# GEOGRAPHY OF SYMPLECTIC 4-MANIFOLDS ADMITTING LEFSCHETZ FIBRATIONS AND THEIR INDECOMPOSABILITY

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Abstract. In this paper, we show that for a given finitely presented group $G$, there exist integers $h_G \geq 0$ and $n_G \geq 4$ such that for all $h \geq h_G$ and $n \geq n_G$, and for all $0 \leq i \leq 2n-2$, there exists a genus-$(2h+n-1)$ Lefschetz fibration on a minimal symplectic 4-manifold with $(\chi, c_1^2) = (n, i)$ whose fundamental group is isomorphic to $G$. We also prove that such a fibration cannot be decomposed as a fiber sum for $1 \leq i \leq 2n-2$ if $h > (5n-3)/2$. In addition, we give a relation among the genus of the base space of a ruled surface admitting a Lefschetz fibration, the number of blow-ups and the genus of the Lefschetz fibration.

## 1. Introduction

Briefly speaking, a genus-$g$ Lefschetz fibration is a smooth fibration of a 4-manifold over $\mathbb{S}^2$ with regular fiber diffeomorphic to a closed orientable surface of genus $g$, which may admit certain singular fibers. In 4-dimensional topology, Lefschetz fibrations are fundamental and important objects to study. From the result of [20], after some blow-ups, any closed symplectic 4-manifolds admit Lefschetz fibrations. Conversely, it was shown in [29] that a 4-manifold admitting a Lefschetz fibration has a symplectic structure if the fibers are nontrivial in homology. This is a generalization of an earlier work of Thurston in [55] (more details can be found in [44, 29]).

In this paper, we study the geography of minimal symplectic 4-manifolds admitting Lefschetz fibrations (see Section 1.1) and discuss the indecomposability of the Lefschetz fibrations (see Section 1.2). Moreover, we investigate Lefschetz fibrations on blow-ups of ruled surfaces (see Section 1.3).

More precise definitions of the terms in this introduction and statements of the main results will be given in Section 2.

### 1.1. Lefschetz fibrations violating the Noether inequality.

Let $\sigma(X)$ and $e(X)$ be the signature and the Euler characteristic of a closed oriented smooth 4-manifold $X$, respectively, and we set $\chi(X) := (\sigma(X) + e(X))/4$ (the holomorphic Euler characteristic) and $c_1^2(X) = 3\sigma(X) + 2e(X)$ (the first Chern number). Note that $\chi(X) \in \mathbb{Z}$ if $X$ is a complex surface or a symplectic 4-manifold. The geography problem for complex surfaces (resp. symplectic 4-manifolds) is the characterization of pairs $(\chi, c_1^2)$ corresponding to minimal complex surfaces (resp. minimal symplectic 4-manifolds).

It is well-known that every minimal complex surface of general type satisfies $\chi > 0$, $c_1^2 > 0$ and $2\chi - 6 \leq c_1^2 \leq 9\chi$ (see, for example [13]). The last two inequalities are called the *Noether* inequality and the *Bogomolov–Miyaoka–Yau* inequality, respectively.

By using a result of Taubes [54], Liu [42] showed that every minimal symplectic 4-manifold with $b_2^+ > 1$ satisfies $c_1^2 \geq 0$. It remains open whether any minimal symplectic 4-manifold with $c_1^2 \geq 0$ satisfies $\chi \geq 0$ (more strongly, $e \geq 0$ since $c_1^2 + e = 12\chi$) and the Bogomolov–Miyaoka–Yau inequality. On the other hand, Fintushel and Stern [26] showed that for $n-1 \geq 3$, there exists a genus-$(n-1)$ Lefschetz fibration over $\mathbb{S}^2$ whose total space is a simply connected, minimal symplectic 4-manifold with $(\chi, c_1^2) = (n, n-3)$. Since this pair, $(\chi, c_1^2) = (n, n-3)$, satisfies $c_1^2 = \chi - 3$, this symplectic 4-manifold violates the Noether inequality. Moreover, Gompf and Stipsicz gave a simply connected minimal symplectic 4-manifold with $x = \chi$ and $y = c_1^2$ for most pairs $(x, y)$ satisfying $y < 2x - 6$ (see [29]). Although the examples in [29] admit Lefschetz pencil structures, it is not clear from the construction that they admit Lefschetz fibration structures. Furthermore, it is not clear if more recent exotic symplectic 4-manifolds constructed via symplectic connected sums, knot surgeries and Luttinger surgeries [1, 7, 8] admit Lefschetz fibration structures. For more about results concerning the geography of symplectic 4-manifolds, see for example [29].

In this paper, we give a generalization of the result of [26] to $(\chi, c_1^2) = (n, i)$ for $n - 1 \geq 3$ and $0 \leq i \leq 2n - 1$ (Theorem A in Section 2.3). Moreover, we generalize Theorem A to Lefschetz fibrations with arbitrary fundamental groups. More precisely, for a given finitely presented group $G$, there exist integers $h_G \geq 0$ and $n_G \geq 4$ such that for all $h \geq h_G$ and $n \geq n_G$, and for $0 \leq i \leq 2n - 2$, there exists a genus-$(2h + n - 1)$ Lefschetz fibration on a minimal symplectic 4-manifold with $(\chi, c_1^2) = (n, i)$ whose fundamental group is isomorphic to $G$ (see Theorem B in Section 2.3). This result is also a generalization of a result of the first author and Ozbagci [6].

By the Enriques-Kodaira classification of complex surfaces (see, for example, [13]), we see that there exists no minimal complex surface with $0 < c_1^2 < 2\chi - 6$. Hence, the manifolds in Theorems A and B violating the Noether inequality cannot admit any complex structure with either orientation. As a corollary, we obtain nonholomorphic genus-$g$ Lefschetz fibrations for $g \geq 3$ (see Corollary 2 in Section 2.3). On the other hand, every genus-2 Lefschetz fibration satisfies the Noether inequality (see Section 8). Nakamura [46] studied the geography of genus-2 Lefschetz fibrations.

### 1.2. Indecomposable Lefschetz fibrations with minimal total spaces.

The *fiber sum* is one of the most important and natural operations to construct new genus-$g$ Lefschetz fibrations obtained by "summing" given genus-$g$ Lefschetz fibrations. It was shown by Stipsicz [52], and independently by Smith [50], that every Lefschetz fibration over $\mathbb{S}^2$ with a $(-1)$-section is indecomposable with respect to fiber sum. Note that the total spaces of Lefschetz fibrations with a $(-1)$-section are nonminimal. Based on the above-mentioned result of [52], Stipsicz conjectured that if a Lefschetz fibration is decomposable, then its total space is minimal. This was proved by Usher [56] (see also [48, 14]). On the other hand, it was shown in [4] that the converse of the above-mentioned Stipsicz's conjecture for genus-2 Lefschetz fibrations is false. We generalize this result (see Theorem C, Corollaries 3 and 4 in Section 2.3). More precisely, the fibrations in Theorem B are indecomposable under the condition $h > (5n-3)/2$ (see Theorem C in Section 2.3).

We would like to emphasize that as far as the authors know, the monodromies of all known indecomposable Lefschetz fibrations with minimal total spaces have not been known and there have been no explicit examples for fiber genus $g \geq 3$. In this paper, we give such examples for fiber genus $g \geq 21$ (see Corollary 3 in Section 2.3) and the monodromies of the fibrations in Theorem C.

To the best of our knowledge, all known explicit examples of indecomposable Lefschetz fibrations with minimal total spaces are constructed by Xiao [57], and the fiber genera of all the fibrations are two. Note that their total spaces are not simply connected (see Proposition 43 in Section 8). From Theorems B and C, we obtain indecomposable genus-$g$ Lefschetz fibrations with minimal and simply connected total spaces for $g \geq 21$ (see Corollary 3 in Section 2.3).

As far as the authors know, up to isomorphism, there have been constructed only finitely many examples of indecomposable genus-$g$ Lefschetz fibrations whose total spaces are minimal for $g = 2$, and there have been no such examples for $g \geq 3$. We give infinitely many isomorphism classes of such genus-$g$ Lefschetz fibrations for each $g \geq 28$ (see Corollary 4 in Section 2.3). It was conjectured in [52] that if a Lefschetz fibration is indecomposable, then it has a $(-1)$-section. However, there have been constructed some indecomposable genus-$g$ Lefschetz fibrations with nonminimal total spaces and no $(-1)$-sections for each $g \geq 2$ ($g = 2$ [49], $g = 2, 3$ [15] and $g \geq 2$ [16]), but the number of such examples is finite up to isomorphism as far as the authors know. On the other hand, from Corollary 4, there are infinitely many indecomposable genus-$g$ Lefschetz fibrations with minimal total spaces for each $g \geq 28$.

1.3. **Lefschetz fibrations on ruled surfaces.** Ruled surfaces play an important role in the theory of Lefschetz fibrations. In fact, using (the monodromies of) Lefschetz fibrations on blow-ups of ruled surfaces, many interesting examples have been obtained (see, for example, [47, 37, 38, 36, 35, 3, 4]). In this paper, we give a relation among the genus of the base space of a ruled surface admitting a Lefschetz fibration, the number of blow-ups and the genus of the Lefschetz fibration (see Proposition 6 in Section 2.3). We prove Theorem C using Proposition 6.

