Asymptotic existence theorem for formal solutions with singularities of nonlinear partial differential equations via multisummability

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Abstract. In this paper, we consider the summability of formal solutions with singularities (such as logarithmic singularities, functional power singularities, etc.) of nonlinear partial differential equations in the complex domain. The main result is as follows: when a formal solution with singularities is given, under appropriate assumptions related to the formal solution, the equation has a true solution that admits the given formal solution as an asymptotic expansion. The proof is done by constructing a new formal solution that is equivalent to the given formal solution in the asymptotic sense and is multisummable in a suitable direction. The assumptions are stated in terms of the Newton polygon associated with the given formal solution.

1. Introduction

In the study of differential and partial differential equations, we often encounter the situation where we have a formal solution that is not convergent. One way to guarantee the existence of a true solution that admits the formal solution as an asymptotic expansion is to show the summability (or multisummability) of the formal solution, and many authors have discussed the summability of formal solutions in the case where the formal solution is a formal power series solution. We can refer to Balser [2], Balser-Braaksma-Ramis-Sibuya [6], Braaksma [8, 9], etc. in the case of ordinary differential equations, and to Lutz-Miyake-Schäffke [17], Balser [4], Michalik [18], Ouchi [20, 21], Tahara-Yamazawa [27], etc. in the case of partial differential equations. However, the case where the formal solution is not a formal power series solution had not been well studied until now.

For a formal solution \( \hat{u}(t, x) \), we say that the asymptotic existence theorem is valid for \( \hat{u}(t, x) \), if there is a true solution that admits \( \hat{u}(t, x) \) as an asymptotic expansion (as \( t \to 0 \)). See Wasow [29] (Theorem 12.1), Balser [5], etc.

In this paper, we will study the asymptotic existence theorem for general formal solutions (which are not necessarily formal power series solutions) of nonlinear partial differential equations in the complex domain, from the viewpoint of the multisummability method.

The motivation comes from the following examples. We write \( \mathbb{N} = \{0, 1, 2, \ldots\} \), \( \mathbb{N}^* = \{1, 2, \ldots\} \), and we denote by \( \mathcal{O}_R \) the set of all holomorphic functions on \( D_R = \{x \in \mathbb{C} : |x| < R\} \). For \( x \in \mathbb{R} \) we denote by \([x]\) the integer part of \( x \).

Example 1.1. Let \( (t, x) \in \mathbb{C}^2 \), and let us consider the following linear or nonlinear

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2020 Mathematics Subject Classification. Primary 35C10; Secondary 35A01, 35C20, 35G20.

Key Words and Phrases. asymptotic existence theorem, multisummability, formal solution with singularities, nonlinear partial differential equation.
partial differential equations:

\[ \partial_t^2 u - a(x)(\partial_t u)^2 = u + (\partial_x^2 u)^n + t^p \partial_t^q u + f(t, x), \tag{1.1} \]
\[ t\partial_t^2 u = u + (\partial_x^2 u)^n + t^p \partial_t^q u + f(t, x), \tag{1.2} \]
\[ t\partial_x^2 u - (\lambda(x) - 1)\partial_t u = u + (\partial_x^2 u)^n + t^p \partial_t^q u + f(t, x), \tag{1.3} \]
\[ t^2 \partial_t^2 u + x\partial_x u = u + a(x) u^2 + (\partial_x^2 u)^n + t^p \partial_t^q u + f(t, x), \tag{1.4} \]

where \( a(x) \in \mathcal{O}_R, \lambda(x) \in \mathcal{O}_R, n \in \mathbb{N}^*, p \in \mathbb{N}, q \in \mathbb{N}^*, \) and \( f(t, x) \) is a holomorphic function in a neighborhood of \((t, x) = (0, 0)\). We know the following results. For details, see section 10.

(1)(Formal solutions of (1.1)). If \( p \geq q - 1 \) and \( a(0) \neq 0 \), for any \( \varphi(x) \in \mathcal{O}_R \) the equation (1.1) has a unique formal solution of the form

\[ \hat{u}_1(t, x) = -\frac{\log t}{a(x)} + \varphi(x) + b(x)t + \sum_{i \geq 2, |n/2| \geq j \geq 0} \varphi_{i,j}(x)(\log t)^j \]

with suitable \( b(x) \in \mathcal{O}_{R_1}, \varphi_{i,j}(x) \in \mathcal{O}_{R_1} \) \((i \geq 2, j \geq 0)\) and \( R_1 > 0 \).

(2)(Formal solutions of (1.2)). If \( p \geq q \) holds, for any \( \varphi_0(x), \varphi_1(x) \in \mathcal{O}_R \) the equation (1.2) has a unique formal solution of the form

\[ \hat{u}_2(t, x) = \varphi_0(x) + \varphi_1(x)t + a_{1,1}(x)t \log t + \sum_{i \geq 2, j \geq 0} \varphi_{i,j}(x)(\log t)^j \]

with suitable \( a_{1,1}(x) \in \mathcal{O}_{R_1}, \varphi_{i,j}(x) \in \mathcal{O}_{R_1} \) \((i \geq 2, j \geq 0)\) and \( R_1 > 0 \).

(3)(Formal solutions of (1.3)). If \( p \geq q, \lambda(0) \notin \mathbb{N}^* \) and \( \Re \lambda(0) > 0 \) hold, for any \( v_0(x), \varphi(x) \in \mathcal{O}_R \) the equation (1.3) has a unique formal solution of the form

\[ \hat{u}_3(t, x) = \sum_{i \geq 0} v_i(x) t^i + \varphi(x) t^{\lambda(x)} + \sum_{(i, l, j) \in S} \varphi_{i,l,j}(x) t^{i + l + \lambda(j)} (\log t)^j \]

with \( S = \{(i, l, j) \in \mathbb{N}^3; i + l \geq 2, l \geq 1, 2(i + l - 1) \geq j\} \), and suitable \( v_i(x) \in \mathcal{O}_{R_1}, \varphi_{i,l,j}(x) \in \mathcal{O}_{R_1} \) \((i, l, j) \in S\) and \( R_1 > 0 \).

If \( n = 1 \), (1.3) is a linear partial differential equation and the formal solution \( \hat{u}_3(t, x) \) is reduced to the form

\[ \hat{u}_3(t, x) = \sum_{i \geq 0} v_i(x) t^i + \varphi(x) t^{\lambda(x)} + \sum_{i \geq 1, |l| \geq 0} \varphi_{i,l,j}(x) t^{i + l + \lambda(j)} (\log t)^j. \]

In this case, Yamazawa [30] has shown the multisummability of this formal solution in a suitable sense.

(4)(Formal solutions of (1.4)). If \( n \geq 2, p \geq q + 1, f(0, x) \equiv 0 \) and \( (\partial_t \partial_x f)(0, 0) \neq 0 \) hold, for any \( b_1 \in \mathbb{C} \) the equation (1.4) has a unique formal solution of the form

\[ \hat{u}_4(t, x) = b(x)t + cx \log t + \sum_{i \geq 2, j \geq 0} \varphi_{i,j}(x)(\log t)^j \]

with suitable \( b(x) \in \mathcal{O}_{R_1} \) satisfying \( (\partial_x b)(0) = b_1, c = (\partial_t \partial_x f)(0, 0), \varphi_{i,j}(x) \in \mathcal{O}_{R_1} \) \((i \geq 2, j \geq 0)\) and \( R_1 > 0 \).
As is seen above, many linear and nonlinear partial differential equations have formal solutions with singularities at \( t = 0 \) such as \( t^{\lambda(x)} \) and \( \log t \). It is well-known that divergent formal solutions with logarithmic singularities also appear at an irregular singularity of linear ordinary differential equations. Formal solutions containing terms with singularities as above can be also obtained from the asymptotic expansions of solutions in Ouchi [22].

Recently, Yamazawa [30] has discussed the multisummability of formal solutions with \( \log t \) singularities of some linear partial differential equations like (1.3) (with \( n = 1 \)), by setting \( y = \log t \) and reducing the problem to the one for formal power series solutions with a parameter \( y \). However, in the other cases we had no results up to now about the existence of a true solution that has the above formal solution with singularities as an asymptotic expansion.

Thus, the purpose of this paper is to give a systematic study of the problem given below, so that we have a true solution \( u_i(t, x) \) that admits \( \hat{u}_i(t, x) \) as an asymptotic expansion (as \( t \to 0 \)) in all cases (1.1)~(1.4).

**Problem 1.2.** Suppose that we have a formal solution \( \hat{u}(t, x) \) with singularities at \( t = 0 \). Then, can we show the existence of a true solution that admits \( \hat{u}(t, x) \) as an asymptotic expansion (as \( t \to 0 \))? .

### 2. Formulation

Let \( t \) be the variable in \( \mathbb{C}_t \) (or in \( \mathcal{R}(\mathbb{C}_t \setminus \{0\}) \)) the universal covering space of \( \mathbb{C}_t \setminus \{0\} \), and let \( x = (x_1, \ldots, x_K) \) be the variable in \( \mathbb{C}^K_x \). For an open subset \( U \) of \( \mathbb{C}^K_x \) denoted by \( \mathcal{O}(U) \) the set of all holomorphic functions on \( U \). For \( R > 0 \) we write \( D_R = \{ x \in \mathbb{C}^K_x \mid |x_i| < R \ (i = 1, \ldots, K) \} \), and set \( \mathcal{O}_R = \mathcal{O}(D_R) \). For \( \alpha = (\alpha_1, \ldots, \alpha_K) \in \mathbb{N}^K \) we write \( |\alpha| = \alpha_1 + \cdots + \alpha_K \).

Let \( m \in \mathbb{N}^* \), \( N = \# \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^K \mid j + |\alpha| \leq m \} \) (where \( \#A \) denotes the cardinal of a set \( A \)), and let \( Z = \{ Z_{j,\alpha} \}_{j+|\alpha| \leq m} \) be the variable in \( \mathbb{C}^N_z \).

We note that by multiplying (1.1) \~ (1.3) by \( t^2 \) or \( t \), equations (1.1) \~ (1.4) are expressed in the form:

\[
(t\partial_t)(t\partial_t - 1)u - a(x)(t\partial_x u)^2 = t^2 u + t^2 (\partial_x^2 u)^n + t^{p+2-q} |t\partial_t|_q u + t^2 f(t, x),
\]
\[
(t\partial_t)(t\partial_t - 1)u = tu + t(\partial_x^2 u)^n + t^{p+1-q} |t\partial_t|_q u + t f(t, x),
\]
\[
(t\partial_t)(t\partial_t - \lambda(x))u = tu + t(\partial_x^2 u)^n + t^{p+1-q} |t\partial_t|_q u + t f(t, x),
\]
\[
(t\partial_t)(t\partial_t - 1)u + x \partial_x u = u + a(x)u^2 + (\partial_x^2 u)^n + t^{p-q} |t\partial_t|_q u + f(t, x),
\]
where \( |t\partial_t|_q = t\partial_t(t\partial_t - 1) \cdots (t\partial_t - q + 1) \) for \( q \in \mathbb{N}^* \).

With these forms in mind, in this paper we will consider a nonlinear partial differential equation

\[
F \left( t, x, \{(t\partial_t)^2 \partial_x^n u\}_{j+|\alpha| \leq m} \right) = 0
\]
under the following assumption: \( F(t, x, Z) \) is a holomorphic function in a neighborhood of the origin of \( \mathbb{C}_t \times \mathbb{C}^K_x \times \mathbb{C}^N_z \).

Let \( I = (\theta_1, \theta_2) \) be a non-empty open interval, \( r > 0 \) and \( R > 0 \). We write \( S_1 = \{ t \in \mathbb{R} \mid \}

\[
\lambda = \log t \to 0 \}
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\lambda \to 0 \}
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\lambda = 0 \to 0 \}
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\lambda \to 0 \}
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\lambda = 0 \to 0 \}
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\lambda \to 0 \}
\[ R(\mathcal{C}_t \setminus \{0\}); \theta_1 < \arg t < \theta_2 \} \text{ and } S_I(r) = \{ t \in S_I; 0 < |t| < r \}. \] For a formal series
\[ \hat{u}(t, x) = \sum_{n \geq 1} u_n(t, x) \quad (2.2) \]
and for \( N \geq 1 \) we denote by \( \hat{u}_N(t, x) \) the \( N \)-finite sum of this formal series \( \hat{u}(t, x) \), that is,
\[ \hat{u}_N(t, x) = \sum_{1 \leq n \leq N} u_n(t, x). \quad (2.3) \]
For a function \( f(t, x) \in \mathcal{O}(S_I(r) \times D_R) \) we write
\[ \| f(t) \|_R = \sup_{x \in D_R} |f(t, x)|. \]

**Definition 2.1.** We say that a formal series \( \hat{u}(t, x) \) in (2.2) is a formal solution of (2.1) on \( S_I(r) \times D_R \) if the following three conditions 1), 2) and 3) are satisfied:
1) \( u_n(t, x) \in \mathcal{O}(S_I(r) \times D_R) \) \( (n = 1, 2, \ldots) \).
2) There is a sequence \( 0 < \mu_1 < \mu_2 < \cdots \) such that \( \| (t \partial_t)^j \partial_x^\alpha u_n(t) \|_R = O(|t|^{\mu_n}) \) \( (as \ S_I \ni t \to 0) \)
holds for any \( j + |\alpha| \leq m \) and \( n \geq 1 \).
3) For any \( A > 0 \) there is an \( N_0 \in \mathbb{N}^* \) such that
\[ \left\| F\left(t, x, \{(t \partial_t)^j \partial_x^\alpha \hat{u}_N \}_{j + |\alpha| \leq m}\right)\right\|_R = O(|t|^A) \] \( (as \ S_I \ni t \to 0) \)
holds for any \( N \geq N_0 \).

Our problem 1.2 is solved if the following problem is positive.

**Problem 2.2.** Let \( \hat{u}(t, x) \) be a formal solution of (2.1) on \( S_I(r) \times D_R \). Then, is this formal solution multisummable (in the sense of Definition 3.3 given in section 3) in a suitable direction?

However, this formulation is not very good. The reason is as follows. We note that the function \( e^{-1/\sigma t} \) is flat at \( t = 0 \) uniformly on \( I \) if \( I \subseteq (-\pi/2\sigma, \pi/2\sigma) \) holds, that is, if \( \sigma > 0 \) is sufficiently small. Then, if \( \hat{u}(t, x) = \sum_{n \geq 1} u_n(t, x) \) is a formal solution of (2.1) on \( S_I(r) \times D_R \), for any \( C_n > 0 \) \( (n = 1, 2, \ldots) \) the formal series
\[ \hat{u}(t, x) = \sum_{n \geq 1} (u_n(t, x) + C_n e^{-1/\sigma t}) \]
is also a formal solution of (2.1) on \( S_I(r) \times D_R \). Therefore, by taking \( C_n > 0 \) \( (n = 1, 2, \ldots) \) suitably we can construct a formal solution that is not multisummable in any Gevrey class.

By this reason, Problem 2.2 is not affirmative in general. In order to avoid this situation, let us introduce the notion of equivalence of two formal solutions.
Definition 2.3. Let 
\[ \hat{u}(t, x) = \sum_{n \geq 1} u_n(t, x) \quad \text{and} \quad \hat{w}(t, x) = \sum_{n \geq 1} w_n(t, x) \]
be two formal solutions of (2.1) on \( S_I(r) \times D_R \). We say that \( \hat{u}(t, x) \) and \( \hat{w}(t, x) \) are equivalent (in the asymptotic sense) if for any \( A > 0 \) there is a \( N_0 \in \mathbb{N}^* \) such that
\[ \left\| (t \partial_t)^j \partial_x^\alpha (\hat{u}_N - \hat{w}_N) \right\|_R = O(|t|^A) \quad (\text{as} \ S_I \ni t \to 0) \]
holds for any \( j + |\alpha| \leq m \) and any \( N \geq N_0 \).

Using this notion, it would be more reasonable to consider

Problem 2.4. Let \( \hat{u}(t, x) \) be a formal solution of (2.1) on \( S_I(r) \times D_R \). Then, is there a formal solution that is equivalent to \( \hat{u}(t, x) \) and is multisummable (in the sense of Definition 3.3) in a suitable direction?

In this paper, we will consider this problem. Roughly speaking, our main results are as in (I) and (II) given below: the precise statements will be given in Theorems 4.7, 4.8 and 4.10 in section 4.

(I) (Solution to Problem 2.4). If a formal solution \( \hat{u}(t, x) \) of (2.1) is given on \( S_I(r) \times D_R \) for a sufficiently large interval \( I \) and if the Newton polygon of (2.1) along the given formal solution \( \hat{u}(t, x) \) satisfies some suitable conditions, then the equation (2.1) has a new formal solution \( \hat{w}(t, x) \) on \( S_{I_1}(r_1) \times D_{R_1} \) (where \( I_1 \in I, 0 < r_1 < r \) and \( 0 < R_1 < R \)) that is equivalent to the original formal solution \( \hat{u}(t, x) \) and is multisummable (in the sense of Definition 3.3) in a suitable direction in the \( t \)-variable.

(II) (Asymptotic existence theorem). Under the situation in (I), the equation (2.1) has a true solution \( w(t, x) \) on \( S_{I_2}(r_2) \times D_{R_2} \) (where \( I_2 \in I_1, 0 < r_2 < r_1 \) and \( 0 < R_2 < R_1 \)) that admits \( \hat{u}(t, x) \) as an asymptotic expansion (as \( t \to 0 \)) in the following sense: for any \( A > 0 \) there is an \( N_0 \in \mathbb{N}^* \) such that
\[ \left\| w(t, x) - \hat{u}_N(t, x) \right\|_{R_2} = O(|t|^A) \quad (\text{as} \ S_{I_2} \ni t \to 0) \]
holds for any \( N \geq N_0 \).

If the given formal solution \( \hat{u}(t, x) \) is a formal power series solution in \( t \) with coefficients holomorphic in \( x \), we can take \( \hat{w}(t, x) = \hat{u}(t, x) \) and the above result (I) gives a slight generalization of that in Ouchi [21]. Our interest in this paper lies in the treatment of formal solutions with singularities.

The rest part of this paper is organized as follows. Since our arguments are based on the summability theory, in the next section 3 we will briefly review some basic tools and definitions in the multisummability method. In section 4, we will state our main theorem which gives a sufficient condition for the equation to have a new formal solution that is equivalent to the given formal solution and is multisummable. In this theorem, the Newton polygon associated with the given formal solution plays an important role to describe our sufficient condition. The proof of main theorem will be given in sections
In section 9 we present some supplementary results, and in section 10 we give the details of the calculations in examples (1.1)−(1.4). The last section 11 is the addendum to section 3, in which proofs of some results in section 3 for which no good literature can be found are given.

In the proof, the result on the analytic continuation of solutions of a convolution equation in the Borel plane is very important. However, this has already been studied in Tahara [24], and the result there can be used in this paper.

3. On multisummability

In this section, we will recall some basic results on Laplace transform, Borel transform, multisummability of formal series of functions, and the convolution of two functions. For the details of these topics and the proofs of some results, readers can refer to Braaksma [8, 9], Balser [2, 3], Ouchi [20, 21] and Loday-Richaud [16]. See also section 2 of Tahara-Yamazawa [27].

If $K = (\phi_1, \phi_2)$ is an open interval, we write $|K| = \phi_2 - \phi_1$: for $a > 0$ we write $K + [a] = (\phi_1 - a, \phi_2 + a)$ and $K - [a] = (\phi_1 + a, \phi_2 - a)$ (if $0 < a < |K|/2$).

3.1. Laplace transform

Let $k > 0$, $I = (\theta_1, \theta_2)$ be a non-empty open interval, and $U$ be an open subset of $\mathbb{C}^K$. We denote by $E_k(S_I \times U)$ the set of all holomorphic functions $f(\xi, x)$ on $S_I \times U$ satisfying the estimate

$$|f(\xi, x)| \leq A|\xi|^{-k} \exp(b|\xi|^{\theta}) \quad \text{on } S_I \times U$$

for some $A > 0$, $a > 0$ and $b > 0$. For $f(\xi, x) \in E_k(S_I \times U)$ we define the $k$-Laplace transform $L_k[f](t, x)$ of $f(\xi, x)$ with respect to $\xi$ by the following:

$$L_k[f](t, x) = \int_{I}^{\infty} \exp(-(\xi/t)^{\theta}) f(\xi, x) d\xi^k,$$

where $\theta$ and $t$ are taken so that $\theta \in I$ and $|\theta - \arg t| < \pi/2k$ hold. Then, for any $\epsilon > 0$ there is an $\epsilon > 0$ such that $L_k[f](t, x)$ is well-defined as a holomorphic function on $S_{I+|\pi/2k-\epsilon|}(r) \times U$ and that the estimate $|L_k[f](t, x)| \leq C|t|^\alpha$ is valid on $S_{I+|\pi/2k-\epsilon|}(r) \times U$ for some $C > 0$ (where $a$ is the constant in (3.1)).

3.2. Borel transform

If $F(t, x)$ is a holomorphic function on $S_{I+|\pi/2k|}(r) \times U$ for some $r > 0$ and if the estimate

$$|F(t, x)| \leq C|t|^\alpha \quad \text{on } S_{I+|\pi/2k|}(r) \times U$$

holds for some $C > 0$ and $a > 0$, we define $k$-Borel transform $B_k[F](\xi, x)$ of $F(t, x)$ with respect to $t$ by the following:

$$B_k[F](\xi, x) = \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{C}^K} \exp((\xi/t)^{\theta}) F(t, x) dt^{-k}, \quad (\xi, x) \in S_I \times U,$$
where \( \mathcal{C}(\xi) \) is a contour in \( S_{t+\pi/2k}(r) \) that starts from \( 0e^{-\sqrt{-1}(\arg \xi + \pi/2k + \delta_2)} \) and ends to \( 0e^{-\sqrt{-1}(\arg \xi - \pi/2k - \delta_1)} \) with \( 0 < \delta_2 < \min\{\theta_2 - \arg \xi, \pi/k\} \) and \( 0 < \delta_1 < \min\{\arg \xi - \theta_1, \pi/k\} \) in \( S_{t+\pi/2k}(r) \). Then, \( B_{k}[F](\xi, x) \) is well-defined as a holomorphic function on \( S_{t} \times U \), and we have the following estimate: for any \( \epsilon > 0 \) (sufficiently small) and \( b > 1/r^k \) there is an \( A > 0 \) such that \( |B_k[F](\xi, x)| \leq |A|\xi|^{a-k}\exp(b|\xi|^k) \) on \( S_{t-|\cdot|} \times U \). As is proved in subsection 11.1, the dependence of \( A \) on \( C \) and \( a \) is as follows.

**Lemma 3.1.** For any \( \epsilon > 0 \), \( b > 1/r^k \) and \( \eta > 0 \) there is a \( K > 0 \) that satisfies the following: if \( F(t, x) \) satisfies the estimate (3.2) for some \( C > 0 \) and \( a \geq \eta \) we have

\[
|B_k[F](\xi, x)| \leq \frac{KC}{\Gamma(a/k)} |\xi|^{a-k}\exp(b|\xi|^k) \quad \text{on} \quad S_{t-|\cdot|} \times U.
\]

The following inversion formulas are well-known:

\[
(B_k \circ L_k[f])(\xi, x) = f(\xi, x) \quad \text{and} \quad (L_k \circ B_k[F])(t, x) = F(t, x)
\]

under a suitable condition. Also, in subsection 7.2 we use the following identity:

\[
B_k[(x^{k+1}\partial_x)^jF](\xi, x) = (k^{j+1})!B_k[F](\xi, x), \quad j = 1, 2, \ldots.
\]

Let \( u_n(t, x) \) \( (n = 1, 2, \ldots) \) be holomorphic functions on \( S_{t+\pi/2k}(r) \times U \) satisfying the estimates \( |u_n(t, x)| \leq M_n |t|^{\mu_n} \) on \( S_{t+\pi/2k}(r) \times U \) for some \( M_n > 0 \) and \( \mu_n > 0 \) with \( 0 < \mu_1 < \mu_2 < \cdots \) and \( \mu_n \rightarrow \infty \) as \( n \rightarrow \infty \). Then, for the formal series

\[
\hat{u}(t, x) = \sum_{n \geq 1} u_n(t, x)
\]

we define the formal \( k \)-Borel transform \( \hat{B}_k[\hat{u}](\xi, x) \) of \( \hat{u}(t, x) \) by the following:

\[
\hat{B}_k[\hat{u}](\xi, x) = \sum_{n \geq 1} B_k[u_n](\xi, x).
\]

(3.3)

If \( u_n(t, x) = u_n(x)t^{\mu_n} \) \( (n = 1, 2, \ldots) \), the formal \( k \)-Borel transform (3.3) is just the usual one:

\[
\hat{B}_k[\hat{u}](\xi, x) = \sum_{n \geq 1} u_n(x) \frac{\xi^{\mu_n-k}}{\Gamma(\mu_n/k)}.
\]

**3.3. Acceleration operator**

Let \( 0 < k_1 < k_2 \). We define the operator \( A_{k_2, k_1} \) by

\[
A_{k_2, k_1}[F](\xi, x) = B_{k_2} \circ L_{k_1}[F](\xi, x), \quad f(\xi, x) \in E_{k_1}(S_t \times U);
\]

(3.4)

this operator is called the acceleration operator (or accelerator), which was introduced by Ecalle [11]. By replacing \( L_{k_1} \) and \( B_{k_2} \) in (3.4) with their integral representations and then by changing the order of integration, we have

\[
A_{k_2, k_1}[f](\xi, x) = \int_0^\infty e^{\sqrt{-1}\theta} g_{k_2, k_1}(\xi, \tau)f(\tau, x)d\tau^{k_1},
\]

(3.5)
where \( \theta \in I \) and

\[
G_{k_2,k_1}(\xi,\tau) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}(\xi)} \exp((\xi/t)^{k_2} - (\tau/t)^{k_1}) dt^{-k_2}
\]

which is the \( k_2 \)-Borel transform of \( \exp(-\tau/t)^{k_1} \) with respect to \( t \). Thus, instead of (3.4) we can adopt (3.5) as a definition of the acceleration operator \( A_{k_2,k_1} \). We define \( \kappa > 0 \) by the relation \( 1/\kappa = 1/k_1 - 1/k_2 \). We have \( k_1 < \kappa \).

**Proposition 3.2.** Let \( f(\xi, x) \in \mathcal{O}(S_I \times U) \) satisfy the estimate

\[
|f(\xi, x)| \leq A|\xi|^{a-1} \exp(c|\xi|^b) \quad \text{on } S_I \times U
\]

for some \( A > 0, a > 0 \) and \( c > 0 \). Then, for any sufficiently small \( \epsilon > 0 \) there is an \( r > 0 \) such that \( A_{k_2,k_1}[f](\xi, x) \) is well-defined as a holomorphic function on \( S_{I + |\pi/2\kappa - \epsilon|}(r) \times U \) and that the estimate

\[
|A_{k_2,k_1}[f](\xi, x)| \leq C|\xi|^{a-k_2} \quad \text{on } S_{I + |\pi/2\kappa - \epsilon|}(r) \times U
\]

is valid for some \( C > 0 \).

### 3.4. Multisummability

Let \( q \in \mathbb{N}^* \), and \( 0 < k_1 < k_2 < \cdots < k_q < k_{q+1} = \infty \). Define \( \kappa_i > 0 \) \((1 \leq i \leq q)\) by the relation \( 1/\kappa_i = 1/k_i - 1/k_{i+1} \). We have \( \kappa_i > k_i \) \((1 \leq i \leq q - 1)\) and \( \kappa_q = k_q \).

Let \( I_0 \) be a non-empty open interval, and \( \delta > 0 \). Let \( u_n(t,x) \) \((n = 1, 2, \ldots)\) be holomorphic functions on \( S_{I_0 + |\pi/2k_i|}(\delta) \times U \) satisfying the estimates

\[
|u_n(t,x)| \leq M_n|t|^{\mu_n} \quad \text{on } S_{I_0 + |\pi/2k_i|}(\delta) \times U, \quad n = 1, 2, \ldots
\]

(3.6)

for some \( M_n > 0 \) and \( 0 < \mu_1 < \mu_2 < \cdots \) with \( \mu_n \longrightarrow \infty \) \((as n \longrightarrow \infty)\). Set

\[
\hat{u}(t,x) = \sum_{n \geq 1} u_n(t,x).
\]

Then, the formal \( k_1 \)-Borel transform \( w_1(\xi, x) \) of \( \hat{u}(t, x) \) is given by

\[
w_1(\xi, x) = \hat{B}_{k_1}[\hat{u}](\xi, x) = \sum_{n \geq 1} B_{k_1}[u_n](\xi, x).
\]

(3.7)

By Lemma 3.1 we know that \( B_{k_1}[u_n](\xi, x) \) \((n = 1, 2, \ldots)\) are holomorphic functions on \( S_{I_0} \times U \) and satisfy the following estimates: for any \( I_1 \subseteq I_0 \) and \( b > 1/\delta k_1 \) there is a constant \( K > 0 \) such that

\[
|B_{k_1}[u_n](\xi, x)| \leq \frac{K M_n}{\Gamma(\mu_n/k_1)} |\xi|^{|\mu_n-k_1|} \exp(b|\xi|^k) \quad \text{on } S_{I_1} \times U, \quad n = 1, 2, \ldots
\]

**Definition 3.3.** Let \( I \) be a non-empty open interval. We say that the series \( \hat{u}(t, x) \) is \((k_q, \ldots, k_1)\)-summable (or \((k_q, \ldots, k_1)\)-multisummable) in \( I \)-direction in \( t \) (uniformly in \( x \in U \)), if there are intervals \( I_i \) \((i = 1, 2, \ldots, q)\) with \( I_q = I \) and \( I_i \subseteq I \) \((1 \leq i \leq q - 1)\)
and if the following two requirements 1) and 2) are satisfied.

