A GENERALIZED JOIN THEOREM FOR REAL ANALYTIC SINGULARITIES

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ABSTRACT. Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^2, \mathbf{0}_2)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^2, \mathbf{0}_2)$ be real analytic map germs of independent variables, where $n, m \geq 2$. Then the pair (f_1, f_2) of f_1 and f_2 defines a real analytic map germ from $(\mathbb{R}^{n+m}, \mathbf{0}_{n+m})$ to $(\mathbb{R}^4, \mathbf{0}_4)$. We assume that f_1 and f_2 satisfy the a_f-condition at $\mathbf{0}_2$. Let g be a strongly non-degenerate mixed polynomial of 2 complex variables which is locally tame along vanishing coordinate subspaces. A mixed polynomial g defines a real analytic map germ from $(\mathbb{C}^2, \mathbf{0}_4)$ to $(\mathbb{C}, \mathbf{0}_2)$. If we identify $\mathbb{C}$ with $\mathbb{R}^2$, then g also defines a real analytic map germ from $(\mathbb{R}^4, \mathbf{0}_4)$ to $(\mathbb{R}^2, \mathbf{0}_2)$. Then the real analytic map germ $f : (\mathbb{R}^n \times \mathbb{R}^m, \mathbf{0}_{n+m}) \to (\mathbb{R}^2, \mathbf{0}_2)$ is defined by the composition of g and (f_1, f_2), i.e., $f(\mathbf{x}, \mathbf{y}) = (g \circ (f_1, f_2))(\mathbf{x}, \mathbf{y}) = g(f_1(\mathbf{x}), f_2(\mathbf{y}))$, where $(\mathbf{x}, \mathbf{y})$ is a point in a neighborhood of $\mathbf{0}_{n+m}$.

In this paper, we first show the existence of the Milnor fibration of f. We next show a generalized join theorem for real analytic singularities. By this theorem, the homotopy type of the Milnor fiber of f is determined by those of f_1, f_2 and g. For complex singularities, this theorem was proved by A. Némethi. As an application, we show that the zeta function of the monodromy of f is also determined by those of f_1, f_2 and g.

1. Introduction

Let $f_1 : (\mathbb{C}^n, \mathbf{0}_{2n}) \to (\mathbb{C}, \mathbf{0}_2)$ and $f_2 : (\mathbb{C}^m, \mathbf{0}_{2m}) \to (\mathbb{C}, \mathbf{0}_2)$ be holomorphic function germs of independent variables $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{w} = (w_1, \dots, w_m)$. Here $\mathbf{0}_{2N}$ is the origin of $\mathbb{C}^N$. The join theorem for complex singularities is the following.

Theorem 1 (The join theorem). *Let f be a holomorphic function germ on a neighborhood of the origin of $\mathbb{C}^{n+m}$ such that $f(\mathbf{z}, \mathbf{w}) = f_1(\mathbf{z}) + f_2(\mathbf{w})$. Then the Milnor fiber of f is homotopy equivalent to the join of the Milnor fibers of f_1 and f_2 and the monodromy of f is equal to the join of the monodromies of f_1 and f_2 up to homotopy.*

The join theorem was algebraically proved by M. Sebastiani and R. Thom for isolated singularities [27]. So the join theorem is often called the Thom–Sebastiani theorem. M. Oka showed this for weighted homogeneous singularities [17]. For general complex singularities, this was proved by K. Sakamoto [26].

Let $\varphi : (\mathbb{R}^N, \mathbf{0}_N) \to (\mathbb{R}^p, \mathbf{0}_p)$ be a real analytic map germ, where $N \geq p \geq 2$, and $\mathbf{0}_N$ and $\mathbf{0}_p$ are the origins of $\mathbb{R}^N$ and $\mathbb{R}^p$ respectively. In general, real analytic singularities may not admit Milnor fibrations. To show the existence of the Milnor fibration of φ, we assume that φ satisfies the following conditions. Let ε be a small positive real number. Set $V(\varphi) = \varphi^{-1}(\mathbf{0}_p) \cap B_\varepsilon^N$, where $B_\varepsilon^N = \{\mathbf{x} \in \mathbb{R}^N \mid \|\mathbf{x}\| \leq \varepsilon\}$. In this paper, B_ε^N is used for the disk in the domain Euclidean space. A real analytic map germ $\varphi : (\mathbb{R}^N, \mathbf{0}_N) \to (\mathbb{R}^p, \mathbf{0}_p)$ is *locally surjective near the origin* if there exists a positive real number ε such that for any $\mathbf{x} \in V(\varphi)$ and for any neighborhood W of $\mathbf{x}$, the image $\varphi(W)$ is a neighborhood of $\mathbf{0}_p$. We also assume that $V(\varphi)$ has codimension p at the origin. Let $\mathcal{S}$ be a stratification of $V(\varphi)$. The map φ satisfies the *a_f-condition with respect to $\mathcal{S}$* if $B_\varepsilon^N \setminus V(\varphi)$ contains no critical points and satisfies the following condition: Take any sequence p_ν of points in $B_\varepsilon^N \setminus V(\varphi)$ converging to some $p_\infty \in M$, where M is a stratum in $\mathcal{S}$ and suppose

2020 *Mathematics Subject Classification.* Primary 32S55; Secondary: 58K05, 58K10.
Key words and phrases. Milnor fibration, a_f-condition, zeta function of monodromy

that $T_{p_\nu}\varphi^{-1}(\varphi(p_\nu))$ converges to τ in the Grassmanian space. Then $T_{p_\infty}M$ is a subspace of τ. Assume that a real analytic map germ $\varphi : (\mathbb{R}^N, \mathbf{0}_N) \to (\mathbb{R}^p, \mathbf{0}_p)$ satisfies the a_f-condition with respect to $\mathcal{S}$. We say ε is an *a_f-stable radius for φ with respect to $\mathcal{S}$* if it satisfies the following: Each sphere $S^{N-1}_{\varepsilon'} = \{\mathbf{x} \in \mathbb{R}^N \mid \|\mathbf{x}\| = \varepsilon'\}, 0 < \varepsilon' \leq \varepsilon$, intersects transversely with any stratum in $\mathcal{S}$ and $\mathbf{0}_p$ is the unique critical value of $\varphi|_{B^N_\varepsilon} : B^N_\varepsilon \to \mathbb{R}^p$.

Since $\varphi : (\mathbb{R}^N, \mathbf{0}_N) \to (\mathbb{R}^p, \mathbf{0}_p)$ is a real analytic map germ, we may assume that a stratification $\mathcal{S}$ of $V(\varphi)$ is a Whitney stratification. See [8] for further information. By using the same arguments as in the proof of [14, Corollary 2.9] and the proof of [3, Lemma 3.2], we may assume that $S^{N-1}_{\varepsilon'}$ intersects transversely with any stratum in $\mathcal{S}$ for $0 < \varepsilon' \leq \varepsilon$. Assume that φ satisfies the following conditions:

(a-i) φ has an isolated critical value at the origin, $\mathrm{codim}_{\mathbb{R}} V(\varphi) = p$ and φ is locally surjective on $V(\varphi)$ near the origin,

(a-ii) φ satisfies the a_f-condition with respect to $\mathcal{S}$.

Take an a_f-stable radius ε for φ with respect to $\mathcal{S}$. By using the same argument as in the proof of [22, Proposition 11], we can show that there exists a positive real number δ such that S^{N-1}_ε intersects transversely with $\varphi^{-1}(\eta)$ for any $\eta \neq 0$ with $|\eta| \leq \delta \ll \varepsilon$. By the above conditions and the Ehresmann fibration theorem [32], we may assume that

$$\varphi : B^N_\varepsilon \cap \varphi^{-1}(D^p_\delta \setminus \{\mathbf{0}_p\}) \to D^p_\delta \setminus \{\mathbf{0}_p\}$$

is a locally trivial fibration, where $D^p_\delta = \{\mathbf{w} \in \mathbb{R}^p \mid \|\mathbf{w}\| \leq \delta\}$. The isomorphism class of the above fibration does not depend on the choice of ε and δ. We call this fibration the *stable tubular Milnor fibration of φ*.

Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^p, \mathbf{0}_p)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^p, \mathbf{0}_p)$ be real analytic map germs, where $n, m \geq p \geq 2$. Put $n_1 = n$ and $n_2 = m$. Set $V(f_j) = f_j^{-1}(\mathbf{0}_p) \cap B^{n_j}_\varepsilon$ for $0 < \varepsilon \ll 1$ and $j = 1, 2$. We assume that stratifications $\mathcal{S}_j$ of $V(f_j)$ is given and $\{\mathbf{0}_{n_j}\}$ is a stratum in $\mathcal{S}_j$ for $j = 1, 2$. We also assume that f_j satisfies the condition (a-i) and that f_j satisfies the a_f-condition with respect to $\mathcal{S}_j$ for $j = 1, 2$. Take a common a_f-stable radius ε for f_1 and f_2 and take a sufficiently small δ, $0 < \delta \ll \varepsilon$ such that $f_j^{-1}(\eta)$ intersects transversely with $S^{n_j-1}_\varepsilon$ for $j = 1, 2$, for all $\eta \neq 0$ with $|\eta| \leq \delta$. Set $U_j(\varepsilon, \delta) = \{\mathbf{x} \in B^{n_j}_\varepsilon \mid \|f_j(\mathbf{x})\| \leq \delta\}$ for $j = 1, 2$. By the above conditions and the Ehresmann fibration theorem [32], we may assume that

$$f_j : U_j(\varepsilon, \delta) \setminus V(f_j) \to D^p_\delta \setminus \{\mathbf{0}_p\}$$

is the stable tubular Milnor fibration of f_j for $j = 1, 2$. Put $V(f_1 + f_2) = (f_1 + f_2)^{-1}(\mathbf{0}_p) \cap (U_1(\varepsilon, \delta) \times U_2(\varepsilon, \delta))$. We take the stratification $\mathcal{S}_{f_1+f_2}$ of $V(f_1 + f_2)$ as follows:

$$\mathcal{S}_{f_1+f_2} = (\mathcal{S}_1 \times \mathcal{S}_2) \sqcup \{V(f_1 + f_2) \setminus (V(f_1) \times V(f_2))\}.$$

By using $\mathcal{S}_{f_1+f_2}$, we can show that $f_1 + f_2$ also satisfies the conditions (a-i) and (a-ii) [1, Proposition 5.2]. Note that $(f_1 + f_2)^{-1}(\eta) \cap (U_1(\varepsilon, \delta) \times U_2(\varepsilon, \delta))$ is homotopy equivalent to $(f_1 + f_2)^{-1}(\eta) \cap B^{n+m}_{\varepsilon'}$, where $0 < |\eta| \ll \varepsilon' \ll 1$ [9, Lemma 7]. Then we can show that the fiber of the tubular Milnor fibration of $f_1 + f_2$ is homotopy equivalent to the join of the fibers of the tubular Milnor fibrations of f_1 and f_2. Moreover, if $p = 2$, the monodromy of the tubular Milnor fibration of $f_1 + f_2$ is equal to the join of the monodromies of f_1 and f_2 up to homotopy [9, Theorem 2]. L. H. Kauffman and W. D. Neumann studied fiber structures and Seifert forms of links defined by tame isolated singularities of real analytic map germs of independent variables [10]. The definition of tame singularities appears in [10, p. 372]. For mixed weighted homogeneous singularities, the join theorem was proved by J. L. Cisneros-Molina [4].

In [16], A. Némethi studied a generalized join theorem for complex analytic singularities. Let $\phi : (\mathbb{C}^n, \mathbf{0}_{2n}) \to (\mathbb{C}, \mathbf{0}_2)$ be a complex analytic map germ. We can consider a complex analytic map germ as a real analytic map germ $(\Re\phi, \Im\phi) : (\mathbb{R}^{2n}, \mathbf{0}_{2n}) \to (\mathbb{R}^2, \mathbf{0}_2)$. It is known that there exists a stratification $\mathcal{S}_\phi$ of $\phi^{-1}(0)\cap B^{2n}_\varepsilon$ such that ϕ satisfies the a_f-condition with respect to $\mathcal{S}_\phi$ for $0<\varepsilon\ll 1$. See [11, Section 6.4].

