac{q_1}{p_1} > 1$$

and

$$\beta_5 = \frac{q_2}{p_2} < 1.$$

Then, one can deduce that

$$\beta_1 = \frac{q_1 + q_2 - 1}{p_1 + p_2} > \frac{q_2}{p_2} = \beta_5,$$

which contradicts the convexity of Q_{10}.

Case 2.1.2. $y'_3 - y'_2 = 0$ *and* $x'_3 - x'_2 = 2$. By (6) and convexity we have $x'_4 - x'_5 = 2$, $y'_4 - y'_5 = 1$, $\beta_4 = \frac{1}{2}$,

$$\beta_3 = \frac{q_1}{p_1} > \frac{1}{2}, \tag{9}$$

$$\beta_5 = \frac{q_2}{p_2} < \frac{1}{2} \tag{10}$$

and

$$\beta_1 = \frac{q_1+q_2-1}{p_1+p_2} < \frac{q_2}{p_2}. \tag{11}$$

By (7) and (10) one can deduce that

$$\begin{aligned} 3 &\le p_2 \le 7, \\ 1 &\le p_1 \le 5 \end{aligned} \tag{12}$$

and

$$1 \le q_2 \le 3.$$

On the other hand, by (11), (10) and (12) we get

$$q_1 < p_1 \cdot \frac{q_2}{p_2} + 1 < \frac{1}{2} \cdot p_1 + 1$$

and therefore

$$1 \le q_1 \le 3.$$

Then, it can be verified that the only integer groups (p_1, q_1, p_2, q_2) satisfying (7), (9), (10) and (11) are $(1,1,3,1)$, $(1,1,4,1)$, $(1,1,5,1)$, $(1,1,6,1)$, $(1,1,7,1)$, $(1,1,5,2)$, $(1,1,6,2)$, $(1,1,7,2)$ and $(1,1,7,3)$. By checking the areas of their corresponding decagons, keeping the subcase conditions in mind, there are only two Q_{10} satisfying (5). Namely, the one with vertices $\mathbf{u}_1' = (0,1)$, $\mathbf{u}_2' = (4,2)$, $\mathbf{u}_3' = (6,2)$, $\mathbf{u}_4' = (5,1)$, $\mathbf{u}_5' = (3,0)$, $\mathbf{u}_6' = -\mathbf{u}_1'$, $\mathbf{u}_7' = -\mathbf{u}_2'$, $\mathbf{u}_8' = -\mathbf{u}_3'$, $\mathbf{u}_9' = -\mathbf{u}_4'$ and $\mathbf{u}_{10}' = -\mathbf{u}_5'$, which indeed produces fivefold lattice tiles, and the one with vertices $\mathbf{u}_1' = (0,1)$, $\mathbf{u}_2' = (5,2)$, $\mathbf{u}_3' = (7,2)$, $\mathbf{u}_4' = (6,1)$, $\mathbf{u}_5' = (4,0)$, $\mathbf{u}_6' = -\mathbf{u}_1'$, $\mathbf{u}_7' = -\mathbf{u}_2'$, $\mathbf{u}_8' = -\mathbf{u}_3'$, $\mathbf{u}_9' = -\mathbf{u}_4'$ and $\mathbf{u}_{10}' = -\mathbf{u}_5'$, which indeed produces sixfold lattice tiles. Clearly, by the linear transformation

$$\begin{cases} x' = \frac{1}{2}(x-2y), \\ y' = \frac{1}{2}y, \end{cases}$$

the first decagon is equivalent to the fivefold one stated in the lemma and the second one is equivalent to the first type of the sixfold ones (as shown by Figure 5 and Figure 6).

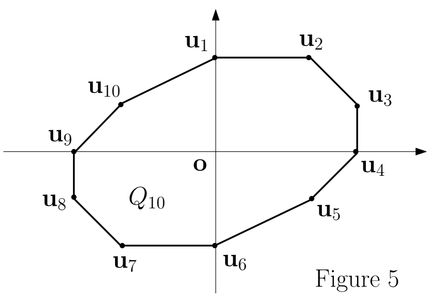

Figure 5

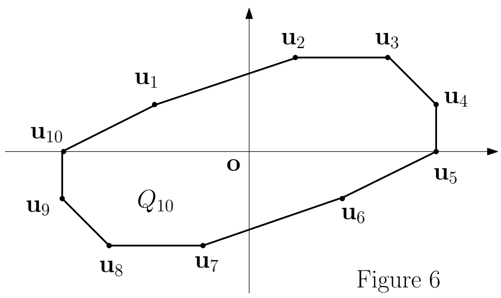

Figure 6

Case 2.1.3. $y_3' - y_2' = 0$ *and* $x_3' - x_2' = 3$. By (6) and convexity we have $x_4' - x_5' = 3$, $y_4' - y_5' = 1$, $\beta_4 = \frac{1}{3}$,

$$\beta_3 = \frac{q_1}{p_1} > \frac{1}{3}, \tag{13}$$

$$\beta_5 = \frac{q_2}{p_2} < \frac{1}{3} \tag{14}$$

and

$$\beta_1 = \frac{q_1+q_2-1}{p_1+p_2} < \frac{q_2}{p_2}. \tag{15}$$

Restricted by (7), similar to the previous case, it can be deduced that the only integer solutions (p_1, q_1, p_2, q_2) for (13), (14) and (15) are $(1,1,4,1)$, $(1,1,5,1)$, $(1,1,6,1)$, $(2,1,4,1)$, $(2,1,5,1)$ and $(2,1,6,1)$. Then one can deduce

$$\mathrm{vol}(Q_{10}) \ge 25$$

for all these cases, which contradicts (5).

Case 2.1.4. $y_3' - y_2' = 0$ *and* $x_3' - x_2' = 4$. Then, one can easily deduce that $\beta_4 = \frac{1}{4}$,

$$\beta_3 = \frac{q_1}{p_1} > \frac{1}{4},$$

$$\beta_5 = \frac{q_2}{p_2} < \frac{1}{4}$$

and

$$\beta_1 = \frac{q_1 + q_2 - 1}{p_1 + p_2} < \frac{q_2}{p_2}.$$

Restricted by (7), similar to Case 2.1.3 one can deduce that the only integer solutions (p_1, q_1, p_2, q_2) for these inequalities are $(1, 1, 5, 1)$, $(2, 1, 5, 1)$, and $(3, 1, 5, 1)$. Then we have

$$\mathrm{vol}(Q_{10}) \geq 25$$

for all these cases, which contradicts (5).

Case 2.1.5. $y_3' - y_2' = 0$ *and* $x_3' - x_2' \geq 5$. Then, one can easily deduce that $\beta_4 \leq \frac{1}{5}$, $p_2 \geq 6$ and

$$x_3' \geq 5 + 6 > 10,$$

which contradicts the restriction of (7).

Case 2.1.6. $y_3' - y_2' = 1$ *and* $x_3' - x_2' = 1$. Then, by convexity we get

$$\alpha > \beta_1 > \beta_2 = 1,$$

which contradicts the assumption of (8).

Case 2.1.7. $y_3' - y_2' = 1$ *and* $x_3' - x_2' = 2$. By (4) and convexity we get $x_4' - x_5' = 2$, $y_4' - y_5' = 2$, $\beta_4 = 1$,

$$\beta_3 = \frac{q_1}{p_1} > 1$$

and

$$\beta_5 = \frac{q_2}{p_2} < 1.$$

Then, it can be deduced that

$$\beta_1 = \frac{q_1 + q_2 - 1}{p_1 + p_2} > \frac{q_2}{p_2} = \beta_5,$$

which contradicts the convexity assumption of Q_{10}.

Case 2.1.8. $y_3' - y_2' = 1$ *and* $x_3' - x_2' = 3$. Then we have $x_4' - x_5' = 3$, $y_4' - y_5' = 2$, $\beta_2 = \frac{1}{3}$ and $\beta_4 = \frac{2}{3}$.