1) The formal series \( w_1(\xi, x) \) is convergent on \( S_{I_1}(r_1) \times U \) for some \( r_1 > 0 \) and it defines a holomorphic function on \( S_{I_1}(r_1) \times U \).

2) From \( i = 1 \) to \( i = q \), we have the following properties. The function \( w_i(\xi, x) \) has an analytic continuation \( w_i^*(\xi, x) \) on \( S_{I_i} \times U \) satisfying the estimate

\[
|w_i^*(\xi, x)| \leq A_i|\xi|^{|\mu_1-k_i|}\exp(c_i|\xi|^{|\kappa_i|}) \quad \text{on } S_{I_i} \times U
\]  

(3.8)

for some \( A_i > 0 \) and \( c_i > 0 \): then, in case \( i < q \) we set

\[
w_{i+1}(\xi, x) = A_{k_{i+1}, k_i}[w_i^*](\xi, x)
\]

which is a holomorphic function on \( S_{I_{i+1}}(r_{i+1}) \times U \) for some \( r_{i+1} > 0 \) (see the explanation given below).

The situation is as follows:

\[
w_1(\xi, x) = \mathcal{B}_{k_1}[\tilde{u}](\xi, x) \quad \text{on } S_{I_1}(r_1) \times U,
\]

\[
w_2(\xi, x) = A_{k_2, k_1}[w_1^*](\xi, x) \quad \text{on } S_{I_2}(r_2) \times U,
\]

\[\ldots\]

\[
w_q(\xi, x) = A_{k_q, k_{q-1}}[w_{q-1}^*](\xi, x) \quad \text{on } S_{I_q}(r_q) \times U.
\]

If \( w_q^*(\xi, x) \) is a holomorphic function on \( S_{I_q} \times U \) and it satisfies (3.8), by Proposition 3.2 we see: for any \( \epsilon > 0 \) there is an \( r > 0 \) such that \( w_{i+1}(\xi, x) = A_{k_{i+1}, k_i}[w_i^*](\xi, x) \) is well-defined on \( S_{I_{i+1}}(r_{i+1}) \times U \) and satisfies

\[
|w_{i+1}(\xi, x)| \leq C_{i+1}|\xi|^{|\mu_1-k_{i+1}|} \quad \text{on } S_{I_{i+1}}(r_{i+1}) \times U
\]

for some \( C_{i+1} > 0 \). Since \( I_{i+1} \in I_i + [\pi/2\kappa_i, \pi/2\kappa_i - \epsilon] \) is supposed, we have \( I_{i+1} \subset I_i + [\pi/2\kappa_i - \epsilon] \) for some \( \epsilon > 0 \). This shows that the function \( w_{i+1}(\xi, x) \) is well-defined on \( S_{I_{i+1}}(r_{i+1}) \times U \) for some \( r_{i+1} > 0 \). Thus, the requirement 2) above makes sense.

Since \( \kappa_q = k_q \) and \( I_q = I \) hold, by (3.8) with \( i = q \) we have

\[
|w_q^*(\xi, x)| \leq A_q|\xi|^{|\mu_1-k_q|}\exp(c_q|\xi|^{|k_q|}) \quad \text{on } S_I \times U.
\]

Hence, we can define the \( k_q \)-Laplace transform of \( w_q^*(\xi, x) \) as

\[
u^*(t, x) = \mathcal{L}_{k_q}[w_q^*](t, x).
\]

This function \( u^*(t, x) \) is called the \((k_q, \ldots, k_1)\)-sum (or \((k_q, \ldots, k_1)\)-multisum) of \( \tilde{u}(t, x) \).

As is stated in subsection 3.1, for any \( \epsilon > 0 \) there is an \( r > 0 \) such that \( u^*(t, x) \) is well-defined as a holomorphic function on \( S_{I+[\pi/2\kappa_q-\epsilon]}(r) \times U \). Roughly, it is given by

\[
u^*(t, x) = \mathcal{L}_{k_q} \circ A_{k_q, k_{q-1}} \circ \cdots \circ A_{k_2, k_1}[w_1](t, x),
\]

where \( w_1(\xi, x) \) is the one in (3.7).

As in the case of formal power series in \( t \), we have the following result: a sketch of the proof is given in subsection 11.2.
Proposition 3.4. Let \( u_n(t, x) \in \mathcal{O}(S_{H+|\pi/2t|}(\delta) \times U) \) \((n = 1, 2, \ldots)\), and suppose that they satisfy the estimates (3.6) with \( M_n = A h^{\mu_n} \Gamma(\mu_n/k_1) \) \((n = 1, 2, \ldots)\) for some \( A > 0, h > 0 \) and \( \mu_n > 0 \) \((n = 1, 2, \ldots)\). Suppose also that \( 0 < \mu_1 < \mu_2 < \cdots, \mu_n \to \infty \) (as \( n \to \infty \)),

\[
\lim_{n \to \infty} (\mu_n - \mu_{n-1})^{1/\mu_n} > 0 \quad \text{and} \quad \sum_{n \geq 1} \eta^{\mu_n} < \infty
\]  

(3.9)

for some \( \eta > 0 \). If the formal series

\[
\hat{u}(t, x) = \sum_{n \geq 1} u_n(t, x)
\]

is \((k_0, \ldots, k_1)\)-multisummable in \( I\)-direction, the \((k_0, \ldots, k_1)\)-multisum \( u^*(t, x) \) of \( \hat{u}(t, x) \) satisfies the following asymptotic relation: for any sufficiently small \( \epsilon > 0 \) there are \( C > 0, H > 0 \) and \( r > 0 \) such that

\[
\left| u^*(t, x) - \sum_{n=1}^{N-1} u_n(t, x) \right| \leq CH^N \Gamma(\mu_N/k_1) |t|^{\mu_N}
\]

holds on \( S_{I+|\pi/2t| \epsilon}(r) \times U \) for any \( N \geq 1 \).

In the case \( q = 1, (k_1)\)-multisummability is called \( k_1\)-summability. If \( \hat{u}(t, x) \) is a formal power series in \( t \) with holomorphic coefficients, the above definition coincides with the usual one.

3.5. Convolution

Let \( k > 0 \). For two functions \( f(\xi, x) \) and \( g(\xi, x) \) on \( S_I \times U \) \((\text{resp. on } S_I(r) \times U)\) we define the \( k\)-convolution \( (f \ast_k g)(\xi, x) \) of \( f(\xi, x) \) and \( g(\xi, x) \) with respect to \( \xi \) by

\[
(f \ast_k g)(\xi, x) = \int_0^\xi f(\tau, x) g((\xi^k - \tau^k)^{1/k}, x) d\tau^k
\]

for \((\xi, x) \in S_I \times U \) \((\text{resp. for } (\xi, x) \in S_I(r) \times U)\). For basic properties, one can refer to \([2, 3, 8, 20, 24]\). The following formulae are very important:

\[
L_k[f \ast_k g](t, x) = L_k[f](t, x) \times L_k[g](t, x),
\]

\[
\mathcal{B}_k[F \times G](\xi, x) = (\mathcal{B}_k[F] *_{k} \mathcal{B}_k[G])(\xi, x).
\]

Let \( \kappa \geq k \). For \( a > 0 \) and \( c \geq 0 \) we set

\[
\phi_a(\xi; c) = \frac{\left| \xi \right|^{a-k}}{\Gamma(a/k)} \exp(c|\xi|^c), \quad \xi \in \mathbb{R}(C \setminus \{0\}).
\]

The following result is very useful for estimating \( k\)-convolutions.

Lemma 3.5. Let \( f(\xi, x) \in \mathcal{O}(S_I \times U) \) and \( g(\xi, x) \in \mathcal{O}(S_I \times U) \); then we have \( (f \ast_k g)(\xi, x) \in \mathcal{O}(S_I \times U) \). If they satisfy the estimates \(|f(\xi, x)| \leq A \phi_a(\xi; c) \) and \(|g(\xi, x)| \leq B \phi_b(\xi; c)\) on \( S_I \times U \) for some \( A > 0, a > 0, B > 0 \) and \( b > 0 \), we have the estimate \(|(f \ast_k g)(\xi, x)| \leq AB \phi_{a+b}(\xi; c)\) on \( S_I \times U \).
4. Main results

In this section, we give a precise statement of the main results of this paper. First we define an analogue of the order of zeros of a function at \( t = 0 \), then we define the Newton polygon of the equation along a given formal solution \( \hat{u}(t,x) \) and the characteristic polynomials on the boundary of the Newton polygon, and finally we give a statement of our main results.

For simplicity, we write \( \Lambda = \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^K : j + |\alpha| \leq m\} \). Let \( r_0 > 0 \), \( R_0 > 0 \), \( \rho_0 > 0 \) and set \( \mathcal{K} = \{(t,x,Z) \in \mathbb{C} \times \mathbb{C}_2^K \times \mathbb{C}^N : |t| \leq r_0, |x| \leq R_0, |Z| \leq \rho_0\} \), where \( |x| = \max_{1 \leq j \leq K} |x_i| \) and \( |Z| = \max_{(j,\alpha) \in \Lambda} |Z_{j,\alpha}| \).

Recall that we are considering the equation

\[
F(t,x,\{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha) \in \Lambda}) = 0 \tag{4.1}
\]

under the assumptions:

(A1) \( F(t,x,Z) \) is a holomorphic function in a neighborhood of \( \mathcal{K} \).

(A2) Equation (4.1) has a formal solution \( \hat{u}(t,x) \) on \( S_J(r) \times D_R \) (in the sense of Definition 2.1) for some open interval \( J = (\theta_1, \theta_2) \), \( r > 0 \) and \( R > 0 \).

4.1. An analogue of the order of zeros

In the study of the summability of formal power series solutions, the notion of the order of zeros of a holomorphic function or the valuation of a formal power series plays an important role. In this section, we will define a similar notion for a bounded function on \( S_J(r) \times D_R \).

For an open interval \( J, r > 0 \) and \( R > 0 \) we denote by \( \mathcal{O}^b(S_J(r) \times D_R) \) the set of all bounded holomorphic functions on \( S_J(r) \times D_R \).

**Definition 4.1.** (1) For \( f(t,x) \in \mathcal{O}^b(S_J(r) \times D_R) \) we define a variant \( \gamma_t(f) \) of the order of zero of \( f(t,x) \) at \( t = 0 \) by

\[
\gamma_t(f) = \sup\{a \geq 0 : \|f(t)\|_R = O(|t|^a) \quad \text{(as } S_J \ni t \rightarrow 0)\}.
\]

(2) We say that the order of zero of \( f(t,x) \) at \( t = 0 \) is well-defined, if \( f(t,x) \) is expressed in the form

\[
f(t,x) = \varphi(x)t^p + O(|t|^{p+\epsilon}) \quad \text{(as } S_J \ni t \rightarrow 0)\]

uniformly on \( D_R \) for some \( \varphi(x) \in \mathcal{O}(D_R) \) with \( \varphi(x) \not\equiv 0 \), \( p \geq 0 \) and \( \epsilon > 0 \). In this case, the value \( p \) is called the order of zero of \( f(t,x) \) at \( t = 0 \).

By the definition we see that if \( p \) is the order of zero of \( f(t,x) \) at \( t = 0 \) then we have \( \gamma_t(f) = p \). However, the converse is not true in general, as is seen in the following example.

**Example 4.2.** (1) Let \( f(t,x) = \varphi(x)e^{-1/t} \) and \( J = (-\pi/4, \pi/4) \); then we have \( \gamma_t(f) = \infty \).

(2) Let \( f(t,x) = \varphi(x)t \); then we have \( \gamma_t(f) = 1 \). In this case, the order of zero of \( f(t,x) \) at \( t = 0 \) is well-defined and it is equal to 1.
(3) Let \( f(t, x) = \varphi(x)t \log t \); then we have \( \gamma_t(f) = 1 \). In this case, the order of zero of \( f(t, x) \) at \( t = 0 \) is not well-defined.

4.2. In the case \( G(t, x, \Theta \hat{u}) \)

Let \( G(t, x, Z) \) be a holomorphic function in a neighborhood of \( K \). Let

\[
\hat{u}(t, x) = \sum_{n \geq 1} u_n(t, x)
\]

be the given formal solution of (4.1) on \( S_J(r) \times D_R \), and let \( 0 < \mu_1 < \mu_2 < \cdots \) be the sequence appearing in Definition 2.1. For simplicity, we write \( \Theta = \Theta(t, x, \hat{u}(t, x)) \). Then, we have the following results.

**Proposition 4.3.** We set

\[
p = \lim_{N \to \infty} \gamma_t(G(t, x, \Theta \hat{u}_N))
\]

(where \( \hat{u}_N \) is as in (2.3)). Then, we have the following results.

1. If \( p = \infty \), for any \( A > 0 \) there is an \( N_0 \in \mathbb{N}^* \) such that \( \gamma_t(G(t, x, \Theta \hat{u}_N)) > A \) holds for any \( N \geq N_0 \).

2. If \( 0 \leq p < \infty \), there is an \( N_0 \in \mathbb{N}^* \) such that \( p = \gamma_t(G(t, x, \Theta \hat{u}_N)) \) holds for any \( N \geq N_0 \).

**Proof.** Let us show (1). Suppose \( p = \infty \), and let \( A > 0 \). Then, we can find an \( N_0 \in \mathbb{N}^* \) so that \( \gamma_t(G(t, x, \Theta \hat{u}_N)) > A \) and \( \mu_{N_0+1} > A \) hold. By Taylor expansion, for any \( N \geq N_0 + 1 \) we have

\[
G(t, x, \Theta \hat{u}_N) = G(t, x, \Theta \hat{u}_{N_0} + \Theta\left( \sum_{N_0+1 \leq n \leq N} u_n(t, x) \right))
\]

\[
= G(t, x, \Theta \hat{u}_{N_0}) + \sum_{(j, \alpha) \in A} \sum_{N_0+1 \leq n \leq N} O(1)((t \partial_t)^j \partial_x^\alpha u_n(t, x))
\]

\[
= O(|t|^{A\varepsilon}) + \sum_{N_0+1 \leq n \leq N} O(|t|^{\mu_n}) \quad (\text{as } S_J \ni t \to 0)
\]

for some \( \varepsilon > 0 \). Since \( \mu_n > A \) holds for \( N_0 + 1 \leq n \leq N \), we have the condition \( \gamma_t(G(t, x, \Theta \hat{u}_N)) > A \). This proves (1).

Next, let us show (2). Suppose \( 0 \leq p < \infty \). Then, we can find an \( N_0 \in \mathbb{N}^* \) so that \( \gamma_t(G(t, x, \Theta \hat{u}_N)) < p + 1 \) and \( \mu_{N_0+1} > p + 1 \) hold. By Taylor expansion, for any \( N \geq N_0 + 1 \) we have

\[
G(t, x, \Theta \hat{u}_N) = G(t, x, \Theta \hat{u}_{N_0} + \Theta\left( \sum_{N_0+1 \leq n \leq N} u_n(t, x) \right))
\]

\[
= G(t, x, \Theta \hat{u}_{N_0}) + O(|t|^{p+1}) \quad (\text{as } S_J \ni t \to 0).
\]

If we set \( p_0 = \gamma_t(G(t, x, \Theta \hat{u}_N)) \) we have \( p_0 < p + 1 \) and so we have \( p_0 = \gamma_t(G(t, x, \Theta \hat{u}_N)) \).
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for any $N \geq N_0$. This implies that

$$p_0 = \lim_{N \to \infty} \gamma_t(G(t,x,\Theta \hat{u}_N)) = p.$$  

Thus, we have $p = \gamma_t(G(t,x,\Theta \hat{u}_N))$ for any $N \geq N_0$. This proves (2). □

By the proof we see: if $0 \leq p < \infty$, the order of zero of $G(t,x,\Theta \hat{u}_N)$ at $t = 0$ (for $N \geq N_0$) is well-defined if and only if it is well-defined for $G(t,x,\Theta \hat{u}_N)$. Hence, we have:

**Corollary 4.4.** In the case $0 \leq p < \infty$, one of the following two cases occur:

(i) There are $N_0 \in \mathbb{N}^*$, $\varphi(x) \in \mathcal{O}(D_R)$ with $\varphi(x) \not\equiv 0$, $p \geq 0$ and $\epsilon > 0$ such that for any $N \geq N_0$ we have

$$G(t,x,\Theta \hat{u}_N) = \varphi(x)t^p + O(|t|^{p+\epsilon}) \quad \text{(as } S_J \ni t \to 0) \text{ uniformly on } D_R.$$  

(ii) There is an $N_0 \in \mathbb{N}^*$ such that for any $N \geq N_0$ the order of zero of $G(t,x,\Theta \hat{u}_N)$ at $t = 0$ is not well-defined.

**Definition 4.5.** In the case (i), we say that the order of zero of $G(t,x,\Theta \hat{u}_N)$ at $t = 0$ is well-defined.

4.3. Newton polygon

Let $\hat{u}(t,x)$ be the given formal solution of (4.1). We set

$$p_{j,\alpha} = \lim_{N \to \infty} \gamma_t((\partial F/\partial Z_{j,\alpha})(t,x,\Theta \hat{u}_N)), \quad (j,\alpha) \in \Lambda :$$

we have $0 \leq p_{j,\alpha} \leq \infty ((j,\alpha) \in \Lambda)$. Set

$$\Lambda(\hat{u}) = \{(j,\alpha) \in \Lambda ; 0 \leq p_{j,\alpha} < \infty\},$$

$$\mathcal{Z}(\hat{u}) = \{(j,\alpha) \in \Lambda(\hat{u}) ; \text{ the order of zero of } (\partial F/\partial Z_{j,\alpha})(t,x,\Theta \hat{u})$$

$$\text{at } t = 0 \text{ is well-defined}\}.$$  

If $(j,\alpha) \in \Lambda(\hat{u})$, we have

$$p_{j,\alpha} = \gamma_t((\partial F/\partial Z_{j,\alpha})(t,x,\Theta \hat{u}_N(t,x)))$$

for any sufficiently large $N$. If $(j,\alpha) \in \mathcal{Z}(\hat{u})$, there are $a^0_{j,\alpha}(x) \in \mathcal{O}(D_R)$ with $a^0_{j,\alpha}(x) \not\equiv 0$ and $\epsilon > 0$ such that

$$(\partial F/\partial Z_{j,\alpha})(t,x,\Theta \hat{u}_N(t,x)) = a^0_{j,\alpha}(x)t^{p_{j,\alpha}} + O(t^{p_{j,\alpha}+\epsilon}) \quad \text{(4.2)}$$

(as $S_J \ni t \to 0$) uniformly on $D_R$ for any sufficiently large $N$.

For $(a,b) \in \mathbb{R}^2$ we write $C(a,b) = \{(x,y) \in \mathbb{R}^2 ; x \leq a, y \geq b\}$, and for $a \in \mathbb{R}$ we write $C(a,\infty) = \emptyset$. Under the assumption

$$\Lambda(\hat{u}) \neq \emptyset, \quad (4.3)$$

the Newton polygon $\mathcal{N}((4.1),\hat{u})$ of the equation (4.1) along the formal solution $\hat{u}(t,x)$ is
defined by the convex hull of the union of sets $C(j + |\alpha|, p_{j,\alpha}) \ (j, \alpha) \in \Lambda)$ in $\mathbb{R}^2$; that is,

$$\mathcal{N}((4.1), \hat{u}) = \text{the convex hull of } \bigcup_{(j, \alpha) \in \Lambda} C(j + |\alpha|, p_{j,\alpha}).$$

The figure of $\mathcal{N}((4.1), \hat{u})$ can be drawn as in Figure 1.

As is seen in Figure 1, we set $p = \min\{p_{j,\alpha} \ ; \ (j, \alpha) \in \Lambda\}$, the vertices of $\mathcal{N}((4.1), \hat{u})$ consists of $p^* + 1$ points

$$(l_0, e_0), \ (l_1, e_1), \ (l_2, e_2), \ \cdots, \ (l_{p^*-1}, e_{p^*-1}), \ (l_{p^*}, e_{p^*}),$$

and the boundary of $\mathcal{N}((4.1), \hat{u})$ consists of a horizontal half line $\Gamma_0$, $p^*$-segments $\Gamma_1, \Gamma_2, \ldots, \Gamma_{p^*}$, and a vertical half line $\Gamma_{p^*+1}$. We denote the slope of $\Gamma_i$ by $k_i$ ($i = 0, 1, 2, \ldots, p^* + 1$); then we have

$$k_0 = 0 < k_1 < k_2 < \cdots < k_{p^*} < k_{p^*+1} = \infty.$$
Note that $\Gamma$ is a part of the boundary of $\mathcal{N}((4.1), \hat{u})$ that excludes the part $\Gamma_{p^*+1} \setminus \{(l_{p^*}, e_{p^*})\}$. The most important assumption in this paper is:

$$(j, \alpha) \in \Lambda(\Gamma) \implies \alpha = 0 \in \mathbb{N}^K. \quad (4.4)$$

This means that on the boundary $\Gamma$ the linearized equation of (4.1) along the given formal solution $\hat{u}(t, x)$ behaves like an ordinary differential equation in $t$. By (4.4) we have $\Lambda(\Gamma) = \{(j_1, 0), (j_2, 0), \ldots, (j_0, 0)\}$ with $0 \leq j_1 < j_2 < \cdots < j_0 = (l_{p^*})$. In addition, we have $(l_{0}, 0) \in \Lambda(\Gamma)$ and $p_{l_{i}, 0} = e_i$ for $i = 0, 1, \ldots, p^*$.

We consider Problem 2.4 under (A1), (A2), (4.3), (4.4) and the following conditions.

(a1) $|J| = \theta_2 - \theta_1 > \pi/k_1$, where $J$ is the one in (A2).

(a2) $(m, 0) \in \Lambda(i)$.

(a3) $e_i \in \mathbb{Q}$ ($i = 0, 1, \ldots p^*$).

(a4) If $(j, 0) \in \Lambda(\Gamma)$ we have $(j, 0) \in \mathcal{P}(\hat{u})$.

(a5) $a_{l_{i}, 0}^0(0) \neq 0$ ($i = 0, 1, \ldots, p^*$), where $a_{l_{i}, 0}^0(0)$ is the one in (4.2).

Under (4.4), the condition (a2) is equivalent to the condition $l_{p^*} = m$. The condition (a3) means that every vertex of the Newton polygon is a rational point and so we have $k_i \in \mathbb{Q}$ ($1 \leq i \leq p^*$). By the condition (a4), if $(j, 0) \in \Lambda(\Gamma)$ we have $a_{j, 0}^0(x) \in \mathcal{O}(D_R)$ with $a_{j, 0}^0(x) \neq 0$ and $\epsilon_{j, 0} > 0$ such that

$$$(\partial F/\partial Z_j)(t, x, \Theta\hat{u}_N(t, x)) = a_{j, 0}^0(x)t^{p_j, 0} + \mathcal{O}(t^{p_j, 0 + \epsilon_{j, 0}}) \quad (4.5)$$

(as $S_J \ni t \to 0$) for any sufficiently large $N$. We may suppose: $\epsilon_{j, 0} \in \mathbb{Q}$ ($(j, 0) \in \Lambda(\Gamma)$). Besides, if $(j, 0) \in \Lambda(\Gamma)$ we have $p_{j, 0} \in \mathbb{Q}$ and

$$p_{j, 0} = \begin{cases} p, & \text{if } j \leq l_0, \\ e_i, & \text{if } j = l_i \text{ for some } i, \\ e_{i-1} + k_i(j - l_{i-1}), & \text{if } l_{i-1} < j < l_i \end{cases} \quad (4.6)$$

(we note that $y = e_{i-1} + k_i(x - l_{i-1})$ is a line containing the segment $\Gamma_i$).

In addition, if $(j, \alpha) \in \Lambda \setminus \Lambda(\Gamma)$ and if $N \in \mathbb{N}$ is sufficiently large we have the expression

$$(\partial F/\partial Z_j, \alpha)(t, x, \Theta\hat{u}_N(t, x)) = \mathcal{O}(t^{p_{j, \alpha}})$$

(as $S_J \ni t \to 0$) for some $p_{j, \alpha}^* > 0$ (which is independent of $N$) satisfying

$$p_{j, \alpha}^* = \begin{cases} p, & \text{if } j + |\alpha| \leq l_0, \\ e_i, & \text{if } j + |\alpha| = l_i \text{ for some } i, \\ e_{i-1} + k_i(j + |\alpha| - l_{i-1}), & \text{if } l_{i-1} < j + |\alpha| < l_i. \end{cases} \quad (4.7)$$

Without loss of generality, we may suppose: $p_{j, \alpha}^* \in \mathbb{Q}$.  

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4.5. Characteristic polynomial on $\Gamma_i$

In the case $p^* \geq 1$, let us define the characteristic polynomial on $\Gamma_i$ ($i = 1, 2, \ldots, p^*$).

For $i = 1, 2, \ldots, p^*$, we set

$$
\Lambda(\Gamma_i) = \{(j, 0) \in \Lambda(\Gamma); (j, p_j, 0) \in \Gamma_i\},
$$

$$
p_i(\lambda, x) = \sum_{(j, 0) \in \Lambda(\Gamma_i)} a_{j, 0}(x) \lambda^{l_i - l_{i-1}} = a_{l_i, 0}(x) \lambda^{l_i - l_{i-1}} + \cdots + a_{l_{i-1}, 0}(x),
$$

$$
P_i(\lambda, x) = \sum_{(j, 0) \in \Lambda(\Gamma_i)} a_{j, 0}(x) \lambda^j = p_i(\lambda, x) \times \lambda^{l_i - l_{i-1}}.
$$

We note: $(l_{i-1}, 0) \in \Lambda(\Gamma_i)$, $(l_i, 0) \in \Lambda(\Gamma_i)$, $a_{l_i, 0}(0) \neq 0$ and $a_{l_{i-1}, 0}(0) \neq 0$ (by the condition $(a_5)$). We also note: if $(j, 0) \in \Lambda(\Gamma_i)$ we have $p_j, 0 = e_{i-1} + k_i(j - l_{i-1})$. We call $p_i(\lambda, x)$ the characteristic polynomial on $\Gamma_i$ and we denote by

$$
\lambda_{i, 1}, \ldots, \lambda_{i, l_i - l_{i-1}}
$$

the roots of $p_i(\lambda, 0) = 0$ which are called the characteristic roots on $\Gamma_i$ at $x = 0$. Since $a_{l_i, 0}(0) \neq 0$ and $a_{l_{i-1}, 0}(0) \neq 0$ hold, we have

$$
\lambda_{i, d} \neq 0 \quad \text{for all } 1 \leq i \leq p^* \text{ and } 1 \leq d \leq l_i - l_{i-1}.
$$

We set

$$
\Xi_i = \bigcup_{d=1}^{l_i - l_{i-1}} \left\{ \frac{\arg \lambda_{i, d} + 2\pi h}{k_i}; \quad h = 0, \pm 1, \pm 2, \ldots \right\}, \quad 1 \leq i \leq p^*,
$$

$$
\mathcal{Z}_i = \mathbb{R} \setminus \Xi_i, \quad 1 \leq i \leq p^*.
$$

These sets play a very important role in determining the direction in which the formal solution $\hat{u}(t, x)$ is summable. We denote by $p: \mathbb{R}(\mathbb{C} \setminus \{0\}) \rightarrow \mathbb{C}$ the natural projection. It is easy to see:

**Lemma 4.6.** Let $I$ be a non-empty open interval. For $i = 1, 2, \ldots, p^*$, the following two conditions 1) and 2) are equivalent.

1) $I \subseteq \mathcal{Z}_i$.

2) $0 < |I| < 2\pi/k_i$ and $\lambda_{i, d} \in \mathbb{C} \setminus \overline{p(S_k, I)}$ for $d = 1, \ldots, l_i - l_{i-1}$.

4.6. Main theorems

Since our equation is defined on $\mathcal{K}$, we always suppose in this paper that the constants $r$ and $R$ are sufficiently small with $0 < r \leq r_0$, $0 < R \leq R_0$ and also $0 < R \leq 1$.

In the case $p^* = 0$, the solution to Problem 2.4 is as follows:

**Theorem 4.7 (The case $p^* = 0$).** Suppose the conditions $(A_1)$ and $(A_2)$. Let

$$
\hat{u}(t, x) = \sum_{n \geq 1} u_n(t, x)
$$

be a formal solution of (4.1) on $S_J(r) \times D_R$ (for some $J$, $r > 0$ and $R > 0$). Suppose the conditions (4.3), $p^* = 0$, (4.4) and $(a_2)$, $(a_4)$, $(a_5)$. Take any $J_1 \in J$ and $0 < R_1 < R$. 


Then, we can find a new formal solution \( \tilde{w}(t, x) \) on \( S_{J_1}(r_1) \times D_{R_1} \) for some \( r_1 > 0 \), that satisfies the following:

1. \( \tilde{w}(t, x) \) is equivalent to \( \hat{u}(t, x) \) on \( S_{J_1}(r_1) \times D_{R_1} \).
2. \( \tilde{w}(t, x) \) is convergent to a true holomorphic solution \( w(t, x) \) of (4.1) on \( S_{J_1}(\delta) \times D_{R_1} \) for some \( \delta > 0 \).