Let $f_1 : (\mathbb{C}^n, \mathbf{0}_{2n}) \to (\mathbb{C}, \mathbf{0}_2), f_2 : (\mathbb{C}^m, \mathbf{0}_{2m}) \to (\mathbb{C}, \mathbf{0}_2)$ and $g : (\mathbb{C}^2, \mathbf{0}_4) \to (\mathbb{C}, \mathbf{0}_2)$ be complex analytic map germs of independent variables. Then the complex analytic map germ $f : (\mathbb{C}^n \times \mathbb{C}^m, \mathbf{0}_{2n+2m}) \to (\mathbb{C}, \mathbf{0}_2)$ is defined by the composition of g and (f_1, f_2), i.e., $f(\mathbf{x}, \mathbf{y}) = (g \circ (f_1, f_2))(\mathbf{x}, \mathbf{y}) = g(f_1(\mathbf{x}), f_2(\mathbf{y}))$. Let F_1, F_2 and F_g be the Milnor fibers of f_1, f_2 and g respectively. For $0 < \delta \ll 1$, we denote the disk in the domain Euclidean space of g by $D^4_\delta = \{(z_1, z_2) \in \mathbb{C}^2 \mid \|(z_1, z_2)\| \leq \delta\}$. Let $b_g \subset D^4_\delta$ be a bouquet of circles with base point $*$. We assume that b_g is a deformation retract of the fiber of the stable tubular Milnor fibration of $g|_{D^4_\delta}$ and $b_g \cap \{z_1z_2 = 0\} = \emptyset$. Then the map $(f_1, f_2) : (f_1, f_2)^{-1}(b_g) \to b_g$ is a locally trivial fibration with fiber $F_1 \times F_2$. See [15, 16]. Set $\tilde{F}_1 = V(f_1) \times F_2$ and $\tilde{F}_2 = F_1 \times V(f_2)$. Némethi showed that the Milnor fiber of f has the homotopy type of the space obtained from $(f_1, f_2)^{-1}(b_g)$ by gluing to $(f_1, f_2)^{-1}(*)$ l_1 copies of $\tilde{F}_1$ and l_2 copies of $\tilde{F}_2$, where l_1 is the number of points of $\{(0, z_2) \in D^4_\delta \cap g^{-1}(\tilde{\delta})\}$ and l_2 is the number of points of $\{(z_1, 0) \in D^4_\delta \cap g^{-1}(\tilde{\delta})\}$ for $0 < \tilde{\delta} \ll \delta \ll 1$ [16].

To study a generalization of the join theorem for real analytic singularities, we consider strongly non-degenerate mixed functions. Let $g = (g_1, g_2) : (\mathbb{R}^{2n}, \mathbf{0}_{2n}) \to (\mathbb{R}^2, \mathbf{0}_2)$ be a real analytic map germ with real $2n$ variables $x_1, \dots, x_n$ and $y_1, \dots, y_n$. Then (g_1, g_2) is represented by a complex-valued function of variables $\mathbf{z} = (z_1, \dots, z_n)$ and $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$ as

$$g(\mathbf{z}, \bar{\mathbf{z}}) := g_1\Big(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2\sqrt{-1}}\Big) + \sqrt{-1}g_2\Big(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2\sqrt{-1}}\Big).$$

Here any complex variable z_j of $\mathbb{C}^n$ is represented by $x_j + \sqrt{-1}y_j$ and $\bar{z}_j$ is the complex conjugate of z_j for $j = 1, \dots, n$. Then the map $g : (\mathbb{C}^n, \mathbf{0}_{2n}) \to (\mathbb{C}, \mathbf{0}_2)$ is called a *mixed function*. Oka introduced the notion of Newton boundaries of mixed functions and the concept of strong non-degeneracy. Let g be a strongly non-degenerate mixed function which is locally tame along vanishing coordinate subspaces. Then there exists a stratification $\mathcal{S}_{can}$ of $g^{-1}(\mathbf{0}_2)$ such that g satisfies the a_f-condition with respect to $\mathcal{S}_{can}$. See [22] and Section 2. By [22, Lemma 14], g also satisfies the condition (a-i).

Assume that $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^2, \mathbf{0}_2)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^2, \mathbf{0}_2)$ satisfy the conditions (a-i) and (a-ii). Let g be a strongly non-degenerate mixed polynomial of 2 complex variables which is locally tame along vanishing coordinate subspaces. By using $\mathcal{S}_{can}$, we can take an a_f-stable radius δ for g and take a sufficiently small $\tilde{\delta}$, $0 < \tilde{\delta} \ll \delta$ such that $g : D^4_\delta \cap g^{-1}(D^2_{\tilde{\delta}} \setminus \{\mathbf{0}_2\}) \to D^2_{\tilde{\delta}} \setminus \{\mathbf{0}_2\}$ is a locally trivial fibration. Let $b_g \subset D^4_\delta$ be a bouquet of circles with base point $*$. Assume that b_g is a deformation retract of the fiber of the stable tubular Milnor fibration of g and $b_g \cap \{z_1z_2 = 0\} = \emptyset$. By the local triviality of $g : D^4_\delta \cap g^{-1}(D^2_{\tilde{\delta}} \setminus \{\mathbf{0}_2\}) \to D^2_{\tilde{\delta}} \setminus \{\mathbf{0}_2\}$, the map $(f_1, f_2) : (f_1, f_2)^{-1}(b_g) \to b_g$ is a locally trivial fibration with fiber $F_1 \times F_2$. See the proof of Theorem 3. If we identify $\mathbb{C}$ with $\mathbb{R}^2$, then g also defines a real analytic map germ from $(\mathbb{R}^4, \mathbf{0}_4)$ to $(\mathbb{R}^2, \mathbf{0}_2)$. Then the real analytic map germ $f : (\mathbb{R}^n \times \mathbb{R}^m, \mathbf{0}_{n+m}) \to (\mathbb{R}^2, \mathbf{0}_2)$ is defined by $f(\mathbf{x}, \mathbf{y}) = (g \circ (f_1, f_2))(\mathbf{x}, \mathbf{y}) = g(f_1(\mathbf{x}), f_2(\mathbf{y}))$, where $(\mathbf{x}, \mathbf{y})$ is a point in a neighborhood of $\mathbf{0}_{n+m}$. In general, f is not strongly non-degenerate. To show the existence of the Milnor fibration of f, we need to prove that f satisfies the a_f-condition. Take a common a_f-stable radius ε for f_1 and f_2 and take a sufficiently small δ' such that $f_j : U_j(\varepsilon, \delta') \setminus V(f_j) \to D^2_{\delta'} \setminus \{\mathbf{0}_2\}$ is the stable tubular Milnor fibration of f_j for $j = 1, 2$. Set $V(f) = f^{-1}(\mathbf{0}_2) \cap (U_1(\varepsilon, \delta') \times U_2(\varepsilon, \delta'))$. We define

$$\mathcal{S}'(1) := \begin{cases} \{(U_1(\varepsilon, \delta') \setminus V(f_1)) \times M_2 \mid M_2 \in \mathcal{S}_2\} & \{1\} \in \mathcal{I}_v(g), \\ \emptyset & \{1\} \notin \mathcal{I}_v(g), \end{cases}$$

$$\mathcal{S}'(2) := \begin{cases} \{M_1 \times (U_2(\varepsilon, \delta') \setminus V(f_2)) \mid M_1 \in \mathcal{S}_1\} & \{2\} \in \mathcal{I}_v(g), \\ \emptyset & \{2\} \notin \mathcal{I}_v(g). \end{cases}$$

The definition of $\mathcal{I}_v(g)$ will be explained in Section 2. Then put $\mathcal{S}' = \mathcal{S}'(1) \cup \mathcal{S}'(2)$ and we define the stratum $V(f)'$ of $V(f)$ and the stratification $\mathcal{S}_f$ of $V(f)$ as follows.

$$V(f)' = V(f) \setminus \bigcup_{N \in (\mathcal{S}_1 \times \mathcal{S}_2) \cup \mathcal{S}'} N,$$
$$\mathcal{S}_f = (\mathcal{S}_1 \times \mathcal{S}_2) \cup \mathcal{S}' \cup \{V(f)'\}.$$

By using $\mathcal{S}_f$, we can show the following theorem.

Theorem 2. *Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^2, \mathbf{0}_2)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^2, \mathbf{0}_2)$ be real analytic map germs of independent variables $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$, where $n, m \geq 2$. Assume that f_1 and f_2 satisfy the conditions (a-i) and (a-ii). Let g be a strongly non-degenerate mixed polynomial of 2 complex variables which is locally tame along vanishing coordinate subspaces. Then the real analytic map germ $f = g \circ (f_1, f_2)$ satisfies the a_f-condition with respect to $\mathcal{S}_f$.*

By Theorem 2, we can show that f admits the Milnor fibration. To study Némethi's theorem for f, we assume that f_1, f_2 and g satisfy the above conditions and add the assumption (A) on g. See Section 4.

Theorem 3. *Let $f = g \circ (f_1, f_2) : (\mathbb{R}^n \times \mathbb{R}^m, \mathbf{0}_{n+m}) \to (\mathbb{R}^2, \mathbf{0}_2)$ be the real analytic map germ as in Theorem 2. Assume that g satisfies the assumption (A) in Section 4. Let $b_g \subset D^4_\delta$ be a bouquet of circles with base point $*$. Assume that b_g is a deformation retract of the fiber of the stable tubular Milnor fibration of g and $b_g \cap \{z_1 z_2 = 0\} = \emptyset$. Set $\tilde{F}_1 = V(f_1) \times F_2$ and $\tilde{F}_2 = F_1 \times V(f_2)$. Then the Milnor fiber F_f of f is homotopy equivalent to the space obtained from $(f_1, f_2)^{-1}(b_g)$ by gluing to $(f_1, f_2)^{-1}(*)$ l_1 copies of $\tilde{F}_1$ and l_2 copies of $\tilde{F}_2$, where l_1 is the number of points of $\{(0, z_2) \in D^4_\delta \cap g^{-1}(\tilde{\delta})\}$ and l_2 is the number of points of $\{(z_1, 0) \in D^4_\delta \cap g^{-1}(\tilde{\delta})\}$ for $0 < \tilde{\delta} \ll \delta \ll 1$.*

As an application of Theorem 3, the monodromy of f is determined by those of f_1, f_2 and g. Then we can calculate the zeta function of the monodromy of f by using the Alexander polynomial of the link determined by $g^{-1}(0)$ and the zeta function of the monodromy of f_j for $j = 1, 2$. See Section 6.

This paper is organized as follows. In Section 2 we give the definition of strongly non-degenerate mixed functions. In Section 3 we prove Theorem 2 and the existence of the Milnor fibration of f. In Section 4 we study homeomorphisms of Milnor fibers of mixed polynomials of 2 complex variables. In Section 5 we prove Theorem 3. In Section 6 we study the zeta function of the monodromy of f.

The author would like to thank Mutsuo Oka for precious comments and fruitful suggestions. He also thanks to the referee and the editor for careful reading of the manuscript and several accurate comments.