## 2. Statements of the main results

In this section, we state the main results. For that, we review the basics of 4-manifolds and Lefschetz fibrations. Throughout this paper, unless otherwise stated, all manifolds are assumed to be oriented. Moreover, if we say that two manifolds are diffeomorphic then we mean that they are orientation preservingly diffeomorphic.

### 2.1. **4-manifolds.**

Let $X$ be a closed, connected, oriented and smooth 4-manifold. The symmetric bilinear form

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$$

defined by counting intersections with signs of oriented surfaces representing homology classes of $X$ is called the *intersection form* of $X$. We write $b_2^+(X)$ (resp. $b_2^-(X)$) for the number of positive (resp. negative) eigenvalues of the intersection form $Q_X$ after diagonalizing it over $\mathbb{R}$. It is well-known that $Q_X$ is unimodular (i.e., $\det Q_X = \pm 1$), and therefore we see that the second Betti number $b_2(X)$ is $b_2^+(X)+b_2^-(X)$. The signature $\sigma(X)$ of $X$ is $b_2^+(X)-b_2^-(X)$. Note that the Euler characteristic of $X$ is $2-2b_1(X)+b_2^+(X)+b_2^-(X)$ by the Poincáre duality, where $b_1(X)$ is the first Betti number of $X$.

A *symplectic* manifold is a $2n$-manifold together with a *symplectic form* $\omega$, that is, $\omega$ is a differential 2-form that is closed and nondegenerate. We say that a 4-manifold is a *complex surface* if it admits a $\mathbb{C}^2$-atlas with holomorphic transition functions.

The connected sum $X\sharp\overline{\mathbb{CP}}^2$ is called the *blow-up* of $X$, where $\overline{\mathbb{CP}}^2$ is the complex projective plane $\mathbb{CP}^2$ with the opposite orientation. It is a well-known fact that the blow-up of a symplectic 4-manifold (resp. complex surface) is also a symplectic 4-manifold (resp. complex surface).

We say that $X$ is *smoothly minimal* if it does not contain any smoothly embedded spheres of self-intersection $-1$ (equivalently, it is not the connected sum of another manifold with $\overline{\mathbb{CP}}^2$). A 2-dimensional submanifold $S$ in a symplectic 4-manifold with a symplectic form $\omega$ is *symplectic* if $\omega|_S$ is a symplectic form on $S$. A symplectic 4-manifold (resp. complex surface) is said to be *minimal* if it does not contain any smoothly embedded spheres of self-intersection $-1$ which are symplectic (resp. complex) submanifolds of the ambient manifold. From a basic fact proved using Taubes' Seiberg–Witten theory [54, 41, 39], a symplectic 4-manifold (resp. complex surface) is minimal if and only if it is smoothly minimal.

A *rational surface* is a smooth 4-manifold diffeomorphic to $\mathbb{S}^2\times\mathbb{S}^2$ or $\mathbb{CP}^2\sharp m\overline{\mathbb{CP}}^2$ with $m\geq 0$. A *ruled surface* over a Riemann surface $\Sigma_h$ of genus $h\geq 0$ is a smooth orientable $\mathbb{S}^2$-bundle over $\Sigma_h$. Note that up to diffeomorphism, there are only two orientable $\mathbb{S}^2$-bundles over $\Sigma_h$. One is the trivial bundle $\Sigma_h\times\mathbb{S}^2$ and the other is the nontrivial bundle $\Sigma_h\widetilde{\times}\mathbb{S}^2$. In particular, the nontrivial bundle $\mathbb{S}^2\widetilde{\times}\mathbb{S}^2$ is diffeomorphic to $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$. Moreover, $(\Sigma_h\times\mathbb{S}^2)\sharp\overline{\mathbb{CP}}^2$ and $(\Sigma_h\widetilde{\times}\mathbb{S}^2)\sharp\overline{\mathbb{CP}}^2$ are diffeomorphic for $h\geq 0$.

2.2. **Lefschetz fibrations.** Let $X$ be a closed, connected, oriented and smooth 4-manifold. A smooth map $f : X \to \mathbb{S}^2$ is called a genus-$g$ *Lefschetz fibration* if a regular fiber of $f$ is diffeomorphic to $\Sigma_g$ and for each critical point $p$ and its image $f(p)$, there are complex local coordinate charts agreeing with the orientations of $X$ and $\mathbb{S}^2$ with respect to which $f$ is of the form

$$f(z_1, z_2) = z_1 z_2.$$

Throughout this paper, we assume that $f$ is injective on the set $C$ of critical points and *relatively minimal*, i.e., no fiber contains a sphere of self-intersection $-1$.

For a genus-$g$ Lefschetz fibration, any fiber containing a critical point is called a *singular fiber*, which is obtained by collapsing a simple closed curve, called the *vanishing cycle*, in a nearby regular fiber to the critical point. We say that a singular fiber is *separating* (resp. *nonseparating*) if the corresponding vanishing cycle is a separating (resp. nonseparating) curve on the regular fiber. For a genus-$g$ Lefschetz fibration $X \to \mathbb{S}^2$ with $m$ singular fibers, we have the formula $e(X) = 4 - 4g + m$. We say that a genus-$g$ Lefschetz fibration $X \to \mathbb{S}^2$ is *trivial* if it has no singular fibers, and in this case, $X$ is diffeomorphic to $\Sigma_g \times \mathbb{S}^2$ for $g \geq 2$.

Two Lefschetz fibrations $f_1 : X_1 \to \mathbb{S}^2$ and $f_2 : X_2 \to \mathbb{S}^2$ are said to be *isomorphic* if there exist orientation preserving diffeomorphisms $H : X_1 \to X_2$ and $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $f_2 \circ H = h \circ f_1$. Note that if $f_1$ is isomorphic $f_2$, then the number of singular fibers and the genus of a regular fiber of $f_1$ are equal to those of $f_2$.

For a Lefschetz fibration $f : X \to \mathbb{S}^2$, a map $s : \mathbb{S}^2 \to X$ is called a $(-k)$-section of $f$ if $f \circ s = \mathrm{id}_{\mathbb{S}^2}$ and the self-intersection number of the homology class $[s(\mathbb{S}^2)]$ in $H_2(X; \mathbb{Z})$ is equal to $-k$.

For $i = 1, 2$, let $f_i : X_i \to \mathbb{S}^2$ be two genus-$g$ Lefschetz fibrations. We remove a fibered neighborhood of a regular fiber $F_i$ from each fibration and glue the resulting 4-manifolds along their boundaries using a fiber-preserving and orientation-reversing diffeomorphism $\phi : F_1 \times S^1 \to F_2 \times S^1$. The result is a new genus-$g$ Lefschetz fibration $f$ on $X := X_1 \sharp_\phi X_2$ called the *fiber sum* of $f_1$ and $f_2$. A Lefschetz fibration is called *indecomposable* if it cannot be expressed as a fiber sum of nontrivial Lefschetz fibrations.

It is a well-known fact that the rational surface $\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2$ admits a nontrivial genus-1 Lefschetz fibration. The *elliptic surface* $E(n)$ is the $n$-fold fiber sum of this fibration for $n \geq 1$. Kas [32], and independently Moishezon [45], showed that if a 4-manifold admits a nontrivial genus-1 Lefschetz fibration, then it is diffeomorphic to $E(n)$ for some $n$ (see also [43]).

By Theorem 10.2.18 and Remark 10.2.22 in [29], we see that if a genus-$g$ Lefschetz fibration $f : X \to \mathbb{S}^2$ is nontrivial, admits a section or satisfies $g \geq 2$, then $X$ is a symplectic manifold and the fibers are symplectic submanifolds.

2.3. **Main results.** The first main result of the present paper is the following.

**Theorem A.** *Let $n-1 \geq 3$. For $0 \leq i \leq 2n-1$, there exists a genus-$(n-1)$ Lefschetz fibration $f_i : X_i \to \mathbb{S}^2$ such that*

(a) *$X_i$ is minimal, $\chi(X_i) = n$, $c_1^2(X_i) = i$ and $\pi_1(X_i) = 1$,*
(b) *$f_i$ admits a $(-2)$-section.*

To state Theorem B, we need to introduce the following notation.

**Definition 1.** For a finite set $\{x_1, x_2, \ldots, x_N\}$, let $F_N$ denote the free group of rank $N$ freely generated by $\{x_1, x_2, \ldots, x_N\}$. For $w \in F_N$, we define $\ell(w)$, called the *syllable length* of $w$, to be

$$\ell(w) = \min\{s \mid w = x_{i_1}^{m_1} x_{i_2}^{m_2} \cdots x_{i_s}^{m_s},\ 1 \leq i_j \leq N,\ m_j \in \mathbb{Z}\}.$$

Let $G = \langle x_1, x_2, \ldots, x_N \mid r_1, r_2, \ldots, r_k \rangle$ be a finitely presented group with $N$ generators and $k$ relations. Define $\ell_k = \max\{\ell(r_i) \mid 1 \leq i \leq k\}$. If $k = 0$, we define $\ell_0 = 1$ ($\ell_k$ depends on the presentation, and our definition of $\ell_k$ differs from that of [38]). We always assume that the relators $r_i$ are cyclically reduced, that is, none of its cyclic permutations contains subwords of the form $x_\nu x_\nu^{-1}$ or $x_\nu^{-1} x_\nu$ for $\nu = 1, 2, \ldots, N$.