On one hand, by (7) it follows that $p_2 \leq 6$. On the other hand, by $\beta_2 < \beta_1 < \beta_5 < \beta_4$ it follows that

$$\frac{1}{3} < \frac{q_2}{p_2} < \frac{2}{3}.$$

Thus, the integer pair (p_2, q_2) has only five choices $(2, 1)$, $(4, 2)$, $(5, 2)$, $(5, 3)$ and $(6, 3)$. Then, by checking

$$\frac{q_1}{p_1} > \frac{2}{3},$$

$$\frac{1}{3} < \frac{q_1 + q_2 - 1}{p_1 + p_2} < \frac{q_2}{p_2}$$

and

$$p_1 + p_2 \leq 7,$$

it can be deduced that the only candidates for (p_1, q_1, p_2, q_2) are $(1, 1, 4, 2)$, $(1, 1, 5, 3)$, $(2, 2, 5, 3)$ and $(1, 1, 6, 3)$. In fact, the only candidate satisfying (5) is the one with vertices $\mathbf{u}_1' = (0, 1)$, $\mathbf{u}_2' = (5, 3)$, $\mathbf{u}_3' = (8, 4)$, $\mathbf{u}_4' = (7, 3)$, $\mathbf{u}_5' = (4, 1)$, $\mathbf{u}_6' = -\mathbf{u}_1'$, $\mathbf{u}_7' = -\mathbf{u}_2'$, $\mathbf{u}_8' = -\mathbf{u}_3'$, $\mathbf{u}_9' = -\mathbf{u}_4'$ and $\mathbf{u}_{10}' = -\mathbf{u}_5'$, satisfying

$$\mathrm{vol}(Q_{10}) = 22.$$

This decagon indeed produces sixfold lattice tiles. Clearly, it is equivalent to the second type of the sixfold ones (as shown in Figure 7) stated in the lemma under the linear transformation

$$\begin{cases} x' = \frac{1}{2}y, \\ y' = \frac{1}{2}(x-2y). \end{cases}$$

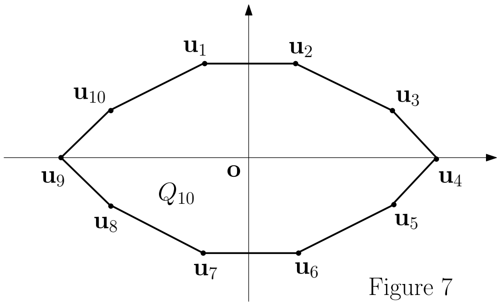

Figure 7

Case 2.1.9. *$y_3' - y_2' = 1$ and $x_3' - x_2' = 4$.* By (6), (7) and convexity it can be deduced that $p_2 \le 5$, $\beta_4 = \frac{1}{2}$ and $\beta_5 < \beta_4$. Consequently, we have $\beta_5 = \frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ or $\frac{2}{5}$. Thus, by $\beta_2 = \frac{1}{4}$ and $\beta_2 < \beta_1 < \beta_5$ we get

$$\frac{1}{4} < \frac{q_1 + q_2 - 1}{p_1 + p_2} < \frac{2}{5}. \tag{16}$$

By (7) we have $p_1 + p_2 \le 6$ and therefore (16) has only one solution $(p_1, q_1, p_2, q_2) = (1, 1, 5, 2)$. However, for such decagon we have

$$\mathrm{vol}(Q_{10}) = 29,$$

which contradicts (5).

Case 2.1.10. *$y_3' - y_2' = 1$ and $x_3' - x_2' = 5$.* Then by (7) and convexity we have

$$p_1 + p_2 \le 5$$

and

$$\frac{1}{5} < \frac{q_1 + q_2 - 1}{p_1 + p_2} < \frac{2}{5}.$$

In fact, these inequalities have no positive integer solution.

Case 2.1.11. *$y_3' - y_2' = 1$ and $x_3' - x_2' \ge 6$.* It follows by (7) that $p_2 \le 3$. Then we get both $\beta_4 \le \frac{1}{3}$ and $\beta_5 \ge \frac{1}{3}$, which contradicts the convexity of Q_{10}.

Case 2.1.12. *$y_3' - y_2' = 2$ and $x_3' - x_2' = 3$.* Then by (6) and convexity we get $\beta_4 = 1$ and $\beta_1 < \beta_5$. However the two inequalities

$$\frac{q_1}{p_1} > \beta_4 = 1$$

and

$$\frac{q_1 + q_2 - 1}{p_1 + p_2} < \frac{q_2}{p_2}$$

have no integer solution.

Case 2.1.13. *$y_3' - y_2' = 2$ and $x_3' - x_2' = 4$.* Then by (6) and convexity we get $\beta_2 = \frac{1}{2}$, $\beta_4 = \frac{3}{4}$, $\beta_2 < \beta_5 < \beta_4$ and therefore

$$\frac{1}{2} < \frac{q_2}{p_2} < \frac{3}{4}. \tag{17}$$

Clearly, by (7) we have $p_2 \le 5$ and therefore (17) has two groups of integer solutions $(p_2, q_2) = (3, 2)$ or $(5, 3)$. Then, the two inequalities $p_1 + p_2 \le 6$ and

$$\frac{1}{2} < \frac{q_1 + q_2 - 1}{p_1 + p_2} < \frac{q_2}{p_2}$$

have one group of integer solution $(p_1, q_1, p_2, q_2) = (2, 2, 3, 2)$. Unfortunately, then we have

$$\mathrm{vol}(Q_{10}) = 25,$$

which contradicts (5).

Case 2.1.14. *$y'_3 - y'_2 = 2$ and $x'_3 - x'_2 = 5$.* Then by (6) and convexity we get $\beta_2 = \frac{2}{5}$, $\beta_4 = \frac{3}{5}$, $\beta_2 < \beta_5 < \beta_4$ and therefore

$$\frac{2}{5} < \frac{q_2}{p_2} < \frac{3}{5}. \tag{18}$$

Clearly, by (7) we have $p_2 \leq 4$ and therefore (18) has two groups of integer solutions $(p_2, q_2) = (2, 1)$ or $(4, 2)$. Then, one can deduce that $p_1 + p_2 \leq 5$ and

$$\frac{2}{5} < \frac{q_1 + q_2 - 1}{p_1 + p_2} < \frac{q_2}{p_2}$$

have no integer solution.

Case 2.1.15. *$y'_3 - y'_2 = 2$ and $x'_3 - x'_2 = 6$.* Then by (6) and convexity we get $\beta_5 < \beta_4 = \frac{1}{2}$ and therefore $\beta_5 = \frac{1}{3}$, which contradicts the fact

$$\beta_5 > \beta_1 > \beta_2 = \frac{1}{3}.$$

Case 2.1.16. *$y'_3 - y'_2 = 2$ and $x'_3 - x'_2 \geq 7$.* Then by (6) and convexity we get $\beta_4 \leq \frac{3}{7}$ and $\beta_5 \geq \frac{1}{2}$, which contradicts the convexity of Q_{10}.

Case 2.1.17. *$y'_3 - y'_2 = 3$ and $x'_3 - x'_2 = 4$.* Then by (6) and convexity we have $p_2 \leq 5$, $\beta_2 = \frac{3}{4}$ and $\beta_4 = 1$. Then we have

$$\beta_3 = \frac{q_1}{p_1} > 1$$

and therefore

$$\beta_1 = \frac{q_1 + q_2 - 1}{p_1 + p_2} > \frac{q_2}{p_2} = \beta_5,$$

which contradicts the convexity of Q_{10}.

Case 2.1.18. *$y'_3 - y'_2 = 3$ and $x'_3 - x'_2 = 5$.* Then by (6) and convexity we have $p_2 \leq 4$, $\beta_2 = \frac{3}{5}$, $\beta_4 = \frac{4}{5}$ and $\beta_2 < \beta_5 < \beta_4$. The inequalities $p_2 \leq 4$ and

$$\frac{3}{5} < \frac{q_2}{p_2} < \frac{4}{5}$$

have two solutions $(p_2, q_2) = (3, 2)$ or $(4, 3)$. Then

$$\frac{3}{5} < \frac{q_1 + q_2 - 1}{p_1 + p_2} < \frac{q_2}{p_2}$$

has no solution satisfying $p_1 + p_2 \leq 5$.

Case 2.1.19. *$y'_3 - y'_2 = 3$ and $x'_3 - x'_2 = 6$.* Then by (6) and convexity we get $p_2 \leq 3$, $\beta_2 = \frac{1}{2}$, $\beta_4 = \frac{2}{3}$ and $\beta_2 < \beta_5 < \beta_4$. Then the inequalities $p_2 \leq 3$ and

$$\frac{1}{2} < \frac{q_2}{p_2} < \frac{2}{3}$$

have no solution.