2. Strongly non-degenerate mixed functions

In this section, we introduce a class of mixed functions which admit tubular Milnor fibrations and spherical Milnor fibrations given by Oka in [20]. Let $g(\mathbf{z}, \bar{\mathbf{z}})$ be a mixed function, i.e., $g(\mathbf{z}, \bar{\mathbf{z}})$ is a function expanded in a convergent power series of variables $\mathbf{z} = (z_1, \dots, z_n)$ and $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$,

$$g(\mathbf{z}, \bar{\mathbf{z}}) := \textstyle\sum_{\nu, \mu} c_{\nu,\mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu,$$

where $\mathbf{z}^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n)$ (respectively $\bar{\mathbf{z}}^\mu = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n)$). The *Newton polygon* $\Gamma_+(g; \mathbf{z}.\bar{\mathbf{z}})$ is defined by the convex hull of

$$\textstyle\bigcup_{(\nu,\mu)} \{(\nu + \mu) + \mathbb{R}^n_+ \mid c_{\nu,\mu} \neq 0\},$$

where $\nu+\mu$ is the sum of the multi-indices of $\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$, i.e., $\nu+\mu=(\nu_1+\mu_1,\dots,\nu_n+\mu_n)$. The *Newton boundary* $\Gamma(g;\mathbf{z},\bar{\mathbf{z}})$ is the union of compact faces of $\Gamma_+(g;\mathbf{z},\bar{\mathbf{z}})$. Let $\mathbb{Z}_+$ be the set of non-negative integers. For any non-zero weight vector $P={}^t(p_1,\dots,p_n)\in(\mathbb{Z}_+)^n$, we define a linear function ℓ_P on $\Gamma_+(g;\mathbf{z},\bar{\mathbf{z}})$ as follows:

$$\xi=(\xi_1,\dots,\xi_n)\mapsto\textstyle\sum_{j=1}^n p_j\xi_j.$$

We denote the minimal value of ℓ_P by $d(P)$ and put $\Delta(P)=\{\xi\in\Gamma_+(g;\mathbf{z},\bar{\mathbf{z}})\mid\ell_P(\xi)=d(P)\}$. Let Δ and P be a face of $\Gamma_+(g;\mathbf{z},\bar{\mathbf{z}})$ and a non-zero weight vector respectively, then we define

$$g_\Delta(\mathbf{z},\bar{\mathbf{z}})=\textstyle\sum_{\nu+\mu\in\Delta}c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu,\quad g_P(\mathbf{z},\bar{\mathbf{z}})=\sum_{\nu+\mu\in\Delta(P)}c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu.$$

The mixed functions g_Δ and g_P are called the *face function of f of the face* Δ and the *face function of f of the weight vector P* respectively.

The *strong non-degeneracy of mixed functions* is defined from the Newton boundary as follows: let Δ be a face of $\Gamma(g;\mathbf{z},\bar{\mathbf{z}})$. If $g_\Delta:\mathbb{C}^{*n}\to\mathbb{C}$ has no critical points, and g_Δ is surjective when $\dim\Delta\geq 1$, we say that $g(\mathbf{z},\bar{\mathbf{z}})$ is *strongly non-degenerate* for Δ, where $\mathbb{C}^{*n}=\{\mathbf{z}=(z_1,\dots,z_n)\mid z_j\neq 0, j=1,\dots,n\}$. If $g(\mathbf{z},\bar{\mathbf{z}})$ is strongly non-degenerate for any Δ, we say that $g(\mathbf{z},\bar{\mathbf{z}})$ is *strongly non-degenerate*. If $g((0,\dots,0,z_j,0,\dots,0),(0,\dots,0,\bar{z}_j,0,\dots,0))\not\equiv 0$ for each $j=1,\dots,n$, then we say that $g(\mathbf{z},\bar{\mathbf{z}})$ is *convenient*.

For a subset $I\subset\{1,\dots,n\}$, we set

$$\mathbb{C}^I=\{(z_1,\dots,z_n)\in\mathbb{C}^n\mid z_i=0,i\notin I\},\quad\mathbb{C}^{*I}=\{(z_1,\dots,z_n)\in\mathbb{C}^n\mid z_i\neq 0\Leftrightarrow i\in I\}.$$

Note that $\mathbb{C}^{*\emptyset}=\{\mathbf{0}_{2n}\}$. Put $g^I=g|_{\mathbb{C}^I}$. Then we define the subsets of $\{I\mid I\subset\{1,\dots,n\}\}$ as follows:

$$\mathcal{I}_{nv}(g)=\{I\subset\{1,\dots,n\}\mid g^I\not\equiv 0\},\quad\mathcal{I}_v(g)=\{I\subset\{1,\dots,n\}\mid g^I\equiv 0\}.$$

If $I\in\mathcal{I}_v(g)$, $\mathbb{C}^I$ is called a *vanishing coordinate subspace*. For $I\in\mathcal{I}_v(g)$, we define the distance function on $\mathbb{C}^I$ by $\rho_I(\mathbf{z})=\sqrt{\sum_{i\in I}|z_i|^2}$. Let $\pi_I:\mathbb{C}^n\to\mathbb{C}^I$ be the projection and put $\mathbf{z}_I=\pi_I(\mathbf{z})$. We say that g is *locally tame along the vanishing coordinate subspace* $\mathbb{C}^I$ if there exists a positive real number r_I such that for any $\mathbf{a}_I=(\alpha_i)_{i\in I}\in\mathbb{C}^{*I}$ with $\rho_I(\mathbf{a}_I)\leq r_I$ and for any non-zero weight vector $P={}^t(p_1,\dots,p_n)$ with $I(P)=\{i\mid p_i=0\}=I$, $g_P|_{\mathbf{z}_I=\mathbf{a}_I}$ is strongly non-degenerate as a function of $\{z_j\mid j\in I^c\}$. A mixed function g is said to be *locally tame* if g is locally tame for any vanishing coordinate subspace. If a strongly non-degenerate mixed function g is convenient or locally tame for any vanishing coordinate subspace, g has both tubular and spherical Milnor fibrations and also two fibrations are isomorphic [20, 22]. The definition of spherical Milnor fibrations appears in Section 5.1 of the present paper. Moreover $g^{-1}(0)\cap B_\varepsilon^{2n}$ has the following stratification.

Theorem 4 ([22]). *Let g be a strongly non-degenerate mixed polynomial. Assume that g is locally tame for any vanishing coordinate subspace. Let ε be a positive real number which satisfies the following conditions:*

- *there exists a positive real number $\delta(\varepsilon)$ such that $g^{-1}(\eta)$ has no singularities in B_ε^{2n} for any non-zero η with $|\eta|\leq\delta(\varepsilon)$,*
- *$\varepsilon\leq\min\{r_I\mid I\in\mathcal{I}_v(g)\}$.*

Set

$$\mathcal{S}_{can}:=\{g^{-1}(0)\cap\mathbb{C}^{*I},\mathbb{C}^{*I}\setminus(g^{-1}(0)\cap\mathbb{C}^{*I})\mid I\in\mathcal{I}_{nv}(g)\}\cup\{\mathbb{C}^{*I}\mid I\in\mathcal{I}_v(g)\}.$$

Then g satisfies the a_f-condition with respect to $\mathcal{S}_{can}$ in B_ε^{2n}.

Let g_t be an analytic family of strongly non-degenerate mixed polynomials which are locally tame along vanishing coordinate subspaces. Assume that the Newton boundary of g_t is constant for $0\leq t\leq 1$. C. Eyral and M. Oka showed that the topological type of $(V(g_t),\mathbf{0}_{2n})$ is constant for any t and their tubular Milnor fibrations are equivalent [6].

3. The existence of the Milnor fibration of f

Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^2, \mathbf{0}_2)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^2, \mathbf{0}_2)$ be real analytic map germs of independent variables, where $n, m \geq 2$. For a small positive real number ε, we take a stratification $\mathcal{S}_j$ of $f_j^{-1}(\mathbf{0}_2) \cap B_\varepsilon^{n_j}$ with $n_1 = n$ and $n_2 = m$. Suppose that f_j satisfies the conditions (a-i) and (a-ii) with respect to $\mathcal{S}_j$ for $j = 1, 2$. Take a positive real number δ' that is sufficiently smaller than ε. Then we may assume that $f_j : U_j(\varepsilon, \delta') \setminus V(f_j) \to D^2_{\delta'} \setminus \{\mathbf{0}_2\}$ is a locally trivial fibration for $j = 1, 2$. Let g be a strongly non-degenerate mixed polynomial of 2 complex variables which is locally tame along vanishing coordinate subspaces. Then the real analytic map germ $f : (\mathbb{R}^n \times \mathbb{R}^m, \mathbf{0}_{n+m}) \to (\mathbb{R}^2, \mathbf{0}_2)$ is defined by $f(\mathbf{x}, \mathbf{y}) = g(f_1(\mathbf{x}), f_2(\mathbf{y}))$, where $(\mathbf{x}, \mathbf{y})$ is a point in a neighborhood of $\mathbf{0}_{n+m}$. In this section, we prove the existence of the Milnor fibration of f.

Lemma 1. *The origin $\mathbf{0}_2$ is an isolated critical value of f.*

Proof. For any $(\mathbf{x}, \mathbf{y}) \in (U_1(\varepsilon, \delta') \times U_2(\varepsilon, \delta')) \setminus V(f)$, we show that the rank of $Jf(\mathbf{x}, \mathbf{y})$ is equal to 2, where Jf is the Jacobian matrix of f. Set $g_1 = \Re g, g_2 = \Im g, z_{j1} = \Re z_j$ and $z_{j2} = \Im z_j$ for $j = 1, 2$. Put

$$G_1 = \begin{pmatrix} \frac{\partial g_1}{\partial z_{11}} & \frac{\partial g_1}{\partial z_{12}} \\ \frac{\partial g_2}{\partial z_{11}} & \frac{\partial g_2}{\partial z_{12}} \end{pmatrix}, \quad G_2 = \begin{pmatrix} \frac{\partial g_1}{\partial z_{21}} & \frac{\partial g_1}{\partial z_{22}} \\ \frac{\partial g_2}{\partial z_{21}} & \frac{\partial g_2}{\partial z_{22}} \end{pmatrix}.$$

Since $f = g \circ (f_1, f_2)$, the Jacobian matrix Jf of f is equal to

$$\begin{pmatrix} G_1 & G_2 \end{pmatrix} \begin{pmatrix} Jf_1 & O' \\ O & Jf_2 \end{pmatrix} = \begin{pmatrix} G_1 Jf_1 & G_2 Jf_2 \end{pmatrix},$$

where O is the $2 \times n$ zero matrix and O' is the $2 \times m$ zero matrix .

Suppose that $f_1(\mathbf{x}) = \mathbf{0}_2, \{2\} \in \mathcal{I}_{nv}(g)$ and $f_2(\mathbf{y}) \neq \mathbf{0}_2$. Since g is a strongly non-degenerate mixed polynomial, there exists a weight vector P such that $g(0, z_2)$ is given by

$$g(0, z_2) = g_P(z_2) + (\text{higher terms})$$

and $g_P : \mathbb{C}^{*\{2\}} \to \mathbb{C}$ has no critical points. Thus for any sufficiently small $\varepsilon > 0$, $g|_{\mathbb{C}^{*\{2\}}} : \mathbb{C}^{*\{2\}} \to \mathbb{C}$ also does not have critical points for z_2, $|z_2| \leq \varepsilon$. By the condition (a-i), $\operatorname{rank} Jf_2(\mathbf{y}) = \operatorname{rank} G_2 = 2$. Thus $\operatorname{rank} Jf(\mathbf{x}, \mathbf{y})$ is equal to 2. If $f_1(\mathbf{x}) \neq \mathbf{0}_2, f_2(\mathbf{y}) = \mathbf{0}_2$ and $\{1\} \in \mathcal{I}_{nv}(g)$, by using the same argument, we can show that $\operatorname{rank} Jf(\mathbf{x}, \mathbf{y})$ is equal to 2.

Assume that $(\mathbf{x}, \mathbf{y})$ satisfies $f_1(\mathbf{x}) \neq \mathbf{0}_2, f_2(\mathbf{y}) \neq \mathbf{0}_2$ and $f(\mathbf{x}, \mathbf{y}) = g(f_1(\mathbf{x}), f_2(\mathbf{y})) \neq \mathbf{0}_2$. Since f_1, f_2 and g have an isolated critical value at the origin, we have

$$\operatorname{rank} Jf_1(\mathbf{x}) = \operatorname{rank} Jf_2(\mathbf{y}) = \operatorname{rank} Jg(f_1(\mathbf{x}), f_2(\mathbf{y})) = 2.$$

If $\operatorname{rank} G_1 = 2$ or $\operatorname{rank} G_2 = 2$, then the rank of $Jf(\mathbf{x}, \mathbf{y})$ is equal to 2.