**Theorem B.** *Let $G$ be a group with the presentation in Definition 1. Suppose that two nonnegative integers $n$ and $h$ satisfy $n-1 \geq 3$, $2n-8 \geq k$ and $h \geq N + \ell_k - 1$. Then, for any $0 \leq i \leq 2n-2$, there is a genus-$(2h+n-1)$ Lefschetz fibration $f_{G,i} : Y_i \to \mathbb{S}^2$ such that*

(a) *$Y_i$ is minimal, $\chi(Y_i) = n$, $c_1^2(Y_i) = i$ and $\pi_1(Y_i) \cong G$,*
(b) *$f_{G,i}$ admits a $(-2)$-section.*

As a corollary to Theorems A and B, we obtain the following result.

**Corollary 2.** *For $1 \leq i \leq 2n-7$, the Lefschetz fibrations $f_i$ and $f_{G,i}$ in Theorems A and B are nonholomorphic, respectively.*

We prove Theorems A and B and Corollary 2 in Section 5.2. The Lefschetz fibrations in Theorems A and B are constructed by applying "lantern substitutions" (corresponding to the rational blowdown surgeries along spheres of self-intersection $-4$) to the monodromy of the fiber sum of Lefschetz fibrations on rational or ruled surfaces. Some different examples of Lefschetz fibrations via this approach are constructed in [9, 3, 5], which motivated us to carry out this work.

The following is the third main result, whose proof is given in Section 3.

**Theorem C.** *In the notation of Theorem B, we suppose that $h > (5n-3)/2$. Then, for $i = 1, 2, \ldots, 2n-2$, $f_{G,i}$ is indecomposable.*

From Theorems B and C, we obtain Corollaries 3 and 4, which are proved in Section 5.2.

**Corollary 3.** *Let $h \geq 9$ and $n-1 \geq 3$ (therefore $2h+n-1 \geq 21$). Then, up to isomorphism, there are at least $2n-2$ genus-$(2h+n-1)$ Lefschetz fibrations such that they are indecomposable and that the total spaces are minimal and simply connected.*

**Corollary 4.** *For $g \geq 28$, up to isomorphism, there are infinitely many genus-$g$ Lefschetz fibrations such that they are indecomposable and that the total spaces are minimal.*

**Remark 5.** Theorem A is a generalization of the result of [26] mentioned in Section 1.1. The authors do not know whether Lefschetz fibrations in Theorem A are indecomposable or not. On the other hand, the fibrations given in [26], which are mentioned in the introduction, are decomposable.

Finally, we state the following proposition. The proof is given in Section 3.

**Proposition 6.** *Let $m, n$ and $h$ be nonnegative integers satisfying $h > n$ (therefore $h > 0$). Then, $R_h \sharp m\overline{\mathbb{CP}}^2$ admits a nontrivial genus-$(2h+n-1)$ Lefschetz fibration over $\mathbb{S}^2$ if and only if $m = 4n$ and $m, n \neq 0$, where $R_h$ is a ruled surface over a Riemann surface $\Sigma_h$ of genus $h$.*

**Remark 7.** The condition $h > n$ is sharp. In fact, if $h \leq n$, then there are examples of genus-$(2h+n-1)$ Lefschetz fibrations on $(\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2$ for $m \neq 4n$. For example, Xiao [57] gave a genus-2 Lefschetz fibration on $(\Sigma_2 \times \mathbb{S}^2)\sharp 3\overline{\mathbb{CP}}^2$ (i.e., $h = n = 1$) whose monodromy was given in [17], and Altunöz [10] constructed a genus-$3k$ Lefschetz fibration on $(\Sigma_k \times \mathbb{S}^2)\sharp 6\overline{\mathbb{CP}}^2$ (i.e., $h = k \leq n = 2k+1$). Counterexamples (to Proposition 6) in the case $h = 0$ can be found in [2, 16].

**Remark 8.** The condition that a Lefschetz fibration is nontrivial is essential. For example, $T^2 \times \mathbb{S}^2$ is obviously a (trivial) genus-1 Lefschetz fibration (i.e., $1 = h > n = 0$ and $m = 0$).

Proposition 6 is a generalization of the results in Section 4 of [53] and Lemma 3.1 in [11]. Theorem C is proved using Proposition 6.

2.4. **Outline of the paper.** The outline of the paper is as follows. In Section 3, we prove Proposition 6 and Theorem C. Section 4 presents preliminaries for the proofs of Theorems A and B. In Section 5, the proofs of Theorem A and Corollaries 2–4 are given. We also prove Theorem B, except for the part of $\pi_1(Y_i) \cong G$. The proof of Theorem B is completed in Section 6. In Section 7, we prove Propositions 25 and 26 in Section 4.3. In the last section, we make some remarks on genus-2 Lefschetz fibrations.

## 3. Proofs of Proposition 6 and Theorem C

It is well-known that there is a correspondence between certain words in mapping class groups and Lefschetz fibrations, but we only use 4-manifold theory in the proofs of Proposition 6 and Theorem C. For this reason, we first prove Proposition 6 and Theorem C.

Let $\Sigma_h \widetilde{\times} \mathbb{S}^2$ be the nontrivial $\mathbb{S}^2$-bundle over $\Sigma_h$. Since $(\Sigma_h \times \mathbb{S}^2)\sharp\overline{\mathbb{CP}}^2$ is diffeomorphic to $(\Sigma_h \widetilde{\times} \mathbb{S}^2)\sharp\overline{\mathbb{CP}}^2$, the proof of Proposition 6 is divided into two cases: *Case* 1. $R_h = \Sigma_h \times \mathbb{S}^2$ and $m \geq 0$ (Lemma 9); *Case* 2. $R_h = \Sigma_h \widetilde{\times} \mathbb{S}^2$ and $m = 0$ (Lemma 11).

**Lemma 9.** *Let $m, n$ and $h$ be nonnegative integers satisfying $h > n$ (therefore $h > 0$). The 4-manifold $(\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2$ admits a nontrivial genus-$(2h+n-1)$ Lefschetz fibration over $\mathbb{S}^2$ if and only if $m = 4n$ and $m, n \neq 0$.*

*Proof.* The "if" part follows from the genus-$(2h+n-1)$ Lefschetz fibration on $(\Sigma_h \times \mathbb{S}^2)\sharp 4n\overline{\mathbb{CP}}^2$ given in [27] (and its monodromy was given in [30, 58]).

We show the "only if" part. For a nontrivial genus-$g$ Lefschetz fibration $X \to \mathbb{S}^2$, the inequality $4(b_1(X)-g)+b_2^-(X) \leq 5b_2^+(X)$ was given in Lemma 3.2 in [53]. This gives $4(2h-g)+m \leq 4$ for $X = (\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2$, and therefore $m \leq 4n$ by $g = 2h+n-1$.

In the rest of the proof, we prove that $m \geq 4n$, which is equivalent to that the 4-manifold $(\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2$ does not admit any genus-$(2h+n-1)$ Lefschetz fibrations for $m < 4n$.

Let $m < 4n$. We denote by $F$ a regular fiber of a nontrivial genus-$g$ Lefschetz fibration on $(\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2$.

Suppose that $m > 0$, and let $e_i$ be the homology class in $H_2((\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2;\mathbb{Z})$ of a complex projective line $\mathbb{CP}^1$ of the $i$-th blow-up (i.e., in the $i$-th $\overline{\mathbb{CP}}^2$ summand), which satisfies $e_i \cdot e_i = -1$, $i = 1, 2, \ldots, m$. The composition of the blow-down $\pi : (\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2 \to \Sigma_h \times \mathbb{S}^2$ and the ruling $p : \Sigma_h \times \mathbb{S}^2 \to \Sigma_h$ gives a smooth map $(p \circ \pi)|_F : F \to \Sigma_h$.

Let $\mu$ and $\nu$ be the homology classes of the trivial section and a fiber of the ruling $p$, respectively. We orient the section and the fiber of $p$ so that $\mu \cdot \nu = 1$ (and $\mu \cdot \mu = \nu \cdot \nu = 0$). After choosing an orientation on $F$, we set $[F] = a\mu + b\nu + \sum_{i=1}^{m} k_i e_i$ in $H_2((\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2;\mathbb{Z})$ for some integers $a, b$ and $k_i$. Then, the degree $d$ of $(p \circ \pi)|_F : F \to \Sigma_h$ is equal to $a$. Here, we consider a nonseparating singular fiber $F_s$. Note that the existence of such a fiber is guaranteed from Theorem 1.3 in [51] (see also [40, 12]). Let $\overline{F}_s$ be the normalization of $F_s$, that is, $\overline{F}_s$ is a Riemann surface obtained by separating the two sheets which meet at the node of $F_s$. We denote by $g(\overline{F}_s)$ the genus of $\overline{F}_s$, and therefore $g(\overline{F}_s) = g-1$. Since $(p \circ \pi)|_F : F \to \Sigma_h$ and $(p \circ \pi)|_{F_s} : F_s \to \Sigma_h$ have the same degree by $[F] = [F_s]$ and the degree of the normalization map $q : \overline{F}_s \to F_s$ is equal to 1, the composition $(p \circ \pi)|_{F_s} \circ q : \overline{F}_s \to \Sigma_h$ has degree $d$. Therefore, by Kneser's inequality [34], we obtain

$$\{2h + (n-1) - 1\} - 1 = (g-1) - 1 = g(\overline{F}_s) - 1 \geq |d|(h-1).$$

We note that $h > 1$, since if $h = 1$, then we obtain $n \geq 1$, which contradicts the assumption $h > n$. By the assumption $h > n$, we have $3(h-1) > |d|(h-1)$, and therefore we conclude that $|d| = |a| \leq 2$.