Case 2.1.20. *$y'_3 - y'_2 = 3$ and $x'_3 - x'_2 = 7$.* Then by (6) and convexity we get $p_2 \leq 2$, $\beta_2 = \frac{3}{7}$, $\beta_4 = \frac{4}{7}$ and $\beta_2 < \beta_5 < \beta_4$. Then the inequalities $p_2 \leq 2$ and

$$\frac{3}{7} < \frac{q_2}{p_2} < \frac{4}{7}$$

have one solution $(p_2, q_2) = (2, 1)$. However, then

$$\frac{3}{7} < \frac{q_1}{p_1 + 2} < \frac{1}{2}$$

has no solution.

Case 2.1.21. *$y'_3 - y'_2 = 3$ and $x'_3 - x'_2 \geq 8$.* Then by (6) and convexity we get $p_2 = 1$, $\beta_5 \geq 1$ and $\beta_4 \leq \frac{1}{2}$, which contradicts the convexity of Q_{10}.

Case 2.1.22. *$y'_3 - y'_2 = 4$ and $x'_3 - x'_2 = 5$.* Then by (6) and convexity we have $p_2 \le 4$, $\beta_2 = \frac{4}{5}$, $\beta_4 = 1$ and $\beta_2 < \beta_5 < \beta_4$. The inequalities $p_2 \le 4$ and

$$\frac{4}{5} < \frac{q_2}{p_2} < 1$$

have no common integer solution.

Case 2.1.23. *$y'_3 - y'_2 = 4$ and $x'_3 - x'_2 = 6$.* Then by (6) and convexity we get $p_2 \le 3$, $\beta_2 = \frac{2}{3}$, $\beta_4 = \frac{5}{6}$ and $\beta_2 < \beta_5 < \beta_4$. The inequalities $p_2 \le 3$ and

$$\frac{2}{3} < \frac{q_2}{p_2} < \frac{5}{6}$$

have no common integer solution.

Case 2.1.24. *$y'_3 - y'_2 = 4$ and $x'_3 - x'_2 = 7$.* Then by (6) and convexity we get $p_2 \le 2$, $\beta_2 = \frac{4}{7}$, $\beta_4 = \frac{5}{7}$ and $\beta_2 < \beta_5 < \beta_4$. The inequalities $p_2 \le 2$ and

$$\frac{4}{7} < \frac{q_2}{p_2} < \frac{5}{7}$$

have no common integer solution.

Case 2.1.25. *$y'_3 - y'_2 = 4$ and $x'_3 - x'_2 \ge 8$.* Then by (6) and convexity we get $p_2 = 1$, $\beta_5 \ge 1$ and $\beta_4 \le \frac{5}{8}$, which contradicts the convexity of Q_{10}.

Case 2.1.26. *$y'_3 - y'_2 = 5$ and $x'_3 - x'_2 = 6$.* Then by (6) and convexity we get $p_2 \le 3$, $\beta_2 = \frac{5}{6}$, $\beta_4 = 1$ and $\beta_2 < \beta_5 < \beta_4$. The inequalities $p_2 \le 3$ and

$$\frac{5}{6} < \frac{q_2}{p_2} < 1$$

have no common integer solution.

Case 2.1.27. *$y'_3 - y'_2 = 5$ and $x'_3 - x'_2 = 7$.* Then by (6) and convexity we get $p_2 \le 2$, $\beta_2 = \frac{5}{7}$, $\beta_4 = \frac{6}{7}$ and $\beta_2 < \beta_5 < \beta_4$. The inequalities $p_2 \le 2$ and

$$\frac{5}{7} < \frac{q_2}{p_2} < \frac{6}{7}$$

have no common integer solution.

Case 2.1.28. *$y'_3 - y'_2 = 5$ and $x'_3 - x'_2 \ge 8$.* Then by (6) and convexity we get $p_2 = 1$, $\beta_5 \ge 1$ and $\beta_4 \le \frac{6}{8}$, which contradicts the convexity of Q_{10}.

Case 2.1.29. *$y'_3 - y'_2 = 6$ and $x'_3 - x'_2 = 7$.* Then by (6) and convexity we get $p_2 \le 2$, $\beta_2 = \frac{6}{7}$, $\beta_4 = 1$ and $\beta_2 < \beta_5 < \beta_4$. The inequalities $p_2 \le 2$ and

$$\frac{6}{7} < \frac{q_2}{p_2} < 1$$

have no common integer solution.

Case 2.1.30. *$y'_3 - y'_2 = 6$ and $x'_3 - x'_2 \ge 8$.* Then by (6) and convexity we get $p_2 = 1$, $\beta_5 \ge 1$ and $\beta_4 \le \frac{7}{8}$, which contradicts the convexity of Q_{10}.

Case 2.1.31. *$y'_3 - y'_2 = 7$ and $x'_3 - x'_2 \ge 8$.* Then by (6) and convexity we get $p_2 = 1$, $\beta_5 \ge 1$ and $\beta_4 \le 1$, which contradicts the convexity of Q_{10}.

Case 2.2. *All $\mathbf{u}'_i$ are even multiplicative.* Then all $\mathbf{u}_i$ belong to Λ. It follows by Lemma 1 that $\frac{1}{2}P_{10} + \Lambda$ is a k-fold lattice tiling with

$$k = \mathrm{vol}\left(\tfrac{1}{2}P_{10}\right) \le \frac{6}{4} = \frac{3}{2},$$

which contradicts Lemma 2.

Case 2.3. *All $\mathbf{u}'_i$ are multiplicative, $\mathbf{u}'_1$ is odd multiplicative.* Without loss of generality, guaranteed by uni-modular linear transformations, we take $\mathbf{u}'_1 = (0, 2q+1)$, where q is a positive integer.

By Lemma 4 it follows that

$$x'_4 - x'_5 = x'_3 - x'_2.$$

Therefore, by convexity and reflection we may assume that

$$x_3' \geq x_i', \qquad i = 1, 2, \ldots, 10.$$

Let T' denote the lattice triangle with vertices $\mathbf{u}_1'$, $\mathbf{u}_2'$ and $\mathbf{u}_3'$, let Q denote the lattice quadrilateral with vertices $\mathbf{u}_3'$, $\mathbf{u}_4'$, $\mathbf{u}_5'$ and $\mathbf{u}_6'$, and let T denote the lattice triangle with vertices $\mathbf{u}_1'$, $\mathbf{u}_3'$ and $\mathbf{u}_6'$, as shown in Figure 3. It follows from (5) and Pick's theorem that

$$\mathrm{vol}(T) \leq \frac{1}{2}\Big(23 - 2\big(\mathrm{vol}(T') + \mathrm{vol}(Q)\big)\Big) \leq 10$$

and therefore

$$x_3' = \frac{2 \cdot \mathrm{vol}(T)}{2(2q+1)} \leq \left\lfloor \frac{10}{3} \right\rfloor = 3.$$

It is assumed that all $\mathbf{u}_i'$ are multiplicative. Therefore by convexity we have

$$x_2' = x_5' = 2$$

and

$$x_3' = x_4' = 3.$$

Then, we have

$$\mathrm{vol}(Q_{10}) \geq 3 \cdot (2(2q+1) + 3) \geq 27,$$

which contradicts (5).

As a conclusion of all these cases, Lemma 10 is proved. $\square$

Lemma 11. *For every centrally symmetric convex octagon P_8 we have*

$$\tau^*(P_8) \geq 5,$$

where the equality holds if and only if, under a suitable affine linear transformation, P_8 has its vertices at $\mathbf{v}_1 = (-\alpha, -\frac{3}{2})$, $\mathbf{v}_2 = (1-\alpha, -\frac{3}{2})$, $\mathbf{v}_3 = (1+\alpha, -\frac{1}{2})$, $\mathbf{v}_4 = (1-\alpha, \frac{1}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{4}$, or with vertices $\mathbf{v}_1 = (\beta, -2)$, $\mathbf{v}_2 = (1+\beta, -2)$, $\mathbf{v}_3 = (1-\beta, 0)$, $\mathbf{v}_4 = (\beta, 1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, $\mathbf{v}_8 = -\mathbf{v}_4$, where $\frac{1}{4} < \beta < \frac{1}{3}$. Furthermore

$$\tau^*(P_8) = 6$$

if and only if (under a suitable affine linear transformation) P_8 has its vertices at $\mathbf{v}_1 = (-\alpha, -2)$, $\mathbf{v}_2 = (1-\alpha, -2)$, $\mathbf{v}_3 = (1+\alpha, -1)$, $\mathbf{v}_4 = (1-\alpha, 0)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{6}$.