In the case \( p^* \geq 1 \), we let \( k_i (1 \leq i \leq p^*) \) be as in Figure 1. For \( i < j \) we define \( \kappa_{i,j} > 0 \) by the relation: \( 1/\kappa_{i,j} = 1/k_i - 1/k_j \). Also, we set \( \kappa_{p^*, p^*} = \infty \) and so \( 1/\kappa_{p^*, p^*} = 0 \).

As before, for \( K = (\phi_1, \phi_2) \) and \( a > 0 \) we write \( K + [a] = (\phi_1 - a, \phi_2 + a) \). Let \( U \) be an open subset of \( \mathbb{R} \) that can be expressed as a disjoint union of non-empty open intervals \( K_j (j = 1, 2, \ldots) \); then we have \( U = \bigcup_{j \geq 1} K_j \). In this case, the notation \( I \in U \oplus [a] \) means that \( I \in K_j + [a] \) holds for some \( j \geq 1 \). For an open set \( A \) we write \( I \in A \cap (U \oplus [a]) \) if \( I \in A \) and \( I \in U \oplus [a] \) hold. Similarly, we write

\[
I \in A \cap (U_1 \oplus [a_1]) \cap (U_2 \oplus [a_2])
\]

if \( I \in A \), \( I \in U_1 \oplus [a_1] \) and \( I \in U_2 \oplus [a_2] \) hold.

Then, we have the following solution to Problem 2.4:

**Theorem 4.8 (The case \( p^* \geq 1 \)).** Suppose the conditions (A1) and (A2). Let \( \hat{u}(t, x) \) be a formal solution of (4.1) on \( S_{J_1}(r) \times D_R \) (for some \( J, r > 0 \) and \( R > 0 \)). Suppose the conditions (4.3), \( p^* \geq 1 \), (4.4) and (a1) \( \sim (a_3) \). Take any \( J_1 \in J \) (with \( |J_1| > \pi/k_1 \) and \( 0 < R_1 < R \)). Then, we can find a new formal solution \( \tilde{w}(t, x) \) on \( S_{J_1}(r_1) \times D_{R_1} \) for some \( r_1 > 0 \), that satisfies the following:

1. \( \tilde{w}(t, x) \) is equivalent to \( \hat{u}(t, x) \) on \( S_{J_1}(r_1) \times D_{R_1} \).
2. For any non-empty open interval \( I \) satisfying \( I + [\pi/2k_1] \in J_1 \) and

\[
I \in \bigcap_{i=1}^{p^*} \mathcal{I}_i \oplus [\pi/2\kappa_{i,p^*}],
\]

there is a \( \rho > 0 \) such that the series \( \hat{w}(t, x) \) is \( (k_{p^*}, \ldots, k_1) \)-multisummable in \( I \)-direction in \( t \) (uniformly in \( x \in D_\rho \)). In addition, the \( (k_{p^*}, \ldots, k_1) \)-multisum \( \hat{w}(t, x) \) is a true holomorphic solution of (4.1) on \( S_{J_1}(\delta) \times D_{R_1} \) for some \( \delta > 0 \).

If \( p^* = 1 \), (4.8) is nothing but \( I \in \mathcal{I}_1 \). If \( p^* \geq 2 \), as is proved in [Proposition 4.6 in [27]], the meaning of the condition (4.8) is as follows:

**Proposition 4.9.** Let \( I \) be a non-empty open interval. The following two conditions 1) and 2) are equivalent:

1. \( I \) satisfies (4.8).
2. We have \( I \in \mathcal{I}_{p^*} \), and we can find non-empty open intervals \( I_i \in \mathcal{I}_i \) (\( i = 1, 2, \ldots, p^* - 1 \)) such that

\[
I_{i+1} = I_i + [\pi/2\kappa_{i,i+1}], \quad i = 1, 2, \ldots, p^* - 1
\]

with \( I_{p^*} = I \). In addition, we may suppose that \( I_i \cap I \) (\( 1 \leq i \leq p^* - 1 \)) hold.

In the case \( p^* = 0 \), under the situation in Theorem 4.7 we set \( J_2 = J_1 \) and \( \rho = R_1 \).
In the case \( p^* \geq 1 \), under the situation in Theorem 4.8 we set \( J_2 = I + [\pi/2k_p] \). Then, as a consequence of Theorems 4.7 and 4.8, we have

**Theorem 4.10 (Asymptotic existence theorem).** Let \( \hat{u}(t,x) \) be a formal solution of (4.1). Then, under the situation in Theorem 4.7 (in the case \( p^* = 0 \)) or Theorem 4.8 (in the case \( p^* \geq 1 \)), we have a true holomorphic solution \( u(t,x) \) on \( S_{J_2}(\delta) \times D_R \) that admits \( \hat{u}(t,x) \) as an asymptotic expansion (as \( t \to 0 \)) in the following sense: for any \( A > 0 \) there is an \( N_0 \) such that for any \( N \geq N_0 \) we have

\[
\| u(t,x) - \hat{u}_N(t,x) \|_p = O(|t|^A) \quad (as \ S_{J_2} \ni t \to 0).
\]

5. A reduction

In order to prove Theorems 4.7 and 4.8, let us do a reduction of the problem. Recall that we have a formal solution \( \hat{u}(t,x) = \sum_{n \geq 1} u_n(t,x) \) of (4.1) on \( S_J(r) \times D_R \). Let \( \mu_n > 0 \) \( (n = 1,2,\ldots) \) be the sequence appearing in Definition 2.1. As before, we write

\[
\hat{u}_N(t,x) = \sum_{n=1}^N u_n(t,x) \quad (N = 1,2,\ldots).
\]

[Step 1]. Set

\[
C_0(\lambda,x) = \sum_{(j,0) \in \Lambda(T_n)} a_{j,0}(x) \lambda^j
\]

(where \( a_{j,0}(x) \) are the ones in (4.5)), and let \( \lambda_{0,1}(x),\ldots,\lambda_{0,q}(x) \) be the roots of \( C_0(\lambda,x) = 0 \). Let \( p = \min \{ p_{j,\alpha} : (j,\alpha) \in \Lambda \} \) be as before.

First, we take any \( c > 0 \), and take \( q \in \mathbb{N}^* \) sufficiently large so that the following two conditions are satisfied:

\[
q \geq \max \left[ \max_{1 \leq i \leq p^*-1} \left( (2k_i + k_{i+1})m + p \right), \ 2k_p m + p + 1 \right], \quad (5.1)
\]

\[
q \geq \Re \lambda_{0,d}(x) + c \quad \text{on} \ D_R, \quad 1 \leq d \leq l_0. \quad (5.2)
\]

Next, we choose a rational number \( A_0 > 0 \) so that \( A_0 > p+q \) holds. Then, we take \( N_0 \in \mathbb{N} \) sufficiently large so that \( \mu_{N_0+1} > q, \| F(t,x,\Theta \hat{u}_{N_0}) \|_R = O(|t|^{A_0}) \) (as \( S_J \ni t \to 0 \)), and (4.5), (4.6), (4.7) are well-defined for \( N = N_0 \).

Under this situation, we divide our formal solution into

\[
\hat{u}(t,x) = \hat{u}_{N_0}(t,x) + t^q \tilde{V}(t,x) \quad \text{with} \ \tilde{V}(t,x) = \sum_{n \geq N_0+1} t^{-q} u_n(t,x).
\]

Then, \( \tilde{V}(t,x) \) is a formal solution of

\[
F\left( t,x,\Theta \hat{u}_{N_0}(t,x) + \Theta t^q v \right) = 0
\]

with respect to the unknown function \( v(t,x) \), that is,

\[
0 = F(t,x,\Theta \hat{u}_{N_0}) + t^q \sum_{(j,\alpha) \in \Lambda} \left( \partial F/\partial Z_{j,\alpha}(t,x,\Theta \hat{u}_{N_0}) \right) (t\partial_t + q)^j \partial_x^\alpha v
\]

\[
+ \sum_{|\nu| \geq 2} t^{|\nu|} \frac{1}{\nu!} \left( \partial F/\partial Z_{\nu} \right) (t,x,\Theta \hat{u}_{N_0}) \prod_{(j,\alpha) \in \Lambda} \left( (t\partial_t + q)^j \partial_x^\alpha v \right)^{\nu_{j,\alpha}},
\]

where \( Z_{\nu} = \sum_{\nu_{j,\alpha} \geq 1} \nu_{j,\alpha} \partial_x^\alpha \).
where $\nu = \{\nu_{j,\alpha}\}_{(j,\alpha) \in \Lambda} \in \mathbb{N}^N$, $|\nu| = \sum_{(j,\alpha) \in \Lambda} \nu_{j,\alpha}$ and $\nu! = \prod_{(j,\alpha) \in \Lambda} \nu_{j,\alpha}!$. Therefore, by setting

$$ f^*(t, x) = \frac{1}{t^p q} F(t, x, \Theta \tilde{u}_{N_0}), $$

$$ a_{j,\alpha}^*(t, x) = (-1) \times \frac{1}{t^p} (\partial F/\partial Z_{j,\alpha})(t, x, \Theta \tilde{u}_{N_0}), \quad (j, \alpha) \in \Lambda, $$

$$ b^*_j(t, x) = \frac{1}{\nu!} (\partial^{\nu} F/\partial Z^\nu)(t, x, \Theta \tilde{u}_{N_0}), \quad |\nu| \geq 2 $$

and by cancelling the factor $t^{p+q}$ we have the equation

$$ \sum_{(j,\alpha) \in \Lambda} a^*_{j,\alpha}(t, x)(t\partial_t + q)^j \partial_x^\nu v $$

$$ = f^*(t, x) + \sum_{|\nu| \geq 2} b^*_j(t, x) t^{|\nu|-p} \prod_{(j,\alpha) \in \Lambda} ((t\partial_t + q)^j \partial_x^\nu v)^{\nu_{j,\alpha}}. \quad (5.3) $$

Since $\tilde{V}(t, x)$ is a formal solution of this equation, instead of treating (4.1) it is sufficient to treat the equation (5.3) with respect to the unknown function $v(t, x)$.

From now, by the notation $O(t^n)$ we mean that it is a holomorphic function $g(t, x)$ on $S_J(r) \times D_R$ satisfying $\|g(t)\|_R = O(|t|^n)$ (as $S_J \ni t \rightarrow 0$).

[Step 2]. By taking $r$ a smaller one if necessary, we see that $f^*(t, x)$, $a^*_{j,\alpha}(t, x)$ ($j, \alpha \in \Lambda$) and $b^*_j(t, x)$ ($|\nu| \geq 2$) are all holomorphic on $S_J(r) \times D_R$, that $|f^*(t, x)| \leq F|t|^{A_0-p-q}$ on $S_J(r) \times D_R$ for some $F \geq 0$, and that $|b^*_j(t, x)| \leq B^*_j$ on $S_J(r) \times D_R$ for some $B^*_j \geq 0$ ($|\nu| \geq 2$) satisfying

$$ \sum_{|\nu| \geq 2} B^*_j X^{|\nu|} \in \mathbb{C}[X] $$

(where $\mathbb{C}[X]$ denotes the ring of convergent power series in $X$). In addition, we have

$$ \mathcal{N}((4.1), \tilde{u})) = (0, p) = \text{the convex hull of } \bigcup_{(j,\alpha) \in \Lambda} C(j + |\alpha|, \gamma_t(a^*_{j,\alpha})), $$

and so we have

**Lemma 5.1.**  (1) If $(j, 0) \in \Lambda(\Gamma)$ we have the expression

$$ a^*_{j,0}(t, x) = a^0_{j,0}(x)t^{p_j,0-p} + O(t^{p_j,0-p + \epsilon_j,0}), $$

where $a^0_{j,0}(x)$, $p_{j,0}$ and $\epsilon_{j,0} > 0$ are the same as in (4.5). By (a$_{5}$) we have $a^0_{j,0}(0) \neq 0$ for $i = 0, 1, \ldots, p^*$.

(2) If $(j, \alpha) \in \Lambda \setminus \Lambda(\Gamma)$ we have $a^*_{j,\alpha}(t, x) = O(t^{p^*_{j,\alpha}-p})$ for some $p^*_{j,\alpha} \in \mathbb{Q}$ satisfying (4.7).

[Step 3]. Since the left-hand side of (5.3) (L.H.S. of (5.3)) is expressed as

$$ \text{L.H.S. of (5.3)} = \sum_{(j,0) \in \Lambda(\Gamma_0)} (a^0_{j,0}(x) + O(t^{\epsilon_j,0}))(t\partial_t + q)^j v $$

In sections 6–8, we will prove Theorems 4.7 and 4.8, by solving this problem. In the next section 6, we construct a formal solution \( \hat{v}(t, x) \) such that the formal series

\[
\hat{v}(t, x) = \hat{u}_{N_0}(t, x) + t^p \hat{v}(t, x)
\]

is a formal solution of (4.1) which is equivalent to the original formal solution \( \hat{u}(t, x) \), and show that this formal solution \( \hat{v}(t, x) \) satisfies a Gevrey type estimate. In the case \( p^* = 0 \), Theorem 4.7 can be obtained immediately from this estimate. In the case \( p^* \geq 1 \),
in section 7 we show that the formal Borel transform \( w_1(\xi, x) = \hat{B}_k[\hat{v}](\xi, x) \) of the formal solution \( \hat{v}(t, x) \) satisfies a convolution equation in the Borel plane, and in section 8, by using a result in [24], we show that this formal solution \( \hat{v}(t, x) \) is multisummable in the direction \( I \) (satisfying (4.7)). This leads us to Theorem 4.8.

6. On a new formal solution of (5.4)

In this section, we construct a new formal solution \( \hat{v}(t, x) \) of the equation (5.4), and investigate some properties of \( \hat{v}(t, x) \).

We set \( \mu_0 = A_0 - p - q; \) then \( \mu_0 \) is a positive rational number and

\[
|f^*(t, x)| \leq F|t|^\mu_0 \quad \text{on } S_J(r) \times D_R
\]

for some \( F > 0 \). Since \( c_i (0 \leq i \leq p^* \) in (a3), \( k_i (0 \leq i \leq p^* \), \( \gamma_{j, \alpha} ((j, \alpha) \in \Lambda) \) and \( \eta_{j,0} ((j, 0) \in \Lambda(\Gamma) \setminus \Lambda(\Gamma_0)) \) in (b1) are all rational numbers, we can take an \( L \in \mathbb{N}^* \) such that \( \mu_0 \in \mathbb{N}^*/L, c_i \in \mathbb{N}/L \) (0 \( \leq i \leq p^* \), \( k_i \in \mathbb{N}/L \) (0 \( \leq i \leq p^* \), \( \gamma_{j,\alpha} \in \mathbb{N}^*/L ((j, \alpha) \in \Lambda) \) and \( \eta_{j,0} \in \mathbb{N}^*/L ((j, 0) \in \Lambda(\Gamma) \setminus \Lambda(\Gamma_0)) \) hold. We note: \( p = c_0 \in \mathbb{N}/L, \) and \( q \in \mathbb{N}^* \subset \mathbb{N}^*/L. \)

6.1. Results on (5.4)

First, to clarify the purpose of this section, we describe the main result for equation (5.4) without proof. We set

\[
m_\nu = \max\{j + |\alpha| ; \nu_{j, \alpha} > 0\} \quad \text{for } \nu = \{\nu_{j, \alpha}\}_{j+|\alpha| \leq m} \in \mathbb{N}^N,
\]

\[
A_a = \{(j, \alpha) \in \Lambda; j + |\alpha| \geq l_0 + 1, a_{j,\alpha}(t, x) \neq 0\};
\]

\[
A_b = \{\nu \in \mathbb{N}^N ; |\nu| \geq 2, m_\nu \geq l_0 + 1, b_\nu(t, x) \neq 0\};
\]

\[
s_a = 1 + \max_{(j, \alpha) \in A_a} \left( \frac{j + |\alpha| - l_0}{\gamma_{j,\alpha}} \right);
\]

\[
s_b = 1 + \max_{\nu \in A_b} \left( \frac{m_\nu - l_0}{(q(|\nu| - 1) - p) + \mu_0(|\nu| - 1)} \right),
\]

and set \( s_0 = \max\{s_a, s_b\} \). The purpose of this section is to prove the following result.

**Theorem 6.1.** (1) We can construct a series of functions

\[
\hat{v}(t, x) = \sum_{n \geq 1} v_n(t, x), \quad v_n(t, x) \in \mathcal{O}(S_J(r) \times D_R) \quad (n = 1, 2, \ldots)
\]

such that \( \hat{v}(t, x) \) is a formal solution of (5.4) on \( S_{J_1}(r_1) \times D_{R_1} \) for any \( J_1 \in J, 0 < r_1 < r \) and \( 0 < R_1 < R \).

(2) In addition, the functions \( v_n(t, x) \) (\( n \geq 1 \)) constructed in (1) satisfy the following:

for any \( J_1 \in J, 0 < r_1 < r \) and \( 0 < R_1 < R \), there are constants \( C > 0 \) and \( h > 0 \) such that

\[
|{(t\partial_t)^i \partial_x^v v_n(t, x)}| \leq Ch^n(n!)^{s_0-1}|t|^{1/L} \quad \text{on } S_{J_1}(r_1) \times D_{R_1}
\]

holds for any \( (j, \alpha) \in \Lambda \) and \( n \geq 1. \)
is another formal solution of (5.4) on \( S_{J_1}(r_1) \times D_{R_1} \), then it is equivalent to \( \hat{v}(t, x) \) on \( S_{J_2}(r_2) \times D_{R_2} \) for any \( J_2 \in J_1, \ 0 < r_2 < r_1 \) and \( 0 < R_2 < R_1 \).

The proof of this result will be given in subsections 6.3 ~ 6.6. By this theorem we have

**Corollary 6.2.**
1. If \( p^* = 0 \) holds, for any \( J_1 \in J \) and \( 0 < R_1 < R \) the formal solution \( \hat{v}(t, x) \) is convergent on \( S_{J_1}(\delta) \times D_{R_1} \) for some \( 0 < \delta < r \). This proves Theorem 4.7.
2. If \( p^* \geq 1 \) holds, for any \( J_1 \in J \) and \( 0 < R_1 < R \) the series

\[
\sum_{n \geq 1} v_n(t, x) \Gamma(n/Lk_1)
\]

is convergent on \( S_{J_1}(\delta) \times D_{R_1} \), for some \( 0 < \delta < r \).

**Proof.** Let us show (1). In the case \( p^* = 0 \), we have \( l_0 = m \) and so \( \Delta_a = \emptyset \) and \( \Delta_b = \emptyset \). This means that \( s_a = 1 \) and \( s_b = 1 \) hold, that is, \( s_0 = 1 \) holds. Therefore, by (2) of Theorem 6.1 we have the estimate

\[
|(t\partial_t)^j \partial_x^n v_n(t, x)| \leq C n!^{1/Lk_1} |t|^{n/L} \text{ on } S_{J_1}(r_1) \times D_{R_1}
\]

for any \((j, \alpha) \in \Lambda \) and \( n \geq 1 \). This proves (1).

Next, let us show (2). By the conditions \( c_0 = p, (b_1) \) and \( (b_2) \) (with \( i = 1 \)) we have \((l_1, 0) \in \Delta_a \) and \( s_a = 1 + 1/k_1 \). By (5.1) we have \( q > k_1m + p \) and so we have

\[
\frac{1}{k_1} > \frac{m}{q-p} > \frac{m_0 - l_0}{q(|\nu| - 1) - p + \mu_0(|\nu| - 1)}, \quad |\nu| \geq 2;
\]

this implies \( s_b < 1 + 1/k_1 \). Therefore, we have \( s_0 - 1 = 1/k_1 \), and

\[
|(t\partial_t)^j \partial_x^n v_n(t, x)| \leq C n^{1/Lk_1} |t|^{n/L} \text{ on } S_{J_1}(r_1) \times D_{R_1}
\]

for any \((j, \alpha) \in \Lambda \) and \( n \geq 1 \). This proves (2).  

6.2. **Settings (again)**

To simplify the notation, we set

\[
C(\lambda, x) = C_0(\lambda + q, x), \quad \mu = \mu_0L \in \mathbb{N}^+, \quad \sigma_{j, \alpha} = \gamma_{j, \alpha}L \in \mathbb{N}^+ ((j, \alpha) \in \Lambda), \quad q_\nu = (q(|\nu| - 1) - p)L \in \mathbb{N}^+ \ (|\nu| \geq 2), \quad \beta_\nu(t, x) = b_\nu(t, x) u^{q(|\nu| - 1) - p} \quad (|\nu| \geq 2) \text{ and } d = 1/L.
\]

Then, the equation (5.4) is written as

\[
C(t\partial_t, x)v + \sum_{(j, \alpha) \in \Lambda} a_{j, \alpha}(t, x)(t\partial_t)^j \partial_x^n v
= f^*(t, x) + \sum_{|\nu| \geq 2} \beta_\nu(t, x) \prod_{(j, \alpha) \in \Lambda} (t\partial_t)^j \partial_x^n v)^{\nu, \alpha}.
\]  

(6.3)
In the proof given later, we deal with this equation, instead of (5.4).

By (5.2) we know that the roots \( \lambda_1(x), \ldots, \lambda_{l_0}(x) \) of \( C(\lambda, x) = 0 \) in \( \lambda \) satisfies

\[
\text{Re} \lambda_i(x) \leq -c \quad \text{on} \quad D_R, \quad 1 \leq i \leq l_0.
\]  

(6.4)

As before, \( f^*(t, x), a_{j,\alpha}(t, x) \) \( ((j, \alpha) \in \Lambda) \) and \( \beta_\nu(t, x) \) \( (|\nu| \geq 2) \) are holomorphic functions on \( S_J(r) \times D_R \) and we have the following conditions:

\[
|f^*(t, x)| \leq F |t|^{d_{\mu}} \quad \text{on} \quad S_J(r) \times D_R,
\]

\[
|a_{j,\alpha}(t, x)| \leq A_{j,\alpha} |t|^{d_{\sigma_{j,\alpha}}} \quad \text{on} \quad S_J(r) \times D_R, \quad (j, \alpha) \in \Lambda,
\]

\[
|\beta_\nu(t, x)| \leq B_\nu |t|^{d_{\eta_\nu}} \quad \text{on} \quad S_J(r) \times D_R, \quad |\nu| \geq 2,
\]

\[
\sum_{|\nu| \geq 2} B_\nu e^{\nu_r X} |\nu| \in C[z, X].
\]

If \( a_{j,\alpha}(t, x) \equiv 0 \) we can set \( A_{j,\alpha} = 0 \), and also if \( \beta_\nu(t, x) \equiv 0 \) we can set \( B_\nu = 0 \). Under these notations, we set

\[
s_{1,\alpha} = 1 + \max\left[ 0, \max_{(j, \alpha) \in \Delta_\alpha} \left( \frac{j + |\alpha| - l_0}{\sigma_{j,\alpha}} \right) \right],
\]

\[
s_{1,b} = 1 + \max\left[ 0, \max_{\nu \in \Delta_\nu} \left( \frac{m_\nu - l}{\eta_\nu + \mu(|\nu| - 1)} \right) \right],
\]

and set \( s_{1,0} = \max\{s_{1,\alpha}, s_{1,b}\} \).

Then, to prove (6.2) it is enough to show that

\[
|(t\partial_t)^j \partial_x^\alpha v_n(t, x)| \leq Ch^n n!^{1 - j} |t|^{d_n} \quad \text{on} \quad S_{J_1}(r_1) \times D_{R_1}
\]

holds for any \( (j, \alpha) \in \Lambda, n \geq 1 \) and \( s \geq s_{1,0} \).

### 6.3. Some basic tools

To prove Theorem 6.1, let us prepare two lemmas. Let \( J = (\theta_1, \theta_2) \) with \( 0 < |J| < \infty, \ 0 < r \leq \infty \) and \( R > 0 \). We define \( d(t) \) and \( d(t, \rho) \) by

\[
d(t) = \min\{ \log r - \log |t|, \theta_2 - \arg t, \arg t - \theta_1 \}, \quad t \in S_J(r),
\]

\[
\frac{1}{d(t, \rho)} = \frac{|J|/2}{d(t)(R - \rho)}, \quad (t, \rho) \in S_J(r) \times (0, R).
\]

Since \( 0 < R \leq 1 \) is supposed and \( d(t) \leq |J|/2 \) holds, we have \( 1/d(t, \rho) > 1 \).

**Lemma 6.3.** Let \( u(t, x) \) be a holomorphic function on \( S_J(r) \times D_R \), and suppose that

\[
\|u(t)\|_\rho \leq \frac{A |t|^n}{d(t, \rho)^a}, \quad \forall (t, \rho) \in S_J(r) \times (0, R)
\]

holds for some \( A > 0, n \geq 0 \) and \( a \geq 0 \). Then, we have

\[
\|t^{\partial_t} u(t)\|_\rho \leq \frac{(a + 1) c(2/|J|) + n) A |t|^n}{d(t, \rho)^{a + 1}}, \quad \forall (t, \rho) \in S_J(r) \times (0, R),
\]
\[ \|\partial_{x_i} u(t)\|_\rho \leq \frac{(a + 1)eA|t|^n}{d(t, \rho)^{n+1}}, \quad \forall(t, \rho) \in S_J(r) \times (0, R), \quad i = 1, \ldots, K. \]

Proof. In the case \( d(t, \rho) = (R - \rho) \), the estimate of \( \|\partial_{x_i} u(t)\|_\rho \) is well-known as Nagumo’s lemma (see Walter [28], Nagumo [19], or [Lemma 5.1.4 in Hörmander [14]]). In the case \( d(t, \rho) = d(t) \) and \( n = 0 \), the estimate of \( \|\partial_{x_i} u(t)\|_\rho \) is just the one given in [Lemma 4.2 in Bacani-Tahara [1]]. In the present case, the proof is as follows.

In the case \( n = 0 \), by [Lemma 4.2 in [1]] and the condition \( 1/(R - \rho) > 1 \) we have

\[
\|t\partial_t u(t)\|_\rho \leq \frac{(|J|/2)^a(1 + a)eA}{d(t, \rho)^{a+1}(R - \rho)^a} \lesssim \frac{(1 + a)e(2/|J|)A}{d(t, \rho)^{a+1}}
\]

for any \((t, \rho) \in S_J(r) \times (0, R)\). In the case \( n > 0 \), by setting \( w(t, x) = t^{-n}u(t, x) \) we can reduce it to the case \( n = 0 \), and we have

\[
\|t\partial_t u(t)\|_\rho = \|t^n(t\partial_t w(t) + nw(t))\|_\rho \leq |t|^n(\|t\partial_t w(t)\|_\rho + n\|w(t)\|_\rho) \\
\leq |t|^n \left( \frac{(a + 1)e(2/|J|)A}{d(t, \rho)^{a+1}} + \frac{nA}{d(t, \rho)^a} \right) \lesssim \frac{(a + 1)e(2/|J|)A|t|^n}{d(t, \rho)^{a+1}}
\]

for any \((t, \rho) \in S_J(r) \times (0, R)\). Similarly, by the usual Nagumo’s lemma and the condition \( d(t) \leq |J|/2 \) we have

\[
\|\partial_{x_i} u(t)\|_\rho \leq \frac{(|J|/2)^a(a + 1)eA|t|^n}{d(t, \rho)^{a+1}(R - \rho)^a} \lesssim \frac{(a + 1)eA|t|^n}{d(t, \rho)^{a+1}}
\]

for any \((t, \rho) \in S_J(r) \times (0, R)\) and \( i = 1, \ldots, K \). \( \square \)

Next, let us consider the equation

\[ C(t\partial_t, x)w = g(t, x) \quad \text{on} \quad S_J(r) \times D_R \quad (6.6) \]

under the condition (6.4) (with \( c > 0 \)). Since \( R > 0 \) is sufficiently small, we may suppose that \( \|1/a_{0, 0}(x)\|_R \leq c_0 \) for some \( c_0 > 0 \). We denote by \( X(S_J(r) \times D_R) \) the set of all functions \( g(t, x) \in \mathcal{O}(S_J(r) \times D_R) \) satisfying the following property: for any \( J_1 \in J, 0 < r_1 < r \) and \( 0 < R_1 < R \) this function \( g(t, x) \) is bounded on \( S_{J_1}(r_1) \times D_{R_1} \).