Since $g = 2h+n-1 > 1$ (from $h > 1$ and $h > n \geq 0$), by Theorem 10.2.18 in [29], the Lefschetz fibration equips $(\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2$ with a symplectic form $\omega$ for which $F$ is a symplectic submanifold (here, we orient $F$ so that $\omega|_F > 0$). Moreover, there exists a compatible almost complex structure $J$ for which $F$ is a pseudo-holomorphic submanifold (or an embedded $J$-holomorphic curve). See Section 10.1 in [29] for the definitions and Lemma 3.1 in [40] for the proof. Then, by the same argument as in the proof of Lemma 4.2 in [40], we have $\nu \cdot [F] = a \geq 0$. This can be proved directly using Proposition 3.2 in [59] stating that for any compatible (or more generally, tamed) almost complex structure $J$ on $R_h \sharp m\overline{\mathbb{CP}}^2$ with $m \geq 0$, the homology class $\nu$ of a fiber of the ruling $R_h \to \Sigma_h$ satisfies that $\nu \cdot [C] \geq 0$ for any embedded $J$-holomorphic curve $C$ (or more generally, any $J$-holomorphic subvariety $C$) in $R_h \sharp m\overline{\mathbb{CP}}^2$ if $h \geq 1$, where $R_h$ is a ruled surface over $\Sigma_h$.

By $[F]^2 = 0$ (being a fiber of a Lefschetz fibration), we get $2ab - \sum_{i=1}^m k_i^2 = 0$. Since the symplectic structure on $(\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2$ is essentially unique (up to diffeomorphism and symplectic deformation) [41], we can assume that $F$ is a symplectic submanifold with respect to the standard symplectic structure, and hence, satisfies the adjunction formula. This gives

$$\begin{aligned}
2g - 2 &= K \cdot [F] + [F]^2 \\
&= \left( -2\mu + (2h-2)\nu + \sum_{i=1}^m e_i \right) \cdot (a\mu + b\nu + \sum_{i=1}^m k_i e_i) + 0 \\
&= 2ah - 2a - 2b - \sum_{i=1}^m k_i,
\end{aligned}$$

where $K$ is the canonical class, which (together with $2ab = \sum_{i=1}^m k_i^2$ and $a \geq 0$) will provide the desired contradiction. If $a = 0$, then we have $k_i = 0$ as well, and consequently $[F] = b\nu$. By tubing $|b|$ disjoint copies of a sphere representing $\nu$, we see that $[F]$ is represented by an embedded sphere. This contradicts that $g > 0$ and that $F$, which is a symplectic surface, realizes the minimum genus in its homology class. In the case $a = 1$, we get $g + n - 1 = -2b - \sum_{i=1}^m k_i$ (since $2h + n - 1 = g$); using $2b = \sum_{i=1}^m k_i^2$ this yields $g + n - 1 - \frac{m}{4} = -\sum_{i=1}^m \left(k_i + \frac{1}{2}\right)^2$, providing a contradiction since $g > 0$ and $m < 4n$. For $a = 2$, the resulting equality is $2n = -2b - \sum_{i=1}^m k_i$, which (together with $4b = \sum_{i=1}^m k_i^2$) gives $0 < 4n - m = -\sum_{i=1}^m (k_i + 1)^2$, which provides another contradiction.

Suppose that $m = 0$, and hence, $\Sigma_h \times \mathbb{S}^2$ admits a nontrivial genus-$g$ Lefschetz fibration. We set $[F] = a\mu + b\nu$. Then, by an argument similar to the case $m > 0$, we obtain $h > 1$, $0 \leq a \leq 2$, $2ab = 0$ and $2g - 2 = 2ah - 2a - 2b$. Since we have $[F] = b\nu$ if $a = 0$, an argument similar to the case $m > 0$ shows that $a \neq 0$, and hence $b = 0$. If $a = 1$, then we get $g = h$. By $g = 2h + n - 1$ and the assumption that $h > n \geq 0$, we obtain $h = 1$ and $n = 0$, which contradicts $h > 1$. In the case $a = 2$, we get $g = 2h - 1$. Then, applying an argument similar to the case $m > 0$, we have

$$g(\overline{F}_s) - 1 = (g-1) - 1 = 2h - 3 \geq |d|(h-1) = |a|(h-1) = 2(h-1),$$

which is impossible.

From the arguments above, we see that the 4-manifold $(\Sigma_h \times \mathbb{S}^2)\sharp m\overline{\mathbb{CP}}^2$ does not admit any genus-$(2h+n-1)$ Lefschetz fibrations for $m < 4n$.

Finally, suppose that $m = n = 0$, and hence, $\Sigma_h \times \mathbb{S}^2$ admits a nontrivial genus-$(2h-1)$ Lefschetz fibration. Then, by an argument similar to the case $m > 0$, we have $g(\overline{F}_s)-1 = 2h-3 \geq |d|(h-1) = |a|(h-1)$, and hence $h > 1$ (by $h > n = 0$) and $|a| \leq 1$. An argument similar to the case $m > 0$ again shows $0 \leq a$ (and therefore $a = 0, 1$), $2ab = 0$ and $2g - 2 = 2ah - 2a - 2b$. Moreover, applying an argument similar to the case $m = 0$, we have $a \neq 0, 1$, a contradiction.

This proves the lemma. □

**Remark 10.** The "only if" part of the proof of Lemma 9 is based on that of Lemma 4.4 in [53] and that of Proposition 4.4 in [40]. The argument of [53] requires the assumption that Lefschetz fibrations admit a section. In [14, 11], the proofs without requiring the assumption were given. The proof of Lemma 9 also does not need the existence of a section.

**Lemma 11.** *Let $n$ and $h$ be nonnegative integers satisfying $h > n$ (therefore $h > 0$). Then, the nontrivial $\mathbb{S}^2$-bundle $\Sigma_h \widetilde{\times} \mathbb{S}^2$ over $\Sigma_h$ cannot admit a nontrivial genus-$(2h+n-1)$ Lefschetz fibration over $\mathbb{S}^2$.*

*Proof.* Suppose that $\Sigma_h \widetilde{\times} \mathbb{S}^2$ admits a nontrivial genus-$g$ Lefschetz fibration over $\mathbb{S}^2$. We denote by $[F]$ the homology class of a regular fiber $F$ of this Lefschetz fibration. Let $\mu$ and $\nu$ be the homology classes of the section with self-intersection number 1 and a fiber of the ruling $p : \Sigma_h \widetilde{\times} \mathbb{S}^2 \to \Sigma_h$, repsectively. We orient the section and the fiber of $p$ so that $\mu \cdot \nu (= \mu \cdot \mu) = 1$ (and $\nu \cdot \nu = 0$). After choosing an orientation on $F$, we set $[F] = a\mu + b\nu$ for some integers $a$ and $b$. By an argument similar to the proof of Lemma 9, we obtain $0 \leq a \leq 2$. Since $[F]^2 = 0$, we get $a^2 + 2ab = a(a+2b) = 0$. The Lefschetz fibration equips $\Sigma_h \widetilde{\times} \mathbb{S}^2$ with a symplectic form $\omega$ for which $F$ is a symplectic submanifold. We orient $F$ so that $\omega|_F > 0$. Since the symplectic structure on $\Sigma_h \widetilde{\times} \mathbb{S}^2$ is essentially unique (up to diffeomorphism and symplectic deformation) [41], we can assume that $F$ is a symplectic submanifold with respect to the standard symplectic structure, and thus, satisfies the adjunction formula. This gives

$$\begin{aligned} 2g - 2 &= K \cdot [F] + [F]^2 \\ &= (-2\mu + (2h-1)\nu) \cdot (a\mu + b\nu) + 0 \\ &= 2ah - 3a - 2b, \end{aligned}$$

where $K$ is the canonical class, which (together with $a(a+2b) = 0$) will provide the desired contradiction. We see that $a \neq 0$ by an argument similar to the case of $m < 4n$ and $m \neq 0$ in the proof of Lemma 9. Therefore, we have $a = -2b$. If $a = 1$, then we get $1 = -2b$, a contradiction. In the case of $a = 2$ (therefore $b = -1$), we have $g = 2h - 1$. By an argument similar to the case of $m < 4n$, $m = 0$ and $a = 2$ in the proof of Lemma 9, we obtain $a \leq 1$, a contradiction.

This finishes the proof. $\square$

By Lemmas 9 and 11, we obtain Proposition 6.

The following lemma immediately follows from the definition of the fiber sum operation and the Novikov additivity for the signature.