Proof. Let P_8 be a centrally symmetric convex octagon centered at the origin, let $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_8$ be the eight vertices of P_8 enumerated in an anti-clock order, let G_i denote the edge with vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$, where $\mathbf{v}_9 = \mathbf{v}_1$, and let $\mathbf{u}_i$ denote the midpoint of G_i. For convenience, we write $\mathbf{v}_i = (x_i, y_i)$ and $\mathbf{u}_i = (x_i', y_i')$. Without loss of generality, by Lemma 2 we may assume that $\Lambda = \mathbb{Z}^2$ and $P_8 + \Lambda$ is a five or sixfold lattice tiling. Then, we have

$$\tau^*(P_8) = \mathrm{vol}(P_8) = 5 \text{ or } 6. \tag{19}$$

Based on Lemma 3, by a uni-modular transformation, we may assume that $G_1 \cap \frac{1}{2}\Lambda \neq \emptyset$ and $\mathbf{v}_2 - \mathbf{v}_1 = (k, 0)$, where k is a positive integer. If $k > 1$, we define P_8' to be the octagon with vertices $\mathbf{v}_1' = \mathbf{v}_1 + (\frac{k-1}{2}, 0)$, $\mathbf{v}_2' = \mathbf{v}_2 + (\frac{1-k}{2}, 0)$, $\mathbf{v}_3' = \mathbf{v}_3 + (\frac{1-k}{2}, 0)$, $\mathbf{v}_4' = \mathbf{v}_4 + (\frac{1-k}{2}, 0)$, $\mathbf{v}_5' = \mathbf{v}_5 + (\frac{1-k}{2}, 0)$, $\mathbf{v}_6' = \mathbf{v}_6 + (\frac{k-1}{2}, 0)$, $\mathbf{v}_7' = \mathbf{v}_7 + (\frac{k-1}{2}, 0)$ and $\mathbf{v}_8' = \mathbf{v}_8 + (\frac{k-1}{2}, 0)$, as shown by Figure 8. By Lemma 1 it can be shown that $P_8' + \Lambda$ is a multiple lattice tiling of $\mathbb{E}^2$ and therefore

$$\tau^*(P_8') \leq \mathrm{vol}(P_8') \leq \mathrm{vol}(P_8) - 3 = 3,$$

which contradicts Lemma 2. Thus, we have $\mathbf{v}_2 - \mathbf{v}_1 = (1, 0)$.

Apply Lemma 1 successively to G_1, G_2, G_3 and G_4, one can deduce that all $2y_2$, $y_3 - y_2$, $y_4 - y_3$ and $y_5 - y_4$ are positive integers. Therefore, we have

$$y_2 = y_1 \leq -\frac{3}{2}.$$

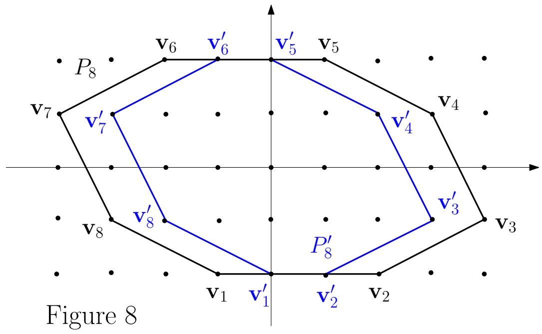

Figure 8

On the other hand, if $y_2 = y_1 \leq -3$ and let P denote the parallelogram with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_5$ and $\mathbf{v}_6$, it can be deduced by convexity that

$$\mathrm{vol}(P_8) > \mathrm{vol}(P) \geq 6,$$

which contradicts the assumption of (19). Thus, to prove the theorem it is sufficient to deal with the three cases

$$y_2 = y_1 = -\frac{3}{2},\ -2,\ -\frac{5}{2}.$$

Case 1. $y_2 = y_1 = -\frac{3}{2}$. In this case,

$$y_{i+1} - y_i = 1$$

must hold for all $i = 2, 3$ and 4. Then, it follows by Lemma 1 that all the midpoints of G_2, G_3 and G_4 belong to $\frac{1}{2}\Lambda$. Furthermore, by a uni-modular transformation

$$\begin{cases} x' = x - ky, \\ y' = y, \end{cases}$$

with a suitable integer k, we may assume that $-\frac{5}{4} \leq x_1 < \frac{1}{4}$.

If G_2 is vertical, then x_2 is an integer or an half integer. Consequently, we have $x_1 \in \frac{1}{2}\mathbb{Z}$. Therefore x_1 only can be -1, $-\frac{1}{2}$ or 0. By considering three subcases with respect to $x_1 = -1$, $-\frac{1}{2}$ or 0, it can be deduced that there is no octagon of this type satisfying Lemma 1. For example, when $x_1 = -\frac{1}{2}$, by Lemma 1 and convexity we have $\mathbf{v}_1 = \left(-\frac{1}{2}, -\frac{3}{2}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, -\frac{3}{2}\right)$, $\mathbf{v}_3 = \left(\frac{1}{2}, -\frac{1}{2}\right)$, $\mathbf{v}_4 = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ and $\mathbf{v}_8 = -\mathbf{v}_4$. Then, P_8 is no longer an octagon but a parallelogram.

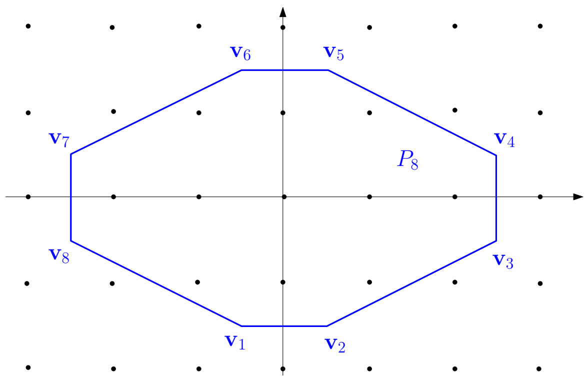

Figure 9

If G_3 is vertical, then x_1 must be an integer or an half integer as well. Therefore, it only can be -1, $-\frac{1}{2}$ or 0. By considering three subcases with respect to $x_1 = -1$, $-\frac{1}{2}$ or 0, it can be deduced that

$$\mathrm{vol}(P_8) \geq 7,$$

which contradicts the assumption of (19). For example, when $x_1 = -\frac{1}{2}$, by Lemma 1 and convexity we have $\mathbf{v}_1 = \left(-\frac{1}{2}, -\frac{3}{2}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, -\frac{3}{2}\right)$, $\mathbf{v}_3 = \left(\frac{1}{2} + k, -\frac{1}{2}\right)$, $\mathbf{v}_4 = \left(\frac{1}{2} + k, \frac{1}{2}\right)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$,

$\mathbf{v}_7 = -\mathbf{v}_3$ and $\mathbf{v}_8 = -\mathbf{v}_4$, where k is a positive integer. Then, as shown by Figure 9, it can be deduced that

$$\mathrm{vol}(P_8) = 3 + 4k \geq 7.$$

If none of the three edges G_2, G_3 and G_4 is vertical, by convexity it is sufficient to deal with the following three subcases.

Subcase 1.1. $x_3' > \max\{x_2', x_4'\}$. Then we replace the eight vertices $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$, $\mathbf{v}_6$, $\mathbf{v}_7$, $\mathbf{v}_8$, $\mathbf{v}_1$ and $\mathbf{v}_2$ by $\mathbf{v}_3' = (x_3', -\frac{1}{2})$, $\mathbf{v}_4' = (x_3', \frac{1}{2})$, $\mathbf{v}_5' = (2x_4' - x_3', \frac{3}{2})$, $\mathbf{v}_6' = (2x_4' - x_3' - 1, \frac{3}{2})$, $\mathbf{v}_7' = -\mathbf{v}_3'$, $\mathbf{v}_8' = -\mathbf{v}_4'$, $\mathbf{v}_1' = -\mathbf{v}_5'$ and $\mathbf{v}_2' = -\mathbf{v}_6'$, respectively (as shown by Figure 10). In practice, one first makes G_3 vertical and then changes the other vertices successively. Clearly, this process does not change the area of the polygon. Then one can deduce that $x_3' \geq \frac{3}{2}$ and therefore

$$\mathrm{vol}(P_8) = 3 \cdot 2x_3' - (2x_3' - 1) = 4x_3' + 1 \geq 7,$$

which contradicts the assumption of (19).