Suppose that $\operatorname{rank} G_1 < 2$ and $\operatorname{rank} G_2 < 2$. Set

$$\left(\frac{\partial g_1}{\partial z_{11}}, \frac{\partial g_1}{\partial z_{12}}, \frac{\partial g_1}{\partial z_{21}}, \frac{\partial g_1}{\partial z_{22}} \right) = (a_1, a_2, b_1, b_2).$$

Since $\operatorname{rank} Jg(f_1(\mathbf{x}), f_2(\mathbf{y})) = 2$, we may assume that $(a_1, a_2, b_1, b_2) \neq (0, 0, 0, 0)$. If $(a_1, a_2) \neq (0, 0)$ and $(b_1, b_2) \neq (0, 0)$, then there exist real numbers r and s such that $Jg(f_1(\mathbf{x}), f_2(\mathbf{y})) = (G_1\ G_2)$ is equal to

$$\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ ra_1 & ra_2 & sb_1 & sb_2 \end{pmatrix}.$$

Since $\operatorname{rank} Jg(f_1(\mathbf{x}), f_2(\mathbf{y})) = 2$, $r \neq s$. Set $f_j = (f_{j1}, f_{j2})$, where f_{j1} and f_{j2} are real-valued functions for $j = 1, 2$. Then $Jf(\mathbf{x}, \mathbf{y})$ is equal to

$$\begin{pmatrix} a_1 df_{11} + a_2 df_{12} & b_1 df_{21} + b_2 df_{22} \\ r(a_1 df_{11} + a_2 df_{12}) & s(b_1 df_{21} + b_2 df_{22}) \end{pmatrix}.$$

Here df_{jk} is the gradient of a smooth function f_{jk} for $j = 1, 2$ and $k = 1, 2$. Since rank $Jf_1(\mathbf{x}) =$ rank $Jf_2(\mathbf{y}) = 2$, we have

$$a_1 df_{11} + a_2 df_{12} \neq O, \quad b_1 df_{21} + b_2 df_{22} \neq O'.$$

Note that r is not equal to s. Thus the rank of $Jf(\mathbf{x}, \mathbf{y})$ is equal to 2.

If $(a_1, a_2) \neq (0, 0)$ and $(b_1, b_2) = (0, 0)$, then $Jg(f_1(\mathbf{x}), f_2(\mathbf{y})) = (G_1\ G_2)$ is equal to

$$\begin{pmatrix} a_1 & a_2 & 0 & 0 \\ ra_1 & ra_2 & b_1' & b_2' \end{pmatrix}.$$

Hence $Jf(\mathbf{x}, \mathbf{y})$ is equal to

$$\begin{pmatrix} a_1 df_{11} + a_2 df_{12} & O' \\ r(a_1 df_{11} + a_2 df_{12}) & b_1' df_{21} + b_2' df_{22} \end{pmatrix}.$$

Since rank $Jf_1(\mathbf{x}) =$ rank $Jf_2(\mathbf{y}) = 2$, $a_1 df_{11} + a_2 df_{12} \neq O$ and $b_1' df_{21} + b_2' df_{22} \neq O'$. Thus rank $Jf(\mathbf{x}, \mathbf{y})$ is equal to 2. If $(a_1, a_2) = (0, 0)$ and $(b_1, b_2) \neq (0, 0)$, then by using the same argument, we can show that rank $Jf(\mathbf{x}, \mathbf{y}) = 2$. □

Proof of Theorem 2. We use Curve Selection Lemma to prove the assertion. Let $(\mathbf{x}(t), \mathbf{y}(t)) \in U_1(\varepsilon, \delta') \times U_2(\varepsilon, \delta')$ be an arbitrary real analytic curve such that $(\mathbf{x}(0), \mathbf{y}(0)) \in f^{-1}(\mathbf{0}_2)$ and $f(\mathbf{x}(t), \mathbf{y}(t)) \neq \mathbf{0}_2$ for $t \neq 0$. It is enough to check that the a_f-condition is satisfied along this curve. Put $(\mathbf{a}, \mathbf{b}) = (\mathbf{x}(0), \mathbf{y}(0))$. Then $(\mathbf{a}, \mathbf{b})$ belongs to one of $\{V(f)'\}, N, N_1', N_2'$, where $N \in \mathcal{S}_1 \times \mathcal{S}_2$ and $N_j' \in \mathcal{S}'(j)$ for $j = 1, 2$. So we divide the proof into four cases:

(1) $(\mathbf{a}, \mathbf{b}) \in V(f)', \mathbf{a} \notin V(f_1)$ and $\mathbf{b} \notin V(f_2)$,
(2) $(\mathbf{a}, \mathbf{b}) \in M_1 \times M_2$,
(3) $\mathbf{a} \in M_1$ and $\mathbf{b} \notin V(f_2)$,
(4) $\mathbf{a} \notin V(f_1)$ and $\mathbf{b} \in M_2$,

where $M_j \in \mathcal{S}_j$ for $j = 1, 2$. Since f_1 and f_2 satisfy the condition (a-i) and g is strongly non-degenerate, in case (1), $(\mathbf{a}, \mathbf{b})$ is a regular point of f. In case (2), since f_1 and f_2 satisfy the condition (a-ii), we have

$$\begin{aligned} &\lim_{t \to 0} T_{(\mathbf{x}(t), \mathbf{y}(t))} f^{-1}\big(f(\mathbf{x}(t), \mathbf{y}(t))\big) \\ &\supset \lim_{t \to 0} \Big(T_{\mathbf{x}(t)} f_1^{-1}\big(f_1(\mathbf{x}(t))\big) \times T_{\mathbf{y}(t)} f_2^{-1}\big(f_2(\mathbf{y}(t))\big)\Big) \\ &\supset T_{(\mathbf{a}, \mathbf{b})} M_1 \times M_2. \end{aligned}$$

Case (3) is divided into two cases:

(3-1) $\{2\} \in \mathcal{I}_v(g)$, i.e., $(\mathbf{a}, \mathbf{b}) \in \mathcal{S}'(2)$,
(3-2) $\{2\} \notin \mathcal{I}_v(g)$, i.e., $(\mathbf{a}, \mathbf{b}) \in V(f)'$.

In case (3-1), by using Theorem 4, we can show that there exist vectors

$$\mathbf{v}_{g,1}(t) = (g_{1,1}, g_{1,2}, 0, 0)t^r + (\text{higher terms}),$$

$$\mathbf{v}_{g,2}(t) = (g_{2,1}, g_{2,2}, 0, 0)t^{r'} + (\text{higher terms})$$

such that rank $\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} = 2$ and $\lim_{t \to 0} T_{(f_1(\mathbf{x}(t)), f_2(\mathbf{y}(t)))} g^{-1}\big(g(f_1(\mathbf{x}(t)), f_2(\mathbf{y}(t)))\big)$ is orthogonal to $(g_{1,1}, g_{1,2}, 0, 0)$ and $(g_{2,1}, g_{2,2}, 0, 0)$. Since f_1 satisfies the a_f-condition with respect to $\mathcal{S}_1$, there exist vectors

$$\mathbf{v}_{f_1,1}(t) = \mathbf{a}_1 t^s + (\text{higher terms}), \ \mathbf{v}_{f_1,2}(t) = \mathbf{a}_2 t^s + (\text{higher terms})$$

such that

$$\lim_{t \to 0} T_{\mathbf{x}(t)} f_1^{-1}\big(f_1(\mathbf{x}(t))\big) = \mathbf{a}_1^{\perp} \cap \mathbf{a}_2^{\perp} \supset T_{\mathbf{x}(0)} M_1,$$

where $\mathbf{a}_j^{\perp} = \{\mathbf{v} \in \mathbb{R}^n \mid (\mathbf{v}, \mathbf{a}_j) = 0\}$ for $j = 1, 2$. Up to scalar multiplications, we may assume that $\mathbf{v}_{f_1,1}(t)$ and $\mathbf{v}_{f_1,2}(t)$ are equal to df_{11} and df_{12} respectively. Note that $\mathbf{v}_{g,1}(t)$ and $\mathbf{v}_{g,2}(t)$ are linear combinations of dg_1 and dg_2 for $t \neq 0$. See the proof of [22, Theorem 20]. Since f is the composition of g and (f_1, f_2), $T_{(\mathbf{x}(t),\mathbf{y}(t))} f^{-1}\big(f(\mathbf{x}(t), \mathbf{y}(t))\big)$ is orthogonal to

$$\begin{pmatrix} \mathbf{v}_{g,1}(t) \\ \mathbf{v}_{g,2}(t) \end{pmatrix} \begin{pmatrix} \mathbf{v}_{f_1,1}(t) & 0 \cdots 0 \\ \mathbf{v}_{f_1,2}(t) & 0 \cdots 0 \\ 0 \cdots 0 & df_{21}(\mathbf{y}(t)) \\ 0 \cdots 0 & df_{22}(\mathbf{y}(t)) \end{pmatrix} = \begin{pmatrix} (g_{1,1}\mathbf{a}_1 + g_{1,2}\mathbf{a}_2, 0, \dots, 0)t^{r+s} + (\text{higher terms}) \\ (g_{2,1}\mathbf{a}_1 + g_{2,2}\mathbf{a}_2, 0, \dots, 0)t^{r'+s} + (\text{higher terms}) \end{pmatrix}.$$

Thus $\lim_{t\to 0} T_{(\mathbf{x}(t),\mathbf{y}(t))} f^{-1}\big(f(\mathbf{x}(t), \mathbf{y}(t))\big)$ is orthogonal to the following vectors:

$$(g_{1,1}\mathbf{a}_1 + g_{1,2}\mathbf{a}_2, 0, \dots, 0), \quad (g_{2,1}\mathbf{a}_1 + g_{2,2}\mathbf{a}_2, 0, \dots, 0).$$

Since rank $\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} = 2$, $\mathbf{v} \in \mathbb{R}^n$ is orthogonal to $\mathbf{a}_1$ and $\mathbf{a}_2$ if and only if $\mathbf{v}$ is orthogonal to $g_{1,1}\mathbf{a}_1 + g_{1,2}\mathbf{a}_2$ and $g_{2,1}\mathbf{a}_1 + g_{2,2}\mathbf{a}_2$. Thus we have the following inclusion relation:

$$\begin{aligned} &\lim_{t\to 0} T_{(\mathbf{x}(t),\mathbf{y}(t))} f^{-1}\big(f(\mathbf{x}(t), \mathbf{y}(t))\big) \\ &\supset \lim_{t\to 0} T_{\mathbf{x}(t)} f_1^{-1}\big(f_1(\mathbf{x}(t)\big) \times T_{\mathbf{y}(t)}(U_2(\varepsilon, \delta') \setminus V(f_2)) \\ &\supset T_{\mathbf{a}} M_1 \times T_{\mathbf{b}}(U_2(\varepsilon, \delta') \setminus V(f_2)). \end{aligned}$$

Thus f satisfies the a_f-condition with respect to $\mathcal{S}'(2)$. If $\{2\} \notin \mathcal{I}_v(g)$, by using the same argument as in the proof of Lemma 1, the rank of $Jf(\mathbf{a}, \mathbf{b})$ is equal to 2. Thus $(\mathbf{a}, \mathbf{b})$ is a regular point of f. Case (4) follows from case (3) by interchanging the variables z_1 and z_2. $\square$

Example 1. Consider $g(\mathbf{z}, \bar{\mathbf{z}}) = (d_1 + c_1\sqrt{-1})z_1|z_2|^2$. Put $\mathbf{w} = (c_1 + d_1\sqrt{-1}, z_2) \in \mathbb{C}^2$, where $c_1 + d_1\sqrt{-1} \neq 0$. Then the normalized gradient of $\Re g$ is given by

$$\frac{1}{\sqrt{c_1^2 + d_1^2}}(d_1, -c_1, 0, 0).$$

When $z_2 \to 0$, we have

$$\lim_{z_2\to 0} T_{\mathbf{w}} g^{-1}(g(\mathbf{w})) \subset \lim_{z_2\to 0} T_{\mathbf{w}}(\Re g)^{-1}(\Re g(\mathbf{w})) \not\supset \mathbb{C} \times \{0\}.$$

Hence g does not satisfy the a_f-condition with respect to $\mathcal{S}_{can}$ [22]. Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^2, \mathbf{0}_2)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^2, \mathbf{0}_2)$ be real analytic map germs of independent variables, where $n, m \geq 2$. Assume that f_1 and f_2 satisfy the conditions (a-i) and (a-ii). In this case, $G_1(\mathbf{w})Jf_1(\mathbf{x}) \neq O$, where $\mathbf{x} \in f_1^{-1}(c_1 + d_1\sqrt{-1})$. Set $\mathbf{y} \in f_2^{-1}(z_2)$. Then we have

$$\lim_{z_2\to 0} T_{(\mathbf{x},\mathbf{y})} f^{-1}(f(\mathbf{x}, \mathbf{y})) \not\supset T_{\mathbf{x}}(U_1(\varepsilon, \delta) \setminus V(f_1)) \times T_{\mathbf{y}} \mathcal{S}_2.$$

Thus $f = g \circ (f_1, f_2)$ does not satisfy the a_f-condition with respect to $\mathcal{S}_f$.