**Lemma 12.** *Suppose that $f : X \to \mathbb{S}^2$ is a fiber sum of two genus-$g$ Lefschetz fibrations $f_1 : X_1 \to \mathbb{S}^2$ and $f_2 : X_2 \to \mathbb{S}^2$. Then, $e(X) = e(X_1) + e(X_2) + 4(g-1)$ and $\sigma(X) = \sigma(X_1) + \sigma(X_2)$. Therefore,*

$$\chi(X) = \chi(X_1) + \chi(X_2) + (g-1), \quad c_1^2(X) = c_1^2(X_1) + c_1^2(X_2) + 8(g-1).$$

**Proposition 13.** *We set $g = 2h+n-1$, where $h$ and $n$ are positive integers and $n \geq 2$. Let $f : X \to \mathbb{S}^2$ be a genus-$g$ Lefschetz fibration with $\sigma(X) = -8n + i$ and $e(X) = 12n - i$ for $i = 1, 2, \dots, 2n-2$. If $h > (5n-3)/2$ (therefore $g > 1$), then $f$ is indecomposable for any $i$.*

*Proof.* Let $i = 1, 2, \dots, 2n-2$. Suppose that $f : X \to \mathbb{S}^2$ is a fiber sum of two nontrivial genus-$g$ Lefschetz fibrations $f_1 : Z_1 \to \mathbb{S}^2$ and $f_2 : Z_2 \to \mathbb{S}^2$. By the assumption and Lemma 12, we have

$$i = c_1^2(X) = c_1^2(Z_1) + c_1^2(Z_2) + 8(g-1).$$

Since every nontrivial genus-$g$ Lefschetz fibration $Y \to \mathbb{S}^2$ satisfies $c_1^2(Y) \geq 4(1-g)$ (see Lemma 3.2 in [51]), we set $c_1^2(Z_j) = i_j + 4(1-g)$ for $j = 1, 2$, where $i_j$ is a nonnegative integer. Therefore, we obtain $i = i_1 + i_2$. By $i \leq 2n-2 \leq 2g-2 (= 2(2h+n-1)-2)$ and $i_j \geq 0$, we see that $i_j \leq 2g-2$ for $j = 1, 2$, and hence

$$c_1^2(Z_j) = i_j + 4(1-g) \leq 2(1-g).$$

This gives that $Z_1$ and $Z_2$ are rational or ruled surfaces from Theorem 1 in [40]. Therefore, $\sigma(Z_j) \leq 0$, except in the case that $Z_j$ is diffeomorphic to $\mathbb{CP}^2$. Since $c_1^2(W) \geq 0$ for $W = \mathbb{CP}^2, \mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}, \mathbb{S}^2 \times \mathbb{S}^2$ and $g > 1$, we see that $Z_j$ is diffeomorphic to $R_{h_j} \sharp n_j \overline{\mathbb{CP}^2}$ for $j = 1, 2$, where $n_j$ and $h_j$ are nonnegative integers and $R_{h_j}$ is a ruled surface over $\Sigma_{h_j}$.

By $Z_j \cong R_{h_j} \sharp n_j \overline{\mathbb{CP}^2}$, we have $e(Z_j) = 4 - 4h_j + n_j$ and $\sigma(Z_j) = -n_j$. Moreover, by $12n - i = e(X) = e(Z_1) + e(Z_2) + 4(g-1)$, $-8n + i = \sigma(X) = \sigma(Z_1) + \sigma(Z_2)$ and $g = 2h + n - 1$ (from the assumption and Lemma 12) , we get

$$n_1 + n_2 = 8n - i,$$
$$h_1 + h_2 = 2h = g - n + 1.$$

For simplicity, suppose that $h_1 \leq h_2$. Then, we have $2h = g - n + 1 = h_1 + h_2 \leq 2h_2$, hence $h \leq h_2$ and $g \leq 2h_2 + n - 1$. Here, note that $2h_2 \leq g$ by Proposition 4.4 in [40]. Therefore, we set $g = 2h_2 + k - 1$ for $k = 1, 2, \dots, n$, and $Z_2 (\cong R_{h_2} \sharp n_2 \overline{\mathbb{CP}^2})$ admits a nontrivial genus-$g$ Lefschetz fibration, where $g = 2h_2 + k - 1$. Since $h_2 \geq h > (5n-3)/2 \geq n$ by $n \geq 2$, we have $h_2 > k$ by

$n \geq k$. Therefore, Proposition 6 gives $n_2 = 4k$. By $h_1 + h_2 = 2h = g - n + 1$ and $g = 2h_2 + k - 1$, we have $g = 2h_1 + (2n - k) - 1$, and $Z_1 (\cong R_{h_1} \sharp n_1 \overline{\mathbb{CP}}^2)$ admits a nontrivial genus-$g$ Lefschetz fibration, where $g = 2h_1 + (2n-k) - 1$. Here, by $2h_2 \leq g$, $h_1 + h_2 = 2h$ and $g = 2h + n - 1$, we obtain

$$\begin{aligned} 2h_2 \leq g = 2h + n - 1 &\Longleftrightarrow 2h_1 + 2h_2 \leq 2h_1 + 2h + n - 1 \\ &\Longleftrightarrow 4h \leq 2h_1 + 2h + n - 1, \end{aligned}$$

and therefore $h - (n-1)/2 \leq h_1$. Moreover, using the assumption $h > (5n-3)/2$, we get $2n - 1 < h_1$, and therefore, $2n - k < h_1$ for $k = 1, 2, \ldots, n$. Therefore, we obtain $n_1 = 4(2n - k)$ by Proposition 6. From the argument above, we have $n_1 + n_2 = 4(2n - k) + 4k = 8n$, which contradicts $n_1 + n_2 = 8n - i$.

This finishes the proof. □

We prove Theorem C.

*Proof of Theorem C.* Let $h$ and $n$ be positive integers, and let $n \geq 2$. Using $\chi(X) = (\sigma(X) + e(X))/4$ and $c_1^2(X) = 3\sigma(X) + 2e(X)$, it follows from Proposition 13 that every genus-$(2h + n - 1)$ Lefschetz fibration $f : X \to \mathbb{S}^2$ with $\chi(X) = n$ and $c_1^2(X) = i$ is indecomposable for $i = 1, 2, \ldots, 2n - 2$ if $h > (5n - 3)/2$. Therefore, we see that the Lefschetz fibrations in Theorem B satisfying the condition $h > (5n - 3)/2$ are indecomposable, which proves Theorem C. □

## 4. Preliminaries for Theorems A and B

4.1. **Mapping class groups and positive factorizations.** The mapping class group arguments are used for the construction of Lefschetz fibrations in Theorems A and B, and 4-manifold theory is used for the minimality of the Lefschetz fibrations.

Let $\Sigma_g^b$ be the compact oriented surface obtained by removing $b$ disjoint open disks from $\Sigma_g$. The *mapping class group* of $\Sigma_g^b$, denoted by $\Gamma_g^b$, is the group of isotopy classes of orientation preserving self-diffeomorphisms of $\Sigma_g^b$. We assume that diffeomorphisms and isotopies fix the points of the boundary. To simplify notation, we write $\Sigma_g = \Sigma_g^0$ and $\Gamma_g = \Gamma_g^0$. Elements of $\Gamma_g^b$ are called *mapping classes*. For $\phi_1$ and $\phi_2$ in $\Gamma_g^b$, the notation $\phi_1\phi_2$ means that we first apply $\phi_2$ and then $\phi_1$. Let $t_c$ be the Dehn twist about a simple closed curve $c$ on $\Sigma_g^b$. Note that $t_{\phi(c)} = \phi t_c \phi^{-1}$ for a mapping class $\phi$ in $\Gamma_g^b$ and $t_c t_d = t_d t_c$ if $c$ is disjoint from a simple closed curve $d$ on $\Sigma_g^b$. We say that a mapping class $\phi$ in $\Gamma_g^b$ is a *half twist* about $c$ if it satisfies $\phi^2 = t_c$. If a mapping class $\phi$ in $\Gamma_g^b$ can be written as a product $t_{v_n} \cdots t_{v_2} t_{v_1}$ of Dehn twists about simple closed curves $v_1, \ldots, v_n$ on $\Sigma_g^n$, then the word $t_{v_n} \cdots t_{v_2} t_{v_1}$ is called a *positive factorization* of $\phi$.

Let us consider a genus-$g$ Lefschetz fibration $f : X \to \mathbb{S}^2$ with $n$ singular fibers. The monodromy of a genus-$g$ Lefschetz fibration $f : X \to \mathbb{S}^2$

comprises a positive factorization of id in $\Gamma_g$ as

$$t_{v_n} \cdots t_{v_2} t_{v_1} = \mathrm{id} \in \Gamma_g,$$

where $v_1, \ldots, v_n$ are the vanishing cycles of the singular fibers. Conversely, we obtain a genus-$g$ Lefschetz fibration over $\mathbb{S}^2$ with the vanishing cycles $v_1, \ldots, v_n$ from the above-mentioned positive factorization in $\Gamma_g$. More details can be found in [44, 29].

Let $\delta$ be the boundary curve of $\Sigma_g^1$. When we consider the genus-$g$ Lefschetz fibration $f$ corresponding to a positive factorization $t_{v_n} \cdots t_{v_2} t_{v_1}$ of id in $\Gamma_g$, a lift of this positive factorization to $\Gamma_g^1$ as

$$t_{v_n'} \cdots t_{v_2'} t_{v_1'} = t_\delta^k$$

shows the existence of a $(-k)$-section of $f$, where $v_i'$ is a simple closed curve on $\Sigma_g^1$ mapped to $v_i$ under the inclusion $\Sigma_g^1 \to \Sigma_g$. Conversely, such a positive factorization of $t_\delta^k$ gives a genus-$g$ Lefschetz fibration with a $(-k)$-section.

The following theorem was given in [33, 44].