Lemma 6.4. For any \( g(t, x) \in X(S_J(r) \times D_R) \) the equation (6.6) has a unique solution \( w(t, x) \in X(S_J(r) \times D_R) \). In addition, if \( g(t, x) \) satisfies

\[
\|g(t)\|_\rho \leq \frac{A|t|^n}{d(t, \rho)^a}, \quad \forall(t, \rho) \in S_J(r) \times (0, R)
\]

for some \( A > 0, n \geq 0 \) and \( a \geq 0 \), the unique solution \( w(t, x) \) satisfies

\[
\|w(t)\|_\rho \leq \frac{c_0 A|t|^n}{(c + n)^a d(t, \rho)^a}, \quad \forall(t, \rho) \in S_J(r) \times (0, R). \quad (6.7)
\]

Proof. The former half is well-known: see Baouendi-Goulaouic [7]. Let us show the estimate (6.7) by using the condition (6.4). Since the equation (6.6) is written in the
under the setting \( g_1(t, x) = g(t, x)/a_{0,0}^0(x) \) the unique solution \( w(t, x) \) is given by the formula
\[
w(t, x) = \mathcal{H}_{\lambda_0} \circ \cdots \circ \mathcal{H}_{\lambda_1}[g_1](t, x),
\]
where \( \mathcal{H}_{\lambda_j} \) \((j = 1, \ldots, l_0)\) are defined by
\[
\mathcal{H}_{\lambda_j}[f](t, x) = \int_0^t \left( \frac{\tau}{t} \right)^{-\lambda_j(x)} f(\tau, x) \frac{d\tau}{\tau}.
\]

Take any \((t, \rho) \in S_J(\tau) \times (0, R)\) and \(x \in D_R\) with \(|x| \leq \rho\). Let \( t = |t|e^{\sqrt{\tau}\phi} \) and \( \tau = ye^{\sqrt{\tau}\phi} \). Then we have \( d(\tau) \geq d(t) \) and so \( d(\tau, \rho) \geq d(t, \rho) \). Hence:
\[
|\mathcal{H}_{\lambda_1}[g_1](t, x)| = \left| \int_0^{|t|} \left( \frac{y}{|t|} \right)^{-\lambda_1(x)} g_1(ye^{i\phi}, x) \frac{dy}{y} \right|
\leq \int_0^{|t|} \left( \frac{y}{|t|} \right)^c \frac{c_0 A y^n}{d(\rho^a, \rho)^a} \frac{dy}{y}
\leq \frac{c_0 A}{d(t, \rho)^a |t|^c} \int_0^{|t|} y^{c+n-1} dy = \frac{c_0 A}{d(t, \rho)^a} \frac{|t|^n}{(c+n)}.
\]
This proves (6.7) in the case \( l_0 = 1 \).

By repeating the same argument we have (6.7) for the general \( l_0 \).

### 6.4. Construction of a formal solution

In this section, we construct a formal solution of (6.3) of the form
\[
v(t, x) = \sum_{n \geq \mu} v_n(t, x), \quad v_n(t, x) = O(|t|^{dn}) \quad (\text{as } t \to 0) \quad (n \geq \mu):
\]
we set \( v_p(t, x) = 0 \) for \( p \leq \mu - 1 \).

By substituting this into the equation (6.3) formally, we have
\[
\sum_{n \geq \mu} C(t\partial_t, x)v_n = f^*(t, x) - \sum_{(j, \alpha) \in \Lambda} a_{j,\alpha}(t, x) \sum_{n \geq \mu} (t\partial_t)^j \partial_x^\alpha v_n
\]
\[
+ \sum_{|\nu| \geq 2} \beta_\nu(t, x) \prod_{(j, \alpha) \in \Lambda} \left( \sum_{n \geq \mu} (t\partial_t)^j \partial_x^\alpha v_n \right)^{\nu_{j,\alpha}} \quad (6.8)
\]
as a formal series. Since \( f^*(t, x) = O(|t|^{dn}) \) \((\text{as } t \to 0)\), \( a_{j,\alpha}(t, x) = O(|t|^{dn}) \) \((\text{as } t \to 0)\) \((j, \alpha) \in \Lambda\) and \( \beta_\nu(t, x) = O(|t|^{dn}) \) \((\text{as } t \to 0)\) \(|\nu| \geq 2\) are supposed, by comparing the terms with the asymptotics \( O(|t|^{dn}) \) \((\text{as } t \to 0)\) in the both sides of (6.8) it will be natural to divide our equation (6.8) into the following recurrence formulas:
\[
C(t\partial_t, x)v_\mu = f^*(t, x) \quad (6.9)
\]
and for \( n \geq \mu + 1 \)
\[
C(t \partial_t, x)v_n = - \sum_{(j, \alpha) \in \Lambda} a_{j, \alpha}(t, x)(t \partial_t)^j \partial_x^\alpha v_{n-\sigma, \alpha}
\]
\[
+ \sum_{|\nu| \geq 2} \beta_{\nu}(t, x) \sum_{|n(\nu)| = n - q_{\nu}} \prod_{(j, \alpha) \in \Lambda} (t \partial_t)^j \partial_x^\alpha v_{n, \alpha(i)}(i), \tag{6.10}
\]
where \( n(\nu) = (n_{j, \alpha}(i); (j, \alpha) \in \Lambda, \ 1 \leq i \leq \nu_{j, \alpha}) \) and
\[
|n(\nu)| = \sum_{(j, \alpha) \in \Lambda} (n_{j, \alpha}(1) + \cdots + n_{j, \alpha}(\nu_{j, \alpha})).
\]

In order to solve this system (6.9) and (6.10) \((n \geq \mu + 1)\) inductively on \( n \), let us define function classes \( \mathcal{F}_m (m \geq 0)\): for \( m \geq 0 \) we denote by \( \mathcal{F}_m \) the set of all holomorphic functions \( u(t, x) \) on \( S_J(r) \times D_R \) satisfying the estimate
\[
\|u(t)\|_{\rho} \leq \frac{A|t|^m}{d(t, \rho)^a}, \quad \forall (t, \rho) \in S_J(r) \times (0, R)
\]
for some \( A > 0 \) and \( a \geq 0 \). Then, by using Lemmas 6.3 and 6.4 we have the following result, which proves the part (1) of Theorem 6.1.

**Proposition 6.5.** The system (6.9) and (6.10) \((n \geq \mu + 1)\) has a unique solution \( v_n(t, x) \in \mathcal{F}_m (n \geq \mu)\). In addition, the formal series
\[
\hat{v}(t, x) = \sum_{n \geq \mu} v_n(t, x)
\]
is a formal solution of the equation (6.3) on \( S_{J_1}(r_1) \times D_{R_1} \) (in the sense of Definition 2.1) for any \( J_1 \in J, 0 < r_1 < r \) and \( 0 < R_1 < R \).

### 6.5. Estimate of the formal solution

In order to show the part (2) of Theorem 6.1, it is sufficient to prove the following result, which yields the estimate (6.5).

**Proposition 6.6.** Let \( \hat{v}(t, x) \) be the formal solution constructed in Proposition 6.5. Then, there are \( C > 0 \) and \( h > 0 \) such that
\[
||(t \partial_t)^j \partial_x^\alpha v_n(t)||_{\rho} \leq \frac{Ch^n n!^{s-1} |t|^{dn}}{d(t, \rho)^{m(2n-1)}}, \quad \forall (t, \rho) \in S_J(r) \times (0, R) \tag{6.11}
\]
holds for any \((j, \alpha) \in \Lambda, n \geq \mu \) and \( s \geq s_{1.0} \).

**Proof.** Take any \( s \geq s_{1.0} = \max\{s_{1.a}, s_{1.b}\} \). Then, we have
\[
(s-1)\sigma_{j, \alpha} \geq j + |\alpha| - l_0, \quad (j, \alpha) \in \Delta_a, \tag{6.12}
\]
\[
(s-1)(q_{\nu} + \mu(\nu - 1)) \geq m_{\nu} - l_0, \quad \nu \in \Delta_b. \tag{6.13}
\]
Set \( A = c_0 F/(c + d\mu)^m, \sigma = d^{l_0} \) and \( \eta = (2mc(1 + 2/|J|) + d)^m \). Let us consider the
following functional equation with respect to \((Y, z)\):

\[
Y = \frac{A\mu^n}{d(t, \rho)^{m(2n-1)}} z^n + \frac{c_0/\sigma}{d(t, \rho)^m} \left[ \sum_{(j, \alpha) \in \Lambda} \frac{A_{j, \alpha} (\sigma_{j, \alpha} + \mu)^{m-\ell_0}}{d(t, \rho)^{m(2\sigma_{j, \alpha} - 1)}} \sigma_{j, \alpha} (\eta Y) \right.
\]
\[
+ \sum_{|\nu| \geq 2} \frac{B_{\nu} (q_\nu + \mu|\nu|)^m}{d(t, \rho)^{m(2q_\nu + |\nu| - 2)}} \left( \sigma_{j, \alpha} (\eta Y) |\nu| \right) \],
\]

where \((t, \rho) \in S_J(r) \times (0, R)\) is regarded as a parameter. If \((t, \rho)\) is fixed, this is an analytic functional equation in \((Y, z)\), and so by the implicit function theorem we see that (6.14) has a unique holomorphic solution \(Y = Y(z)\) in a neighborhood of \(z = 0\) satisfying \(Y(z) = O(z^\alpha)\) as \(z \to 0\). By expanding it into Taylor series \(Y = \sum_{n \geq \mu} Y_n z^n\), we see that the coefficients \(Y_n\) \((n \geq \mu)\) are determined by the following recurrence formulas:

\[
Y_n = \frac{A\mu^n}{d(t, \rho)^{m(2n-1)}},
\]

and for \(n \geq \mu + 1\)

\[
Y_n = \frac{c_0/\sigma}{d(t, \rho)^m} \left[ \sum_{(j, \alpha) \in \Lambda} \frac{A_{j, \alpha} (\sigma_{j, \alpha} + \mu)^{m-\ell_0}}{d(t, \rho)^{m(2\sigma_{j, \alpha} - 1)}} \eta Y_n - \sigma_{j, \alpha} \right.
\]
\[
+ \sum_{|\nu| \geq 2} \frac{B_{\nu} (q_\nu + \mu|\nu|)^m}{d(t, \rho)^{m(2q_\nu + |\nu| - 2)}} \sum_{|\nu'| = n - q_\nu} \prod_{(j, \alpha) \in \Lambda} (\eta Y_{n_j, n_\alpha(i)}) \right],
\]

(6.16)

(6.15)

(where \(Y_p \equiv 0\) if \(p \leq \mu - 1\)). In addition, by induction on \(n\) we can see that \(Y_n = Y_n(t, \rho)\) has the form

\[
Y_n = \frac{C_n}{d(t, \rho)^{m(2n-1)}}, \quad n \geq \mu,
\]

(6.17)

where \(C_\mu = A\mu^n\) and \(C_n > 0\) \((n \geq \mu + 1)\) are constants which are independent of the parameter \((t, \rho)\). Since \(Y_n\) depends on the parameter \((t, \rho)\), we sometimes write \(Y_n = Y_n(t, \rho)\) (if we hope to emphasize that it depends on \((t, \rho)\)). We have

**Lemma 6.7.** Under the above situation, for any \(n \geq \mu\) we have the following estimates:

\[
\|((tD_t)^j \partial_x^n v_n(t))\| \leq \frac{(n - \mu)^{3s-1}}{n^{m-j-\lambda |\alpha|}} |t|^{dn} (\eta Y_n(t, \rho))
\]

(6.18)

for any \((t, \rho) \in S_J(r) \times D_R\) and \((j, \alpha) \in \Lambda\).

**Proof of Lemma 6.7.** By applying Lemma 6.4 to the equation (6.9) we have

\[
\|v_\mu(t)\| \leq \frac{c_0 F}{(c + d\mu)^{\ell_0}} |t|^{d\mu} = A |t|^{d\mu}, \quad \forall (t, \rho) \in S_J(r) \times D_R.
\]

By applying Lemma 6.3 to this estimate, and by using the conditions \(1/d(t, \rho) > 1\) and
we have
\[\int \frac{\left| (t\partial_t)^j \partial_x^{\alpha} v_\mu(t) \right|^2}{\rho} \leq A \left| \frac{\sum \left| f(t, \rho) \right|^2 (\frac{(n - \sigma_{j,\alpha} - \mu)}{(n - \sigma_{j,\alpha})})^{m - |\alpha|} \right| \leq A \left| \frac{\sum \left| f(t, \rho) \right|^2 (\frac{(n - \sigma_{j,\alpha} - \mu)}{(n - \sigma_{j,\alpha})})^{m - |\alpha|} \right| \leq \frac{1}{\mu^{m - |\alpha|}} \int \frac{\left| f(t, \rho) \right|^2 \times \eta Y_\mu(t, \rho)}{\rho}. \]

for any \((t, \rho) \in S_j(r) \times D_R\) and \((j, \alpha) \in \Lambda\). This proves \((6.18)\) for \(n = \mu\).

Let us show the general case by induction on \(n\). Let \(n \geq \mu + 1\), and suppose that \((6.18)\) (with \(n\) replaced by \(p\)) is already proved for all \(p \leq n - 1\). Then, by the induction hypothesis we have

\[
\| f_n \|_{\rho} := \text{the right-hand side of \((6.10)\)} \| \rho \leq \sum \frac{c_{\sigma_{j,\alpha}} | t |^{d \sigma_{j,\alpha}} \times (n - \sigma_{j,\alpha} - \mu)^{n - |\alpha|}}{(n - \sigma_{j,\alpha})^{m - |\alpha|}} \int \frac{\left| f(t, \rho) \right|^2 (\frac{(n - \sigma_{j,\alpha} - \mu)}{(n - \sigma_{j,\alpha})})^{m - |\alpha|} \right| \leq A \left| \frac{\sum \left| f(t, \rho) \right|^2 (\frac{(n - \sigma_{j,\alpha} - \mu)}{(n - \sigma_{j,\alpha})})^{m - |\alpha|} \right| \leq \frac{1}{\mu^{m - |\alpha|}} \int \frac{\left| f(t, \rho) \right|^2 \times \eta Y_\mu(t, \rho)}{\rho}. \]

We note:

**Lemma 6.8.** In the above situation (with \(A_{j,\alpha} \neq 0\) and \(B_\nu \neq 0\)), we have the following inequalities:

\[
\left(\frac{n - \mu}{n - \sigma_{j,\alpha}}\right)^{s - 1} \leq \left(\sigma_{j,\alpha} + \mu\right)^{m - l_0}, \quad \left(\frac{n - \mu}{n - \sigma_{j,\alpha}}\right)^{s - 1} \leq \left(\sigma_{j,\alpha} + \mu\right)^{m - l_0},
\]

\[
\left(\frac{n - \mu}{n - \sigma_{j,\alpha}}\right)^{s - 1} \leq \left(\sigma_{j,\alpha} + \mu\right)^{m - l_0}, \quad \left(\frac{n - \mu}{n - \sigma_{j,\alpha}}\right)^{s - 1} \leq \left(\sigma_{j,\alpha} + \mu\right)^{m - l_0}.
\]

The proof of this lemma will be given later, but let us admit it for the time being.

By applying \((6.20), (6.21)\) to \((6.19)\), and then by using the conditions \(1/d(t, \rho) > 1\) and \((6.16)\) we have

\[
\| f_n \|_{\rho} \leq \sum \left| f(t, \rho) \right|^2 \times \eta Y_\mu(t, \rho) \left| \frac{\sum \left| f(t, \rho) \right|^2 (\frac{(n - \sigma_{j,\alpha} - \mu)}{(n - \sigma_{j,\alpha})})^{m - |\alpha|} \right| \leq A \left| \frac{\sum \left| f(t, \rho) \right|^2 (\frac{(n - \sigma_{j,\alpha} - \mu)}{(n - \sigma_{j,\alpha})})^{m - |\alpha|} \right| \leq \frac{1}{\mu^{m - |\alpha|}} \int \frac{\left| f(t, \rho) \right|^2 \times \eta Y_\mu(t, \rho)}{\rho}. \]


for any \((t, \rho) \in S_J(r) \times (0, R)\). Hence, by (6.17) we have

\[ \|f_n(t)\|_\rho \leq \frac{(n - \mu)^{s-1}}{n^{m-\mu}} |t|^d |t|^n |C_n| \frac{(\sigma/c_0) C_n}{d(t, \rho)^m(n^{2n-2})}, \quad \forall (t, \rho) \in S_J(r) \times (0, R). \]

By applying Lemma 6.4 to the equation (6.10) and by using \(\sigma = d^{l_0}\) we obtain the estimate of \(v_n(t, x)\):

\[ \|v_n(t)\|_\rho \leq \frac{c_0}{(c + d)^{l_0}} \frac{(n - \mu)^{s-1}}{n^{m-\mu}} |t|^d |t|^n |C_n| \frac{(\sigma/c_0) C_n}{d(t, \rho)^m(n^{2n-2})}, \quad \text{on } S_J(r) \times D_R. \]

Thus, by applying Lemma 6.3 to this estimate we can obtain the estimate (6.18) in the same way as the case \(n = \mu\). This proves Lemma 6.7. \(\square\)

**Proof of Lemma 6.8.** The proof of (6.20) is as follows. If \(j + |\alpha| \leq l_0\), by using the condition \(n - \sigma_{j, \alpha} \geq \mu\) we have

\[ \frac{n^{m-\mu}}{(n - \mu)^{s-1}} \frac{(n - \sigma_{j, \alpha} - \mu)^{s-1}}{(n - \sigma_{j, \alpha})^{m-j-|\alpha|}} \leq \frac{n^{m-\mu}}{(n - \sigma_{j, \alpha})^{m-\mu}} = (1 + \frac{\sigma_{j, \alpha}}{n - \sigma_{j, \alpha}})^{m-\mu} \]

\[ \leq (1 + \frac{\sigma_{j, \alpha}}{\mu})^{m-\mu} = (\frac{\mu}{\mu})^{m-\mu}. \]

If \(l_0 + 1 \leq j + |\alpha| \leq m\) holds, by (6.12) we have \(\sigma_{j, \alpha} (s - 1) \geq j + |\alpha| - l_0\) and so we have

\[ \frac{n^{m-\mu}}{(n - \mu)^{s-1}} \frac{(n - \sigma_{j, \alpha} - \mu)^{s-1}}{(n - \sigma_{j, \alpha})^{m-j-|\alpha|}} \leq \frac{n^{m-\mu}}{(n - \sigma_{j, \alpha})^{m-\mu}} \times \frac{1}{(n - \sigma_{j, \alpha} - \mu + 1)^{\sigma_{j, \alpha}(s-1)}} \]

\[ \leq \frac{n^{m-\mu}}{(n - \sigma_{j, \alpha})^{m-\mu}} \times \frac{1}{(n - \sigma_{j, \alpha})^{j+|\alpha|-l_0}} \]

\[ = (1 + \frac{\sigma_{j, \alpha}}{n - \sigma_{j, \alpha}})^{m-\mu} (1 + \frac{\mu - 1}{n - \sigma_{j, \alpha} - \mu + 1})^{j+|\alpha|-l_0} \]

\[ \leq (1 + \frac{\sigma_{j, \alpha}}{\mu})^{m-\mu} (1 + \frac{\mu - 1}{\mu})^{j+|\alpha|-l_0} = (\frac{\mu}{\mu})^{m-\mu}. \]

This proves (6.20).

Let us show (6.21). We note: if \(n_i \geq 1 \text{ for } i = 1, \ldots, |\nu|\) and \(n_1 + \cdots + n_{|\nu|} = n - q_0\), hold, we have \(n_i \leq (n_1 \cdots n_{|\nu|})\) for \(i = 1, \ldots, |\nu|\) and so \(n - q_0 = n_1 + \cdots + n_{|\nu|} \leq |\nu| (n_1 \cdots n_{|\nu|})\) which yields \(n \leq (q_0 + |\nu|) (n_1 \cdots n_{|\nu|})\), that is,

\[ \frac{1}{n_1 \cdots n_{|\nu|}} \leq \frac{q_0 + |\nu|}{n}. \]
By the same argument we have

\[ \prod_{(j, \alpha) \in \Lambda} \frac{1}{n_{j, \alpha}(i)^{m-j-|\alpha|}} \leq \prod_{(j, \alpha) \in \Lambda} \frac{1}{n_{j, \alpha}(i)^{m_{\nu_{\mu}}}} \leq \left( \frac{(q_{\nu} + |\nu|)}{n} \right)^{m_{\nu_{\mu}}}. \]

Therefore,

\[ \frac{n^{m_{\nu_{\mu}} - L}}{(n - \mu)^{s-1}} \prod_{(j, \alpha) \in \Lambda} \frac{1}{n_{j, \alpha}(i)^{m-j-|\alpha|}} \leq \frac{n^{m_{\nu_{\mu}} - L}}{(n - \mu)^{s-1}} \times (|n(\nu) - \mu|)^{s-1} \times \left( \frac{(q_{\nu} + |\nu|)}{n} \right)^{m_{\nu_{\mu}}} \]

\[ = \frac{n^{m_{\nu_{\mu}} - L}}{(n - \mu)^{s-1}} \times (n - q_{\nu} - \mu|\nu|)^{s-1} \times (q_{\nu} + |\nu|)^{m_{\nu_{\mu}}}. \quad (6.22) \]

If \( m_{\nu_{\mu}} \leq L \) holds, the estimate (6.21) is clear from this estimate. If \( m_{\nu_{\mu}} \geq L + 1 \) holds, by (6.13) we have \( (q_{\nu} + \mu|\nu| - 1)(s-1) \geq m_{\nu_{\mu}} - L \) and so by (6.22) we have

\[ \frac{n^{m_{\nu_{\mu}} - L}}{(n - \mu)^{s-1}} \prod_{(j, \alpha) \in \Lambda} \frac{1}{n_{j, \alpha}(i)^{m-j-|\alpha|}} \]

\[ \leq \left( \frac{n}{n - q_{\nu} - \mu|\nu| + 1} \right)^{m_{\nu_{\mu}} - L} \times (q_{\nu} + |\nu|)^{m_{\nu_{\mu}}} \]

\[ = \left( 1 + \frac{q_{\nu} + \mu|\nu| - 1}{n - q_{\nu} - \mu|\nu| + 1} \right)^{m_{\nu_{\mu}} - L} \times (q_{\nu} + |\nu|)^{m_{\nu_{\mu}}} \]

\[ \leq \left( 1 + \frac{q_{\nu} + \mu|\nu| - 1}{1} \right)^{m_{\nu_{\mu}} - L} \times (q_{\nu} + |\nu|)^{m_{\nu_{\mu}}} \]

\[ = (q_{\nu} + \mu|\nu|)^{m_{\nu_{\mu}} - L} \times (q_{\nu} + |\nu|)^{m_{\nu_{\mu}}} \leq (q_{\nu} + \mu|\nu|)^{m}. \]

This proves (6.21). \( \square \)

Now, let us return to the proof of Proposition 6.6. By (6.17), \( Y_n (n \geq \mu) \) are expressed as \( Y_n = C_n/d(t, \rho)^{(2n-1)} \) \( (n \geq \mu) \). Then, by (6.15) and (6.16) \( (n \geq \mu + 1) \) we see that \( C_n (n \geq \mu) \) are determined by the recurrence formulas: \( C_{\mu} = A_{\mu}^m \) and for \( n \geq \mu + 1 \)

\[ C_n = (c_0/\sigma) \left[ \sum_{(j, \alpha) \in \Lambda} A_{j, \alpha}(\sigma_{j, \alpha} + \mu)^{m_{\nu_{\mu}}} (\eta C_{n-\sigma_{j, \alpha}}) \right. \]

\[ + \sum_{|\nu| \geq 2} B_{\nu}(q_{\nu} + \mu|\nu|)^m \sum_{n(\nu) = n-q_{\nu}} \prod_{(j, \alpha) \in \Lambda} \frac{1}{n_{j, \alpha}(i)^{m_{\nu_{\mu}} - L}} \left( \eta C_{n-j, \alpha}(i) \right). \]

This means that the formal series \( Z(z) = \sum_{n \geq \mu} C_n z^n \) is the unique formal solution of the equation

\[ Z = A_{\mu}^m z^{\mu} + (c_0/\sigma) \left[ \sum_{(j, \alpha) \in \Lambda} A_{j, \alpha}(\sigma_{j, \alpha} + \mu)^{m_{\nu_{\mu}}} z^{\sigma_{j, \alpha}} (\eta Z) \right. \]
\[
+ \sum_{|\nu|\ge 2} B_\nu(q_\nu + \mu|\nu|)^m z^\nu (\eta Z)^{[\alpha]}.
\] (6.23)

Since (6.23) is nothing but an analytic functional equation, by the implicit function theorem we see that the formal solution \(Z(z)\) is convergent in a neighborhood of \(z = 0\). Thus, by Cauchy's inequality we have 
\[C > 0 \text{ and } h > 0 \text{ such that } C_n \le C h^n \] holds for all \(n \ge \mu\). Combining this with (6.17) and (6.18) we have (6.11). This complete the proof of Proposition 6.6. □

6.6. Equivalence of two formal solutions

In order to prove the part (3) of Theorem 6.1, it is enough to show

**Proposition 6.9.** Let

\[
\hat{v}(t, x) = \sum_{n \ge 1} v_n(t, x) \quad \text{and} \quad \hat{w}(t, x) = \sum_{n \ge 1} w_n(t, x)
\]

be two formal solution of (6.3) on \(S_J(r) \times D\). Then, they are equivalent on \(S_{J_1}(r_1) \times D_{R_1}\) for any \(J_1 \subset J, 0 < r_1 < r \) and \(0 < R_1 < R\).

**Proof.** Set

\[G(t, x, \Theta v) = C(t \partial_t, x) v + \sum_{(j, \alpha) \in \Lambda} a_{j, \alpha}(t, x) (t \partial_t)^j \partial_x^\alpha v - f^*(t, x) - R(t, x, \Theta v)\]

with

\[R(t, x, Z) = \sum_{|\nu|\ge 2} \beta_\nu(t, x) Z^\nu,\]

where \(Z^{\nu} = \prod_{(j, \alpha) \in \Lambda} (Z_{j, \alpha})^{\nu_{j, \alpha}}\). Our equation (6.3) is written as \(G(t, x, \Theta v) = 0\).

Let \(\hat{v}(t, x)\) and \(\hat{w}(t, x)\) be two formal solution of (6.3) on \(S_J(r) \times D\). Take any \(A > 0\). Then, we can take \(N_0 \in \mathbb{N}^*\) such that for any \(N \ge N_0\) we have

\[
\|G(t, x, \Theta \hat{v}_N)\|_R = O(|t|^A) \quad \text{(as } S_J \ni t \to 0),
\]

\[
\|G(t, x, \Theta \hat{w}_N)\|_R = O(|t|^A) \quad \text{(as } S_J \ni t \to 0).
\]

We set

\[U_N(t, x) = \hat{v}_N(t, x) - \hat{w}_N(t, x)\]

Then, we have \(U_N(t, x) \in \mathcal{O}(S_J(r) \times D), \|((t \partial_t)^j \partial_x^\alpha U_N(t))\|_R = O(|t|^{\mu_1}) \) (as \(S_J \ni t \to 0\)) for some \(\mu_1 > 0\), and

\[
O(t^A) = G(t, x, \Theta \hat{v}_N) - G(t, x, \Theta \hat{w}_N)
\]

\[
= C(t \partial_t, x) U_N + \sum_{(j, \alpha) \in \Lambda} a_{j, \alpha}(t, x) (t \partial_t)^j \partial_x^\alpha U_N - (R(t, x, \Theta \hat{v}_N) - R(t, x, \Theta \hat{w}_N)).
\]
Therefore, under the setting
\[ g(t, x) = G(t, x, \Theta \hat{v}_N) - G(t, x, \Theta \hat{w}_N), \]
\[ c_{j, \alpha}(t, x) = -a_{j, \alpha}(t, x) + \int_0^1 \frac{\partial R}{\partial z_{j, \alpha}}(t, x, s \Theta \hat{v}_N + (1 - s) \Theta \hat{w}_N) ds, \quad (j, \alpha) \in \Lambda \]
we see that \( U_N(t, x) \) satisfies the following equation
\[ C(t \partial_t, x) U_N = g(t, x) + \sum_{(j, \alpha) \in \Lambda} c_{j, \alpha}(t, x)(t \partial_t)^j \partial^\alpha_x U_N \]
(6.24)
on \( S_j(\delta) \times D_R \) for some \( \delta > 0 \). In addition, we know that \( g(t, x) \) and \( c_{j, \alpha}(t, x) \) \( (j, \alpha) \in \Lambda \) are holomorphic functions on \( S_j(\delta) \times D_R \) and satisfy
\[ |g(t, x)| \leq C|t|^A \quad \text{on} \quad S_j(\delta) \times D_R, \]
\[ |c_{j, \alpha}(t, x)| \leq C_{j, \alpha}|t|^\gamma \quad \text{on} \quad S_j(\delta) \times D_R, \quad (j, \alpha) \in \Lambda \]
for some \( C \geq 0, C_{j, \alpha} \geq 0 \) \((j, \alpha) \in \Lambda\) and \( \gamma > 0 \).
We set
\[ R[U_N] = g(t, x) + \sum_{(j, \alpha) \in \Lambda} c_{j, \alpha}(t, x)(t \partial_t)^j \partial^\alpha_x U_N : \]
our equation (6.24) is written as
\[ C(t \partial_t, x) U_N = R[U_N]. \]
(6.25)
Let \( \mathcal{F}_m \) \((m \geq 0)\) be the function spaces used in Proposition 6.5. By the assumption, we have \( R[U_N] \in \mathcal{F}_{n_1} \) for \( n_1 = \min\{\mu_1 + \gamma, A\} \). Therefore, by applying Lemma 6.4 to (6.25) we have \( U_N \in \mathcal{F}_{n_1} \). Then, by the definition of \( R[U_N] \) we have \( R[U_N] \in \mathcal{F}_{n_2} \) for \( n_2 = \min\{n_1 + \gamma, A\} \). By applying Lemma 6.4 to (6.25) again we have \( U_N \in \mathcal{F}_{n_2} \).
Repeating the same argument we can obtain the condition \( U_N \in \mathcal{F}_A \), that is,
\[ \|U_N(t)\|_\rho \leq B|t|^A \frac{1}{d(t, \rho)^a}, \quad \forall (t, \rho) \in S_j(r) \times (0, R) \]
for some \( B > 0 \) and \( a > 0 \). By applying Lemma 6.3 to this estimate we see that
\[ \|(t \partial_t)^j \partial^\alpha_x U_N(t)\|_{R_1} = O(|t|^A) \quad \text{(as} \quad S_{J_1} \ni t \rightarrow 0) \]
holds for any \( 0 < R_1 < R, J_1 \in J \) and \((j, \alpha) \in \Lambda\).
Since \( A > 0 \) is taken arbitrarily, we have the conclusion that \( \hat{v}(t, x) \) and \( \hat{w}(t, x) \) are equivalent on \( S_{J_1}(r_1) \times D_{R_1} \). This proves Proposition 6.9.