We next consider $g_a(\mathbf{z}, \bar{\mathbf{z}}) = z_1 z_2^a \bar{z}_2$ for $a \geq 2$. Then g_a is a strongly non-degenerate mixed polynomial which is locally tame along vanishing coordinate subspaces. By Theorem 2, $f_a = g_a \circ (f_1, f_2)$ satisfies the a_f-condition with respect to $\mathcal{S}_{f_a}$.

Lemma 2. *Let f be as in Theorem 2. The real analytic map germ f is locally surjective on $V(f)$ near the origin and the codimension of $V(f)$ is equal to* 2.

Proof. Since f_1 and f_2 satisfy the condition (a-i) and g admits the Milnor fibration, f is locally surjective on $\mathcal{S}_1 \times \mathcal{S}_2$. Let $(\mathbf{x}, \mathbf{y})$ be a point of $\mathcal{S}'$. By using the condition (a-i) and the Milnor fibrations of f_1, f_2 and g, we can show the existence of a neighborhood $W_{(\mathbf{x},\mathbf{y})}$ of $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{0}_2$ is an interior point of $f(W_{(\mathbf{x},\mathbf{y})})$.

Since $V(f)'$ is the set of regular points of f, f is locally surjective on $V(f)'$ and the codimension of $V(f)$ is equal to 2. $\square$

By Theorem 2, Lemma 2 and the Ehresmann fibration theorem [32], we can show the following corollary.

Corollary 1. *There exists a positive real number ε_0 such that for any $0 < \varepsilon \leq \varepsilon_0$, there exists a positive real number $\delta(\varepsilon)$ such that*

$$f|_{B_\varepsilon^{n+m} \cap f^{-1}(D_\delta^2 \setminus \{\mathbf{0}_2\})} : B_\varepsilon^{n+m} \cap f^{-1}(D_\delta^2 \setminus \{\mathbf{0}_2\}) \to D_\delta^2 \setminus \{\mathbf{0}_2\}$$

is a locally trivial fibration for $0 < \delta \leq \delta(\varepsilon)$. The isomorphism class of this fibration does not depend on the choice of ε and δ.

4. Homeomorphisms of Milnor fibers of mixed polynomials of 2 complex variables

Let $g = \sum_{\nu,\mu} c_{\nu,\mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ be a strongly non-degenerate mixed polynomial of 2 complex variables which is locally tame along vanishing coordinate subspaces. Put $V = g^{-1}(0)$. Let $\hat{\pi}_1 : X_1 \to \mathbb{C}^2$ be an ordinary blowing–up and $\hat{E}$ be the exceptional divisor of $\hat{\pi}_1$. We denote the strict transform of V by $\hat{V}$. Put $\hat{E}(\hat{V}) = \hat{E} \cap \hat{V} \subset \mathbb{CP}^1$. Let (u_0, v_0) and (u_1, v_1) be the local coordinates of X_1 which satisfy

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_0 v_0 \end{pmatrix} = \begin{pmatrix} u_1 v_1 \\ v_1 \end{pmatrix}.$$

Take a point $[a : b] \in \hat{E}$. By using the local coordinates of X_1, we have

$$[a : b] = \begin{cases} [1 : v_0] & ([a : b] \neq [0 : 1]), \\ [u_1 : 1] & ([a : b] \neq [1 : 0]). \end{cases}$$

Let $q : \mathbb{CP}^1 \to \mathbb{RP}^1$ be the map defined by $q([a : b]) = [|a| : |b|]$. We identify $\mathbb{RP}^1 \setminus \{[0 : 1]\}$ with $\mathbb{R}$ and we assume that

(A) there exist positive real numbers r_1 and r_2 such that $r_1 < r_2$ and $[r_1, r_2] \subset \mathbb{RP}^1 \setminus (\{[0 : 1]\} \cup q(\hat{E}(\hat{V})))$.

Example 2. Let p_1, p_2, q_1 and q_2 be integers such that $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$. We define the S^1-action and the $\mathbb{R}^*$-action on $\mathbb{C}^2$ as follows:

$$s \circ \mathbf{z} = (s^{p_1} z_1, s^{p_2} z_2), \quad r \circ \mathbf{z} = (r^{q_1} z_1, r^{q_2} z_2), \quad s \in S^1, \quad r \in \mathbb{R}^*.$$

If there exists a positive integer d_p such that $g(\mathbf{z}, \bar{\mathbf{z}})$ satisfies

$$g(s^{p_1} z_1, s^{p_2} z_2, \bar{s}^{p_1} \bar{z}_1, \bar{s}^{p_2} \bar{z}_2) = s^{d_p} g(\mathbf{z}, \bar{\mathbf{z}}), \quad s \in S^1,$$

then we say that $g(\mathbf{z}, \bar{\mathbf{z}})$ is a *polar weighted homogeneous polynomial*. Similarly $g(\mathbf{z}, \bar{\mathbf{z}})$ is called a *radial weighted homogeneous polynomial* if there exists a positive integer d_r such that

$$g(r^{q_1} z_1, r^{q_2} z_2, r^{q_1} \bar{z}_1, r^{q_2} \bar{z}_2) = r^{d_r} g(\mathbf{z}, \bar{\mathbf{z}}), \quad r \in \mathbb{R}^*.$$

Polar and radial weighted homogeneous mixed polynomials admit global Milnor fibrations. See [24, 4, 19, 20].

We show that polar and radial weighted homogeneous polynomials satisfy the above assumption (A). Since g is a polar and radial weighted homogeneous polynomial, V is an invariant set for the S^1-action and the $\mathbb{R}^*$-action. Let C be a connected component of $V \setminus \{(0, 0)\}$. Note that $\dim_{\mathbb{R}} V = 2$. Then there exist complex numbers α_1 and α_2 such that

$$C = \{(\alpha_1 r^{q_1} s^{p_1}, \alpha_2 r^{q_2} s^{p_2}) \in \mathbb{C}^2 \mid s \in S^1, r > 0\}.$$

Assume that $\alpha_1 \neq 0$. Let (u_0, v_0) be the local coordinates of X_1 which satisfy

$$z_1 = u_0, \quad z_2 = u_0 v_0.$$

Then the strict transform $\hat{C}$ of C is given by

$$\Big\{\Big(u_0=\alpha_1 r^{q_1}s^{p_1}, v_0=\frac{\alpha_2}{\alpha_1}r^{q_2-q_1}s^{p_2-p_1}\Big)\mid s\in S^1, r\geq 0\Big\}.$$

If q_1 is greater than q_2 and $\alpha_2\neq 0$, $\hat{E}\cap\hat{C}$ is equal to $[0:1]$. Since the number of connected components of $V\setminus\{(0,0)\}$ is finite, $\hat{E}(\hat{V})$ is a finite set. Thus $\hat{E}(\hat{V})$ satisfies the assumption (A).

Set $r_0=\frac{r_1+r_2}{2}$ and $D^2_\upsilon\times D^2_{r_0\upsilon}=\{(z_1,z_2)\in\mathbb{C}^2\mid |z_1|\leq\upsilon, |z_2|\leq r_0\upsilon\}$ for $0<\upsilon\ll 1$. By the assumption (A), there exists a positive real number $\hat{\delta}_1$ such that

$$V\cap(\partial D^2_\upsilon\times\partial D^2_{r_0\upsilon})=\emptyset$$

for $\upsilon\leq\hat{\delta}_1$. Take positive real numbers $\hat{\delta}_0,\hat{\delta}_2$ and δ such that $D^4_{\hat{\delta}_0}, D^4_{\hat{\delta}_2}$ are Milnor balls of g and

$$D^4_{\hat{\delta}_0}\subset D^2_\delta\times D^2_{r_0\delta}\subset D^4_{\hat{\delta}_2}\subset D^2_{\hat{\delta}_1}\times D^2_{r_0\hat{\delta}_1}.$$

Then we can choose small positive real numbers $\tilde{\delta}_1$ and $\tilde{\delta}_2$ such that $\tilde{\delta}_1<\frac{\tilde{\delta}_2}{2}\ll\delta$ and

$$g^{-1}(\tilde{\delta}_k)\cap(\partial D^2_\upsilon\times\partial D^2_{r_0\upsilon})=\emptyset$$

for $k=1,2$ and $\upsilon\leq\hat{\delta}_1$. By [20, Lemma 28] and [22, Lemma 7], there exists a positive real number η_0 such that $g^{-1}(\eta)$ is transversal to ∂D^4_γ for any $\eta\neq 0$ with $|\eta|\leq\eta_0$ and $\hat{\delta}_0\leq\gamma\leq\hat{\delta}_2$.

Assertion 1. *There exists a positive real number η_0' such that the fiber $g^{-1}(\eta)$ is transversal to $\partial(D^2_\upsilon\times D^2_{r_0\upsilon})$ for any $\eta\neq 0, |\eta|\leq\eta_0'$ and $\delta\leq\upsilon\leq\hat{\delta}_1$.*

Proof. Assume that $\{2\}\notin\mathcal{I}_v(g)$. Since g is a locally tame mixed polynomial, by Theorem 4, the singular locus $\Sigma(V)$ of V is $\{(0,0)\}$ or $\{z_2=0\}$. Since g satisfies the assumption (A), the origin is an isolated singularity of $V\cap\{|z_1|\leq|z_2|/r_0\}$. Note that the function $|z_1|^2: V\setminus\Sigma(V)\to\mathbb{R}$ has only finitely many critical values. By using the same argument as in the proof of [14, Corollary 2.9], V and $\{|z_1|=\upsilon\}$ intersect transversely and $V\cap\{|z_1|\leq|z_2|/r_0\}$ is compact. Thus we can show the existence of η_0' such that $g^{-1}(\eta)$ and $\{|z_1|=\upsilon\}$ intersect transversely for $\eta\neq 0$ with $|\eta|\leq\eta_0'$.

If $\{2\}\in\mathcal{I}_v(g)$, we assume that the assertion does not hold for $\partial D^2_\upsilon\times D^2_{r_0\upsilon}$. By [22, Lemma 2] and Curve Selection Lemma, we can find a real analytic curve $\mathbf{z}(t)=(z_1(t),z_2(t))$ and a complex–valued function $\alpha(t)$ such that

- $g(\mathbf{z}(0))=0$ and $g(\mathbf{z}(t))\neq 0$ for $t\neq 0$,
- $z_1(t)=\alpha(t)\overline{\frac{\partial g}{\partial z_1}}(\mathbf{z}(t))+\overline{\alpha}(t)\frac{\partial g}{\partial\bar{z}_1}(\mathbf{z}(t))$ for $t\geq 0$.