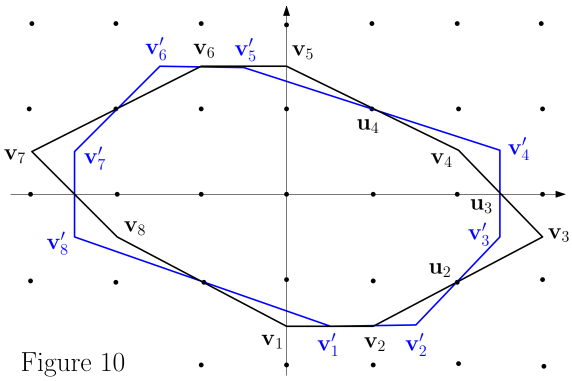

Figure 10

Subcase 1.2. $x_2' > \max\{x_3', x_4'\}$. If $x_3 > x_2$, one can repeat the above process. At the end we get $x_2' \geq 2$ and

$$\mathrm{vol}(P_8) > 3 \cdot 2x_2' - 2(2x_2' - 1) = 2x_2' + 2 \geq 6,$$

which contradicts the assumption of (19). If $x_2 > x_3$, since $-\frac{5}{4} \leq x_1 < \frac{1}{4}$, $\mathbf{u}_2$ only can be $(1, -1)$, $(\frac{1}{2}, -1)$, $(0, -1)$ or $(-\frac{1}{2}, -1)$. Then it can be easily checked that there is no convex octagon of this type satisfying Lemma 1.

Subcase 1.3. $x_2' = x_3' > x_4'$. Then, we replace the eight vertices $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$, $\mathbf{v}_6$, $\mathbf{v}_7$, $\mathbf{v}_8$ and $\mathbf{v}_1$ by $\mathbf{v}_2' = (x_2', -\frac{3}{2})$, $\mathbf{v}_3' = (x_2', -\frac{1}{2})$, $\mathbf{v}_4' = (x_2', \frac{1}{2})$, $\mathbf{v}_5' = 2\mathbf{u}_4 - \mathbf{v}_4'$, $\mathbf{v}_6' = -\mathbf{v}_2'$, $\mathbf{v}_7' = -\mathbf{v}_3'$, $\mathbf{v}_8' = -\mathbf{v}_4'$ and $\mathbf{v}_1' = -\mathbf{v}_5'$, respectively (as shown by Figure 11). In practice, one first makes G_2 and G_3 vertical and then changes the other vertices successively, keeping the rules of Lemma 1. Clearly, this process does not change the area of the polygon, $x_2' \geq 1$ and therefore

$$\mathrm{vol}(P_8) = 3 \cdot 2x_2' - (2x_2' - 1) = 4x_2' + 1 \geq 5,$$

where the equality holds if and only if P_8 with vertices $\mathbf{v}_1 = (-\alpha, -\frac{3}{2})$, $\mathbf{v}_2 = (1 - \alpha, -\frac{3}{2})$, $\mathbf{v}_3 = (1 + \alpha, -\frac{1}{2})$, $\mathbf{v}_4 = (1 - \alpha, \frac{1}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{4}$. They are the first octagon type of the fivefold lattice tiles listed in the lemma.

Case 2. $y_2 = y_1 = -2$. Then, it can be deduced that one of $y_3 - y_2$, $y_4 - y_3$ and $y_5 - y_4$ is two and the others are ones, and the midpoint $\mathbf{u}_i$ must belong to $\frac{1}{2}\Lambda$ whenever $y_{i+1} - y_i = 1$. Furthermore, we may assume that $-\frac{3}{2} \leq x_1 < \frac{1}{2}$ by a uni-modular transformation and assume that G_i is primitive if it is a lattice vector by reduction.

If one of G_2, G_3 and G_4 is vertical, it can be easily deduced that

$$\mathrm{vol}(P_8) \geq 7.$$

For instance, when G_3 is vertical, we have $x_3 - x_2 \geq 1$, $x_4 - x_5 \geq 1$ and thus $x_3 = x_3' = x_4 \geq \frac{3}{2}$. Then, it can be deduced that

$$\mathrm{vol}(P_8) \geq 4 \cdot 2x_3 - 2(2x_3 - 1) = 4x_3 + 2 \geq 8,$$

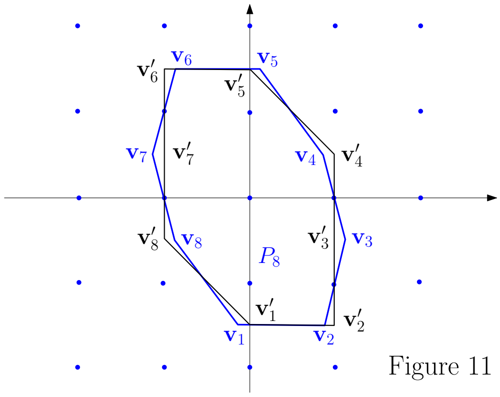

Figure 11

which contradicts the assumption of (19).

Now, we assume that all G_2, G_3 and G_4 are not vertical.

Subcase 2.1. $y_3 - y_2 = 2$ *and* $\mathbf{u}_2 \notin \frac{1}{2}\Lambda$. Then $\mathbf{v}_3 - \mathbf{v}_2 = (k, 2)$ is a lattice vector, where k is a positive integer (when k is negative, one can easily deduce that P_8 cannot be a convex octagon). On the other hand, it follows by the assumption $-\frac{3}{2} \le x_1 < \frac{1}{2}$ that

$$\mathbf{v}_5 - \mathbf{v}_2 = (x, 4),$$

where $-2 < x \le 2$. Let P denote the parallelogram with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_5$ and $\mathbf{v}_6$, and let T denote the triangle with vertices $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_5$, as shown by Figure 12.

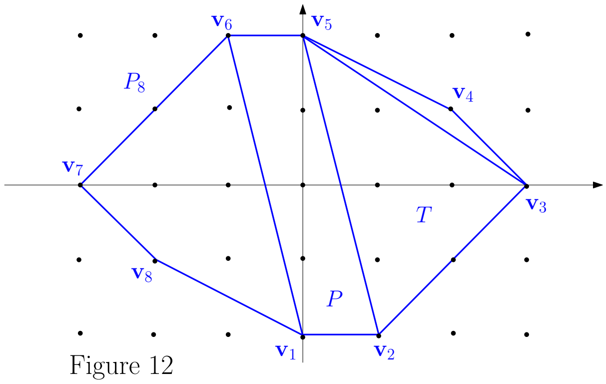

Figure 12

If $k \ge 2$, one can deduce

$$\mathrm{vol}(T) = \frac{1}{2}\begin{vmatrix} k & 2 \\ x & 4 \end{vmatrix} = 2k - x \ge 2$$

and therefore

$$\mathrm{vol}(P_8) > \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(T) \ge 8,$$

which contradicts the assumption of (19).

If $k = x_3 - x_2 = 1$, $G_2 \cap \frac{1}{2}\Lambda \ne \emptyset$ and $\mathbf{u}_2 \notin \frac{1}{2}\Lambda$, one can deduce that $x_2 \in \frac{1}{4}\mathbb{Z}$ and therefore $x_1 \in \frac{1}{4}\mathbb{Z}$. In fact, by checking all the eight cases $x_1 = -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0$ or $\frac{1}{4}$, it can be shown that there is no such octagon satisfying the conditions of Lemma 1. For example, when $x_1 = \frac{1}{4}$, by convexity (as shown by Figure 13) the only candidate for $\mathbf{u}_3$ is $\mathbf{u}_3' = (2, \frac{1}{2})$ and the only candidates for $\mathbf{u}_4$ are $\mathbf{u}_4' = (\frac{1}{2}, \frac{3}{2})$ and $\mathbf{u}_4^* = (1, \frac{3}{2})$. However, no octagon P_8 satisfying Lemma 1 can be constructed from these candidate midpoints.