7. Convolution equations in the Borel plane

From now, we suppose: \( p^* \geq 1 \). Let us consider the equation (5.4). Since the formal solution \( \hat{v}(t, x) \) of (5.4) in Theorem 6.1 satisfies a Gevrey type estimate, we can apply the formal \( k_1 \)-Borel transform to this formal solution and we can get a \( k_1 \)-convolution equation in the Borel plane. In this section, we will make clear the meaning of this
argument.

7.1. Another expression of (5.4)
For $d \geq 0$, we denote by $\mathcal{D}_d(S_f(r) \times D_R)$ the set of all holomorphic functions $f(t, x)$ on $S_f(r) \times D_R$ satisfying the estimate $|f(t, x)| \leq C|t|^d$ on $S_f(r) \times D_R$ for some $C \geq 0$. As before, by $O(t^d)$ we mean that it is a function belonging to $\mathcal{D}_d(S_f(r) \times D_R)$.

Let $P_1(\lambda, x)$ be the polynomial defined in subsection 4.5. For $x \in \mathbb{R}$ we write $[x]_+ = \max\{x, 0\}$. For $l \in \mathbb{N}$ and $\nu = \{\nu_{j, \alpha}\}_{(j, \alpha) \in \Lambda} \in \mathbb{N}^\Lambda$ we write

$$
|\nu|_l = \sum_{(j, \alpha) \in \Lambda} |j + |\alpha| - l| + \nu_{j, \alpha}.
$$

We have

**Proposition 7.1.** By multiplying the equation (5.4) by $t^{k_1 l_0}$, it is expressed in the form

$$
P_1(t^{k_1 + 1} \partial_t) F = t^{k_1 l_0} f^*(t, x) + \sum_{(j, \alpha) \in \Lambda} \mathcal{A}_{1, j, \alpha}(t, x)(t^{k_1 |\alpha|} (t^{k_1 + 1} \partial_t)^{|\alpha|} \partial_x^\alpha v)
$$

$$
+ \sum_{|\nu| \geq 2} c_{1, \nu}(t, x) t^{e_{1, \nu}} \prod_{(j, \alpha) \in \Lambda} (t^{k_1 |\alpha|} (t^{k_1 + 1} \partial_t)^{|\alpha|} \partial_x^\alpha v)^{\nu_{j, \alpha}},
$$

where $\mathcal{A}_{1, j, \alpha}(t, x)$ $(j, \alpha) \in \Lambda$ and $c_{1, \nu}(t, x)$ $(|\nu| \geq 2)$ are suitable holomorphic functions on $S_f(r) \times D_R$, and $e_{1, \nu} = q(|\nu| - 1) - p - k_1 (m |\nu| - l_0) \in \mathbb{N}^* / L \ (|\nu| \geq 2)$. In addition, we have the following properties.

1. There are rational numbers $n_{1, j, \alpha} \in \mathbb{N}^* / L \ (j, \alpha) \in \Lambda$, $H_{1, j, \alpha} \geq 0 \ (j, \alpha) \in \Lambda$, and $C_{1, \nu} \geq 0 \ (|\nu| \geq 2)$ such that

$$
|\mathcal{A}_{1, j, \alpha}(t, x)| \leq H_{1, j, \alpha} |t|^{n_{1, j, \alpha}} \text{ on } S_f(r) \times D_R, \ (j, \alpha) \in \Lambda,
$$

$$
|c_{1, \nu}(t, x)| \leq C_{1, \nu} \text{ on } S_f(r) \times D_R, \ |\nu| \geq 2,
$$

$$
\sum_{|\nu| \geq 2} C_{1, \nu} X^{\nu} \in \mathbb{C}\{X\}.
$$

2. If $p^* \geq 2$, we have

$$
k_2 = \min_{l_1 + 1 \leq j + |\alpha| \leq m, \mathcal{A}_{1, j, \alpha}(t, x) \neq 0} \frac{m_{1, j, \alpha} + k_1 (j + |\alpha| - l_1)}{j + |\alpha| - l_1},
$$

$$
k_2 \leq \frac{c_{1, \nu} + k_1 (\nu)_t + \mu_0 (|\nu| - 1)}{m_{\mu} - l_1}, \text{ if } |\nu| \geq 2, \ m_{\mu} \geq l_1 + 1.
$$

In the above, the condition $e_{1, \nu} > 0$ follows from the condition $q > 2k_1 m + p$ in (5.1). To prove Proposition 7.1, we note:

**Lemma 7.2.** For any $j \geq 0$ we have the formula

$$
t^{k_1 j} (t \partial_t)^j = \sum_{0 \leq h \leq j} M_{j, h} t^{k_1 (j - h)} (t^{k_1 + 1} \partial_t)^h
$$

for some constants $M_{j, h} \ (0 \leq h \leq j)$ satisfying $M_{j, j} = 1 \ (\text{for } j \geq 0)$ and $M_{j, 0} = 0 \ (\text{for }
\[ j \geq 1. \]

**Proof of Proposition 7.1.** We recall that by (b₁) and (b₂) (with \( i = 1 \)) we have \( \gamma_{l_0,0} > 0 \),

\[
t^{k_1l_0} a_{j,0}(t,x) = (a_{j,0}^0(x) + O(t^{n_{1,0}})) t^{k_1j}, \quad (j,0) \in \Lambda(\Gamma_1) \setminus \{(l_0,0)\}
\]

and \( k_1l_0 + \gamma_{j,0} > k_1(j + |\alpha|) \) for \( (j,\alpha) \in \Lambda \setminus \Lambda(\Gamma_1) \). Therefore, under the setting

\[
C_0(\lambda + q, x) = a_{j,0}^0(x)\lambda^0 + \sum_{0 \leq j < l_0} c_{j,0}(x)\lambda^j \quad \text{(with } c_{j,0}(x) \in \mathcal{O}(D_R), \text{)}
\]

\[
a_{1,j,0}(t,x) = t^{-k_1(j+|\alpha|-l_0)} a_{j,0}(t,x), \quad (j,\alpha) \in \Lambda \setminus \Lambda(\Gamma_1),
\]

\[
m_{1,j,0} = \gamma_{j,0} - k_1(j + |\alpha| - l_0), \quad (j,\alpha) \in \Lambda \setminus \Lambda(\Gamma_1),
\]

we have \( m_{1,j,0} \in \mathbb{N}^+ / L \ ( (j,\alpha) \in \Lambda \setminus \Lambda(\Gamma_1) ) \), \( a_{1,j,0}(t,x) \in \mathcal{X}_{m_{1,j,0}}(S_J(r) \times D_R) \ ( (j,\alpha) \in \Lambda \setminus \Lambda(\Gamma_1) ) \), and

\[
t^{k_1l_0} C_0(\partial_t + q, x) v + \sum_{(j,\alpha) \in \Lambda} t^{k_1l_0} a_{j,0}(t,x)(\partial_t)^j \partial_x^\alpha v
\]

\[
= a_{l_0,0}^0(x)(t^{k_1+1}\partial_t)^{l_0} v
\]

\[
+ \sum_{0 \leq h < l_0} \left( a_{l_0,0}^0(x) M_{j,h} + \sum_{h < l_0} c_{j,0}(x) M_{j,h} \right) t^{l_0-h}(t^{k_1+1}\partial_t)^h v
\]

\[
+ O(t^{n_{1,0}}) \sum_{0 \leq h < l_0} M_{l_0,h} t^{l_0-h}(t^{k_1+1}\partial_t)^h v
\]

\[
+ \sum_{(j,0) \in \Lambda(\Gamma_1) \setminus \{(l_0,0)\}} \left[ O(t^{n_{1,0}})(t^{k_1+1}\partial_t)^j v + \sum_{(j,0) \in \Lambda(\Gamma_1) \setminus \{(l_0,0)\}} \left[ O(t^{n_{1,0}})(t^{k_1+1}\partial_t)^j v
\]

\[
+ \sum_{(j,\alpha) \in \Lambda \setminus \Lambda(\Gamma_1)} a_{1,j,0}(t,x) M_{j,h} t^{l_0-h}(t^{k_1+1}\partial_t)^h v \right] \right].
\]

(7.4)

In addition, we have

**Lemma 7.3.** (1) \( n \geq 2 \). We define \( c_{1,\nu}(t,x) \in \mathcal{X}_0(S_J(r) \times D_R) \ (|\nu| = n) \) by the relation

\[
\sum_{|\nu|=n} b_{\nu}(t,x) \prod_{(j,\alpha) \in \Lambda} \left( \sum_{0 \leq h \leq j} M_{j,h} t^{k_1(m-h-|\alpha|)} Z_{h,\alpha} \right)^{\nu_j,\alpha} = \sum_{|\nu|=n} c_{1,\nu}(t,x) Z^\nu
\]

(as polynomials in \( Z \)). Set \( c_{1,\nu} = q(|\nu| - 1) - p - k_1(m|\nu| - l_0) \) (for \( |\nu| = n \)). Then, we have

\[
t^{k_1l_0} \sum_{|\nu|=n} b_{\nu}(t,x) t^{|\nu|-1-p} \prod_{(j,\alpha) \in \Lambda} \left( (t^{l_0})^{l_0} \partial_x^\alpha v \right)^{\nu_j,\alpha}
\]

\[
= \sum_{|\nu|=n} b_{\nu}(t,x) t^{k_1l_0} \prod_{(j,\alpha) \in \Lambda} \left( \sum_{0 \leq h \leq j} M_{j,h} t^{k_1(m-h-|\alpha|)} \left[ t^{k_1|\alpha|} (t^{k_1+1}\partial_t)^h \partial_x^\alpha v \right] \right)^{\nu_j,\alpha}
\]
(2) We have constants $C_{1,\nu} \geq 0 \ (|\nu| \geq 2)$ that satisfy the following properties: $|c_{1,\nu}(t,x)| \leq C_{1,\nu}$ on $S_2 \times D_R \ (|\nu| \geq 2)$ and $\sum_{|\nu|\geq 2} C_{1,\nu} X^{[\nu]} \in \mathbb{C}\{X\}$.

Proof of Lemma 7.3. Since $e_{1,\nu}$ depends only on $|\nu|$, (1) is verified by Lemma 7.2 and the setting $Z_{h,\alpha} = t^{k_1|\alpha|}(t^{k_1+1}\partial_1)^{\nu} \partial^{\nu}_2 v ((h, \alpha) \in \Lambda)$. Let us show (2). Take $M > 0$ so that $M_{j,h} \leq M$ for any $0 \leq h \leq j \leq m$. Then, as polynomials in $Z$ we have

\[
\sum_{|\nu|\geq 2} |c_{1,\nu}(t,x)|Z^{\nu} \ll \sum_{|\nu|\geq 2} |b_{\nu}(t,x)| \prod_{(j,\alpha)\in\Lambda} \left( \sum_{0 \leq h \leq j} |M_{j,h}|t^{k_1(m-h-|\alpha|)} Z_{h,\alpha} \right)^{\nu_{j,\alpha}} = \sum_{|\nu|\geq 2} C_{1,\nu} Z^{\nu}
\]

for some $C_{1,\nu} \geq 0$. Hence, we have $|c_{1,\nu}(t,x)| \leq C_{1,\nu}$ on $S_2 \times D_R \ (|\nu| \geq 2)$. By the definition of $C_{1,\nu} \geq 0 \ (|\nu| \geq 2)$ and by the condition $\sum_{|\nu|\geq 2} B_{\nu} X^{[\nu]} \in \mathbb{C}\{X\}$ (in section 5) we have

\[
\sum_{|\nu|\geq 2} C_{1,\nu} X^{[\nu]} = \sum_{|\nu|\geq 2} B_{\nu} \prod_{(j,\alpha)\in\Lambda} \left( M \sum_{0 \leq h \leq j} r^{k_1(m-h-|\alpha|)} X \right)^{\nu_{j,\alpha}} \in \mathbb{C}\{X\}.
\]

This proves the condition (2). \(\square\)

Thus, by (7.4) and Lemma 7.3 we have the equality (7.1). The condition (1) of Proposition 7.1 also follows from (7.4) and Lemma 7.3. In general, the rational numbers $n_{1,j,\alpha} ((j, \alpha) \in \Lambda)$ are not the same as $m_{1,j,\alpha} ((j, \alpha) \in \Lambda)$, but we can still take them so that $n_{1,j,\alpha} \in \mathbb{N}^*/L ((j, \alpha) \in \Lambda)$ holds, as is seen below.

Let us show (2) of Proposition 7.1. Suppose $p^* \geq 2$. By (b2) with $i = 2$ we have

\[
\gamma_{j,0} = k_2(j-l_1) + c_1 - p, \quad \text{if} \ (j,0) \in \Lambda(\Gamma_2),
\]

\[
\gamma_{j,\alpha} > k_2(j + |\alpha| - l_1) + c_1 - p, \quad \text{if} \ (j,\alpha) \in \Lambda \setminus \Lambda(\Gamma_2).
\]

Since $m_{1,j,\alpha} = \gamma_{j,\alpha} - k_1(j + |\alpha| - l_0) \ (\gamma_{j,\alpha} \in \Lambda \setminus \Lambda(\Gamma_1))$ and $c_1 - p = k_1(l_1 - l_0)$ hold, we have

\[
m_{1,j,0} = (k_2 - k_1)(j-l_1), \quad \text{if} \ (j,0) \in \Lambda(\Gamma_2) \setminus \{(l_1,0)\},
\]

\[
m_{1,j,\alpha} > (k_2 - k_1)(j + |\alpha| - l_1), \quad \text{if} \ (j,\alpha) \in \Lambda \setminus \Lambda(\Gamma_1 \cup \Gamma_2).
\]

Under these conditions, let us see how to determine $n_{1,j,\alpha}$ (for $j + |\alpha| \geq l_1 + 1$).

Let $(h, \alpha) \in \Lambda$ with $h + |\alpha| \geq l_1 + 1$. Then, by (7.4) we see that $\mathcal{A}_{1,h,\alpha}(t,x)$ is given by

\[
\mathcal{A}_{1,h,\alpha}(t,x) = \sum_{(j,\alpha)\in\Lambda, j\geq h} a_{1,j,\alpha}(t,x) M_{j,h} t^{k_1(j-h)}.
\]

Therefore, if $\mathcal{A}_{1,h,\alpha}(t,x) \neq 0$, we can set

\[
n_{1,h,\alpha} = \min\{ m_{1,j,\alpha} + k_1(j-h) ; (j,\alpha) \in \Lambda, j \geq h \},
\]
and if \( \mathcal{A}_{1,h,0}(t, x) \equiv 0 \) we can take any \( n_{1,h,0} \in \mathbb{N}^* / L \).

In the case \((h, 0) \in \Lambda(\Gamma_2)\) with \( h \geq l_1 + 1 \), by (7.6) we have

\[
\begin{align*}
m_{1,j,0} + k_1(j - h) &= m_{1,h,0} = (k_2 - k_1)(h - l_1), \quad \text{if } j = h, \\
m_{1,j,0} + k_1(j - h) &\geq (k_2 - k_1)(j - l_1) + k_1(j - h) \\
&> (k_2 - k_1)(h - l_1), \quad \text{if } j > h,
\end{align*}
\]

d and so by (b1) we have

\[
\mathcal{A}_{1,h,0}(t, x) = (a^0_{h,0}(x) + O(t^{\eta}))t^{(k_2 - k_1)(h - l_1)} + O(t^{(k_2 - k_1)(h - l_1) + \eta})
\]

for some \( \eta > 0 \). Since \( a^0_{h,0}(x) \neq 0 \) we have \( \mathcal{A}_{1,h,0}(t, x) \neq 0 \) and \( n_{1,h,0} = (k_2 - k_1)(h - l_1) \).

In the case \((h, \alpha) \in \Lambda \setminus \Lambda(\Gamma_2)\) with \( h + |\alpha| \geq l_1 + 1 \), by (7.6) we have

\[
\begin{align*}
m_{1,j,0} + k_1(j - h) &= m_{1,h,0} > (k_2 - k_1)(h + |\alpha| - l_1), \quad \text{if } j = h, \\
m_{1,j,0} + k_1(j - h) &\geq (k_2 - k_1)(j + |\alpha| - l_1) + k_1(j - h) \\
&> (k_2 - k_1)(h + |\alpha| - l_1), \quad \text{if } j > h,
\end{align*}
\]

d and so \( n_{1,h,\alpha} > (k_2 - k_1)(h + |\alpha| - l_1) \).

Thus, we have seen that \( \mathcal{A}_{1,h,0}(t, x) \neq 0 \) holds for any \((h, 0) \in \Lambda(\Gamma_2)\) with \( h \geq l_1 + 1 \), and that

\[
\begin{align*}
k_2 &= \frac{n_{1,h,0} + k_1(h - l_1)}{h - l_1}, \quad \text{if } (h, 0) \in \Lambda(\Gamma_2), h \geq l_1 + 1, \\
k_2 &< \frac{n_{1,h,\alpha} + k_1(h + |\alpha| - l_1)}{h + |\alpha| - l_1}, \quad \text{if } (h, \alpha) \in \Lambda \setminus \Lambda(\Gamma_2), h + |\alpha| \geq l_1 + 1.
\end{align*}
\]

This leads us to (7.2).

Since \( c_1,\nu = q(\nu - 1) - p - k_1(m|\nu| - l_0) \) and \( q \geq (2k_1 + k_2)m + p \) (by (5.1)) hold, the condition (7.3) is verified as follows:

\[
\begin{align*}
\frac{c_1,\nu + k_1(\nu)}{m_\nu} + \mu_0 |\nu - 1| &> \frac{c_1,\nu + \mu_0}{m} \geq q(\nu - 1) - p - k_1m|\nu| + \mu_0 \\
&= \frac{(q - k_1m)|\nu| - q - p + \mu_0}{m} \\
&\geq \frac{(q - k_1m)2 - q - p + \mu_0}{m} = \frac{q - 2k_1m - p + \mu_0}{m} > k_2.
\end{align*}
\]

Thus, the condition (2) of Proposition 7.1 is proved. \( \square \)

### 7.2. From (5.4) to a convolution equation

As in subsection 3.5, we define the \( k_1 \)-convolution of \( f(\xi, x) \) and \( g(\xi, x) \) with respect to \( \xi \) by

\[
(f *_{k_1} g)(\xi, x) = \int_0^\xi f(\tau, x)g((\xi^{k_1} - \tau^{k_1})^{1/k_1}, x)\, d\tau^{k_1}.
\]
We write \( u^{*0} = 1, \ u^{*1} = u, \ u^{*2} = u *_{k_1} u, \ u^{*3} = u *_{k_1} u *_{k_2} u \), and so on. We often write

\[
\prod_{i=1}^{n} u_i = u_1 *_{k_1} u_2 *_{k_1} \cdots *_{k_1} u_n.
\]

For \((j, \alpha) \in \Lambda\) we write

\[
\mathcal{M}_{j, \alpha}[W] = \begin{cases} 
\xi_{j, \alpha}^{k_1} & \text{if } |\alpha| > 0, \\
\Gamma(|\alpha|) \cdot (k_1 \xi_{j, \alpha})^j W, & \text{if } |\alpha| = 0.
\end{cases}
\]

We note that \(\mathcal{M}_{j, \alpha}[W]\) is nothing but the \(k_1\)-Borel transform of

\[
\xi_{j, \alpha}^{k_1} \times (k^{k_1+1} \partial_j)^j U \quad \text{under } W = B_{k_1}[U].
\]

Recall that in section 6 we have set \(d = 1/L, \mu_0 \in \mathbb{N}^*/L \) (in (6.1)), and \(\mu = L \mu_0 \in \mathbb{N}^*\).

**Proposition 7.4.** Let

\[
\hat{v}(t, x) = \sum_{n \geq 1} v_n(t, x)
\]

be the formal solution of (5.4) constructed in (1) of Theorem 6.1. Take any open interval \(I\) satisfying \(I + [\pi/2k_1] \subseteq J\), and any \(0 < R_1 < R\). Then, we have the following results.

1. The \(k_1\)-Borel transforms \(B_k[v_n](\xi, x) \ (n \geq 1)\) are well-defined as holomorphic functions on \(S_I \times D_{R_1}\) and the sum

\[
w_1(\xi, x) = \sum_{n \geq 1} B_k[v_n](\xi, x)
\]

is uniformly convergent on \(S_I(\delta) \times D_{R_1}\) for some \(\delta > 0\), and satisfies \(|w_1(\xi, x)| \leq A|\xi|^{-k_1+1} \) on \(S_I(\delta) \times D_{R_1}\) for some \(A > 0\).

2. In addition, \(w_1(\xi, x)\) satisfies the following convolution partial differential equation

\[
P_t(\xi_{k_1} *_{\alpha} x)w_1 = B_{k_1}[\xi_{k_1} *_{\alpha} f^*(\xi, x)\] + \sum_{(j, \alpha) \in \Lambda} B_{k_1}[\mathcal{M}_{j, \alpha}[\xi_k, x] *_{k_1} \mathcal{M}_{j, \alpha}[\partial^2 \xi w_1]
\]

\[+ \sum_{|\alpha| \geq 2} B_{k_1}[c_1, u^{*_{\alpha}}(\xi, x) *_{k_1} \prod_{(j, \alpha) \in \Lambda} \mathcal{M}_{j, \alpha}[\partial^2 w_1]]^*_{k_1} *_{k_1} \quad (7.7)
\]

on \(S_I(\delta) \times D_{R_1}\), which is just the equation obtained by applying \(k_1\)-Borel transform \(B_{k_1}\) to (7.1).

**Proof.** Let us show (1). Take \(J_1\) and \(r_1\) so that \(I + [\pi/2k_1] \subseteq J_1 \subseteq J\) and \(0 < r_1 < r\). By (2) of Theorem 6.1 (and (2) of Corollary 6.2) we know that \(v_n(t, x) \ (n \geq 1)\) satisfy

\[
|v_n(t, x)| \leq C h \Gamma(n/Lk_1)|t|^{n/L} \quad \text{on } S_{J_1}(r_1) \times D_{R_1}, \ n \geq 1
\]

for some \(C > 0\) and \(h > 0\). Therefore, by Lemma 3.1 we see that \(B_k[v_n](\xi, x) \ (n \geq 1)\)
are well-defined as holomorphic functions on $S_I \times D_R$, and satisfy
\[ |B_{k_1}[v_n](\xi, x)| \leq KCh^n|\xi|^{n/k_1} \exp(b|\xi|^{k_1}) \quad \text{on} \quad S_I \times D_R, \quad n \geq 1 \]
for some $K > 0$ and $b > 0$ which are independent of $n$.

Here, we recall that by the construction of $\hat{v}(t, x)$ in subsection 6.4, we have $v_n(t, x) = 0$ for $1 \leq n \leq \mu - 1$. Hence, by taking $\delta > 0$ so that $h\delta^{1/L} < 1$ holds we have
\[
\sum_{n \geq 1} |B_{k_1}[v_n](\xi, x)| = \sum_{n \geq \mu} |B_{k_1}[v_n](\xi, x)| \\
\leq KCh^n|\xi|^{\mu/k_1} \sum_{n \geq \mu} (h|\xi|^{1/L})^{n-\mu} \exp(b|\xi|^{k_1}) \\
\leq A|\xi|^{\mu/k_1} \quad \text{on} \quad S_I(\delta) \times D_R,
\]
for some $A > 0$. Since $\mu_0 = \mu/L$, this proves the part (1) of Proposition 7.4.

Next, let us show (2). Usually, in the case of formal power series solutions, the convolution equation in the Borel plane is derived by the following principle: if $f(\xi) \in \mathbb{C}[\xi]$ and $g(\xi) \in \mathbb{C}[\xi]$ satisfy $f(\xi) = g(\xi)$ as formal power series, then we have $f(\xi) = g(\xi)$ as holomorphic functions in a neighborhood of $\xi = 0 \in \mathbb{C}$.

In the present case, we will use the following principle: if $X_n = Y_n \quad (n = 1, 2, \ldots)$ hold, and if $\sum_{n \geq 1} X_n$ and $\sum_{n \geq 1} Y_n$ are convergent, then we have the equality
\[
\sum_{n \geq 1} X_n = \sum_{n \geq 1} Y_n.
\]

By the construction of $\hat{v}(t, x)$ in subsection 6.4, the functions $v_n(t, x) \quad (n \geq 1)$ are determined by $v_n(t, x) = 0$ (for $1 \leq n \leq \mu - 1$), and recurrence formulas: (6.9) and (6.10) ($n \geq \mu + 1$). Therefore, by applying $B_{k_1}$ to the recurrence formulas, and then by using $t^{k_1}a_\nu(t, x) = b_\nu(t, x)t^{\nu_1+\nu_k}m[\nu]$ and Lemma 7.2 we have the following equalities:
\[
B_{k_1}[t^{k_1+\nu}C_{\nu}] = B_{k_1}[t^{k_1+\nu}f^+] \tag{7.8}
\]
and for $n \geq \mu + 1$
\[
B_{k_1}[t^{k_1+\nu}C_{\nu}] = -\sum_{(j, \alpha) \in \Lambda} B_{k_1}[t^{k_1+\nu_j}(\partial_t^j)^{\nu_j}v_n-\sigma_{j, \alpha}] + \sum_{|\nu| \geq 2} \sum_{|\nu_0| = n - q_0} B_{k_1}[t^{\nu_j+b_\nu} \prod_{(j, \alpha) \in \Lambda} \prod_{i=1} \left( \sum_{0 \leq h \leq j} t^{k_1(m-h-|\alpha|)}M_{j,h}[t^{k_1+1}\sigma^h]^{\nu_j+a_{j,\alpha}(i)} \right)], \tag{7.9}
\]

where $C = C(t, \partial_t, x)$, $a_{j, \alpha} = a_{j, \alpha}(t, x)$ ($j, \alpha \in \Lambda$) and $b_\nu = b_\nu(t, x) \quad (|\nu| \geq 2)$. Since (7.9) depends on $n$, we write this as (7.9) $n$.

Since (5.4) is decomposed into a system (6.9) and (6.10) ($n \geq \mu + 1$), formally we have (5.4) = (6.9) + $\sum_{n \geq \mu+1}$ (6.10) and so we have formally
\[
B_{k_1}[t^{k_1+\nu}] = B_{k_1}[t^{k_1+\nu} \times (6.9)] + \sum_{n \geq \mu+1} B_{k_1}[t^{k_1+\nu} \times (6.10)].
\]
Hence, to show the part (2) of Proposition 7.4, it is enough to prove

$$(7.8) + \sum_{n \geq \mu + 1} (7.9)_n = (7.7).$$

Formally, this is verified as follows. For the linear part, we apply the same calculation as in (7.4) to each (7.9)$_n$, use the formula

$$B_{k_1}[k_1^1|\alpha|(k_1+1)^k \partial_x^n v_n](\xi,x) = \mathcal{M}_{1,h,\alpha} \left[ \partial_x^n B_{k_1}[v_n](\xi,x) \right], \quad n \geq \mu$$

and then take the summation. For the nonlinear part, we change the order of the summations and then apply (7.5).

In order to give a substantial meaning to this formal verification, we need to show that every sum appearing in the argument is absolutely and uniformly convergent on $S_f(\delta) \times D_R$. To show this, we need only Lemma 3.1, the estimate (6.2) in Theorem 6.1 and a job of estimating. Since it is only a simple work, we may omit the details. \qed

7.3. A result on (7.7)

Let us recall a result in Tahara [24], where convolution partial differential equations of type (7.7) have been well studied.