**Theorem 14** ([33, 44])**.** *Let $f_i : X_i \to \mathbb{S}^2$ be a genus-$g$ Lefschetz fibration corresponding to a positive factorization $\rho_i$ of* id *in $\Gamma_g$ for $g \geq 2$ $(i = 1, 2)$. Then, $f_2$ is isomorphic to $f_1$ if and only if $\rho_2$ is obtained from $\rho_1$ by applying a finite series of elementary transformations*

$$t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1} \;\leftrightarrow\; t_{v_n} \cdots t_{v_{i+2}} t_{v_i} t_{t_{v_i}^{-1}(v_{i+1})} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1},$$

$$t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1} \;\leftrightarrow\; t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{t_{v_i}(v_{i-1})} t_{v_i} t_{v_{i-2}} \cdots t_{v_1},$$

*and simultaneous conjugations*

$$t_{v_n} \cdots t_{v_2} t_{v_1} \;\leftrightarrow\; t_{\phi(v_n)} \cdots t_{\phi(v_2)} t_{\phi(v_1)}$$

*for any $\phi$ in $\Gamma_g$.*

It is well-known that when we apply a cyclic permutation to the positive factorization of id (resp. $t_\delta^k$) corresponding to a Lefschetz fibration, the Lefschetz fibration corresponding to the resulting positive factorization of id (resp. $t_\delta^k$) is the same as the original one. For this reason, if a positive factorization $\rho_2$ of id in $\Gamma_g$ (resp. $t_\delta^k$ in $\Gamma_g^1$) is obtained from a positive factorization $\rho_1$ of id in $\Gamma_g$ (resp. $t_\delta^k$ in $\Gamma_g^1$) by applying a finite series of elementary transformations, simultaneous conjugations and cyclic permutations, then we write

$$\rho_1 \equiv \rho_2.$$

Finally, we present a fundamental lemma to compute the fundamental group of the total space of a Lefschetz fibration.

**Lemma 15** (cf.[29])**.** *Let $f : X \to S^2$ be a genus-$g$ Lefschetz fibration with a section and corresponding to a positive factorization $t_{v_n} \cdots t_{v_2} t_{v_1}$ of* id *in $\Gamma_g$. Then, the fundamental group $\pi_1(X)$ is isomorphic to the quotient of $\pi_1(\Sigma_g)$ by the normal subgroup generated by $v_1, \ldots, v_n$.*

4.2. **Minimality.** Let $X$ and $Y$ be symplectic 4-manifolds, and let $V_X \subset X$ and $V_Y \subset Y$ be embedded symplectic surfaces of genus $g \geq 0$ whose homology classes satisfy $[V_X]^2 + [V_Y]^2 = 0$. We denote by $\mathcal{N}V_X$ (resp. $\mathcal{N}V_Y$) the open disk normal bundle of $V_X$ in $X$ (resp. $V_Y$ in $Y$). For any orientation-reversing diffeomorphism $\psi : \partial\mathcal{N}V_X \to \partial\mathcal{N}V_Y$ between the boundaries of $\mathcal{N}V_X$ and $\mathcal{N}V_Y$ that is lifted from an orientation-preserving diffeomorphism from $V_X$ to $V_Y$, the *symplectic sum* (or *symplectic fiber sum*) of $X$ and $Y$ along $V_X$ and $V_Y$ is defined as

$$X\#_{V_X=V_Y}Y = (X - \mathcal{N}V_X) \cup_\psi (Y - \mathcal{N}V_Y).$$

It was shown in [28] that there is a natural isotopy class of symplectic structures on $X\#_{V_X=V_Y}Y$ extending the symplectic structures on $X - \mathcal{N}V_X$ and $Y - \mathcal{N}V_Y$. The minimality of symplectic sums is described by the following theorem. We will use this theorem to verify that the total spaces of our Lefschetz fibrations are minimal symplectic 4-manifolds.

**Theorem 16** ([56], [21])**.** *In the notation above, let $M$ be the symplectic sum of $X$ and $Y$ along $V_X$ and $V_Y$. Then, the following holds.*

- (i) *If $X \setminus V_X$ or $Y \setminus V_Y$ contains an embedded symplectic sphere of self-intersection $-1$, then $M$ is not minimal.*
- (ii) *If one of the summands is $\mathbb{CP}^2$ with $V_{\mathbb{CP}^2}$ an embedded sphere of self-intersection $4$ in the class $[V_{\mathbb{CP}^2}] = 2[H] \in H_2(\mathbb{CP}^2;\mathbb{Z})$ and the other summand (for definiteness, say $X$) has at least $2$ disjoint embedded symplectic spheres $E_i$ of self-intersection $-1$ each meeting $V_X$ positively and transversely in a single point with $[E_i]\cdot[V_X] = 1$, where $[H]$ is the homology class of the complex projective line $H = \{[x : y : z] \in \mathbb{CP}^2 \mid x = 0\}$, then $M = X\#_{V_X=V_{\mathbb{CP}^2}}\mathbb{CP}^2$ is not minimal.*
- (iii) *If one of the summands (for definiteness, say $Y$) is an $\mathbb{S}^2$-bundle over a genus $g$ surface and $V_Y$ is a section of this bundle, then $M$ is minimal if and only if $X$ is minimal.*
- (iv) *In all other cases $M$ is minimal.*

As a corollary, Usher showed the following result.

**Corollary 17** ([56],[14], ([48], for the case $g = 2$))**.** *Let $f_i$ be a genus-$g$ Lefschetz fibration for $i = 1, 2$. Then, the total space of a genus-$g$ Lefschetz fibration obtained by fiber summing $f_1$ and $f_2$ is minimal.*

Next, we present a technique to get a new Lefschetz fibration from a given Lefschetz fibration.

**Definition 18.** Let $x, y, z$ be the interior curves on a subsurface $\Sigma_0^4$ in $\Sigma_g^m$ as in Figure 1, where $m$ is a nonnegative integer, and let $a, b, c, d$ be the boundary curves of $\Sigma_0^4$ as in the figure. Then, the *lantern relation*

$$t_a t_b t_c t_d = t_x t_y t_z$$

holds in $\Gamma_g^m$ (see [19, 31]).

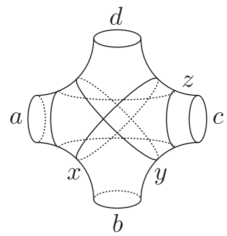


FIGURE 1. The curves $a, b, c, d, x, y, z$ on $\Sigma_0^4$.

Let $t_a t_b t_c t_d$ be a product of four Dehn twists satisfying the lantern relation $t_a t_b t_c t_d = t_x t_y t_z$. If there is a genus-$g$ Lefschetz fibration corresponding to a positive factorization $\rho := t_{v_n} \cdots t_{v_{i+1}} t_a t_b t_c t_d t_{v_i} \cdots t_{v_1}$ of id, we get a new genus-$g$ Lefschetz fibration corresponding to a positive factorization $\rho' := t_{v_n} \cdots t_{v_{i+1}} t_x t_y t_z t_{v_i} \cdots t_{v_1}$ of id by the lantern relation $t_a t_b t_c t_d = t_x t_y t_z$. Then, we say that $\rho'$ is obtained by applying a *lantern substitution* to $\rho$.

It was shown in [23] that a lantern substitution corresponds to a rational blowdown along a sphere of self-intersection $-4$ (see [25] for the definition).

**Theorem 19** ([23])**.** *Let $\rho$ and $\rho'$ be positive factorizations of* id *in $\Gamma_g$, and let $X$ and $X'$ be the total spaces of Lefschetz fibrations corresponding to $\rho$ and $\rho'$, respectively. If $\rho'$ is obtained by applying a lantern substitution to $\rho$, then $X'$ is a rational blowdown of $X$ along a sphere of self-intersection $-4$. Therefore, $\sigma(X') = \sigma(X) + 1$ and $e(X') = e(X) - 1$.*

The following lemma is useful to show that the total spaces of Lefschetz fibrations in Theorems A and B are minimal.

**Lemma 20.** *In the notation of Theorem 19, if $X$ is minimal, then $X'$ is also minimal.*

*Proof.* Since we can apply a lantern substitution to $\rho$, $\rho$ contains a subword $t_a t_b t_c t_d$ satisfying the lantern relation $t_a t_b t_c t_d = t_x t_y t_z$. By perturbing the Lefschetz fibration $f : X \to \mathbb{S}^2$, we can arrange it so that the critical points corresponding to the vanishing cycles $a, b, c, d$ lie on the same singular fiber. Then, the singular fiber has a component, which is a sphere $S$ of self-intersection $-4$. In addition, $S$ can be assumed to be symplectic with respect to a Gompf–Thurston form (this follows from Corollary 23).

We can view the rational blowdown surgery along a symplectic sphere of self-intersection $-4$ as the symplectic sum: we have $X' = X \#_{S = V_{\mathbb{CP}^2}} \mathbb{CP}^2$, where $V_{\mathbb{CP}^2}$ is an embedded sphere of self-intersection 4 in the class $[V_{\mathbb{CP}^2}] = 2[H] \in H_2(\mathbb{CP}^2; \mathbb{Z})$, $S$ is an embedded symplectic sphere of self-intersection $-4$ in $X$ and $[H]$ is the homology class of the complex projective line $H = \{[x : y : z] \in \mathbb{CP}^2 \mid x = 0\}$. The lemma follows from Theorem 16 implying that $X'$ is a minimal symplectic 4-manifold. $\square$

We show that not only the above-mentioned sphere $S$ of self-intersection $-4$ but also a surface of self-intersection $-n$ which is similarly obtained by perturbing a Lefschetz fibration $f$ is symplectic with respect to a Gompf–Thurston form for any integer $n \geq 2$. This follows from Lemma 21 below.