Subcase 2.2. $y_4 - y_3 = 2$ *and* $\mathbf{u}_3 \notin \frac{1}{2}\Lambda$. Then $\mathbf{v}_4 - \mathbf{v}_3 = (k, 2)$ is a lattice vector, where k is a positive integer (if it is negative, then make a reflection with respect to the x-axis). On the other hand, it follows by the assumption $-\frac{3}{2} \le x_1 < \frac{1}{2}$ that

$$\mathbf{v}_5 - \mathbf{v}_2 = (x, 4),$$

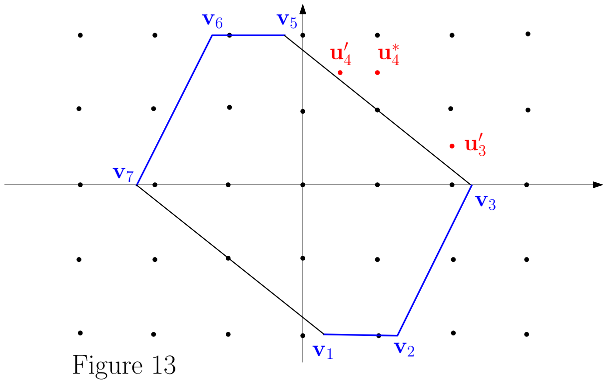

Figure 13

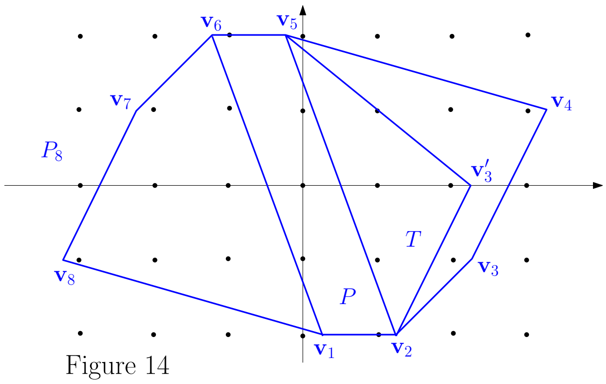

Figure 14

where $-2 < x \leq 2$. Let P denote the parallelogram with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_5$ and $\mathbf{v}_6$, and let T denote the triangle with vertices $\mathbf{v}_2$, $\mathbf{v}_3' = \mathbf{v}_2 + (\mathbf{v}_4 - \mathbf{v}_3)$ and $\mathbf{v}_5$, as shown by Figure 14.

If $k \geq 2$, one can deduce

$$\mathrm{vol}(T) = \frac{1}{2}\begin{vmatrix} k & 2 \\ x & 4 \end{vmatrix} = 2k - x \geq 2$$

and therefore

$$\mathrm{vol}(P_8) > \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(T) \geq 8,$$

which contradicts the assumption of (19).

If $k = x_4 - x_3 = 1$, $G_3 \cap \frac{1}{2}\Lambda \neq \emptyset$ and $\mathbf{u}_3 \notin \frac{1}{2}\Lambda$, one can deduce that $x_3 \in \frac{1}{4}\mathbb{Z}$ and therefore $x_1 \in \frac{1}{4}\mathbb{Z}$. By checking all the eight cases $x_1 = -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0$ or $\frac{1}{4}$, it can be deduced that

$$\mathrm{vol}(P_8) \geq 7.$$

For example, when $x_1 = -\frac{3}{2}$, we define $\mathbf{v}_3' = (\frac{3}{2}, -1)$, $\mathbf{v}_4' = (\frac{5}{2}, 1)$, $\mathbf{v}_7' = (-\frac{3}{2}, 1)$, $\mathbf{v}_8' = (-\frac{5}{2}, -1)$, and define P_8' to be the octagon with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3'$, $\mathbf{v}_4'$, $\mathbf{v}_5$, $\mathbf{v}_6$, $\mathbf{v}_7'$ and $\mathbf{v}_8'$, as shown by Figure 15. By shifting G_3 and G_7, one can deduce $P_8' \subseteq P_8$ and therefore

$$\mathrm{vol}(P_8) \geq \mathrm{vol}(P_8') = 13.$$

Subcase 2.3. *None of the three edges G_2, G_3 and G_4 is vertical and all $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$ belong to $\frac{1}{2}\Lambda$.* Then, it is sufficient to consider the following three situations.

Subcase 2.3.1. $x_3' > \max\{x_2', x_4'\}$. Similar to Subcase 1.1, we get $x_3' \geq \frac{3}{2}$ and therefore

$$\mathrm{vol}(P_8) \geq 4 \cdot 2x_3' - 2(2x_3' - 1) = 4x_3' + 2 \geq 8,$$

which contradicts the assumption of (19).

Subcase 2.3.2. $x_2' > \max\{x_3', x_4'\}$. If $x_3 > x_2$, just like Subcase 1.2, one can get $x_2' \geq \frac{3}{2}$ and

$$\mathrm{vol}(P_8) > 4 \cdot 2x_2' - 3(2x_2' - 1) \geq 6,$$

which contradicts the assumption of (19).

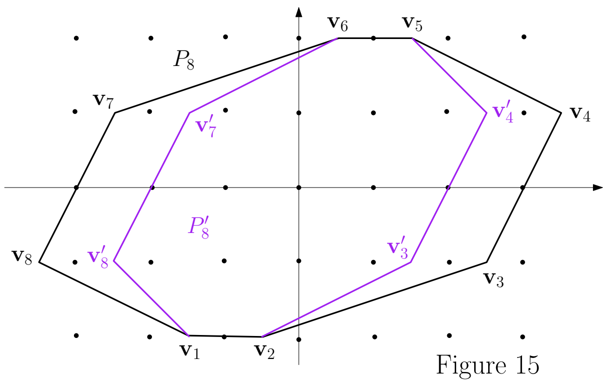

Figure 15

If $x_2 > x_3$ and $y_3 - y_2 = 1$, since $-\frac{3}{2} \leq x_1 < \frac{1}{2}$, $\mathbf{u}_2$ only can be $(1, -\frac{3}{2})$, $(\frac{1}{2}, -\frac{3}{2})$, $(0, -\frac{3}{2})$ or $(-\frac{1}{2}, -\frac{3}{2})$. Then it can be routinely checked that there is no convex octagon of this type satisfying Lemma 1.

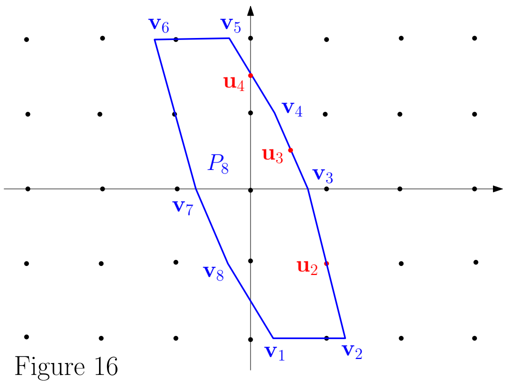

Figure 16

If $x_2 > x_3$ and $y_3 - y_2 = 2$, since $-\frac{3}{2} \leq x_1 < \frac{1}{2}$, $\mathbf{u}_2$ only can be $(1, -1)$, $(\frac{1}{2}, -1)$, $(0, -1)$ or $(-\frac{1}{2}, -1)$. By checking these four cases, it can be shown that there is only one class of such convex octagons satisfying Lemma 1. Namely, the ones satisfying $\mathbf{u}_2 = (1, -1)$, $\mathbf{u}_3 = (\frac{1}{2}, \frac{1}{2})$ and $\mathbf{u}_4 = (0, \frac{3}{2})$, as shown in Figure 16. In other words, they are the octagons with vertices $\mathbf{v}_1 = (\beta, -2)$, $\mathbf{v}_2 = (1+\beta, -2)$, $\mathbf{v}_3 = (1-\beta, 0)$, $\mathbf{v}_4 = (\beta, 1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, $\mathbf{v}_8 = -\mathbf{v}_4$, where $\frac{1}{4} < \beta < \frac{1}{3}$. Then, one can deduce that

$$\mathrm{vol}(P_8) = 5,$$

which is the second type of octagons of the fivefold lattice tiles listed in the lemma.

Subcase 2.3.3. $x'_2 = x'_3 > x'_4$. Similar to Subcase 1.3, one can deduce $x'_2 \geq 1$ and therefore

$$\mathrm{vol}(P_8) \geq 4 \cdot 2x'_3 - 2(2x'_3 - 1) = 4x'_3 + 2 \geq 6,$$

where the equalities hold if and only if P_8 with vertices $\mathbf{v}_1 = (-\alpha, -2)$, $\mathbf{v}_2 = (1-\alpha, -2)$, $\mathbf{v}_3 = (1+\alpha, -1)$, $\mathbf{v}_4 = (1-\alpha, 0)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{6}$ (as shown in Figure 17). This is the octagon type of the sixfold lattice tiles listed in the lemma.