We note that by (6.1) we have

$$|t^{k_1 l_0} f^*(t,x)| \leq F^{k_1 l_0} |t|^\mu \text{ on } S_f(r) \times D_R.$$  

Since $I + [\pi/2k_1] \in I$, by Lemma 3.1, and (1) of Proposition 7.1 we have

$$|B_{k_1}[k_1 l_0 f^*](\xi,x)| \leq \frac{K F^{k_1 l_0}}{\Gamma(\mu_0/k_1)} |\xi|^{\mu_0 - k_1} \exp(c \xi^{k_1}) \text{ on } S_I \times D_R,$$

$$|B_{k_1}[e_{1,\nu} t^{e_{1,\nu}}](\xi,x)| \leq \frac{K C_{1,\nu}}{\Gamma(e_{1,\nu}/k_1)} |\xi|^{e_{1,\nu} - k_1} \exp(c \xi^{k_1}) \text{ on } S_I \times D_R$$

for some $K > 0$ and $c > 0$. Since $\sum_{|\nu| \geq 2} C_{1,\nu} X^{|\nu|} \in C(X)$ and $e_{1,\nu} = q(|\nu| - 1) - p - k_1(m|\nu| - l_0) \in \mathbb{N}^* / L$, we have

$$\sum_{|\nu| \geq 2} K C_{1,\nu} X^{|\nu|} \in C(z,X).$$

Set

$$D_{a} = \{ (j, \alpha) \in A : j + |\alpha| \geq l_1 + 1, e_{1,\nu}(\xi,x) \neq 0 \};$$

$$D_{c} = \{ \nu \in \mathbb{N}^* : |\nu| \geq 2, m_\nu \geq l_1 + 1, e_{1,\nu}(\xi,x) \neq 0 \};$$

$$\sigma_a = 1 + \max \left[ 0, \max_{(j, \alpha) \in D_a} \left( \frac{j + |\alpha| - l_1}{n_1(j, \alpha) + k_1(j + |\alpha| - l_1)} \right) \right];$$

$$\sigma_c = 1 + \max \left[ 0, \max_{\nu \in D_c} \left( \frac{m_\nu - l_1}{e_{1,\nu} + k_1(\nu)l_1 + \mu_0(|\nu| - 1)} \right) \right].$$
and set $\sigma_0 = \max\{\sigma_\alpha, \sigma_c \}$. Now, we define $\kappa > 0$ by the relation

$$1/\kappa = 1/k_1 - (\sigma_0 - 1).$$

By [Theorem 5.1 in [24]] (with $d = 1/L$) we have

**Theorem 7.5.** Suppose that $0 < |I| < 2\pi/k_1$, and that the roots $\lambda_1, \ldots, \lambda_{l_1}$ of $P_1(\lambda, 0) = 0$ satisfy

$$\lambda_i = 0 \quad \text{or} \quad \lambda_i \in \mathbb{C} \setminus \overline{p(S_{k_i,1})} \quad \text{for } i = 1, \ldots, l_1.$$

Then, $w_1(\xi, x)$ in Proposition 7.4 has an analytic continuation $w_1^*(\xi, x)$ on $S_I \times D_{\rho_1}$ for some $\rho_1 > 0$ and we have

$$|w_1^*(\xi, x)| \leq \frac{M_1}{\left| \xi^{k_1+1} \right|^l} \left| \xi \right|^{\mu_0 - k_1} \exp(b_1 |\xi|^\kappa) \quad \text{on } S_I \times D_{\rho_1} \quad (7.10)$$

for some $M_1 > 0$ and $b_1 > 0$.

**7.4. Another expression of (5.4) (again)**

Suppose $p^* \geq 1$. Recall that the Newton polygon of the equation (5.4) is just $N((4.1), \bar{a}) = (0, p)$. We set $h_i = k_il_i - e_i + p \ (1 \leq i \leq p^*)$. Then, we have $h_i \in \mathbb{N} / L$, the line containing $\Gamma_i - (0, p)$ is given by the equation $y = k_ix - h_i$, and so

$$0 \leq h_1 < h_2 < \cdots < h_{p^*}.$$ 

Since $k_i(l_i - l_{i-1}) = e_i - e_{i-1}$ holds, we have also $h_i = k_i(l_i - 1) - e_{i-1} + p$. In particular, we have $h_1 = k_1l_1$.

By the same argument as in the proof of Proposition 7.1 we have

**Proposition 7.6.** (1) Let $1 \leq i \leq p^*$. By multiplying the equation (5.4) by $t^{h_i}$, it is expressed in the form

$$P_1(t^{k_i+1}\partial_t,x)v = t^{h_i}f^*(t,x) + \sum_{(j,\alpha) \in \Lambda} \omega_{i,\alpha}(t,x) \left[ t^{k_i|\alpha|} (t^{k_i+1}\partial_t)^j \partial_x^\alpha v \right]$$

$$+ \sum_{|\nu| \geq 2} c_{i,\nu}(t,x) t^{h_i+|\nu|} \prod_{(j,\alpha) \in \Lambda} \left( t^{k_i|\alpha|} (t^{k_i+1}\partial_t)^j \partial_x^\alpha \right)^{\nu_{j,\alpha}} \quad (7.11)$$

for some suitable $\omega_{i,\alpha}(t,x) \in \mathcal{H}_{n_{i,\alpha}}(S_I(r) \times D_R) \ ((j, \alpha) \in \Lambda)$ with $n_{i,\alpha} \in \mathbb{N} / L$ $(\langle j, \alpha \rangle \in \Lambda)$, $c_{i,\nu}(t,x) \in \mathcal{H}_0(S_I(r) \times D_R) \ (|\nu| \geq 2)$, and $e_{i,\nu} \in \mathbb{N} / L \ (|\nu| \geq 2)$. Besides, we have $|c_{i,\nu}(t,x)| \leq C_{i,\nu}$ on $S_I(r) \times D_R \ (|\nu| \geq 2)$ for some $C_{i,\nu} \geq 0 \ (|\nu| \geq 2)$ with

$$\sum_{|\nu| \geq 2} C_{i,\nu} z^{L_{i,\nu}} X^{|\nu|} \in \mathbb{C}[z, X].$$

(2) If $1 \leq i \leq p^* - 1$, we have

$$k_{i+1} = \min_{l_{i+1} \leq j + |\alpha| \leq m_{i,\alpha} \omega_{i,\alpha}(t,x) \neq 0} \frac{m_{i,j,\alpha} + k_i(j + |\alpha| - l_i)}{j + |\alpha| - l_i},$$
\[ k_{i+1} < \frac{e_{i,v} + k_i(u)_{l_i} + \mu_0(|\nu| - 1)}{m_\nu - l_i}, \quad \text{if } |\nu| \geq 2, \quad m_\nu \geq l_i + 1. \]

(3) Since (7.11) depends on \( i \), we write (7.11). Then, we have the following equalities:

\[ t^{h_{i+1} - h_i} \times (7.11)_i = (7.11)_{i+1}, \quad 1 \leq i \leq p^* - 1. \] 

8. Proof of Theorem 4.8

Suppose \( p^* \geq 1 \), and let us prove Theorem 4.8. Let

\[ \hat{v}(t, x) = \sum_{n \geq 1} v_n(t, x) \]

be the formal solution of (5.4) on \( S_f(r) \times D_R \) constructed in Theorem 6.1. Take any \( J_i \in J \) (with \( |J_i| > \pi/k_1 \) ), \( 0 < r_1 < r \), \( 0 < R_1 < R \), and fix them. Take any non-empty open interval \( I \) satisfying \( I + [\pi/2k_1] \subset J_i \) and (4.8). Then, we have \( I \in \mathcal{I}_p \) and by Proposition 4.9 we can find non-empty open intervals \( I_1, I_2, \ldots, I_{p-1} \) satisfying the conditions \( I_i \subset I \) (1 \( \leq i \leq p^* - 1 \) ) and

\[ I \in \mathcal{I}_p \text{ and } I_{i+1} \in I_i + [\pi/2k_{i+1}] \quad (1 \leq i \leq p^* - 1) \]

under the setting \( I_{p^*} = I \). Therefore, by Lemma 4.6 we have \( 0 < |I_i| < 2 \pi/k_i \) (1 \( \leq i \leq p^* \) ) and

\[ \lambda_{i,d} \in \mathbb{C} \setminus \overline{p(S_{k_i}I_i)} \quad \text{for } d = 1, \ldots, l_i - l_{i-1} \quad (1 \leq i \leq p^*). \]

[Step 1]. We set

\[ w_1(\xi, x) = \sum_{n \geq 1} B_k[v_n](\xi, x). \]

Since \( I_1 \subset I \) holds, by Proposition 7.4 and Theorem 7.5 we know that \( w_1(\xi, x) \) is well-defined as a holomorphic solution of (7.7) on \( S_{\lambda_1}(\delta) \times D_{\rho_1} \) for some \( \delta > 0 \), and that it has an analytic continuation \( w^*_1(\xi, x) \) on \( S_{I_1} \times D_{\rho_1} \) for some \( \rho_1 > 0 \) satisfying the estimate (7.10) for some \( M_1 > 0 \) and \( b_1 > 0 \). 

If \( p^* = 1 \), we have \( l_1 = m \), and so \( D_a = \emptyset \) and \( D_c = \emptyset \) hold in Theorem 7.5. This means that \( \sigma_0 = 1 \) and so \( \kappa = k_1 \) holds. Since \( I_1 = I \), the estimate (7.10) is written as

\[ |w^*_1(\xi, x)| \leq \frac{M_1}{(|\xi|^{k_1} + 1)^{l_1}} |\xi|^{m_0 - k_1} \exp(b_1|\xi|^{k_1}) \quad \text{on } S_{I_1} \times D_{\rho_1}. \]

Thus, we see that our formal solution \( \hat{v}(t, x) \) is \( (k_1) \)-summable in \( I \)-direction in the sense of Definition 3.3. This proves Theorem 4.8 in the case \( p^* = 1 \).

[Step 2]. Let us consider the case \( p^* \geq 2 \). In this case, by (7.2), (7.3) in Proposition 7.1 and by the definition of \( \sigma_0 \) in subsection 7.3, we can see that \( \sigma_0 = 1 + 1/k_2 \) and so \( 1/\kappa = 1/k_1 - 1/k_2 = 1/k_{1,2} \). Therefore, the estimate (7.10) is written as

\[ |w^*_1(\xi, x)| \leq \frac{M_1}{(|\xi|^{k_1} + 1)^{l_1}} |\xi|^{m_0 - k_1} \exp(b_1|\xi|^{k_{1,2}}) \quad \text{on } S_{I_1} \times D_{\rho_1}. \]
and we can define
\[ w_2(\xi, x) = A_{k_2,k_1}[w_1^\ast](\xi, x). \] (8.1)

For \((j, \alpha) \in \Lambda\), we write
\[ \mathcal{M}_{2,j,\alpha}[W] = \begin{cases} \frac{\xi^{k_2|\alpha|} - k_2}{\Gamma(|\alpha|)} *_{k_2} \left((k_2\xi^{k_2})^j W\right), & \text{if } |\alpha| > 0, \\ (k_2\xi^{k_2})^j W, & \text{if } |\alpha| = 0. \end{cases} \]

Then we have:

**Lemma 8.1.** The function \(w_2(\xi, x)\) in (8.1) is well-defined as a holomorphic function on \(S_{I_2}(\delta_2) \times D_{\rho_1}\), for some \(\delta_2 > 0\), and we have the estimate \(|w_2(\xi, x)| \leq A_2|\xi|^{\mu_{k_2} - k_2}\) on \(S_{I_2}(\delta_2) \times D_{\rho_1}\), for some \(A_2 > 0\). In addition, \(w_2(\xi, x)\) satisfies the following equation:

\[ P_2(k_2\xi^{k_2}, x)w_2 = B_{k_2}[h^2 f^\ast](\xi, x) + \sum_{(j, \alpha) \in \Lambda} B_{k_2}[\mathcal{M}_{2,j,\alpha}](\xi, x) *_{k_2} \left(\mathcal{M}_{2,j,\alpha}[\partial_{k_2}^\alpha w_2]\right) \]

\[ + \sum_{|\nu| \geq 2} B_{k_2}[\xi^{2,\nu^2}] (\xi, x) *_{k_2} \prod_{(j, \alpha) \in \Lambda} *_{k_2} \left(\mathcal{M}_{2,j,\alpha}[\partial_{k_2}^\alpha w_2]\right) *_{k_2} \nu_{j,\alpha} \] (8.2)

on \(S_{I_2}(\delta_2) \times D_{\rho_1}\), which is just the equation obtained by applying \(k_2\)-Borel transform to \((7.11)\).

**Proof.** Since \(I_2 \subseteq I_1 + [\pi/2\kappa_{1,2}]\) holds, the former half of this lemma is clear from Proposition 3.2.

Let us show (8.2). Since \(w_1(\xi, x)\) satisfies (7.7) on \(S_{I_1}(\delta) \times D_{\rho_1}\), by the analytic continuation we see that \(w_1^\ast(\xi, x)\) satisfies the equality (7.7) on \(S_{I_1} \times D_{\rho_1}\). Therefore, to show (8.2) it is enough to prove the equality:

\[ (8.2) = \frac{\xi^{(h_2-h_1)-k_2}}{\Gamma((h_2-h_1)/k_2)} *_{k_2} A_{k_2,k_1}[(7.7)] \text{ on } S_{I_2}(\delta_2) \times D_{\rho_1}. \] (8.3)

The proof of this fact is as follows.

1. In the case \(w_1^\ast(\xi, x) \in E_{k_2}(S_{I_1} \times D_{\rho_1})\), the equality (8.3) is verified as follows. In this case, we have \(w_2(\xi, x) = B_{k_2} \circ \mathcal{L}_{k_1}[w_1^\ast]\). Since \(w_1^\ast(\xi, x)\) satisfies (7.7) on \(S_{I_1} \times D_{\rho_1}\), by applying \(\mathcal{L}_{k_1}\) to (7.7) we have the equality (7.1) with \(v = \mathcal{L}_{k_1}[w_1^\ast]\) on \(S_{I_1+|\pi/2k_1-\epsilon|}(r) \times D_{\rho_1}\) for some \(\epsilon > 0\) (sufficiently small) and \(r > 0\), that is, we have (7.11) on \(S_{I_1+|\pi/2k_1-\epsilon|}(r) \times D_{\rho_1}\). Then, let us multiply (7.11) by \(\xi^{h_2-h_1}\); by (7.12) we have the equality (7.11) on \(S_{I_1+|\pi/2k_1-\epsilon|}(r) \times D_{\rho_1}\). Since \(I_2 \subseteq I_1 + [\pi/2\kappa_{1,2}]\) is supposed, we have \(I_2 \subseteq I_1 + [\pi/2\kappa_{1,2} - \epsilon]\) and so by applying \(B_{k_2}\) to this (7.11) we have the equality (8.2) (with \(w_2 = B_{k_2}[v]\) on \(S_{I_2} \times D_{\rho_1}\)).

By summing up, we have \(w_2 = B_{k_2}[v] = B_{k_2}[\mathcal{L}_{k_1}[w_1^\ast]] = A_{k_2,k_1}[w_1^\ast]\), and

\[ (8.2) = B_{k_2}[\xi^{(h_2-h_1) - k_2} \mathcal{L}_{k_1}[(7.7)]] = B_{k_2}[\xi^{(h_2-h_1)}] *_{k_2} \left(B_{k_2} \circ \mathcal{L}_{k_1}[(7.7)]\right) \]

\[ = \frac{\xi^{(h_2-h_1)-k_2}}{\Gamma((h_2-h_1)/k_2)} *_{k_2} A_{k_2,k_1}[(7.7)]. \]
This proves (8.3).

(2) The general case can be proved by approximations. First, we show the equalities:

\[ \frac{\xi^{(h_2-h_1)}-k_2}{\Gamma((h_2-h_1)/k_2)} *_{k_2} A_{k_2,k_1} \left[ P_2(k_2 \xi^{k_2}, x)w_1^* - \sum_{(j,\alpha) \in \Lambda} B_{k_2}[A_{j,\alpha}[\partial_x \partial_y w_1^*]] \right] \]

= \[ P_2(k_2 \xi^{h_2}, x)A_{k_2,k_1} \left[ w_1^* - \sum_{(j,\alpha) \in \Lambda} B_{k_2}[A_{j,\alpha}[\partial_x \partial_y w_1^*]] \right] \] \quad (8.4)

\[ \frac{\xi^{(h_2-h_1)}-k_2}{\Gamma((h_2-h_1)/k_2)} *_{k_2} A_{k_2,k_1} \left[ \sum_{|\nu|=N} B_{k_2}[c_{1,\nu} \epsilon^{1,\nu}] *_{k_1} \prod_{(j,\alpha) \in \Lambda} k_1 \left( A_{j,\alpha}[\partial_x \partial_y w_1^*] \right) \right] \]

= \[ \sum_{|\nu|=N} B_{k_2}[c_{2,\nu} \epsilon^{2,\nu}] *_{k_2} \prod_{(j,\alpha) \in \Lambda} k_2 \left( A_{j,\alpha}[\partial_x \partial_y w_1^*] \right) \] \quad (8.5)

for \( N = 2, 3, \ldots \). The proof is as follows.

Let \( \theta \) be the middle point of \( I_1 \), take a \( \sigma > 0 \) so that \( \sigma > \kappa_2 \) and \( \pi/2\sigma < |I_1|/2 \), and set \( \alpha = e^{-\sqrt{\log \sigma}} \). Take an \( \gamma > 0 \) so that \( 0 < \gamma < \pi/2\sigma \), and set \( I_{1,0} = (\theta - \pi/2\sigma + \gamma, \theta + \pi/2\sigma - \gamma) \); then we have \( I_{1,0} \subseteq I_1 \) and

\[ |\exp(-\eta \alpha \xi^\sigma)| \leq |\exp(-\eta \sin(\gamma \sigma))| |\xi^\sigma| \quad \text{on} \quad S_{I_{1,0}}. \]

By setting \( w_{1,0}(\xi, x) = w_0^*(\xi, x) \exp(-\eta \alpha \xi^\sigma) \) (for \( \eta > 0 \)), we have \( w_{1,0}(\xi, x) \in \mathcal{E}_{k_1}(S_{I_{1,0}} \times D_{\rho_1}) \), \( |w_{1,0}(\xi, x)| \leq |w_0^*(\xi, x)| \) on \( S_{I_{1,0}} \times D_{\rho_1} \), and \( w_{1,0}(\xi, x) \rightarrow w_0^*(\xi, x) \) (as \( \eta \rightarrow 0 \)) for any \( (\xi, x) \in S_{I_{1,0}} \times D_{\rho_1} \).

Then, the above equalities (8.4) and (8.5) are valid on \( S_{I_{1,0} + \pi/2\kappa_2} \times D_{\rho_1} \) for any \( w_{1,0}(\xi, x) \). Since the both sides (8.4) and (8.5) are represented in integral form, if we let \( \eta \rightarrow 0 \), use Lebesgue’s convergence theorem and take into account of Proposition 3.2, we can take \( r > 0 \) such that the equalities (8.4) and (8.5) are valid on \( S_{I_{1,0} + \pi/2\kappa_2}(r) \times D_{\rho_1} \) also for \( w_0^*(\xi, x) \). Since the both sides of (8.4) and (8.5) are well-defined on \( S_{I_{1} + \pi/2\kappa_2}(r) \times D_{\rho_1} \) for \( r > 0 \) (depending on \( \epsilon > 0 \)), by the unique continuation property for holomorphic functions we have the equalities (8.4) and (8.5) on \( S_{I_{1} + \pi/2\kappa_2}(r) \times D_{\rho_1} \). By taking \( \epsilon > 0 \) sufficiently small we have the condition \( I_2 \subseteq I_1 + \pi/2\kappa_2 - \epsilon \). Then, we have the equalities (8.4) and (8.5) on \( S_{I_2}(r) \times D_{\rho_1} \), and so on \( S_{I_2}(\delta_2) \times D_{\rho_1} \) (by retaking \( \delta_2 > 0 \) if necessary).

Next, we show the equalities:

\[ \frac{\xi^{(h_2-h_1)}-k_2}{\Gamma((h_2-h_1)/k_2)} *_{k_2} A_{k_2,k_1} \left[ \sum_{|\nu|=N} B_{k_2}[c_{1,\nu} \epsilon^{1,\nu}] *_{k_1} \prod_{(j,\alpha) \in \Lambda} k_1 \left( A_{j,\alpha}[\partial_x \partial_y w_1^*] \right) \right] \]

= \[ \sum_{N \geq 2} \frac{\xi^{(h_2-h_1)}-k_2}{\Gamma((h_2-h_1)/k_2)} *_{k_2} A_{k_2,k_1} \left[ \sum_{|\nu|=N} B_{k_2}[c_{1,\nu} \epsilon^{1,\nu}] *_{k_1} \prod_{(j,\alpha) \in \Lambda} k_1 \left( A_{j,\alpha}[\partial_x \partial_y w_1^*] \right) \right] \]

= \[ \sum_{N \geq 2} \sum_{|\nu|=N} B_{k_2}[c_{2,\nu} \epsilon^{2,\nu}] *_{k_2} \prod_{(j,\alpha) \in \Lambda} k_2 \left( A_{j,\alpha}[\partial_x \partial_y w_1^*] \right) \] \quad (8.6)

The first equality follows from Lebesgue’s convergence theorem, and the second equality follows from (8.5). The fact that Lebesgue’s convergence theorem can be used is verified by checking that the infinite series in the left-hand side of (8.6) is abso-
for some $M$ in the framework of [24] by using [Lemmas 3.4 and 3.5 in [24]] (under the setting $\phi_0(t; c) = (|t^{a-k_1}/\Gamma(a/k_1)| \exp(c|t^{a-k_1}|))$, and so we may omit the details.

Thus, by (8.4) and (8.6) we have (8.3).

By Lemma 8.1 we saw that $w_2(\xi, x)$ is a holomorphic function on $S_{I_1}(\delta_2) \times D_{\rho_1}$ with $|w_2(\xi, x)| \leq A_2|\xi|^{p_0-k_2}$ on $S_{I_1}(\delta_2) \times D_{\rho_1}$, and it satisfies the equation (8.2). Hence, by applying [Theorem 5.1 in [24]] to (8.2) and by the same argument as in Step 2 we have

**Lemma 8.2.** The function $w_2(\xi, x)$ has an analytic continuation $w_2^*(\xi, x)$ on $S_{I_2} \times D_{\rho_2}$ for some $\rho_2 > 0$, and we have

$$|w_2^*(\xi, x)| \leq \frac{M_2}{(|\xi|^{k_2} + 1)^{2k}} \times |\xi|^{p_0-k_2} \exp(b_2|\xi|^{\kappa_2+3}) \quad \text{on} \quad S_{I_2} \times D_{\rho_2}$$

for some $M_2 > 0$ and $b_2 > 0$.

[Step 3.]. In this way, we can show inductively on $i (1 \leq i \leq p^*)$ that the functions $w_1(\xi, x), \ldots, w_i(\xi, x)$ are well-defined by

$$w_1(\xi, x) = \tilde{B}_{k_1}[\tilde{v}](\xi, x),$$

$$w_2(\xi, x) = A_{k_2, k_1}[w_1^*](\xi, x),$$

$$\ldots \ldots \ldots$$

$$w_i(\xi, x) = A_{k_i, \ldots, k_1}[w_{i-1}^*](\xi, x)$$

and that $w_i(\xi, x)$ has an analytic continuation $w_i^*(\xi, x)$ on $S_{I_i} \times D_{\rho_i}$ satisfying the estimate

$$|w_i^*(\xi, x)| \leq \frac{M_i}{(|\xi|^{k_i} + 1)^{k_i}} \times |\xi|^{p_0-k_i} \exp(b_i|\xi|^{\kappa_i+1}) \quad \text{on} \quad S_{I_i} \times D_{\rho_i}$$

for some $M_i > 0$, $b_i > 0$ and $\rho_i > 0$. In addition, we see that $w_i^*(\xi, x)$ satisfies the equation

$$P_i(k_i \xi^{k_i}, x)w_i^* = B_{k_i}[h^* f^*](\xi, x) + \sum_{(j, \alpha) \in \Lambda} B_{k_i}[\mathcal{M}_{i,j,\alpha}](\xi, x) *_{k_i} (\mathcal{M}_{i,j,\alpha}[\partial^\alpha w_i^*])$$

$$+ \sum_{|v| \geq 2} B_{k_i}[\mathcal{M}_{i,j,\alpha}^v](\xi, x) *_{k_i} \prod_{(j, \alpha) \in \Lambda} (\mathcal{M}_{i,j,\alpha}[\partial^\alpha w_i^*])^{k_i, v_{i,j,\alpha}}$$

on $S_{I_i} \times D_{\rho_i}$, which is just the equation obtained by applying $k_i$-Borel transform to (7.11). In the above, $\mathcal{M}_{i,j,\alpha}$ ($i = (j, \alpha) \in \Lambda$) are defined in the same way as $\mathcal{M}_{1,j,\alpha}$ and $\mathcal{M}_{2,j,\alpha}$.

Finally, in the case $i = p^*$ we have

**Lemma 8.3.** The function $w_{p^*}(\xi, x)$ has an analytic continuation $w_{p^*}^*(\xi, x)$ on $S_{I_{p^*}} \times D_{\rho_{p^*}}$, satisfying the estimate

$$|w_{p^*}^*(\xi, x)| \leq \frac{M_{p^*}}{(|\xi|^{k_{p^*}} + 1)^{k_{p^*}}} \times |\xi|^{p_0-k_{p^*}} \exp(b_{p^*}|\xi|^{\kappa_{p^*}+p^*+1}) \quad \text{on} \quad S_{I_{p^*}} \times D_{\rho_{p^*}}$$
for some $M_{p^*} > 0$, $b_{p^*} > 0$ and $\rho_{p^*} > 0$. In addition, $w_{p^*}(\xi, x)$ satisfies the equation (8.7) for $i = p^*$ on $S_{I_{p^*}} \times D_{p_{p^*}}$.

Since $I = I_{p^*}$ and $\kappa_{p, p^*+1} = k_{p^*}$ hold, the assertion of Lemma 8.3 concludes that our formal series $\hat{v}(t, x)$ in Theorem 6.1 is $(k_{p^*}, \ldots, k_1)$-multisummable in $L$-direction in $t$ (uniformly in $x \in D_\rho$ with $\rho = \rho_{p^*}$).

Since $w_{p^*}(\xi, x)$ satisfies the equation (8.7) (with $i = p^*$), by applying $\mathcal{L}_{k_{p^*}}$ to the both sides of (8.7) (with $i = p^*$) we see that the $(k_{p^*}, \ldots, k_1)$-multisum

$$v(t, x) = \mathcal{L}_{k_{p^*}}[w_{p^*}^*(t, x)]$$  \hspace{1cm} (8.8)

is a true solution of (7.11)$_{p^*}$ and so by cancelling the factor $t^{h_{p^*}}$ we see that $v(t, x)$ is a true solution of (5.4) on $S_{I_{p^*+[\pi/2k_{p^*}+\varepsilon]}}(\delta_\epsilon) \times D_\rho$ for some $\delta_\epsilon > 0$ (depending on $\epsilon > 0$). Thus, by Proposition 3.4 we have

**Lemma 8.4.** $v(t, x)$ in (8.8) is a true solution of (5.4) on $S_{I_{p^*+[\pi/2k_{p^*}+\varepsilon]}}(\delta_\epsilon) \times D_\rho$. If $\epsilon > 0$ is fixed, there are $C > 0$ and $H > 0$ such that

$$|v(t, x) - \hat{v}_{N,1}(t, x)| \leq CH^N \Gamma(N/Lk_1)|t|^{N/L}$$

holds on $S_{I_{p^*+[\pi/2k_{p^*}+\varepsilon]}}(\delta_\epsilon) \times D_\rho$ for any $N \geq 1$.

[Step 4]. By setting

$$\hat{W}(t, x) = \hat{u}_N(t, x) + t^\varrho \hat{v}(t, x),$$

we have a formal solution $\hat{W}(t, x)$ of (4.1) on $S_{I_{p^*}(r_1)} \times D_{R_1}$ which is equivalent to $\hat{u}(t, x)$ and is $(k_{p^*}, \ldots, k_1)$-multisummable in $L$-direction in $t$ (uniformly in $x \in D_\rho$). Thus, by

$$W(t, x) = \hat{u}_N(t, x) + t^\varrho \times v(t, x)$$

we have a true solution $W(t, x)$ of (4.1) on $S_{I_{p^*+[\pi/2k_{p^*}+\varepsilon]}}(\delta_\epsilon) \times D_\rho$.

By replacing $I - [\epsilon]$ by $I$ we have the result in Theorem 4.8. This complete the proof of Theorem 4.8.

[Step 5]. As a consequence, we have

**Corollary 8.5 (Asymptotic existence theorem).** Under the above situation, we have the following result: for any $A > 0$ there is an $N_0 \in \mathbb{N}^*$ such that for any $N \geq N_0$ we have

$$\|W(t, x) - \hat{u}_N(t, x)\|_\rho = O(|t|^A) \quad (\text{as } S_{I_{p^*+[\pi/2k_{p^*}+\varepsilon]}} \ni t \rightarrow 0).$$

By replacing $I - [\epsilon]$ by $I$ we have the result in Theorem 4.10.

9. Supplementary results

In this section, we present two supplementary results, one when (4.4) is replaced by a weaker condition and the other when equation (4.1) is not necessarily holomorphic.
9.1. On the condition (4.4)

Until now, we have treated the equation (4.1) under the condition (4.4). In this subsection, instead of (4.4) we suppose:

\[(j + |\alpha|, p_j, \alpha) \in \Gamma_1 \cup \ldots \cup \Gamma_p, \implies \alpha = 0 \in \mathbb{N}^K. \tag{9.1}\]

In this case, we set \(\Lambda(\Gamma_0) = \{(j, \alpha) \in \Lambda : (j + |\alpha|, p_j, \alpha) \in \Gamma_0\}\) and suppose:

(a6) If \((j, \alpha) \in \Lambda(\Gamma_0)\) we have \((j, \alpha) \in \mathcal{F}(\tilde{u})\).