Put

$$z_j(t)=c_jt^{p_j}+(\text{higher terms}),\ c_j\neq 0 \text{ if } z_j(t)\not\equiv 0,$$
$$\alpha(t)=\alpha_0t^{m'}+(\text{higher terms}),\ \alpha_0\neq 0.$$

To prove the assertion, we may assume that $z_1(t)\not\equiv 0, z_2(t)\not\equiv 0, p_1>0$ and $p_2=0$. Since $\{2\}\in\mathcal{I}_v(g)$ and g is locally tame, $g|_{z_2=c_2}$ is a strongly non-degenerate mixed function of variable z_1 for $|c_2|\leq r_0\hat{\delta}_1$. By using $c_1\neq 0, c_2\neq 0$ and $z_1(t)=\alpha(t)\overline{\frac{\partial g}{\partial z_1}}(\mathbf{z}(t))+\overline{\alpha}(t)\frac{\partial g}{\partial\bar{z}_1}(\mathbf{z}(t))$, we can show that there exists a face Δ of $\Gamma(g;\mathbf{z},\bar{\mathbf{z}})$ such that

$$0=\alpha_0\overline{\frac{\partial g_\Delta}{\partial z_1}}(c_1,c_2)+\overline{\alpha_0}\frac{\partial g_\Delta}{\partial\bar{z}_1}(c_1,c_2).$$

See the proof of [22, Lemma 7]. By the above equation and [19, Proposition 1], c_1 is a singularity of $g_\Delta|_{z_2=c_2}$. This is a contradiction to the strong non-degeneracy of $g|_{z_2=c_2}$. We can apply the same argument for the cylinder $\{|z_2|=r_0\upsilon\}$. Thus there exists a positive real number η_0' such that $g^{-1}(\eta)$ is transversal to $\partial(D^2_\upsilon\times D^2_{r_0\upsilon})$ for any $\eta\neq 0$ with $|\eta|\leq\eta_0'$ and $\delta\leq\upsilon\leq\hat{\delta}_1$. □

Proposition 1. *Let g be a strongly non-degenerate mixed polynomial of 2 complex variables which is locally tame along vanishing coordinate subspaces. Assume that $\hat{E}(\hat{V})$ satisfies the assumption (A). Then $\big(D^2_\delta \times D^2_{r_0\delta}, g^{-1}(\tilde{\delta}_1)\cap(D^2_\delta\times D^2_{r_0\delta})\big)$ is homeomorphic to $(D^4_{\hat{\delta}_2}, g^{-1}(\tilde{\delta}_1)\cap D^4_{\hat{\delta}_2})$.*

Proof. We take $\tilde{\delta}_2$ such that $0 < \tilde{\delta}_2 < \min\{\eta_0, \eta'_0\}$. By Assertion 1, $g^{-1}(\eta)$ is transversal to $\partial(D^2_v \times D^2_{r_0 v})$ for $|\eta| \leq \tilde{\delta}_2$ and $\delta \leq v \leq \hat{\delta}_1$. Thus there exists a vector field $\mathbf{v}(\mathbf{z})$ defined on $g^{-1}(D^2_{\tilde{\delta}_2}) \cap (D^4_{\hat{\delta}_2} \setminus \mathrm{Int}\,(D^2_{\delta/2} \times D^2_{r_0\delta/2}))$ such that $g(\mathbf{h}(t,\mathbf{z}))$ is constant, $|h_1(t,\mathbf{z})|$ and $|h_2(t,\mathbf{z})|$ are monotone increasing, where $\mathbf{h}(t,\mathbf{z}) = (h_1(t,\mathbf{z}), h_2(t,\mathbf{z}))$ is the integral curve of $\mathbf{v}(\mathbf{z})$ with $\mathbf{h}(0,\mathbf{z}) = \mathbf{z}$. So we can define a homeomorphism from $g^{-1}(D^2_{\tilde{\delta}_2})\cap(D^2_\delta\times D^2_{r_0\delta})$ onto $g^{-1}(D^2_{\tilde{\delta}_2})\cap D^4_{\hat{\delta}_2}$ such that this homeomorphism is equal to the identity map on $g^{-1}(D^2_{\tilde{\delta}_2}) \cap (D^2_{\delta/2} \times D^2_{r_0\delta/2})$.

Since g is strongly non-degenerate and locally tame along vanishing coordinate subspaces, there exists a vector field $\mathbf{v}'(\mathbf{z})$ defined on $D^4_{\hat{\delta}_2} \setminus g^{-1}(D^2_{\tilde{\delta}_2/2})$ such that $|g(\mathbf{h}'(t,\mathbf{z}))|$ and $|\mathbf{h}'(t,\mathbf{z})|$ are monotone increasing, where $\mathbf{h}'(t,\mathbf{z})$ is the integral curve of $\mathbf{v}'(\mathbf{z})$ with $\mathbf{h}'(0,\mathbf{z}) = \mathbf{z}$. See [20, 22]. We take $\tilde{\delta}_1$ such that $0 < \tilde{\delta}_1 < \frac{\tilde{\delta}_2}{2} \ll \delta$. Then $\mathbf{h}(t,\mathbf{z})$ and $\mathbf{h}'(t,\mathbf{z})$ induce a homeomorphism from $\big(D^2_\delta \times D^2_{r_0\delta}, g^{-1}(\tilde{\delta}_1) \cap (D^2_\delta \times D^2_{r_0\delta})\big)$ onto $(D^4_{\hat{\delta}_2}, g^{-1}(\tilde{\delta}_1) \cap D^4_{\hat{\delta}_2})$. □

5. Proof of Theorem 3

Let g be a mixed polynomial of 2 complex variables which satisfies the assumptions in Section 4. Let $b_g \subset D^4_\delta$ be a bouquet of circles with base point $*$. Assume that b_g is a deformation retract of the fiber of the stable tubular Milnor fibration of g and $b_g \cap \{z_1z_2 = 0\} = \emptyset$. Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^2, \mathbf{0}_2)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^2, \mathbf{0}_2)$ be real analytic map germs of independent variables as in Section 3. Set $\tilde{F}_1 = V(f_1) \times F_2$ and $\tilde{F}_2 = F_1 \times V(f_2)$.

Take positive real numbers ε and ε_1 such that $\varepsilon < \varepsilon_1$ and ε and ε_1 are common a_f-stable radii for f_1 and f_2. Then there exist a positive real number $\tilde{\eta} \ll \varepsilon$ and a vector field $\mathbf{v}_j$ on $(B^{n_j}_{\varepsilon_1} \setminus \mathrm{Int}\, B^{n_j}_\varepsilon) \cap \{0 < |f_j| \leq \tilde{\eta}\}$ such that

- $(\mathbf{v}_1(\mathbf{x}), \mathbf{x}) < 0$ and $(\mathbf{v}_2(\mathbf{y}), \mathbf{y}) < 0$,
- $\mathbf{v}_1(\mathbf{x})$ is tangent to $f_1^{-1}(f_1(\mathbf{x}))$ and $\mathbf{v}_2(\mathbf{y})$ is tangent to $f_2^{-1}(f_2(\mathbf{y}))$

for $j = 1, 2$. Choose positive real numbers $\tilde{\delta}, \delta$ and δ_1 as in Proposition 1, i.e., $\big(D^2_\delta \times D^2_{r_0\delta}, g^{-1}(\tilde{\delta}) \cap (D^2_\delta \times D^2_{r_0\delta})\big)$ is homeomorphic to $(D^4_{\delta_1}, g^{-1}(\tilde{\delta}) \cap D^4_{\delta_1})$. We also assume that $\delta_1 \ll \tilde{\eta}$. By using $\mathbf{v}_1$ and $\mathbf{v}_2$, we have

(i) $f_j(B^{n_j}_{\hat{\varepsilon}}) \subset D^2_{\delta_1}$ and

$$f_j : B^{n_j}_{\hat{\varepsilon}} \cap f_j^{-1}(D^2_\eta \setminus \{\mathbf{0}_2\}) \to D^2_\eta \setminus \{\mathbf{0}_2\}$$

is a locally trivial fibration for $j = 1, 2, \hat{\varepsilon} = \varepsilon, \varepsilon_1$ and $\eta = \delta, \delta_1$,

(ii) $\big(D^2_\delta \times D^2_{r_0\delta}, g^{-1}(\tilde{\delta}) \cap (D^2_\delta \times D^2_{r_0\delta})\big)$ is homeomorphic to (Milnor ball, $g^{-1}(\tilde{\delta})$),

(iii) there exists a deformation retract

$$\hat{r}_j : (B^{n_j}_{\varepsilon_1} \setminus \mathrm{Int}\, B^{n_j}_\varepsilon) \cap f_j^{-1}(D^2_\delta) \to \partial B^{n_j}_\varepsilon \cap f_j^{-1}(D^2_\delta)$$

such that $\hat{r}_j|_{\partial B^{n_j}_\varepsilon \cap f_j^{-1}(D^2_\delta)} = \mathrm{id}$ and $f_j \circ \hat{r}_j = f_j$.

We take $\tilde{\delta}$ sufficiently small such that

- $g^{-1}(\tilde{\delta})$ is a Milnor fiber in $D^2_{\delta_1} \times D^2_{r_0\delta_1}$ and $D^2_\delta \times D^2_{r_0\delta}$,
- $g^{-1}(\tilde{\delta}) \cap \{z_1z_2 = 0\} \cap (D^2_\delta \times D^2_{r_0\delta}) = g^{-1}(\tilde{\delta}) \cap \{z_1z_2 = 0\} \cap D^4_\delta$.

Lemma 3. *Set $F_{\varepsilon,\tilde{\delta}} = f^{-1}(\tilde{\delta}) \cap (B^n_\varepsilon \times B^m_\varepsilon)$. Then $(f_1, f_2)^{-1}(D^2_\delta \times D^2_{r_0\delta}) \cap F_{\varepsilon,\tilde{\delta}}$ is homotopy equivalent to $F_{\varepsilon,\tilde{\delta}}$.*

Proof. Set $g^{-1}(\tilde{\delta})^\circ = g^{-1}(\tilde{\delta}) \cap \big((D^2_{\delta_1} \times D^2_{r_0\delta_1}) \setminus \mathrm{Int}\,(D^2_\delta \times D^2_{r_0\delta})\big)$. Since the Milnor fibers of g are transversal to small spheres, there exists a deformation retract $D_t : g^{-1}(\tilde{\delta})^\circ \to g^{-1}(\tilde{\delta})^\circ$ such that $D_0 = \mathrm{id}$ and $\mathrm{Im}\, D_1 \in g^{-1}(\tilde{\delta}) \cap \partial\big(D^2_\delta \times D^2_{r_0\delta}\big)$. By the local triviality of (f_1, f_2), there exists a deformation retract

$$\tilde{D}_t : (B^n_{\varepsilon_1} \times B^m_{\varepsilon_1}) \cap (f_1, f_2)^{-1}(g^{-1}(\tilde{\delta})^\circ) \to (B^n_{\varepsilon_1} \times B^m_{\varepsilon_1}) \cap (f_1, f_2)^{-1}(g^{-1}(\tilde{\delta})^\circ)$$

such that $\tilde{D}_t$ is the lifting of D_t, $\tilde{D}_0 = \mathrm{id}$ and $\mathrm{Im}\, \tilde{D}_1 \in (f_1, f_2)^{-1}(g^{-1}(\tilde{\delta}) \cap \partial(D^2_\delta \times D^2_{r_0\delta}))$.

Define $\tilde{r}_j : B^{n_j}_{\varepsilon_1} \cap f_j^{-1}(D^2_\delta) \to B^{n_j}_{\varepsilon_1} \cap f_j^{-1}(D^2_\delta)$ by

$$\tilde{r}_j = \begin{cases} \hat{r}_j, & |\mathbf{x}| \geq \varepsilon, \\ \mathrm{id}, & |\mathbf{x}| \leq \varepsilon. \end{cases}$$

Then the composed map $(\tilde{r}_1 \times \tilde{r}_2) \circ \tilde{D}_t$ defines a deformation retract of $(f_1, f_2)^{-1}(D^2_\delta \times D^2_{r_0\delta}) \cap F_{\varepsilon,\tilde{\delta}}$ in $F_{\varepsilon,\tilde{\delta}}$. □

We take $0 < \varepsilon_1' < \varepsilon$ and $0 < \delta_1' < \delta$. Assume that $(\varepsilon_1', \delta_1')$ has the same properties as (ε, δ) and $\tilde{\delta}$ is sufficiently small. By using the above argument, we can show that the inclusion $(f_1, f_2)^{-1}(D^2_{\delta_1'} \times D^2_{r_0\delta_1'}) \cap F_{\varepsilon_1',\tilde{\delta}} \subset (f_1, f_2)^{-1}(D^2_\delta \times D^2_{r_0\delta}) \cap F_{\varepsilon,\tilde{\delta}}$ is a deformation retract. So we can show the following corollary.