Case 3. $y'_1 = -\frac{5}{2}$. Then all $y_{i+1} - y_i$ are positive integers for $2 \leq i \leq 4$ and their sum is five. By a uni-modular transformation, we may assume that $-\frac{7}{4} \leq x_1 < \frac{3}{4}$. Then we have $\mathbf{v}_5 - \mathbf{v}_2 = (x, 5)$, where $-\frac{5}{2} < x \leq \frac{5}{2}$. Now we consider two subcases.

Subcase 3.1. $\mathbf{u}_i \notin \frac{1}{2}\Lambda$ *holds for one of the indices* $i \in \{2, 3, 4\}$. Then we have $y_{i+1} - y_i = 2$ or 3.

Subcase 3.1.1. $\mathbf{v}_{i+1} - \mathbf{v}_i = (k, 2)$ *and* $|k| \geq 2$. Let Q denote the quadrilateral with vertices $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ and $\mathbf{v}_5$, then we have

$$\mathrm{vol}(Q) > \frac{1}{2}\begin{vmatrix} x & 5 \\ k & 2 \end{vmatrix} = \frac{1}{2}\,|2x - 5k| \geq \frac{5}{2}$$

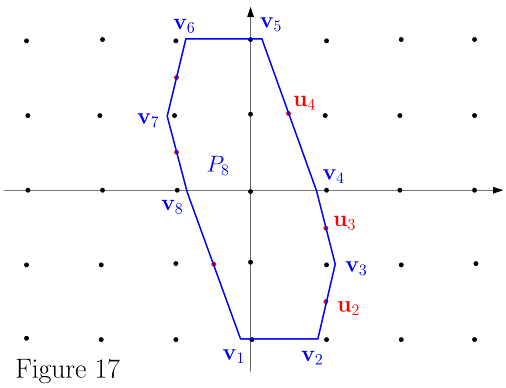

Figure 17

and thus

$$\tau^*(P_8) = \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(Q) \geq 10.$$

Subcase 3.1.2. $\mathbf{v}_{i+1} - \mathbf{v}_i = (1, 2)$. Then we have $x_1 \in \frac{1}{4}\mathbb{Z}$ and therefore $x \in \frac{1}{2}\mathbb{Z}$ and $-2 \leq x \leq \frac{5}{2}$. If $x_1 = -\frac{7}{4}$, then we have $\mathbf{v}_5 - \mathbf{v}_2 = (\frac{5}{2}, 5)$. Applying Pick's theorem to Q and $\frac{1}{2}\Lambda$, we get

$$\mathrm{vol}(Q) > \frac{1}{4}\left(\frac{8}{2} - 1\right) = \frac{3}{4}$$

and thus

$$\tau^*(P_8) = \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(Q) > 6.$$

If $x_1 \neq -\frac{7}{4}$, then we have $x \in \frac{1}{2}\mathbb{Z}$, $-2 \leq x \leq 2$,

$$\mathrm{vol}(Q) > \frac{1}{2}\begin{vmatrix} x & 5 \\ 1 & 2 \end{vmatrix} = \frac{1}{2}|2x - 5| \geq \frac{1}{2}$$

and thus

$$\tau^*(P_8) = \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(Q) > 6.$$

Subcase 3.1.3. $\mathbf{v}_{i+1} - \mathbf{v}_i = (k, 3)$ *and* $|k| \geq 2$. Let Q denote the quadrilateral with vertices $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ and $\mathbf{v}_5$, then we have

$$\mathrm{vol}(Q) > \frac{1}{2}\begin{vmatrix} x & 5 \\ k & 3 \end{vmatrix} = \frac{1}{2}|3x - 5k| \geq \frac{5}{4}$$

and thus

$$\tau^*(P_8) = \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(Q) \geq 7.$$

Subcase 3.1.4. $\mathbf{v}_{i+1} - \mathbf{v}_i = (1, 3)$. Then we have $x_1 \in \frac{1}{6}\mathbb{Z}$ and therefore $x \in \frac{1}{3}\mathbb{Z}$ and $-\frac{7}{3} \leq x \leq \frac{7}{3}$. If $x_1 = -\frac{4}{3}$, then we have $\mathbf{v}_5 - \mathbf{v}_2 = (\frac{5}{3}, 5)$. Applying Pick's theorem to Q and $\frac{1}{2}\Lambda$, we get

$$\mathrm{vol}(Q) > \frac{1}{4}\left(\frac{7}{2} - 1\right) = \frac{5}{8}$$

and thus

$$\tau^*(P_8) = \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(Q) > 6.$$

If $x_1 \neq -\frac{4}{3}$, then we have $x \in \frac{1}{3}\mathbb{Z}$, $x \neq \frac{5}{3}$,

$$\mathrm{vol}(Q) > \frac{1}{2}\begin{vmatrix} x & 5 \\ 1 & 3 \end{vmatrix} = \frac{1}{2}|3x - 5| \geq \frac{1}{2}$$

and thus

$$\tau^*(P_8) = \mathrm{vol}(P) + 2 \cdot \mathrm{vol}(Q) > 6.$$

Subcase 3.2. $\mathbf{u}_i \in \frac{1}{2}\Lambda$ *holds for all* $i \in \{2, 3, 4\}$. Then, it is sufficient to consider the following three situations.

Subcase 3.2.1. $x'_3 > \max\{x'_2, x'_4\}$. Similar to Subcase 1.1, we get $x'_3 \geq \frac{3}{2}$ and therefore

$$\mathrm{vol}(P_8) \geq 5 \cdot 2x'_3 - 3(2x'_3 - 1) = 4x'_3 + 3 \geq 9.$$

Subcase 3.2.2. $x_2' > \max\{x_3', x_4'\}$. If $x_3 > x_2$, just like Subcase 1.2, one can get $x_2' \geq \frac{3}{2}$ and

$$\mathrm{vol}(P_8) > 5 \cdot 2x_2' - 4(2x_2' - 1) \geq 7.$$

When $x_2 > x_3$, we consider the following four situations:

Subcase 3.2.2.1. $y_3 - y_2 = 1$, $y_4 - y_3 = 1$ *and* $y_5 - y_4 = 3$. Then, recalling the assumption that $-\frac{7}{4} \leq x_1 < \frac{3}{4}$, the only possible candidates for $\mathbf{u}_2$ are $(1, -2)$ and $(\frac{3}{2}, -2)$, and the only possible candidates for $\mathbf{u}_3$ are $(\frac{1}{2}, -1)$ and $(1, -1)$. Then there is no $\mathbf{u}_4$ which can satisfy the condition Lemma 1.

Subcase 3.2.2.2. $y_3 - y_2 = 1$, $y_4 - y_3 = 3$ *and* $y_5 - y_4 = 1$. Then, recalling the assumption that $-\frac{7}{4} \leq x_1 < \frac{3}{4}$, the only possible candidates for $\mathbf{u}_2$ are $(1, -2)$ and $(\frac{3}{2}, -2)$, and the only possible candidates for $\mathbf{u}_3$ are $(\frac{1}{2}, 0)$ and $(1, 0)$. Then one can deduce that the possible octagons have to take $\mathbf{u}_2 = (\frac{3}{2}, -2)$, $\mathbf{u}_3 = (1, 0)$ and $\mathbf{u}_4 = (0, 2)$. Unfortunately, for such octagons we have

$$\mathrm{vol}(P_8) = 9.$$

Subcase 3.2.2.3. $y_3 - y_2 = 1$, $y_4 - y_3 = 2$ *and* $y_5 - y_4 = 2$. Then, the only possible candidates for $\mathbf{u}_2$ are $(1, -2)$ and $(\frac{3}{2}, -2)$, and the only possible candidates for $\mathbf{u}_3$ are $(\frac{1}{2}, -\frac{1}{2})$ and $(1, -\frac{1}{2})$. Then one can deduce that the possible octagons have to take $\mathbf{u}_2 = (\frac{3}{2}, -2)$, $\mathbf{u}_3 = (1, -\frac{1}{2})$ and $\mathbf{u}_4 = (0, \frac{3}{2})$. Unfortunately, for such octagons we have