Under the notation in (4.2) we set

\[L(x, \lambda, \zeta) = \sum_{(j, \alpha) \in \Lambda(\Gamma_0)} a_{j, \alpha}^0(x) \lambda^j \zeta^\alpha.\]

Then, by the same argument as in (5.4) we can reduce our problem to a problem on the following equation with respect to \(v\):

\[L(x, t \partial_t + q, \partial_x)v + \sum_{(j, \alpha) \in \Lambda} a_{j, \alpha}(t, x)(t \partial_t)^j \partial_x^\alpha v = f^*(t, x) + \sum_{|\nu| \geq 2} b_{\nu}(t, x)t^{q(|\nu| - 1) - p} \prod_{(j, \alpha) \in \Lambda} ((t \partial_t)^j \partial_x^\alpha v)^{\nu_{j, \alpha}}. \tag{9.2}\]

In order to construct a new formal solution \(\hat{v}(t, x) = \sum_{n \geq 1} v_n(t, x)\) of this equation (9.2), we need to solve the following equation:

\[L(x, t \partial_t + q, \partial_x)w = g(t, x) \in X(S_J(r) \times D_R), \tag{9.3}\]

where \(X(S_J(r) \times D_R)\) is the same as in Lemma 6.4. If the following condition (H) is valid, all the arguments in sections 6, 7 and 8 work also in this case.

(H) (9.3) is uniquely solvable in \(X(S_J(r) \times D_R)\) and we have the same estimate as in Lemma 6.4 (or Lemma 9.2 given below).

Hence, we have

**Theorem 9.1.** Suppose \((A_1), (A_2), (4.3), (9.1), (a_1) \sim (a_6), and (H)\). Then, we have the same results as in Theorems 4.7, 4.8 and 4.10.

9.2. Using a different norm

In section 6 we used the supremum norm \(\|f(t)\|_{\rho}\), but it is also possible to use a different norm. For \(f(t, x) = \sum_{t \geq 0} f(t, x)^l \in \mathcal{O}(S_J(r) \times D_R)\) with \((t, x) \in \mathcal{C}_t \times \mathcal{C}_x\) we set

\[\|f(t)\|_\rho = \sum_{l \geq 0} |f_l(t)|\rho^l\]

which is a formal power series in \(\rho\) with coefficients in \(C^0(S_J(r))\). We write \(\sum_{l \geq 0} a_l \rho^l \ll \sum_{l \geq 0} b_l \rho^l\) if \(|a_l| \leq b_l\) holds for all \(l \geq 0\).

For example, let us consider the case:

\[L(x, \lambda, \zeta) = \lambda(\lambda - 1) + x\zeta - 1, \quad l_0 = 2, \tag{9.4}\]
which comes from Example 10.4. Take \( q \in \mathbb{N}^* \) and \( c > 0 \) so that \( q \geq c + (1 + \sqrt{5})/2 \). Then, for any \( l \in \mathbb{N} \) the roots \( \lambda_{l,1}, \lambda_{l,2} \) of the equation \((\lambda + q)(\lambda + q - 1) + l - 1 = 0\) satisfy

\[
\text{Re} \lambda_{l,i} \leq -c \quad \text{for} \ i = 1, 2. \tag{9.5}
\]

Under this situation, let us consider the equation (9.3). We have

**Lemma 9.2.** For any \( g(t, x) \in X(S_J(r) \times D_R) \) the equation (9.3) has a unique solution \( w(t, x) \in X(S_J(r) \times D_R) \). In addition, if \( g(t, x) \) satisfies

\[
\|g(t)\|_\rho \leq \frac{|A|t^n}{d(t, \rho)^a} \quad \text{on} \ S_J(r) \tag{9.6}
\]

for some \( A > 0, n \geq 0 \) and \( a \geq 0 \), the unique solution \( w(t, x) \) satisfies

\[
\|w(t)\|_\rho \leq \frac{|A|t^n}{(c + n)d(t, \rho)^a} \quad \text{on} \ S_J(r). \tag{9.7}
\]

If we replace the supremum norm \( \|f(t)\|_\rho \) and “\( \leq \)" by the norm \( \|f(t)\|_\rho \) and “\( \ll \)”, respectively, the discussion in section 6 can be modified so that it works also in the framework with \( \|f(t)\|_\rho \) and “\( \ll \)”. This proves Theorem 9.1 in the case with Lemma 9.2.

**Proof of Lemma 9.2.** In the case (9.4), our equation (9.3) is written as

\[
((t \partial_t + q)(t \partial_t + q - 1) + x \partial_x - 1)w = g(t, x). \tag{9.8}
\]

Set

\[
w(t, x) = \sum_{l \geq 0} w_l(t)x^l, \quad g(t, x) = \sum_{l \geq 0} g_l(t)x^l;
\]

then the equation (9.8) is equivalent to

\[
((t \partial_t + q)(t \partial_t + q - 1) + l - 1)w_l = g_l(t), \quad l = 0, 1, 2, \ldots.
\]

Since (9.5) holds, the unique solution \( w_l(t) \) is expressed as

\[
w_l(t) = \mathcal{H}_{x,t} \circ \mathcal{H}_{x,l}[g_l(t)], \quad l = 0, 1, 2, \ldots.
\]

This shows that the equation (9.8) is uniquely solvable in \( X(S_J(r))[[x]] \) (the ring of formal power series in \( \rho \) with coefficients in \( X(S_J(r)) \)).

If \( g(t, x) \in X(S_J(r) \times D_R) \) holds, for any \( J_l \in J, 0 < r_1 < r \) and \( 0 < R_1 < R \) there is an \( M > 0 \) such that \( |g(t, x)| \leq M \) on \( S_{J_l}(r_1) \times D_{R_1} \). Then we have \( |g_l(t)| \leq M/R_1^l \) \( (l = 0, 1, 2, \ldots) \) on \( S_{J_l}(r_1) \), and so by the same argument as in Lemma 6.4 we have \( |w_l(t)| \leq (M/R_1^l)/c^2 \) \( (l = 0, 1, 2, \ldots) \) on \( S_{J_l}(r_1) \). This yields

\[
|w(t, x)| \leq \sum_{l \geq 0} \frac{M}{R_1^l c^2} |x^l| = \frac{M/c^2}{1 - |x|/R_1} \quad \text{on} \ S_{J_l}(r_1) \times D_{R_1}.
\]
Since $J_1, r_1$ and $R_1$ are taken arbitrarily, this shows that $w(t, x) \in X(S_J(r) \times D_R)$ holds. Thus, we have seen that the equation (9.8) is uniquely solvable in $X(S_J(r) \times D_R)$.

Suppose that $g(t, x)$ satisfies (9.6). Since

$$\frac{1}{(R - \rho)^a} = \sum_{i \geq 0} \frac{a(a + 1) \cdots (a + l - 1)}{R^{a+l} l!} \rho^l$$

holds, by (9.6) we have

$$|g(t)| \leq \frac{(|J|/2)^a|t|^n}{d(t)^a} \cdot \frac{a(a + 1) \cdots (a + l - 1)}{R^{a+l} l!}, \quad l = 0, 1, 2, \ldots$$

Therefore, by the same calculation as in the proof of Lemma 6.4 we have

$$|w_l(t)| \leq \frac{1}{(c + n)^2} \frac{(|J|/2)^a|t|^n}{d(t)^a} \cdot \frac{a(a + 1) \cdots (a + l - 1)}{R^{a+l} l!}, \quad l = 0, 1, 2, \ldots$$

Since $l_0 = 2$, this leads us to (9.7).

### 9.3. On the condition $(A_1)$

Let $J_0$ be an open interval with $|J_0|$ being sufficiently large, let $r_0 > 0, R_0 > 0$ and $\rho_0 > 0$ and set $K^* = S_{J_0}(r_0) \times DR_0 \times \{Z \in \mathbb{C}^N; |Z| < \rho_0\}$. Until now, in the condition $(A_1)$ we have supposed that $F(t, x, Z)$ is a holomorphic function in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}^M_Z$. However, by the proof of Theorems 4.7, 4.8 and 9.1 we understand that $(A_1)$ and $(A_2)$ can be replaced by $(A_1)^*$ and $(A_2)^*$ given below.

$(A_1)^*$ $F(t, x, Z)$ is a bounded holomorphic function on $K^*$.

$(A_2)^*$ Equation (4.1) has a formal solution $\hat{u}(t, x)$ on $S_J(r) \times DR$ (in the sense of Definition 2.1) for some open interval $J \subset J_0, r > 0$ and $R > 0$.

**Theorem 9.3.** Theorems 4.7, 4.8, 4.10 and Theorem 9.1 hold even if conditions $(A_1)$ and $(A_2)$ are replaced by conditions $(A_1)^*$ and $(A_2)^*$.

### 10. Examples

In this section, we give some typical examples. Recall that $[x]$ denotes the integer part of $x \in \mathbb{R}$, and $[\ell^q]_q = (\ell^q_1)(\ell^q_2) \cdots (\ell^q_{q+1})$ for $q \in \mathbb{N}^*$.

**Example 10.1.** Let $(t, x) \in \mathbb{C}^2$, and let us consider the following nonlinear partial differential equation:

$$\partial^2_t u - a(x)(\partial_t u)^2 = h(x)u + (\partial^2_x u)^n + t^p \partial^q_t u + f(t, x), \quad (10.1)$$

where $a(x) \in \mathcal{O}_R$ with $a(0) \neq 0, h(x) \in \mathcal{O}_R, n \in \mathbb{N}^*, p \in \mathbb{N}, q \in \mathbb{N}^*$ satisfying $p \geq q - 1$ and $f(t, x)$ is a holomorphic function in a neighborhood of $(t, x) = (0, 0)$. Then, for any $\varphi(x) \in \mathcal{O}_R$ the equation (10.1) has a unique formal solution of the form

$$\hat{u}_1(t, x) = \frac{\log t}{a(x)} + \varphi(x) + b(x)t + \sum_{i \geq 2, [n/i] \geq 2, j \geq 0} \varphi_{i,j}(x)t^i(\log t)^j \quad (10.2)$$
with \( b(x) = (-1)^q (q - 1)! / (2a(x)) \) (if \( p = q - 1 \)), \( b(x) = 0 \) (if \( p > q - 1 \)) and suitable \( \varphi_{i, j}(x) \in \mathcal{O}_R \), \( i \geq 2, \lfloor ni/2 \rfloor \geq j \geq 0 \) for some \( R > 0 \).

(1) The part \(-t \log t / a(x) + \varphi(x)\) of \( \hat{u}(t, x) \) comes from the fact that it is a solution of the equation: \( \partial_t^2 u - a(x)(\partial_t u)^2 = 0 \). The mechanism of the appearance of logarithmic singularities was studied by Tahara-Yamane [25] in a general setting.

(2) If \( q \leq 2 \), this formal solution is convergent in a suitable domain near \((t, x) = (0, 0)\) and defines an analytic solution (see [25]), but if \( q \geq 3 \), it is not convergent in general.

(3) In the case \( q \geq 3 \), we can apply Theorem 4.8 (and Theorem 4.10) under \( \rho^* = 1 \), \( \Xi_1 = \{2\pi j / (q - 2) ; j \in \mathbb{Z}\}, \mathcal{Z}_2 = \mathbb{R} \setminus \Xi_1 \) and \( k_1 = (p + 2 - q) / (q - 2) \). Hence, for any \( I \in \mathcal{Z}_2 \), we have a true solution \( u_1(t, x) \) on \( S_{I + \pi / 2k_1}(\delta) \times D_\rho \) for some \( \delta > 0 \) and \( \rho > 0 \), that admits \( \hat{u}_1(t, x) \) as an asymptotic expansion (as \( t \to 0 \)).

**Proof.** Take any \( \varphi(x) \in \mathcal{O}_R \), and set

\[
  u(t, x) = -\alpha(x) \log t + \varphi(x) + b(x)t + tw(t, x),
\]

where \( \alpha(x) = 1 / a(x) \), \( b(x) = (-1)^q (q - 1)! / (2a(x)) \) (if \( p = q - 1 \)), and \( b(x) = 0 \) (if \( p > q - 1 \)). Then, we can reduce (10.1) to the following equation with respect to \( w \):

\[
  (t \partial_t + 1)(t \partial_t + 2)w = -h(x) \alpha(x) t \log t + (h(x) \varphi(x) + a(x)b(x)t + tf(t, x) + h(x)b(x)t^2
  + c(x) t^{p + 1 - q} + [1]_q b(x)t^{p + 2 - q} + t(-\alpha(x) \log t + \varphi(x) + b(x)t) \n  + h(x)t^2 w + 2a(x)b(x)t(t \partial_t + 1)w + t^{p + 2 - q}[t \partial_t + 1]q + a(x)t([t \partial_t + 1]w)^2
  + \sum_{m=1}^{n} \binom{n}{m} (-\alpha(x) \log t + \varphi(x) + b(x)t) \times t^{n-1} \partial_t^2 w)^m, \tag{10.3}
\]

where \( c(x) = 0 \) (if \( p = q - 1 \)) and \( c(x) = (-1)^q (q - 1)! / a(x) \) (if \( p > q - 1 \)).

Take \( R > 0 \) so that \( \alpha(x) \in \mathcal{O}_{R_1} \) holds. For \( i \geq 1 \) we denote by \( F_i \) the set of all functions \( g(t, x) \) expressed in the form

\[
  g(t, x) = \sum_{j=0}^{[n(i+1)/2]} \phi_j(x) t^i (\log t)^j, \quad \text{with} \quad \phi_j(x) \in \mathcal{O}_{R_i}.
\]

Then, we have the following properties:

1) \( t, t \log t, \ldots, t (\log t)^n \in F_i \).
2) \( (t \partial_t + 1)(t \partial_t + 2)w = g \) is uniquely solvable in \( F_i \) (for \( i \geq 1 \)).
3) \( (t \partial_t) F_i \subseteq F_i \), \( \partial_t F_i \subseteq F_i \), and \( t \times F_i \subseteq F_i \).
4) \( t \times F_i \subseteq F_{i+1} \).
5) \( t^{m+1}(\log t)^{n-m} \times F_i \cdots F_{i_n} \subseteq F_{i_1 + \cdots + i_n + m + 1} \) (for \( 1 \leq m \leq n \)).

By using these properties, we can find \( w_i(t, x) \in F_i \) (i \( \geq 1 \)) inductively on \( i \) so that the formal series

\[
  \hat{w}(t, x) = \sum_{i \geq 1} w_i(t, x)
\]

is a formal solution of (10.3). Thus, we have a formal solution (10.2).
In the case \( q \leq 2 \), we can show the convergence of the formal solution \( \hat{w}(t,x) \) in the same way as in [25]. In the case \( q \geq 3 \), by the condition \( \gamma_i(\hat{w}) \geq 1 \) we see that the points \((1, 1 + \gamma_i((t\partial_t + 1)\hat{w})) \) and \((2, m + 1 + (m - 1)\gamma_i(\partial^2_{x} \hat{w})) \) \((1 \leq m \leq n)\) are located in the interior of the convex hull of the set \( C(2, 0) \cup C(q, p + 2 - q) \) in \( \mathbb{R}^2 \), and so the Newton polygon of (10.3) along the formal solution \( \hat{w}(t,x) \) is given by

\[
\mathcal{N}((10.3), \hat{w}) = \{(x, y) \in \mathbb{R}^2; x \leq q, y \geq 0, y \geq k_1(x - 2)\}
\]

with \( k_1 = (p + 2 - q)/(q - 2) \). Therefore, by applying Theorem 4.8 (and Theorem 4.10) to (10.3) we have the result (3). We note that (10.3) contains \( t \log t \) and \( t^{n+1}(\log t)^j \) (\( 0 \leq j \leq n - m \)); however, by Theorem 9.3 we know it is not a problem.

\[\Box\]

**Example 10.2.** Let \((t, x) \in \mathbb{C}^2\), and let us consider the following linear or nonlinear partial differential equation:

\[
t\partial^2_x u = u + (\partial^2_x u)^n + t^p\partial^4_x u + f(t, x), \tag{10.4}
\]

where \( n \in \mathbb{N}^* \), \( p \in \mathbb{N}^* \), \( q \in \mathbb{N}^* \) satisfying \( p \geq q \) and \( f(t, x) \) is a holomorphic function in a neighborhood of \((t, x) = (0, 0)\). Then, for any \( \varphi_0(x), \varphi_1(x) \in \mathcal{O}_R \) the equation (10.4) has a unique formal solution of the form

\[
\hat{u}_2(t, x) = \varphi_0(x) + \varphi_1(x)t + a_{1,1}(x)t \log t + \sum_{i \geq 2, N_i \geq j \geq 0} \varphi_{i,j}(x)t^i(\log t)^j \tag{10.5}
\]

with \( a_{1,1}(x) = \varphi_0(x) + (\varphi_0^{(2)}(x))^n + f(0, x), N_i = \left\lfloor (n - 1)i + 1/n \right\rfloor \) \((i \geq 2)\), and suitable \( \varphi_{i,j}(x) \in \mathcal{O}_{R_i} \) \((i \geq 2, N_i \geq j \geq 0)\) for some \( R_i > 0 \).

(1) We note that (10.4) is a Fuchsian type partial differential equation with respect to \( t \), and the two characteristic exponents of (10.4) are 0 and 1 which differ by positive integers. In such a case, logarithmic singularities as above often appear in the formal solution.

(2) If \( q \leq 2 \), this formal solution is convergent in a suitable domain near \((t, x) = (0, 0)\) and defines an analytic solution, but if \( q \geq 3 \), it is not convergent in general. In the convergent case, equations like (10.4) whose characteristic exponents differ by positive integers were studied by Tahara-Yamazawa [26] in a general setting.

(3) In the case \( q \geq 3 \), we can apply Theorem 4.8 (and Theorem 4.10) under \( p^* = 1, \left\{ \gamma_1 = \{2\pi j/(q - 2); j \in \mathbb{Z} \}, \mathcal{Z}_1 = \mathbb{R} \setminus \mathcal{Z}_1 \right\} \) and \( k_1 = (p + 1 - q)/(q - 2) \). Hence, for any \( I \in \mathcal{Z}_1 \) we have a true solution \( u_2(t, x) \) on \( S_{\delta + \pi/2\kappa_4} \times D_\rho \) for some \( \delta > 0 \) and \( \rho > 0 \), that admits \( \hat{u}_2(t, x) \) as an asymptotic expansion (as \( t \rightarrow 0 \)).

**Proof.** Take any \( \varphi_0(x) \in \mathcal{O}_R \), and set \( u = \varphi_0(x) + w \). Then, we can reduce (10.4) to the following equation with respect to \( w \):

\[
(t\partial_t)(t\partial_t - 1)w = a_{1,1}(x)t + tw + t^{p+1-q}[t\partial_t]^q w
\]

\[
+ \sum_{m=1}^{n} \left( \frac{n}{m} \right) (\varphi_0^{(2)}(x))^{n-m} t(\partial^2_x w)^m + \sum_{i \geq 1} f_i(x)t^{i+1}, \tag{10.6}
\]

where \( a_{1,1}(x) = \varphi_0(x) + (\varphi_0^{(2)}(x))^n + f(0, x), \) and \( f_i(x) = (\partial_i^2 f)(0, x)/i! \) \((i \geq 1)\).
Take $R_1 > 0$ so that $f_i(x) \in \mathcal{O}_{R_1}$ $(i \geq 1)$ holds. For $i \geq 1$ we denote by $F_i$ the set of all functions $g(t,x)$ expressed in the form

$$g(t,x) = \sum_{j=0}^{(n-1)i+1/n} \phi_j(x)t^j(\log t)^j \text{ with } \phi_j(x) \in \mathcal{O}_{R_1}.$$  

Then, we have the following properties:

1) $t, t \log t \in F_1$.
2) $(t\partial_t)(t\partial_t-1)(\varphi_1(x)t+a(x)t\log t) = a(x)t$ (for any $\varphi_1(x)$ and $a(x)$).
3) $(t\partial_t)(t\partial_t-1)\nu = g$ is uniquely solvable in $F_1$ (for $i \geq 2$).
4) $(t\partial_t)F_i \subset F_{i+1}$, $\partial_x F_i \subset F_i$, and $t \times F_i \subset F_{i+1}$.
5) $t \times F_i \cdots \times F_m \subset F_{i+m+1}$ (for $1 \leq m \leq n$).

By using these properties, for any $\varphi_1(x) \in \mathcal{O}_{R_1}$ we can find $w_i(t,x) \in F_i$ $(i \geq 2)$ inductively on $i$ so that the formal series

$$\hat{w}(t,x) = \varphi_1(x)t + a_1(x)t \log t + \sum_{i \geq 2} w_i(t,x)$$

is a formal solution of (10.6). Thus, we have a formal solution (10.5).

In the case $q \leq 2$, we can show the convergence of the formal solution $\hat{w}(t,x)$ in the same way as in [26]. In the case $q \geq 3$, by using the condition $\gamma_i(\hat{w}) \geq 1$ we can easily see that the Newton polygon of (10.6) along the formal solution $\hat{w}(t,x)$ is given by

$$\mathcal{N}(10.6) \hat{w} = \{(x,y) \in \mathbb{R}^2; x \leq q, y \geq 0, y \geq k_1(x-2)\}$$

with $k_1 = (p + 1 - q)/(q - 2)$. Therefore, by applying Theorem 4.8 (and Theorem 4.10) to (10.6) we have the result (3). \hfill $\square$

**Example 10.3.** Let $(t,x) \in \mathbb{C}^2$, and let us consider the following linear or nonlinear partial differential equation:

$$t\partial_t^2 u - (\lambda(x) - 1)\partial_x u = u + (\partial_x^2 u)^n + t^p \partial_t^p u + f(t,x), \quad (10.7)$$

where $\lambda(x) \in \mathcal{O}_R$, $n \in \mathbb{N}$, $p \in \mathbb{N}$, $q \in \mathbb{N}$ satisfying $p \geq q$, and $f(t,x)$ is a holomorphic function in a neighborhood of $(t,x) = (0,0)$. Then, if $\lambda(0) \not\in \mathbb{N}$ and $\text{Re}\lambda(0) > 0$ hold, for any $v_0(x), \varphi(x) \in \mathcal{O}_R$ the equation (10.7) has a unique formal solution of the form

$$\hat{u}_3(t,x) = \sum_{i \geq 0} v_i(x)t^i + \varphi(x)t^{\lambda(x)} + \sum_{(i,j,l) \in S} \varphi_{i,l,j}(x)t^{i+j+l\lambda(x)}(\log t)^j \quad (10.8)$$

with $S = \{(i,l,j) \in \mathbb{N}^3; i + l \geq 2, l \geq 1, 2(i + l - 1) \geq j\}$ and suitable $v_i(x) \in \mathcal{O}_{R_1}$ $(i \geq 1)$, $\varphi_{i,l,j}(x) \in \mathcal{O}_{R_1}$ $(i,l,j) \in S$ for some $R_1 > 0$.

1. The part $v_0(x) + \varphi(x)t^{\lambda(x)}$ of $\hat{u}_3(t,x)$ comes from the fact that it is a solution of the equation: $t\partial_t^2 u - (\lambda(x) - 1)\partial_x u = 0$. The logarithmic singularity appears from the formula:

$$\partial_x(t^{\lambda(x)}) = (\partial \lambda(x)/\partial x)t^{\lambda(x)} \log t.$$  

2. If $q \leq 2$, as is seen in Gérard-Tahara [12, 13], this formal solution is convergent.
in a suitable domain near \((t, x) = (0, 0)\) and defines an analytic solution, but if \(q \geq 3\), it is not convergent in general.

(3) If \(n = 1\), (10.7) is a linear equation and the formal solution \(\bar{u}_3(t, x)\) is reduced to the form

\[
\bar{u}_3(t, x) = \sum_{i \geq 0} v_i(t)x^i + \varphi(x)t^{\lambda(x)} + \sum_{i \geq 1, 2i \geq 0} \varphi_{i,i}(x)t^{i+\lambda(x)}(\log t)^j.
\]

In this case with \(q \geq 3\), Yamazawa [30] has shown the summability of this formal solution in a suitable sense.

(4) In the general case \(q \geq 3\), we can apply Theorem 4.8 (and Theorem 4.10) under \(p^* = 1, \Xi_1 = \{2\pi j/(q - 2); j \in \mathbb{Z}\}, \mathcal{F}_1 = \mathbb{R}\setminus\Xi_1\) and \(k_1 = (p + 1 - q)/(q - 2)\). Hence, for any \(I \in \mathcal{F}_1\) we have a true solution \(u_3(t, x)\) on \(S_{l+1}x/(2k_1\delta) \times D_{\rho}\) for some \(\delta > 0\) and \(\rho > 0\), that admits \(\bar{u}_3(t, x)\) as an asymptotic expansion (as \(t \to 0\)).

**Proof.** Take any \(v_0(x) \in \mathcal{O}_R\) and set \(u = v_0(x) + w\). Then, we can reduce (10.7) to the following equation with respect to \(w\):

\[
(t\partial_t)(t\partial_t - \lambda(x))w = a_{1,1}(t)x + tw + t^{p+1-q}[t\partial_t]q^w
\]

\[
+ \sum_{m=1}^{n} \left( \frac{n}{m} \right) (v_0^{(2)}(x))^{n-m}t^{\partial_x^2 w^m} + \sum_{i \geq 1} f_i(x)t^{i+1},
\]

where \(a_{1,1}(x) = v_0(x) + (v_0^{(2)}(x))^{n} + f(0, x)\), and \(f_i(x) = (t\partial_t f)(0, x)/i! (i \geq 1)\). Since \(\lambda(x) \in \mathbb{N}^*\) and \(\text{Re}\lambda(x) > 0\) are supposed, by taking \(R_1 > 0\) and \(\delta > 0\) sufficiently small we have the conditions: \(\lambda(x) \not\in \mathbb{N}^*\) and \(\text{Re}\lambda(x) > \delta\) for any \(x \in D_{R_1}\).

For \(i \geq 1\) we set \(F_{i,0} = \mathcal{O}_{R_i} t^i\). For \(i + l \geq 1\) with \(l \geq 1\) we denote by \(F_{i,l}\) the set of all functions \(g(t, x)\) expressed in the form

\[
g(t, x) = \sum_{j=0}^{2(i+l-1)} \phi_j(x)t^{i+l+\lambda(x)}(\log t)^j\quad\text{with } \phi_j(x) \in \mathcal{O}_{R_i}.
\]

Then, we have the following properties:

1) \(F_{0,0} = \mathcal{O}_{R_0} t^\lambda(x)\) and \((t\partial_t)(t\partial_t - \lambda(x))F_{0,0} = \{0\},\)

2) \((t\partial_t)(t\partial_t - \lambda(x))w = g\) is uniquely solvable in \(F_{i,l}\) (for \(i + l \geq 1, (i, l) \neq (0, 1)\)).

3) \((t\partial_t)F_{i,l} \subset F_{i+1,l}\) and \(t \times F_{i,l} \subset F_{i+1,l}\),

4) \(t \times (\partial_t^2 F_{i_1,i_1}) \cdots (\partial_t^2 F_{i_m,i_m}) \subset F_{i_1+i_2+\cdots+i_m+1,i_1+\cdots+i_m}\) (for \(m \geq 1\)).

By using these properties, for any \(\varphi(x) \in \mathcal{O}_{R_1}\) we can find \(w_{i,l}(t, x) \in F_{i,l} (i + l \geq 1, (i, l) \neq (0, 1))\) inductively on \(i + l\) so that the formal series

\[
\hat{w}(t, x) = \varphi(x)t^{\lambda(x)} + \sum_{i \geq 1, 2i \geq 0} w_{i,i}(t, x)
\]

is a formal solution of (10.9). Thus, we have a formal solution (10.8).

In the case \(q \leq 2\), the convergence of the formal solution \(\hat{w}(t, x)\) is already proved in [12]. In the case \(q \geq 3\), by using the condition \(\gamma_t(\hat{w}) \geq \min\{\delta, 1\}\) we can easily see that
the Newton polygon of (10.9) along the formal solution \( \hat{w}(t, x) \) is given by
\[
\mathcal{N}(10.9), \hat{w} = \{(x, y) \in \mathbb{R}^2 ; x \leq q, y \geq 0, y \geq k_1(x - 2)\}
\]
with \( k_1 = (p + 1 - q)/(q - 2) \). Therefore, by applying Theorem 4.8 (and Theorem 4.10) to (10.9) we have the result (4).

Let us give a remark. If \( \lambda(x) \not\in \mathbb{N}^* \), \( \lambda(x) \neq 0 \) and \( i + l\lambda(x) \neq 0 \) (\( i \geq 1, l \geq 1 \)) hold for any \( x \in D_{R_1} \), we can determine a formal series \( \hat{w}(t, x) \) in the same way as above. However, in the case \( \text{Re} \lambda(0) \leq 0 \) it is not, in general, a formal solution of (10.9) in the sense of Definition 2.1.

**Example 10.4.** Let \( (t, x) \in \mathbb{C}^2 \), and let us consider the following nonlinear partial differential equation:
\[
t^2 \partial_t^2 u + x \partial_x u = u + a(x)u^2 + (\partial_x^2 u)^n + t^p \partial_t^p u + f(t, x), \tag{10.10}
\]
where \( a(x) \in \mathcal{O}_{R_1} \), \( n \in \mathbb{N}^* \) with \( n \geq 2 \), \( p \in \mathbb{N}^* \), \( q \in \mathbb{N}^* \) satisfying \( p \geq q + 1 \), and \( f(t, x) \) is a holomorphic function in a neighborhood of \( (t, x) = (0, 0) \). Then, if \( f(0, x) \equiv 0 \) and \( (\partial_t \partial_x f)(0, 0) \neq 0 \) hold, for any \( b_1 \in \mathbb{C} \) the equation (10.10) has a unique formal solution of the form
\[
\hat{u}_4(t, x) = b(x)t + cxt \log t + \sum_{i \geq 2, i \geq j \geq 0} \varphi_{i,j}(x)(t^i)(\log t)^j, \tag{10.11}
\]
with suitable \( b(x) \in \mathcal{O}_{R_1} \), satisfying \( (\partial_x b)(0) = b_1 \), \( c = (\partial_t \partial_x f)(0, 0) \), and \( \varphi_{i,j}(x) \in \mathcal{O}_{R_1} \), \( (i \geq 2, i \geq j \geq 0) \) for some \( R_1 > 0 \).