Corollary 2. *The inclusion $F_{\varepsilon_1',\tilde{\delta}} \subset F_{\varepsilon,\tilde{\delta}}$ is a homotopy equivalence.*

By Corollary 2, we have

Lemma 4. *Let F_f be the Milnor fiber of f. Then $F_{\varepsilon,\tilde{\delta}}$ has the same homotopy type of F_f.*

Proof. We choose sufficiently small positive real numbers $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$. Set $F_{\varepsilon_k,\tilde{\delta}} = f^{-1}(\tilde{\delta}) \cap (B^n_{\varepsilon_k} \times B^m_{\varepsilon_k})$ for $k = 1, 2, 3$. By Corollary 2, The inclusion $F_{\varepsilon_{k+1},\tilde{\delta}} \subset F_{\varepsilon_k,\tilde{\delta}}$ is a homotopy equivalence for $k = 1, 2$. Since the fiber $f^{-1}(\tilde{\delta})$ intersects transversely with $S^{n+m}_{\varepsilon_1}, S^{n+m}_{\varepsilon_2}$ and $S^{n+m}_{\varepsilon_3}$, the inclusion $f^{-1}(\tilde{\delta}) \cap B^{n+m}_{\varepsilon_{k+1}} \subset f^{-1}(\tilde{\delta}) \cap B^{n+m}_{\varepsilon_k}$ is also a homotopy equivalence for $k = 1, 2$. Thus the sequence

$$F_{\varepsilon_1,\tilde{\delta}} \supset f^{-1}(\tilde{\delta}) \cap B^{n+m}_{\varepsilon_1} \supset F_{\varepsilon_2,\tilde{\delta}} \supset f^{-1}(\tilde{\delta}) \cap B^{n+m}_{\varepsilon_2} \supset F_{\varepsilon_3,\tilde{\delta}}$$

defines a homotopy equivalence $F_{\varepsilon,\tilde{\delta}} \to F_f$. See [13, Proposition 1.1] and [9, Lemma 7]. □

Proof of Theorem 3. Consider the following map

$$(f_1, f_2) : (f_1, f_2)^{-1}(D^2_\delta \times D^2_{r_0\delta}) \cap F_{\varepsilon,\tilde{\delta}} \to (D^2_\delta \times D^2_{r_0\delta}) \cap g^{-1}(\tilde{\delta}).$$

This map is locally trivial over $g^{-1}(\tilde{\delta}) \setminus \{z_1 z_2 = 0\}$ with fiber $F_1 \times F_2$.

Let D^2_j be a small neighborhood of a point of $g^{-1}(\tilde{\delta}) \cap \{z_1 z_2 = 0\}$ and γ_j be a path from b_g to D^2_j for $j = 1, \ldots, l_1 + l_2$. Assume that

$$b_g \cap \gamma_j = \{*\}, \quad D^2_j \cap \gamma_j = \{\text{a point}\} \subset \partial D^2_j$$

and

$$D^2_j \cap D^2_{j'} = D^2_j \cap \gamma_{j'} = \emptyset, \quad \gamma_j \cap \gamma_{j'} = \{*\}$$

for $j = 1, \ldots, l_1 + l_2$ and $j \neq j'$. Since (f_1, f_2) is locally trivial over $g^{-1}(\tilde{\delta}) \setminus \{z_1 z_2 = 0\}$, by homotopy lifting property, $F_{\varepsilon,\tilde{\delta}}$ is homotopy equivalent to

$$(f_1, f_2)^{-1}\Big(b_g \cup \big(\textstyle\bigcup_{j=1}^{l_1+l_2} D^2_j\big) \cup \big(\bigcup_{j=1}^{l_1+l_2} \gamma_j\big)\Big).$$

See [29, p. 55]. Let (z_1, z_2) be a point of $g^{-1}(\tilde{\delta}) \cap \{z_1 z_2 = 0\} \cap D_\delta^4$. Then $(f_1, f_2)^{-1}(z_1, z_2)$ is homotopy equivalent to

$$\begin{cases} \tilde{F}_1 & (z_1 = 0), \\ \tilde{F}_2 & (z_2 = 0). \end{cases}$$

Thus F_f has the homotopy type of a space obtained from $(f_1, f_2)^{-1}(b_g)$ by gluing to the fiber $(f_1, f_2)^{-1}(*)$ l_1 copies of $\tilde{F}_1$ and l_2 copies of $\tilde{F}_2$. □

Corollary 3. *Let $f = g \circ (f_1, f_2) : (\mathbb{R}^n \times \mathbb{R}^m, \mathbf{0}_{n+m}) \to (\mathbb{R}^2, \mathbf{0}_2)$ be the real analytic map germ as in Theorem 3. Then the Euler characteristic of the Milnor fiber F_f of f is given by*

$$\chi(F_f) = \chi(F_g \setminus \{z_1 z_2 = 0\})\chi(F_1)\chi(F_2) + l_1\chi(F_2) + l_2\chi(F_1).$$

Remark 1. Let g be a strongly non-degenerate mixed polynomial of 2 complex variables which is locally tame along vanishing coordinate subspaces. Assume that $g' := g|_{z_1=0} = \sum_{\nu,\mu} c_{\nu,\mu} z_2^\nu \bar{z}_2^\mu \not\equiv 0$. Put $g'_\ell = \sum_{\nu+\mu=\ell} c_{\nu,\mu} z_2^\nu \bar{z}_2^\mu$. Then we can write

$$g' = g'_{\underline{d}} + \cdots + g'_{\overline{d}},$$
$$g'_{\underline{d}} = c z_2^a \bar{z}_2^b \prod_{j=1}^s (z_2 + \delta_k \bar{z}_2)^{\mu_k},$$

where $\underline{d} = \min\{\nu + \mu \mid c_{\nu,\mu} \neq 0\}$ and $\overline{d} = \max\{\nu + \mu \mid c_{\nu,\mu} \neq 0\}$. Suppose that all zeros of g' are regular points of g' and $|\delta_k| < 1$ for $k = 1, \dots s$. By [21, Theorem 20], the number of points of $g'^{-1}(0) \cap D_\delta^2$ is equal to $a - b + \sum_{k=1}^s \mu_k$.

5.1. **Spherical Milnor fibrations.** Let Φ be a real analytic map germ which satisfies the conditions (a-i) and (a-ii). We assume that Φ satisfies the following condition:

(a-iii) there exists a positive real number r' such that

$$\Phi/|\Phi| : \partial B_r^N \setminus K_\Phi \to S^{p-1}$$

is a locally trivial fibration and this fibration is isomorphic to the tubular Milnor fibration of Φ, where $K_\Phi = \partial B_r^N \cap \Phi^{-1}(0)$ and $0 < r \leq r'$.

The fibration in (a-iii) is called the *spherical Milnor fibration of* Φ.

Corollary 4. *Let $f = g \circ (f_1, f_2) : (\mathbb{R}^n \times \mathbb{R}^m, \mathbf{0}_{n+m}) \to (\mathbb{R}^2, \mathbf{0}_2)$ be the real analytic map germ as in Theorem 3. Assume that f_1, f_2 and $f = g \circ (f_1, f_2)$ satisfy the condition (a-iii). Let $\overline{F}_j$ be the fiber of the spherical Milnor fibration of f_j for $j = 1, 2$. Then the fiber of the spherical Milnor fibration of f is homotopy equivalent to the space obtained from $(f_1, f_2)^{-1}(b_g)$ by gluing to $(f_1, f_2)^{-1}(*)$ l_1 copies of $\overline{F}_1$ and l_2 copies of $\overline{F}_2$, where l_1 is the number of points of $\{(0, z_2) \in D_\delta^4 \cap g^{-1}(\tilde{\delta})\}$ and l_2 is the number of points of $\{(z_1, 0) \in D_\delta^4 \cap g^{-1}(\tilde{\delta})\}$ for $0 < \tilde{\delta} \ll \delta \ll 1$.*

Proof. By Theorem 3 and the condition (a-iii), the fiber of the spherical Milnor fibration of f is homotopy equivalent to a space obtained from $(f_1, f_2)^{-1}(b_g)$ by gluing to $(f_1, f_2)^{-1}(*)$ l_1 copies of $\tilde{F}_1$ and l_2 copies of $\tilde{F}_2$. Since F_j is diffeomorphic to $\overline{F}_j$, $\tilde{F}_1$ and $\tilde{F}_2$ are homotopy equivalent to $\overline{F}_1$ and $\overline{F}_2$ respectively. This completes the proof. □

6. Zeta functions of monodromies of Milnor fibrations

We assume that a real analytic map germ $\Phi : (\mathbb{R}^{2n}, \mathbf{0}_{2n}) \to (\mathbb{R}^2, \mathbf{0}_2)$ satisfies the conditions (a-i) and (a-ii). Let F_Φ be the fiber of the Milnor fibration of Φ. Set $P_j(\lambda) = \det(\mathrm{Id} - \lambda h_{*,j})$, where $h_{*,j} : H_j(F_\Phi, \mathbb{C}) \to H_j(F_\Phi, \mathbb{C})$ is an isomorphism induced by the monodromy of Φ for $j \geq 0$. Then the *zeta function $\zeta(\lambda)$ of the monodromy* is defined by

$$\zeta(\lambda) = \prod_{j=0}^{2n-2} P_j(\lambda)^{(-1)^{j+1}}.$$

See [14, Section 9] and [18, Chapter I].

In this section, we study the zeta function of the monodromy of f, where $f = g \circ (f_1, f_2)$ is a real analytic map germ as in Theorem 3. Let g be a strongly non-degenerate mixed polynomial of 2 complex variables which is locally tame along vanishing coordinate subspaces. We denote $D = \{(z_1, z_2) \mid z_1 z_2 = 0\} \subset D^2_\delta \times D^2_{r_0\delta}$. Take sufficiently small positive real numbers δ and $\tilde{\delta}$ such that and $\tilde{\delta} \ll \delta$. Consider the following pair of maps

$$g : (D^4_\delta \cap g^{-1}(\partial B^2_{\tilde{\delta}}), D \cap g^{-1}(\partial B^2_{\tilde{\delta}})) \to \partial B^2_{\tilde{\delta}},$$
$$g/|g| : (S^3_\delta \setminus g^{-1}(0), (S^3_\delta \cap D) \setminus g^{-1}(0)) \to S^1,$$

where $S^3_\delta = \{(z_1, z_2) \in \mathbb{C}^2 \mid \|(z_1, z_2)\| = \delta\}$. In [22, Theorem 10], Oka showed that the spherical and the tubular Milnor fibrations of g are fiber homotopy equivalent. Since g satisfies the a_f-condition and $\tilde{\delta}$ is sufficiently small, the fibers of two maps intersect transversely with S^3_δ and D. So the above maps are locally trivial fibrations. Moreover the fibrations are fiber homotopy equivalent.