**Lemma 21.** *Let $n \geq 2$ be an integer. Suppose that $g \geq 2$. Let us consider a genus-$g$ Lefschetz fibration $f : X \to \mathbb{S}^2$ such that $f$ is "not" injective on the set of critical points and has only two types of singular fibers as follows:*

1. *a fiber containing only one singular point,*
2. *a fiber containing $n$ singular points such that the corresponding vanishing cycles $a_1, a_2, \ldots, a_n$ on a regular fiber $F$ are boundary curves of a subsurface $\Sigma_h^n$ of genus $h$ with $n$ boundary components in $F$.*

*Then, $X$ admits a symplectic structure with symplectic fibers.*

The proof is similar to that of Theorem 10.2.18 in [29] except for the part corresponding to Exercise 10.2.19 in [29], so we give the following lemma, which is a generalization of Exercise 10.2.19 in [29].

**Lemma 22.** *In the notation of Lemma 21, there exists a closed 2-form $\zeta$ on $X$ such that $\int_E \zeta > 0$ for any closed surface $E$ contained in a fiber (with the induced orientation), which can be an entire regular fiber.*

*Proof.* Let $[F] \in H_2(X;\mathbb{R})$ denote the homology class of a regular fiber. Note that there is an element $a \in H^2_{dR}(X)$ with $\langle a, [F]\rangle > 0$ since $[F] \neq 0$ in $H_2(X;\mathbb{R})$ by $g \geq 2$. Moreover, it follows immediately from Exercise 10.2.19 in [29] that there exists an element $a \in H^2_{dR}(X)$ with $\langle a, [F]\rangle > 0$ and $\langle a, [E]\rangle > 0$ for any closed surface $E$ in the regular fibers and the singular fibers satisfying the condition (1) in Lemma 21.

Suppose that a singular fiber satisfying the condition (2) in Lemma 21 admits a decomposition into (nonempty) $m+1$ closed surfaces $F_0 \cup F_1 \cup \cdots \cup F_m$, where $F_0$ is the "core" surface corresponding to $\Sigma_h^n$. Note that $[F_{i_1}] \cdot [F_{i_2}] = 0$ for $0 < i_1 < i_2$ since there is no singular point between the two surfaces corresponding to $F_{i_1}$ and $F_{i_2}$ from the property of singular fibers satisfying the condition (2) in Lemma 21. If $F_0$ and $F_i$ intersect transversely at $k_i(> 0)$ points for $i = 1, 2, \ldots, m$, then we see that

- $[F_0] \cdot [F_0] = -n$;
- $[F_0] \cdot [F_i] = -[F_i] \cdot [F_i] = k_i$;
- $[F_{i_1}] \cdot [F_{i_2}] = 0$ for $0 < i_1 < i_2$;
- $[F_0] \cdot [F_0] = \sum_{i=1}^m [F_i] \cdot [F_i]$ (i.e., $n = \sum_{i=1}^m k_i$);
- $0 < \langle a, [F]\rangle = \left\langle a, \left[\bigcup_{j=0}^m F_j\right]\right\rangle = \sum_{j=0}^m \langle a, [F_j]\rangle$.

Then, we obtain the following symmetric matrix $A$:

$$A := ([F_i]\cdot[F_j]) = \begin{pmatrix} -n & k_1 & k_2 & k_3 & \cdots & k_m \\ k_1 & -k_1 & 0 & 0 & \cdots & 0 \\ k_2 & 0 & -k_2 & 0 & \cdots & 0 \\ k_3 & 0 & 0 & -k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & 0 & 0 & 0 & \cdots & -k_m \end{pmatrix}.$$

Moreover, it is easy to check from the fourth equation above that rank $A = m$. Therefore, for any $r_0, r_1, \ldots, r_m \in \mathbb{R}$ satisfying $\sum_{j=0}^m r_j = 0$, we can choose constants $s_0, s_1, \ldots, s_m \in \mathbb{R}$ such that $\left(\sum_{j=0}^m s_j[F_j]\right) \cdot [F_k] = r_k$, $0 \le k \le m$.

Here, we set

$$a' = a + \sum_{j=0}^m s_j PD[F_j] \in H^2(X;\mathbb{R}),$$

where $PD[F_j]$ is the Poincaré dual of $[F_j]$ and $s_j \in \mathbb{R}$. Note that we have $\langle PD[F_j], [S]\rangle = [F_j]\cdot[S]$ for an oriented surface $S$. Since a regular fiber is disjoint from $F_j$ (therefore $[F_j]\cdot[F] = 0$), we see that

$$\langle a', [F]\rangle = \langle a, [F]\rangle + \left\langle \sum_{j=0}^m s_j PD[F_j], [F] \right\rangle = \langle a, [F]\rangle > 0.$$

Moreover, we have

$$\langle a', [F_k]\rangle = \langle a, [F_k]\rangle + \left\langle \sum_{j=0}^m s_j PD[F_j], [F_k] \right\rangle = \langle a, [F_k]\rangle + r_k$$

for $k = 0, 1, \ldots, m$. By choosing $r_k$ suitably, we get $\langle a', [F_k]\rangle > 0$ for any $k$ (for example, we set $r_k = \frac{\langle a,[F]\rangle}{m+1} - \langle a, [F_k]\rangle$, and then $\sum_{k=0}^m r_k = 0$ from $\langle a, [F]\rangle = \sum_{j=0}^m \langle a, [F_j]\rangle$).

Since any closed surface $E$ in the regular fibers and the singular fibers satisfying the condition (1) in Lemma 21 are disjoint from $F_j$ (and therefore $[F_j]\cdot[E] = 0$) for $j = 0, 1, \ldots, m$, we obtain

$$\langle a', [E]\rangle = \langle a, [E]\rangle + \left\langle \sum_{j=0}^m s_j PD[F_j], [E] \right\rangle = \langle a, [E]\rangle > 0.$$

By repeating this argument for every singular fiber satisfying the condition (2) in Lemma 21, we get a cohomology class $\zeta$ with the desired properties. $\square$

Since the proof of Lemma 21 is similar to that of Theorem 10.2.18 in [29], we omit it. From the construction of a symplectic 2-form $\omega_t$ on $X$ (see the proof of Theorem 10.2.18 in [29]), we obtain the following.

**Corollary 23.** *In the notation of Lemma 21 and with the symplectic structure given in the same lemma, any closed surfaces contained in any fibers are symplectic surfaces. In particular, the component corresponding to $\Sigma_h^n$ is a symplectic surface of self-intersection $-n$, and therefore the sphere $S$ of self-intersection $-4$ in the proof of Lemma 20 is symplectic.*

**Remark 24.** From Corollary 23, we see that various surgery operations corresponding to substitution techniques of monodromies of Lefschetz fibrations are symplectic surgery operations (for example, substitutions of a lantern relation, a star relation and a chain relation corresponding to a rational blowdown along a sphere of self-intersection $-4$, a star surgery and a chain surgery, respectively, and so on).

4.3. **Lifts of the hyperelliptic relation and Gurtas' relation.** In this subsection, we present Propositions 25 and 26 below. These propositions express that the Lefschetz fibration arising from the *hyperelliptic relation*, which appears in [18], and Gurtas' Lefschetz fibration given in [30] (and see also [58]) have sections. Using these propositions, we show the existence of sections of Lefschetz fibrations in the main theorems.

**Proposition 25.** *Let $c_1', c_1'', c_2, c_3, \ldots, c_{2n-1}$ be the simple closed curves on $\Sigma_{n-1}^1$ as in Figure 2, and let $\delta$ be the boundary curve of $\Sigma_{n-1}^1$ as in the figure. Then, the product $\overline{\eta}$*

$$\overline{\eta} := t_{c_{2n-1}} t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c_1'} t_{c_1''} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}} t_{c_{2n-1}}$$

*is the half twist $h_\delta$ about $\delta$ such that $h_\delta(c_1') = c_1''$ and $h_\delta(c_i) = c_i$ for $2 \leq i \leq 2n-1$. Therefore, the following holds in $\Gamma_{n-1}^1$:*

$$\overline{\eta}^2 = t_\delta.$$

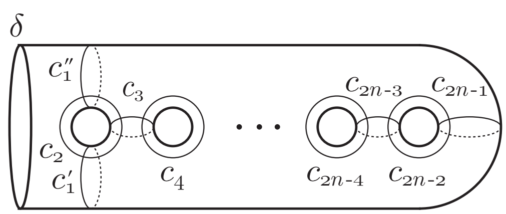


FIGURE 2. The curves $c_1', c_1'', c_2, c_3, \ldots, c_{2n-1}$ on $\Sigma_{n-1}^1$ and the boundary curve $\delta$ of $\Sigma_{n-1}^1$.