$$\mathrm{vol}(P_8) = 7.$$

Subcase 3.2.2.4. $y_3 - y_2 = 2$, $y_4 - y_3 = 1$ *and* $y_5 - y_4 = 2$. Then, the only possible candidates for $\mathbf{u}_2$ are $(1, -\frac{3}{2})$ and $(\frac{3}{2}, -\frac{3}{2})$, and the only possible candidates for $\mathbf{u}_3$ are $(\frac{1}{2}, 0)$ and $(1, 0)$. Then one can deduce that the possible octagons have to take $\mathbf{u}_2 = (\frac{3}{2}, -\frac{3}{2})$, $\mathbf{u}_3 = (1, 0)$ and $\mathbf{u}_4 = (0, \frac{3}{2})$. Unfortunately, for such octagons we have

$$\mathrm{vol}(P_8) = 8.$$

Subcase 3.2.3. $x_2' = x_3' > x_4'$. Similar to Subcase 2.3.3, one can deduce $x_2' \geq 1$ and therefore

$$\mathrm{vol}(P_8) \geq 5 \cdot 2x_2' - 3(2x_2' - 1) = 4x_2' + 3 \geq 7.$$

As a conclusion of all these cases, Lemma 11 is proved. □

4. Proofs of the Theorems

Proof of Theorem 1. Assume that P_{2m} is a centrally symmetric $2m$-gon satisfying $\tau^*(P_{2m}) = 5$. First, by Fedorov's theorem and Lemma 6 we have $4 \leq m \leq 7$. Second, by Lemma 9 and Lemma 8 we get $m \neq 6$ and 7, respectively. When $m = 5$, the theorem follows by the first part of Lemma 10. Finally, when $m = 4$, the theorem follows from the first part of Lemma 11. □

Proof of Theorem 2. Let Q_{10} denote the convex decagon with vertices $\mathbf{u}_1 = (0, 1)$, $\mathbf{u}_2 = (1, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (\frac{3}{2}, 0)$, $\mathbf{u}_5 = (1, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ and $\mathbf{u}_{10} = -\mathbf{u}_5$, let L_i denote the straight line containing $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$, where $\mathbf{u}_{10+i} = \mathbf{u}_i$ and $L_{10+i} = L_i$, let $\mathbf{v}_i'$ denote the common point of L_{i-2} and L_i, and let T_i denote the triangle with vertices $\mathbf{v}_i'$, $\mathbf{u}_i$ and $\mathbf{u}_{i-1}$, as shown by Figure 18.

Assume that P_{10} is a fivefold lattice tile with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_{10}$ satisfying

$$\mathbf{v}_{i+1} - \mathbf{u}_i = \mathbf{u}_i - \mathbf{v}_i$$

and therefore

$$\mathbf{v}_{i+1} = 2\mathbf{u}_i - \mathbf{v}_i, \tag{20}$$

where $\mathbf{v}_{10+i} = \mathbf{v}_i$. Apparently, it follows by convexity that

$$\mathbf{v}_i \in \mathrm{int}(T_i), \quad i = 1,\ 2,\ \ldots,\ 10.$$

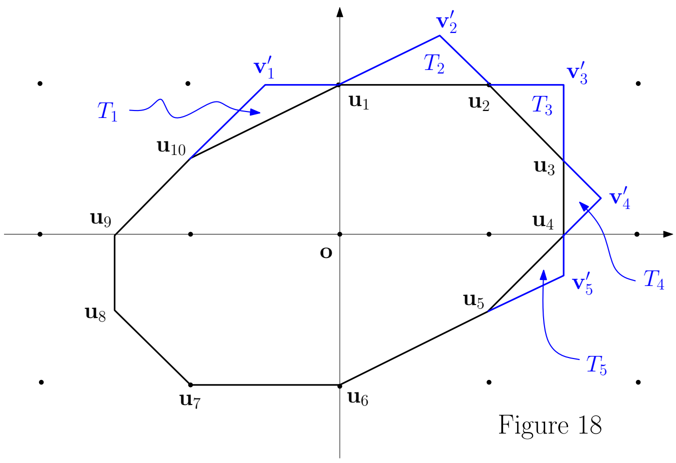

Figure 18

In addition, by (20) we have

$$
\begin{aligned}
&\mathbf{v}_5 \in \operatorname{int}(T_5),\\
&\mathbf{v}_4 \in \big(2\mathbf{u}_4 - \operatorname{int}(T_5)\big) \cap \operatorname{int}(T_4),\\
&\mathbf{v}_3 \in \Big(2\mathbf{u}_3 - \big(2\mathbf{u}_4 - \operatorname{int}(T_5)\big) \cap \operatorname{int}(T_4)\Big) \cap \operatorname{int}(T_3)\\
&\quad = \big(2(\mathbf{u}_3 - \mathbf{u}_4) + \operatorname{int}(T_5)\big) \cap \big(2\mathbf{u}_3 - \operatorname{int}(T_4)\big) \cap \operatorname{int}(T_3),\\
&\mathbf{v}_2 \in \Big(2\mathbf{u}_2 - \big(2\mathbf{u}_3 - \big(2\mathbf{u}_4 - \operatorname{int}(T_5)\big) \cap \operatorname{int}(T_4)\big) \cap \operatorname{int}(T_3)\Big) \cap \operatorname{int}(T_2)\\
&\quad = \big(2(\mathbf{u}_2 - \mathbf{u}_3 + \mathbf{u}_4) - \operatorname{int}(T_5)\big) \cap \big(2(\mathbf{u}_2 - \mathbf{u}_3) + \operatorname{int}(T_4)\big) \cap \big(2\mathbf{u}_2 - \operatorname{int}(T_3)\big) \cap \operatorname{int}(T_2),\\
&\mathbf{v}_1 \in \Big(2\mathbf{u}_1 - \big(2\mathbf{u}_2 - \big(2\mathbf{u}_3 - \big(2\mathbf{u}_4 - \operatorname{int}(T_5)\big) \cap \operatorname{int}(T_4)\big) \cap \operatorname{int}(T_3)\big) \cap \operatorname{int}(T_2)\Big) \cap \operatorname{int}(T_1)\\
&\quad = \big(2(\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3 - \mathbf{u}_4) + \operatorname{int}(T_5)\big) \cap \big(2(\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3) - \operatorname{int}(T_4)\big) \cap\\
&\qquad \big(2(\mathbf{u}_1 - \mathbf{u}_2) + \operatorname{int}(T_3)\big) \cap \big(2\mathbf{u}_1 - \operatorname{int}(T_2)\big) \cap \operatorname{int}(T_1).
\end{aligned}
$$

For convenience, we define

$$
\begin{aligned}
W = &\big(2(\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3 - \mathbf{u}_4) + T_5\big) \cap \big(2(\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3) - T_4\big) \cap\\
&\big(2(\mathbf{u}_1 - \mathbf{u}_2) + T_3\big) \cap \big(2\mathbf{u}_1 - T_2\big) \cap T_1.
\end{aligned}
$$

On the other hand, whenever we take

$$\mathbf{v}_1 \in \operatorname{int}(W)$$

and define $\mathbf{v}_i$ successively by (20), the inverse of the above process and Lemma 4 guarantee that

$$\mathbf{v}_i \in \operatorname{int}(T_i)$$

holds for all $i = 1, 2, \ldots, 10$. Therefore, by Lemma 1 the decagon with them as its vertices is indeed a fivefold lattice tile.

By routine and detailed computation, it can be deduced from its definition that W is a quadrilateral with vertices $\mathbf{w}_1 = (-\frac{1}{2}, 1)$, $\mathbf{w}_2 = (-\frac{1}{2}, \frac{3}{4})$, $\mathbf{w}_3 = (-\frac{2}{3}, \frac{2}{3})$ and $\mathbf{w}_4 = (-\frac{3}{4}, \frac{3}{4})$. Theorem 2 is proved. □

Proof of Theorem 3. Assume that P_{2m} is a centrally symmetric $2m$-gon satisfying $\tau^*(P_{2m}) = 6$. First, by Fedorov's theorem and Lemma 6 we have $4 \le m \le 7$. Second, by Lemma 9 and Lemma 8 we get $m \neq 6$ and 7, respectively. When $m = 5$, the theorem follows by the second part of Lemma 10. Finally, when $m = 4$, the theorem follows from the second part of Lemma 11. □

Proof of Theorem 4. Theorem 4 can be proved just like Theorem 2. □

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Chuanming Zong, Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
Email: cmzong@tju.edu.cn