Let $*$ be a point of $F_g \setminus D$, where F_g is the Milnor fiber of the spherical Milnor fibration $g/|g| : S^3_\delta \setminus g^{-1}(0) \to S^1$. By using the above fibrations, we have the exact sequence of groups:

$$1 \to \pi_1(F_g \setminus D, *) \xrightarrow{i_*} \pi_1(S^3_\delta \setminus (g^{-1}(0) \cup D), *) \xrightarrow{(g/|g|)_*} \mathbb{Z} \to 1,$$

where i is the inclusion $F_g \setminus D \hookrightarrow S^3_\delta \setminus (g^{-1}(0) \cup D)$. Consider

$$A^q = H^q(F_1 \times F_2, \mathbb{C}), \quad G = \pi_1(S^3_\delta \setminus (g^{-1}(0) \cup D), *), \quad H = \pi_1(F_g \setminus D, *)$$

for $q \geq 0$. Since the restricted map $(f_1 \times f_2) : (B^n_\varepsilon \times B^m_\varepsilon) \cap (f_1 \times f_2)^{-1}(D^4_\delta \setminus D) \to D^4_\delta \setminus D$ is a locally trivial fibration, we have a monodromy representation

$$\rho : \pi_1(D^4_\delta \setminus D) = \pi_1(S^3_\delta \setminus D) = \mathbb{Z}^2 \to \mathbf{Aut}(A^q).$$

The generators of $\mathbb{Z}^2$ are chosen such that $(1, 0)$ and $(0, 1)$ are meridians of the link components $\{z_1 = 0\}$ and $\{z_2 = 0\}$ respectively. By the inclusion $S^3_\delta \setminus (g^{-1}(0) \cup D) \hookrightarrow S^3_\delta \setminus D$, A^q becomes a G-module, and by i_* an H-module. Set

$$\mathrm{Der}(H, A^q) = \{d : H \to A^q \mid d(h_1 h_2) = d(h_1) + h_1 d(h_2) \text{ for all } h_1, h_2 \in H\},$$
$$H^0(H, A^q) = (A^q)^H, \quad H^1(H, A^q) = \mathrm{Der}(H, A^q)/\mathrm{Im}\,\delta,$$

where $\delta : A^q \to \mathrm{Der}(H, A^q)$ is defined by $\delta(a)(k) = \rho(k)(a) - a$ for $a \in A^q$ and $k \in H$. Let $h \in G$ be an element such that $(g/|g|)_*(h) = 1$. The automorphism $c_h : H \to H$ is defined by $c_h(k) = h^{-1}kh$ for $k \in H$. Then the maps $\rho(h) : A^q \to A^q$ and $c_h : H \to H$ induce an automorphism of the exact sequence of $\mathbb{C}$-vector spaces:

$$\begin{array}{ccccccccc}
0 \to & H^0(H, A^q) & \to & A^q & \xrightarrow{\delta} & \mathrm{Der}(H, A^q) & \to & H^1(H, A^q) & \to 0 \\
& \downarrow h_0^* & & \downarrow \rho(h) & & \downarrow h_{\mathrm{Der}} & & \downarrow h_1^* & \\
0 \to & H^0(H, A^q) & \to & A^q & \xrightarrow{\delta} & \mathrm{Der}(H, A^q) & \to & H^1(H, A^q) & \to 0.
\end{array}$$

Note that $h_{\mathrm{Der}}(d)(k) = \rho(h)(d(c_h(k)))$ for $d \in \mathrm{Der}(H, A^q)$ and $k \in H$. So h_{Der}, h_0^* and h_1^* are the automorphisms induced by $\rho(h)$ and c_h. See [15, p. 72] and [16, p. 11].

Set $\Delta_h(\lambda) = \det(1 - \lambda\rho(h))$ and $\Delta_{\mathrm{Der}}(\lambda) = \det(1 - \lambda h_{\mathrm{Der}})$. Then by the above exact sequence, h_{Der} is determined by $\rho(h)$ and c_h. So we define

$$\begin{aligned}
(\zeta_{g,D}(\lambda))_q^{(-1)^q} &= \det(1 - \lambda h_0^*)/\det(1 - \lambda h_1^*) \\
&= \Delta_h(\lambda)/\Delta_{\mathrm{Der}}(\lambda).
\end{aligned}$$

The automorphisms h_0^* and h_1^* do not depend on the choice of h. See [28, p. 116]. Thus $(\zeta_{g,D})_q$ is well-defined.

Note that H is a free group. Let $b_1, \dots, b_{\mu(g,D)}$ be generators of H. By using the map

$$\mathrm{Der}(H, A^q) \to (A^q)^{\mu(g,D)}, \delta \mapsto (\delta(b_1), \dots, \delta(b_{\mu(g,D)})),$$

$\mathrm{Der}(H, A^q)$ can be identified with $(A^q)^{\mu(g,D)}$.

Let $\tilde{i}_* : \mathbb{Z}[H] \to \mathbb{Z}[G]$ be the homomorphism induced by i_* and $\tilde{\rho} : \mathbb{Z}[G] \to \mathbb{Z}[\mathbf{Aut}A^q]$ be the homomorphism induced by ρ. The homomorphism of rings $s : \mathbb{Z}[\mathbf{Aut}A^q] \to \mathbf{End}_{\mathbb{C}}A^q$ is defined by

$$s(\textstyle\sum_i c_i[a^i_{jk}]) = [\sum_i c_i a^i_{jk}]_{jk}.$$

Let $\frac{\partial}{\partial b_j} : \mathbb{Z}[H] \to \mathbb{Z}[H]$ be the derivation determined by $\frac{\partial b_i}{\partial b_j} = \delta_{ij}$ for $1 \leq i, j \leq \mu(g, D)$. We denote $c_h(b_i)$ by w_i. Note that h_{Der} is determined by $\rho(h)$ and c_h. By using the derivation rule, we have

$$[h_{\mathrm{Der}}] = \Big[s \circ \tilde{\rho}\Big(h \cdot \tilde{i}_*\Big(\frac{\partial w_i}{\partial b_j}\Big)\Big)\Big]_{ij}.$$

See [15, p. 73]. We set $K_j = S^3_\delta \cap \{z_j = 0\}$ for $j = 1, 2$. Consider the multilink

$$(S^3_\delta, S^3_\delta \cap g^{-1}(0)) = (S^3_\delta, m_1K_1 \cup m_2K_2 \cup m_3K_3 \cup \cdots \cup m_rK_r),$$

where K_j is an oriented knot and $m_j \in \mathbb{Z}$ for $1 \leq j \leq r$. Note that $m_j = 0$ if and only if $g|_{z_j=0} \not\equiv 0$ for $j = 1, 2$. Since g is strongly non-degenerate, $|m_j| = 1$ for $j \geq 3$. Put $L = (S^3_\delta, K_1 \cup K_2 \cup K_3 \cup \cdots \cup K_r)$. Then $(\zeta_{h,D})_q$ can be calculated by the Alexander polynomial of L [15]. We follow the arguments in [15, p. 88–93] and [16, p. 10–11]. The following assertions are similar to those in [16].

Theorem 5. *Let f_1, f_2 and g be real analytic map germs in Theorem 3. Let $H_{j,k} : H_k(F_j, \mathbb{C}) \to H_k(F_j, \mathbb{C})$ be the monodromy matrix induced by the monodromy of f_j for $j = 1, 2$ and $k \geq 0$. Set $E_{q,1} = \bigoplus_{i+j=q}(H_{1,i}) \otimes (I_2)_j$ and $E_{q,2} = \bigoplus_{i+j=q}(I_1)_i \otimes (H_{2,j})$, where $(I_l)_k : H_k(F_l, \mathbb{C}) \to H_k(F_l, \mathbb{C})$ is the identity matrix for $l = 1, 2$ and $k \geq 0$. Then up to multiplication by monomials $\pm\lambda^u$, the zeta function of $f = g(f_1, f_2)$ is determined by*

$$\zeta_f(\lambda) = \zeta_{f_1}(\lambda^{l_2})\zeta_{f_2}(\lambda^{l_1}) \textstyle\prod_q \det \Delta_L(\lambda^{m_1}E_{q,1}, \lambda^{m_2}E_{q,2}, \lambda^{m_3}I, \dots, \lambda^{m_r}I)^{(-1)^q},$$

where $\Delta_L(\lambda_1, \dots, \lambda_r)$ is the Alexander polynomial of L. If $l_j = 0$, then set $\zeta_{f_{(j+1) \bmod 2}}(\lambda^{l_j}) = 1$ for $j = 1, 2$.

Remark 2. Let L_j be the link obtained form L by reversing the orientation of K_j. Then the two Alexander polynomials satisfy

$$\Delta_L(\lambda_1, \dots, \lambda_r) = \epsilon\lambda_j^{u'}\Delta_{L_j}(\lambda_1, \dots, \lambda_j^{-1}, \dots \lambda_r),$$

where $\epsilon = \pm1$ and $u' \in \mathbb{Z}$. We denote the link K_j with the reversed orientation by $-K_j$. Then the associated multiplicity of $-K_j$ is $-m_j$. Thus up to multiplication by monomials, we have

$$\det \Delta_L(\lambda^{m_1}E_{q,1}, \lambda^{m_2}E_{q,2}, \lambda^{m_3}I, \dots, \lambda^{m_j}I, \dots \lambda^{m_r}I)$$
$$= \det \Delta_{L_j}(\lambda^{m_1}E_{q,1}, \lambda^{m_2}E_{q,2}, \lambda^{m_3}I, \dots, \lambda^{-m_j}I, \dots, \lambda^{m_r}I)$$

for any $q \geq 0$.

Example 3. Set $g = z_1z_2 \prod_{j=1}^{k}(z_1^{p_1} + \alpha_j z_2^{p_2}) \prod_{j=k+1}^{k+\ell} \overline{(z_1^{p_1} + \alpha_j z_2^{p_2})}$. Assume that $\alpha_j \neq \alpha_{j'}$ for $j \neq j'$ and $1 \leq j, j' \leq k+\ell$. Then g is a strongly non-degenerate mixed polynomial of 2 complex variables which is locally tame along vanishing coordinate subspaces. By [5], the Alexander polynomial $\Delta_L(\lambda_1, \dots, \lambda_{k+\ell+2})$ is equal to $(\lambda_1^{p_2}\lambda_2^{p_1}\lambda_3^{p_1p_2} \cdots \lambda_{k+\ell+2}^{p_1p_2} - 1)^{k+\ell}$. Therefore the zeta function of the monodromy of f is given by

$$\zeta_f(\lambda) = \textstyle\prod_q \det \Delta_L(\lambda^{m_1}E_{q,1}, \lambda^{m_2}E_{q,2}, \lambda^{m_3}I, \dots, \lambda^{m_{k+\ell+2}}I)^{(-1)^q}$$
$$= \textstyle\prod_{i,j} \det(\lambda^{p_1+p_2+p_1p_2(k-\ell)}(H_{1,i})^{p_2} \otimes (H_{2,j})^{p_1} - I)^{(-1)^{i+j}(k+\ell)}.$$

Example 4. Set $f_1:\mathbb{R}^3\to\mathbb{R}^2, f(x_1,x_2,x_3)=(x_3(x_1^2+x_2^2+x_3^2), x_2-x_1^3), f_2:\mathbb{C}\to\mathbb{C}, f_2(w)=w^2$ and $g:\mathbb{C}^2\to\mathbb{C}, g(z_1,z_2)=z_1^2+z_2^3$. Let $f:\mathbb{R}^3\times\mathbb{C}\to\mathbb{R}^2$ be the real analytic map germ which is defined by $f(x_1,x_2,x_3,w)=g(f_1,f_2)(x_1,x_2,x_3,w)$. By [2], f_1 has an isolated singularity at the origin. Hence f also satisfies the conditions (a-i) and (a-ii). Note that $\zeta_{f_1}(\lambda)=\frac{1}{\lambda-1}, \zeta_{f_2}(\lambda)=\frac{1}{\lambda^2-1}$ and $\det\Delta_L(\lambda_1,\lambda_2,\lambda_3)=\lambda_1^3\lambda_2^2\lambda_3^6-1$. By Theorem 5, $\zeta_f(\lambda)$ is equal to

$$\frac{1}{(\lambda^2-1)(\lambda^6-1)}\det\begin{pmatrix}\lambda^6-1 & 0\\ 0 & \lambda^6-1\end{pmatrix}=\frac{\lambda^6-1}{\lambda^2-1}=\lambda^4+\lambda^2+1.$$

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