(1) We note that this is just the case (9.4) and so we have Lemma 9.2.

(2) The part \( b_1 xt + cxt \log t \) of \( \hat{u}_4(t, x) \) comes from the fact that it is a solution of the equation: \( t^2 \partial_t^2 u + x \partial_x u = u + cxt \). We note that solutions with logarithmic singularities to equations like (10.10) were constructed in Tahara [23]. In the setting of Lastra-Tahara [15], if the non-resonance condition (N) is not satisfied, the equation has a formal solution with logarithmic singularities that is not convergent in general.

(3) If \( q \leq 2 \), we can show in the same way as in [23] that this formal solution is convergent in a suitable domain near \( (t, x) = (0, 0) \) and defines an analytic solution, but if \( q \geq 3 \), it is not convergent in general.

(4) In the case \( q \geq 3 \), we can apply Theorem 9.1 under \( p^* = 1, \mathbb{Z}_1 = \{2\pi j/(q - 2) ; j \in \mathbb{Z}\}, \mathcal{I}_1 = \mathbb{R} \setminus \mathbb{Z}_1 \) and \( k_1 = (p - q)/(q - 2) \). Hence, for any \( I \in \mathcal{I}_1 \) we have a true solution \( u_4(t, x) \) on \( S_{\mathbb{I} + \pi/2 \mathbb{K}_I}(\delta) \times \mathcal{D}_\rho \) for some \( \delta > 0 \) and \( \rho > 0 \), that admits \( \hat{u}_4(t, x) \) as an asymptotic expansion (as \( t \to 0 \)).

**Proof.** We set \( L = t^2 \partial_t^2 + x \partial_x - 1 \) and \( f_i(x) = (\partial_t^i f)(0, x)/i! \) (\( i \geq 1 \)). First, we note that for any \( b_1 \in \mathbb{C} \) the equation \( Lu_1 = f_1(x) t \) has a unique solution
\[
u_1 = b(x)t + f_1'(0)xt \log t, \quad b'(0) = b_1.
\]
Then, by setting \( c = f_1'(0) \) and \( u = b(x)t + cxt \log t + w \) we can reduce (10.10) to the
following equation with respect to \( w \):
\[
Lw = a(x)(b(x)t + cx \log t)^2 + t^{p-q}|t\partial_t|^q(b(x)t + cx \log t) + (b^{(2)}(x))^{p+q}a(x)(b(x)t + cx \log t)w + a(x)w^2 + t^{p-q}|t\partial_t|^qw + \sum_{i=1}^n \frac{n}{m} (b^{(2)}(x))^{n-m} t^{n-m}(t^2 x^m) + \sum_{i=2}^\infty f_i(t)x^i. \tag{10.12}
\]

Take \( R_1 > 0 \) so that \( b(x) \in \mathcal{O}_{R_1} \) and \( f_i(x) \in \mathcal{O}_{R_1} (i \geq 2) \). For \( i \geq 1 \) we denote by \( F_i \) the set of all functions \( g(t,x) \) expressed in the form
\[
g(t,x) = \sum_{j=0}^i \phi_j(x)t^j(\log t)^j \quad \text{with} \quad \phi_j(x) \in \mathcal{O}_{R_1}.
\]

Then, we have the following properties:

1) \( t, t \log t \in F_1 \).
2) \( L \in F_i \) is uniquely solvable in \( F_i \) for \( i \geq 2 \).
3) \( (t\partial_t)F_i \subset F_i \) and \( \partial_x F_i \subset F_i \).
4) \( F_{i_1}, F_{i_2} \subset F_{i_1+i_2} \).

By using these properties, we can find \( w_i(t,x) \in F_i (i \geq 2) \) inductively on \( i \) so that the formal series
\[
\hat{w}(t,x) = \sum_{i \geq 2} w_i(t,x)
\]
is a formal solution of (10.12). Thus, we have a formal solution (10.11).

In the case \( q \leq 2 \), we can show the convergence of the formal solution \( \hat{u}_4(t,x) \) in the same way as in [23]. In the case \( q \geq 3 \), by using the condition \( \gamma_i(\hat{u}_4) \geq 1 \) we can easily see that the Newton polygon of (10.10) along the formal solution \( \hat{u}_4(t,x) \) is given by
\[
\mathcal{N}((10.10), \hat{u}_4) = \{ (x,y) \in \mathbb{R}^2 ; x \leq q, y \geq 0, y \geq k_1(x-2) \}
\]
with \( k_1 = (p-q)/(q-2) \). Therefore, by applying Theorem 9.1 (with Lemma 9.2) to (10.10) we have the result (4). \( \square \)

**Example 10.5.** Let us consider a linear ordinary differential equation
\[
\sum_{i=0}^m a_i(t)(d/dt)^iu = 0,
\]
where \( a_i(t) (0 \leq i \leq m) \) are holomorphic functions in a neighborhood of \( t = 0 \) with \( a_m(t) \neq 0 \). As is well known (see, for example, Coddington-Levinson [10], Wasow [29], etc.), if \( t = 0 \) is an irregular singularity of this equation, we can construct a fundamental system of formal solutions \( \{ \hat{u}_1(t), \ldots, \hat{u}_m(t) \} \) and each \( \hat{u}_i(t) \) has the form
\[
\hat{u}_i = \exp \left( \frac{\alpha_{i,1}}{t^{q_{1,1}}} + \frac{\alpha_{i,2}}{t^{q_{1,2}}} + \cdots + \frac{\alpha_{i,n}}{t^{q_{1,n}}} \right) \times t^\beta H_i(t^1/r_i, \log t)
\]
with \( s_i \in \mathbb{N}^*, \alpha_{i,j} \in \mathbb{C} (j = 1, \ldots, s_i), q_j \in \mathbb{Q} (j = 1, \ldots, s_i) \) with \( q_{i,1} > q_{i,2} > \cdots > q_{i,s_i} \).
In addition, we have $K$ constant holds for any $0 < r < r_0$ with coefficients in $\mathbb{C}$ and $0 < C$ for some ring of formal power series in $t^{1/r}$ and $(\mathbb{C}[t^{1/r}])[X]$ denotes the ring of polynomials in $X$ with coefficients in $\mathbb{C}[t^{1/r}]$.

By setting $\tilde{w}_i(t) = H_i(t, r_i \log t)$ we can see that this $\tilde{w}_i(t)$ is a formal solution of a linear differential equation with holomorphic coefficients. In this situation, we can apply Theorem 4.8 (and Theorem 4.10), and hence we have a true solution $w_i(t)$ that admits $\tilde{w}_i(t)$ as an asymptotic expansion (as $t \to 0$).

11. Proofs of Lemma 3.1 and Proposition 3.4

Since there are no good references about Lemma 3.1 and Proposition 3.4, for the convenience the reader we write here a sketch of the proof.

11.1. Proof of Lemma 3.1

Let $I$ be an open interval, $r_0 > 0$ and $U$ be an open subset of $\mathbb{C}_x^\ast$. Let $F(t, x)$ be a holomorphic function on $S_{I+\pi/2k}(r_0) \times U$ satisfying the estimate

$$|F(t, x)| \leq C|t|^a \quad \text{on } S_{I+\pi/2k}(r_0) \times U$$

(11.1)

for some $C > 0$ and $a > 0$. In subsection 3.2, $B_k[F](\xi, x)$ is defined by

$$B_k[F](\xi, x) = \frac{1}{2\pi \sqrt{-1}} \int_{\mathcal{C}(\xi)} \exp((\xi/t)^k) F(t, x) dt^{-k}.$$

**Lemma 11.1.** Let $\epsilon > 0$ and $0 < \mu < r_0$ be fixed. Then, for any $a > 0$ there is a constant $K_a > 0$ (which is independent of $C$) such that from (11.1) we have the estimate

$$|B_k[F](\xi, x)| \leq \frac{K_a C}{|\mu|^{-k}} \exp((|\xi|/|\mu|)^k) \quad \text{on } S_{I-|\epsilon|} \times U.$$  

(11.2)

In addition, we have $K_a = O(1)$ (as $a \to +\infty$) and $K_a = O(1/a)$ (as $a \to 0$).

**Proof.** Set $f(\xi, x) = B_k[F](\xi, x)$. Take any $\epsilon > 0$ and $0 < \mu < r_0$.

[Step 1]. First, let us show (11.2) for $0 < a < k/2$. Since $\Gamma(a/k) = O(1/a)$ (as $a \to +0$) holds, it is enough to prove that there is an $M > 0$ (which is independent of $C$ and $0 < a < k/2$) such that

$$|f(\xi, x)| \leq M C |\xi|^{-a} \exp((|\xi|/|\mu|)^k) \quad \text{on } S_{I-|\epsilon|} \times U$$

(11.3)

holds for any $0 < a < k/2$. Let us show this now.

We take $0 < \delta/k < \min\{\epsilon, \pi/2k\}$ and $0 < r < r_0$. For $\xi \in S_{I-|\epsilon|}$ we take a contour $\mathcal{C}(\xi)$ as follows: $\mathcal{C}(\xi) = L_1 \cup C_0 \cup L_2$ with

$$L_1 = \{ t = \rho e^{\sqrt{-1}(\arg \xi - \pi/2k + \delta/k)}, \rho : 0 \to r \},$$

$$C_0 = \{ t = r e^{\sqrt{-1} \theta}, \theta : \arg \xi + \pi/2k + \delta/k \to \arg \xi - \pi/2k - \delta/k \},$$

$$L_2 = \{ t = \rho e^{\sqrt{-1}(\arg \xi - \pi/2k - \delta/k)}, \rho : r \to 0 \}.$$ 

Then, we have $|e^{(\xi/t)^k}| \leq e^{(|\xi|/r)^k}$ for any $t \in C_0$, $|e^{(\xi/t)^k}| \leq e^{-(\sin \delta)(|\xi|/\rho)^k}$ for any
If \( t \in L_1 \cup L_2 \), and \( \rho^a \leq r^a \) for any \( t \in L_1 \cup L_2 \). By using these estimates we have

\[
|f(\xi, x)| \leq C \left( \frac{r^a}{(\sin \delta)|\xi|^k} + e^{(\ell(|\xi|)/r)^k r^a - k(\pi + 2\delta)} + \frac{r^a}{(\sin \delta)|\xi|^k} \right),
\]

(11.4)

where the first one comes from the integral on \( L_1 \), the second one from \( C_{00} \), and the third one from \( L_2 \). We note that the estimate (11.4) is valid for any \( r \) satisfying \( 0 < r < r_0 \).

[Step 2.] We take \( \mu_1 \) so that \( \mu < \mu_1 < r_0 \) and \( \mu_1^k \leq 2\mu^k \) hold. Set \( c = (\mu_1/\mu)^k - 1 \geq 0 \); then we have \( 0 < c \leq 1 \). If \( 0 < |\xi| < \mu_1 \), we set \( r = |\xi| \); then by (11.4) we have

\[
|f(\xi, x)| \leq \frac{C|\xi|^{a-k}}{2\pi} \left( \frac{2}{\sin \delta} + e(\pi + 2\delta) \right).
\]

(11.5)

If \( |\xi| \geq \mu_1 \) we set \( r = \mu_1 \); then by (11.4) we have

\[
|f(\xi, x)| \leq \frac{C|\xi|^{a-k}}{2\pi} \left( \frac{2}{\sin \delta} \right)
\]

\[
\leq \frac{C|\xi|^{a-k}}{2\pi} \left( \frac{2}{\sin \delta} + (\pi + 2\delta)e^{(\ell(|\xi|)/\mu_1)^k} \right).
\]

(11.6)

Since we are considering the case \( 0 < a/k < 1/2 \), we have

\[
\left( \frac{|\xi|}{\mu_1} \right)^{k-a} e^{-c(|\xi|/\mu_1)^k} \leq \max_{x>0}(x^{1-a/k}e^{-cx}) = \left( \frac{1-a/k}{c} \right)^{1-a/k} e^{-(1-a/k)}
\]

\[
\leq (1/c)^{1-a/k} \leq \frac{\Gamma(1-a/k)}{\sqrt{2\pi}}
\]

In the above, we have used the Stirling’s formula. Hence, by applying this estimate to (11.6) we have

\[
|f(\xi, x)| \leq \frac{C|\xi|^{a-k}}{2\pi} \left( \frac{2}{\sin \delta} + (\pi + 2\delta)\frac{\Gamma(1/2)}{c\sqrt{2\pi}} \right) \exp((|\xi|/\mu)^k).
\]

(11.7)

Thus, by (11.5) and (11.7) we have the result (11.3). This proves (11.2) for \( 0 < a < k/2 \).

[Step 3.] Next, let us show (11.2) for \( a \geq k/2 \). In this case, we set \( F_1(t, x) = F(t, x)/t^{a-k/4} \) and \( F_2(t) = t^{a-k/4} \). Then, we have \( F(t, x) = F_1(t, x)F_2(t) \) and under the setting \( f_1(\xi, x) = B_6[F_1](\xi, x) \) and \( f_2(\xi) = B_6[F_2](\xi) \) we have

\[
f(\xi, x) = (f_1 \ast_k f_2)(\xi, x) \quad \text{on} \quad S_t \times U.
\]

Since \( |F_1(t, x)| \leq C|t|^{k/4} \) holds on \( S_{t+|\pi/2k|}(r_0) \times U \) and since \( 0 < k/4 < k/2 \) holds, by Step 2 we already know that \( f_1(t, x) \) is estimated as follows:

\[
|f_1(\xi, x)| \leq \frac{K_{k/4}C|\xi|^{k/4-k}}{\Gamma(1/4)} \exp((|\xi|/\mu)^k) \quad \text{on} \quad S_{t-|\xi|} \times U.
\]

Since \( f_2(\xi) = \xi^{(a-k/4)-k}/\Gamma(a/k - 1/4) \) holds, we have

\[
|f_2(\xi)| \leq \frac{|\xi|^{(a-k)/k}}{\Gamma(a/k - 1/4)} \quad \text{on} \quad S_{t-|\xi|}.
\]
Therefore, by Lemma 3.5 we can estimate \((f_1 * f_2)(\xi, x)\) and we have
\[
|f(\xi, x)| \leq \frac{K_{k/4}C_1 a^{-k}}{\Gamma(a/k)} \exp((|\xi|/\mu)^k) \quad \text{on } S_{\delta/r} \times U.
\]
This proves the result (11.2) for \(a \geq k/2\).

11.2. Proof of Proposition 3.4

First, we note:

**Lemma 11.2.** Let \(\mu_n (n = 1, 2, \ldots)\) be a sequence satisfying \(0 < \mu_1 < \mu_2 < \cdots, \mu_n \to \infty \) (as \(n \to \infty\)), and (3.9). Then, we have the following properties:

1. There is a \(\gamma \geq 1\) such that \((\mu_n - \mu_{n-1})^{-1} \leq \gamma^{\mu_n}\) holds for any \(n \geq 2\).
2. Let \(\kappa > 0\). For any \(1 \leq i \leq n - 1\) we have
\[
\Gamma((\mu_n - \mu_i)/\kappa) \Gamma(\mu_i/\kappa) \leq 2\kappa(1 + 1/\mu_1)\gamma^{\mu_n} \Gamma(\mu_n/\kappa).
\]

**Proof.** (1) is clear from the first condition of (3.9). Let \(B(p, q)\) \((p > 0, q > 0)\) be the beta function. It is easy to see that if \(p \geq 1\) we have \(B(p, q) \leq 1/q\), if \(q \geq 1\) we have \(B(p, q) \leq 1/p\), and if \(0 < p < 1\) and \(0 < q < 1\) we have \(B(p, q) \leq (1/2)^{p+q-1}(1/p + 1/q)\). Hence, we have \(B(p, q) \leq 2(1/p + 1/q)\) for all \(p > 0\) and \(q > 0\). Using this estimate we have
\[
\Gamma((\mu_n - \mu_i)/\kappa) \Gamma(\mu_i/\kappa) = \Gamma(\mu_n/\kappa)B((\mu_n - \mu_i)/\kappa, \mu_i/\kappa)
\[
\leq \Gamma(\mu_n/\kappa)B((\mu_n - \mu_{n-1})/\kappa, \mu_1/\kappa)
\[
\leq \Gamma(\mu_n/\kappa) \times 2\left(\frac{\kappa}{\mu_n - \mu_{n-1}} + \frac{\kappa}{\mu_1}\right) \leq \Gamma(\mu_n/\kappa) \times 2\kappa\left(\gamma^{\mu_n} + \frac{1}{\mu_1}\right).
\]
Since \(\gamma^{\mu_n} \geq 1\) holds, we have (2).

Next, let us present two lemmas which are needed in the proof of Proposition 3.4. As before, let \(I\) be an open interval, and let \(U\) be an open subset of \(C^\infty_x\).

**Lemma 11.3.** Let \(k_0 > 0\) and \(k > 0\). Define \(\kappa_1 > 0\) by the relation \(1/\kappa_1 = 1/k_0 + 1/k\).
Let \(f(\xi, x)\) and \(\phi_n(\xi, x)\) \((n = 1, 2, \ldots)\) be holomorphic functions on \(S_{\delta} \times U\) satisfying the following conditions 1), 2) and 3) for some \(A > 0, b > 0, C > 0, h > 0\) and \(\delta > 0\):

1. \(|f(\xi, x)| \leq A|\xi|^{\mu_k - k} \exp(b|\xi|^k)\) on \(S_{\delta} \times U\),
2. \(|\phi_n(\xi, x)| \leq Ch^{\mu_k} \Gamma(\mu_n/k_0)|\xi|^{\mu_n - k} \exp(b|\xi|^k)\) on \(S_{\delta} \times U\) for \(n \geq 1\),
3. \(|f(\xi, x) - \sum_{n=1}^{N-1} \phi_n(\xi, x)| \leq Ch^{\mu_k} \Gamma(\mu_N/k_0)|\xi|^{\mu_N - k}\) on \(S_{\delta} \times U\) for \(N \geq 1\).

Then, for any \(\epsilon > 0\) there are \(M > 0, H > 0\) and \(r > 0\) such that
\[
|\mathcal{L}_{\kappa}f(t, x) - \sum_{n=1}^{N-1} \mathcal{L}_{\kappa}\phi_n(t, x)| \leq MH^{\mu_k} \Gamma(\mu_N/\kappa)|t|^{\mu_N}
\]
holds on \(S_{\delta/r} \times U\) for any \(N \geq 1\).
LEMA 11.4. Let $k_0 > 0$ and $0 < k_1 < k_2$. Define $\kappa > 0$ by the relation $1/\kappa = 1/k_1 - 1/k_2$, and $\kappa_1 > 0$ by the relation $1/\kappa_1 = 1/k_0 + 1/\kappa$. Let $f(\xi, x)$ and $\phi_n(\xi, x)$ ($n = 1, 2, \ldots$) be holomorphic functions on $S_I \times U$ satisfying the following conditions 1), 2) and 3) for some $A > 0$, $b > 0$, $C > 0$, $h > 0$ and $\delta > 0$:

1) $|f(\xi, x)| \leq A|\xi|^{\mu - k_1} \exp(b|\xi|^\nu)$ on $S_I \times U$,

2) $|\phi_n(\xi, x)| \leq Ch^{\mu_0} \Gamma(\mu_n/k_0)|\xi|^{\mu_n - k_1} \exp(b|\xi|^\nu)$ on $S_I \times U$ for $n \geq 1$,

3) $\left|f(\xi, x) - \sum_{n=1}^{N-1} \phi_n(\xi, x)\right| \leq Ch^{\mu_0} \Gamma(\mu_N/k_0)|\xi|^{\mu_N - k_1}$ on $S_I(\delta) \times U$ for $N \geq 1$.

Then, for any $\epsilon > 0$ there are $M > 0$, $H > 0$ and $r > 0$ such that

$$|A_{k_2,k_1}[f](\xi, x) - \sum_{n=1}^{N-1} A_{k_2,k_1}[\phi_n](\xi, x)| \leq MH^{\mu_0} \Gamma(\mu_N/k_1)|\xi|^{\mu_N - k_2}$$

holds on $S_{I+\left[\pi/2k-\epsilon\right]}(r) \times U$ for any $N \geq 1$.

REMARK 11.5. The above result is valid also in the case with $\Gamma(\mu_n/k_0)$, $\Gamma(\mu_N/k_0)$, and $\Gamma(\mu_N/k_1)$ being replaced by $1(= \Gamma(1 + \mu_n/\infty))$, $1(= \Gamma(1 + \mu_N/\infty))$ and $\Gamma(\mu_N/\kappa)$. This corresponds to the case of $k_0 = \infty$, and so we call this case “Lemma 11.4 (with $k_0 = \infty$)”.

We note that Lemma 11.3 corresponds to [Theorem 1 in Section 2.1 in [2]], and Lemma 11.4 corresponds to [Theorem 1 in Section 5.2 in [2]]. By using Lemma 11.2, these results can be proved in the same way as [Theorem 1 in Section 2.1 in [2]], and so we may omit the details.

Proof of Proposition 3.4. Let $u_n(t, x) \in \mathcal{O}(S_{I_0 + \left[\pi/2k_1\right]}(\delta) \times U)$ ($n = 1, 2, \ldots$), and suppose that they satisfy

$$|u_n(t, x)| \leq Ah^{\mu_0} \Gamma(\mu_n/k_1)|t|^{\mu_n}$$
on $S_{I_0 + \left[\pi/2k_1\right]}(\delta) \times U$, $n = 1, 2, \ldots$

for some $A > 0$, $h > 0$ and a sequence $\mu_n > 0$ ($n = 1, 2, \ldots$) with $0 < \mu_1 < \mu_2 < \ldots$, $\mu_n \to \infty$ (as $n \to \infty$) and (3.9). Suppose also that the formal series

$$\tilde{u}(t, x) = \sum_{n \geq 1} u_n(t, x)$$

is ($k_q, \ldots, k_1$)-multisummable in $I$-direction. Let $I_i$ ($i = 1, \ldots, q$) and $\kappa_i$ ($i = 1, \ldots, q - 1$) be as in Definition 3.3; then we have $I_1 \subseteq I_0$ and $I_{i+1} \subseteq I_i + [\pi/2\kappa_i]$ ($i = 1, \ldots, q - 1$).

We define also $\kappa_{i,i} > 0$ by the relation $1/\kappa_{i,i} = 1/k_i - 1/k_i$ ($i = 2, \ldots, q$). We have $\kappa_1 = \kappa_{i,1,2}$, and $I_i \subseteq I_0 + \left[\pi/2\kappa_{i,1}\right]$ ($i = 2, \ldots, q$).

[Step 1]. By the assumption and Lemma 3.1 we see that $B_{k_i}[u_n](\xi, x)$ ($n = 1, 2, \ldots$) are well-defined as holomorphic functions on $S_{I_i} \times U$ and there are $\kappa_i > 0$ and $b_i > 0$ such that

$$|B_{k_i}[u_n](\xi, x)| \leq K_1 Ah^{\mu_0}|\xi|^{\mu_n - k_1} \exp(b_i|\xi|^k_1)$$
on $S_{I_i} \times U$ (11.8)
holds for any \( n \geq 1 \). By taking \( r_1 > 0 \) so that \( hr_1 \leq \eta \) (where \( \eta \) is the constant in (3.9)) we see that the series
\[
w_1(\xi, x) = \sum_{n \geq 1} B_{k_1} [u_n](\xi, x)
\]
is convergent on \( S_{I_1}(r_1) \times U \) and satisfies \(|w_1(\xi, x)| \leq C_1^0|\xi|^{\mu_1 - k_1} \) on \( S_{I_1}(r_1) \times U \) for some \( C_1^0 > 0 \). In addition, by (11.8) we can see that there are \( M_1 > 0 \) and \( H_1 > 0 \) such that
\[
|w_1(\xi, x) - \sum_{n=1}^{N-1} B_{k_1} [u_n](\xi, x)| \leq M_1 H_1^N |\xi|^{\mu_N - k_1} \quad \text{on} \ S_{I_1}(r_1) \times U
\]
holds for any \( N \geq 1 \).

[Step 2]. Since \( w_1(\xi, x) \) has an analytic continuation \( w_1^*(\xi, x) \) on \( S_{I_1} \times U \) satisfying (3.8) (with \( i = 1 \)), we can define \( w_2(\xi, x) = A_{k_2,k_1}[w_1^*](\xi, x) \). Since
\[
A_{k_2,k_1} [B_{k_1} [u_n]](\xi, x) = B_{k_2} [u_n](\xi, x) \quad \text{on} \ S_{I_1 + [\pi/2k_1]} \times U
\]
and since \( I_2 \in I_1 + [\pi/2k_1] \) is supposed, by Proposition 3.2 and Lemma 11.4 (with \( k_0 = \infty \)) we have the following properties:

1) \( w_2(\xi, x) \) is a holomorphic function on \( S_{I_2}(\delta_2) \times U \) for some \( \delta_2 > 0 \) and satisfies \(|w_2(\xi, x)| \leq C_2^0|\xi|^{\mu_1 - k_2} \) on \( S_{I_2}(\delta_2) \times U \) for some \( C_2^0 > 0 \).

2) There are \( M_2 > 0 \), \( H_2 > 0 \) and \( 0 < r_2 \leq \delta_2 \) such that
\[
|w_2(\xi, x) - \sum_{n=1}^{N-1} B_{k_2} [u_n](\xi, x)| \leq M_2 H_2^N \Gamma(\mu_N/\kappa_{1,2}) |\xi|^{\mu_N - k_2} \quad \text{holds on} \ S_{I_2}(r_2) \times U \quad \text{for any} \ N \geq 1.
\]

3) Since \( I_2 \in I_0 + [\pi/2\kappa_{1,2}] \) holds, by Lemma 3.1 and the Stirling’s formula we see that \( B_{k_2} [u_n](\xi, x) \) \((n = 1, 2, \ldots)\) are well-defined as holomorphic functions on \( S_{I_2} \times U \) and there are \( C_2 > 0 \), \( h_2 \) and \( b_2 > 0 \) such that
\[
|B_{k_2} [u_n](\xi, x)| \leq C_2 h_2^N \Gamma(\mu_n/\kappa_{1,2}) |\xi|^{\mu_n - k_2} \exp(b_2 |\xi|^{k_2}) \quad \text{holds on} \ S_{I_2}(r_2) \times U \quad \text{for any} \ n \geq 1.
\]

[Step 3]. Since \( w_2(\xi, x) \) has an analytic continuation \( w_2^*(\xi, x) \) on \( S_{I_2} \times U \) satisfying (3.8) (with \( i = 2 \)) we can define \( w_3(\xi, x) = A_{k_3,k_2} [w_2^*](\xi, x) \).

By repeating the same argument we can define
\[
w_q(\xi, x) = A_{k_q,k_{q-1}} [w_{q-1}^*](\xi, x).
\]
As in Step 2, by Proposition 3.2 and Lemma 11.4 we have the following properties.

1) \( w_q(\xi, x) \) is a holomorphic function on \( S_{I_q}(\delta_q) \times U \) for some \( \delta_q > 0 \) and satisfies \(|w_q(\xi, x)| \leq C_q^0|\xi|^{\mu_1 - k_q} \) on \( S_{I_q}(\delta_q) \times U \) for some \( C_q^0 > 0 \).

2) There are \( M_q > 0 \), \( H_q > 0 \) and \( 0 < r_q \leq \delta_q \) such that
\[
|w_q(\xi, x) - \sum_{n=1}^{N-1} B_{k_q} [u_n](\xi, x)| \leq M_q H_q^N \Gamma(\mu_N/\kappa_{1,q}) |\xi|^{\mu_N - k_q} \quad \text{holds on} \ S_{I_q}(r_q) \times U \quad \text{for any} \ N \geq 1.
\]
holds on \( S_{I_q}(r_q) \times U \) for any \( N \geq 1 \).

3) \( \mathcal{B}_{k_q}[u_n](\xi, x) \) (\( n = 1, 2, \ldots \)) are well-defined as holomorphic functions on \( S_{I_q} \times U \) and there are \( C_q > 0, h_q > 0 \) and \( b_q > 0 \) such that

\[
| \mathcal{B}_{k_q}[u_n](\xi, x) | \leq C_q h_q^{\mu_n} \Gamma(\mu_n/\kappa_{1, q}) |\xi|^{\mu_n-k_q} \exp(b_q |\xi|^{k_q})
\]

holds on \( S_{I_q} \times U \) for any \( n \geq 1 \).

[Step 4.] Since \( w_q(\xi, x) \) has an analytic continuation \( w_q^*(\xi, x) \) on \( S_{I_q} \times U \) satisfying (3.8) (with \( i = q \)), and since \( I_q = I \) and \( \kappa_q = k_q \) hold, we have

\[
| w_q^*(\xi, x) | \leq A_q |\xi|^{\mu_1-k_q} \exp(c_q |\xi|^{k_q}) \quad \text{on} \quad S_I \times U.
\]

Therefore, we can define \( u^*(t, x) = L_{k_q}[w_q^*](\xi, x) \); then by Lemma 11.3 we have the conclusion of Proposition 3.4. \( \square \)

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