**Proposition 26.** *Let $c_1', c_1'', c_2, c_3, \ldots, c_{2n-1}, D_0, D_1, D_2, \ldots, D_{2h}$ be the simple closed curves on $\Sigma_{2h+n-1}^1$ as in Figure 3, and let $\delta$ be the boundary curve* *of $\Sigma^1_{2h+n-1}$ as in the figure. Then, the product $\overline{\theta}$*

$$\overline{\theta} := t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c'_1} t_{c''_1} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}} t_{D_0} t_{D_1} t_{D_2} \cdots t_{D_{2h}} t_{c_{2n-1}}$$

*is the half twist $h'_\delta$ about $\delta$ such that $h'_\delta(c'_1) = c''_1$ and $h'_\delta(c_i) = c_i$ for $2 \leq i \leq 2n-1$ and $h'_\delta(D_j) = D_j$ for $0 \leq j \leq 2h$. Therefore, the following holds in $\Gamma^1_{2h+n-1}$:*

$$\overline{\theta}^2 = t_\delta.$$

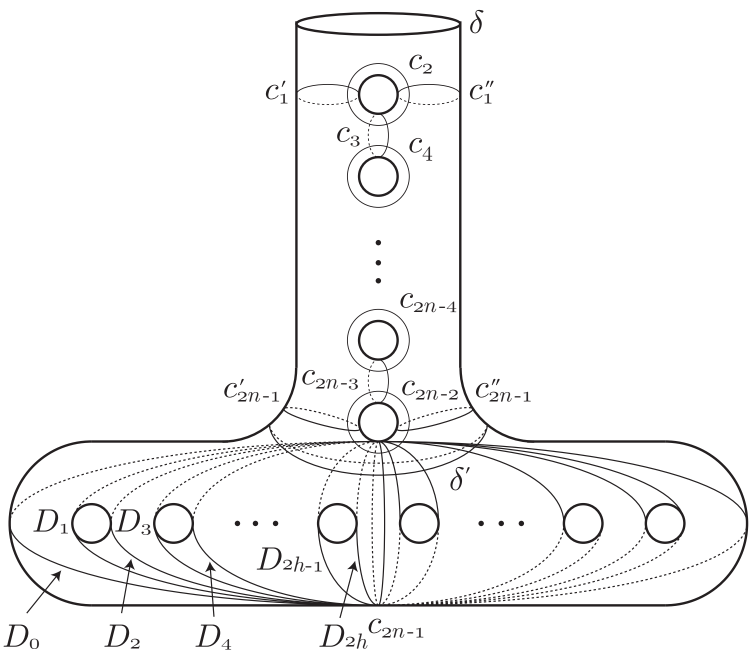


FIGURE 3. The curves $c'_1, c''_1, c_2, c_3, \ldots, c_{2n-1}, c'_{2n-1}, c''_{2n-1}, \delta'$, $D_0, D_1, D_2, \ldots, D_{2h}$ on $\Sigma^1_{2h+n-1}$ and the boundary curve $\delta$ of $\Sigma^1_{2h+n-1}$.

It is easily seen that the relations $\overline{\eta}^2 = t_\delta$ and $\overline{\theta}^2 = t_\delta$ in Propositions 25 and 26 are lifts of the hyperelliptic relation $\eta^2 = \mathrm{id}$ and Gurtas' relation $\theta^2 = \mathrm{id}$ (defined below), respectively. We postpone the proofs of Propositions 25 and 26 until Section 7. The propositions are proved by using the Alexander method, that is, one first fixes a finite set of curves whose complement is a disk on the surface and shows that the images of each curve under the half twist $h_\delta$ and $\overline{\eta}$ (resp. $h'_\delta$ and $\overline{\theta}$) are isotopic.

Let $\eta$ be the positive factorization of the hyperelliptic involution, which is the image of $\overline{\eta}$ under the map $\Gamma^1_{n-1} \to \Gamma_{n-1}$ induced by the inclusion $\Sigma^1_{n-1} \to \Sigma_{n-1}$. Let $Y(n, 0)$ be the total space of the genus-$(n-1)$ Lefschetz fibration corresponding to the positive factorization $\eta^2$ of id in $\Gamma_{n-1}$. We see that $e(Y(n,0)) = 4+4n$ since this fibration has $8n-4$ singular fibers and that $\sigma(Y(n,0)) = -4n$, using Endo's signature formula [22]. It is well-known that $Y(n,0)$ is diffeomorphic to $\mathbb{CP}^2\sharp(4n+1)\overline{\mathbb{CP}}^2$.

Let $\theta$ be Gurtas' positive factorization of a certain involution $\iota$ in $\Gamma_{2h+n-1}$ given in [30], which is the image of $\overline{\theta}$ under the map $\Gamma^1_{2h+n-1} \to \Gamma_{2h+n-1}$ induced by the inclusion $\Sigma^1_{2h+n-1} \to \Sigma_{2h+n-1}$. For the rest of this subsection, we give some remarks on $\theta$ and $\iota$. In [30], Gurtas showed that $\theta$ is a positive factorization of $\iota$ using the Alexander method, and hence $\theta^2$ is a positive factorization of id in $\Gamma_{2h+n-1}$. This fact was also verified in [58], up to Hurwitz equivalence. Let $Y(n,h)$ denote the total space of the Lefschetz fibration corresponding to the positive factorization $\theta^2$ of id in $\Gamma_{2h+n-1}$. It follows that the 4-manifold $Y(n,h)$ has a genus-$(2h+n-1)$ Lefschetz fibration over $\mathbb{S}^2$ with $4h+8n-4$ singular fibers, all of which are induced by nonseparating vanishing cycles. Therefore, the Euler characteristic of the symplectic 4-manifold $Y(n,h)$ is equal to $e(Y(n,h)) = 4-4(2h+n-1)+(4h+8n-4) = 4-4h+4n$. The signature $\sigma(Y(n,h))$ was calculated to be $-4n$ in [58].

We can also describe the Lefschetz fibration $f$ on $Y(n,h)$ corresponding to the positive factorization $\theta^2$ of id from a different viewpoint as follows. Let us take a double branched cover of $\Sigma_h\times\mathbb{S}^2$ along the union of two disjoint copies of $\Sigma_h\times\{pt\}$ and $2n$ disjoint copies of $\{pt\}\times\mathbb{S}^2$. The deck transformation of the double cover of $\Sigma_h$ branched over $2n$ points is the involution $\iota$. Then, we obtain the branched cover with $4n$ singular points corresponding to the number of the intersection points of the two horizontal genus $h$ surfaces and the $2n$ vertical spheres in the branch set. By desingularizing these $4n$ singular points, we get $(\Sigma_h\times\mathbb{S}^2)\sharp 4n\overline{\mathbb{CP}}^2$. Note that by projecting onto the $\mathbb{S}^2$ factor, we obtain a horizontal fibration $f' : (\Sigma_h\times\mathbb{S}^2)\sharp 4n\overline{\mathbb{CP}}^2 \to \mathbb{S}^2$ whose generic fiber is the double cover of $\Sigma_h$, branched over $2n$ points. Thus, the genus of a generic fiber of $f'$ is equal to $n+2h-1$. Moreover, each pair of singular fibers of $f'$, arising from two disjoint copies of $\Sigma_h\times\{pt\}$ in the branch set of the double cover of $\Sigma_h\times\mathbb{S}^2$, can be perturbed into $4n+2h-2$ Lefschetz type singular fibers, which is equivalent to the positive factorization $\theta$ of the involution $\iota$, as shown in the proof of [58]. As an immediate corollary, the 4-manifold $Y(n,h)$ is in fact diffeomorphic to $(\Sigma_h\times\mathbb{S}^2)\sharp 4n\overline{\mathbb{CP}}^2$, and therefore we obtain the Lefschetz fibration $f$.

4.4. **A mapping class $\phi$ and a lantern relation.** We give a lantern relation and a mapping class $\phi$ in $\Gamma^1_{n-1}$ (resp. $\Gamma^1_{2h+n-1}$), which are used to construct Lefschetz fibrations in Theorems A and B.

Suppose that $n-1\geq 3$. Let $v_1', v_1'', w, x, y, z$ be the simple closed curves on $\Sigma^1_{n-1}$ (resp. $\Sigma^1_{2h+n-1}$) as in Figure 4, and let $\delta$ be the boundary curve of $\Sigma^1_{n-1}$ (resp. $\Sigma^1_{2h+n-1}$) as in the figure. The simple closed curves $c_1', c_1'', c_2, \ldots, c_{12}$ as in Figure 4 are the same as those in Figure 2 (resp. Figure 3). Then, we have the following lantern relation

$$t_{c_1'}t_{c_3}t_{c_5}t_{v_1'} = t_xt_yt_z.$$

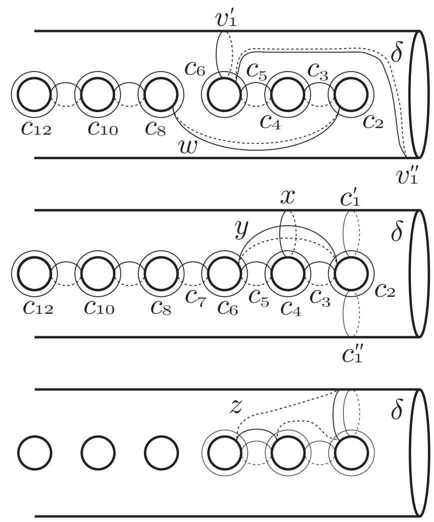


FIGURE 4. The curves $v_1', v_1'', w, x, y, z$ on $\Sigma^1_{n-1}$ (resp. $\Sigma^1_{2h+n-1}$) and the boundary curve $\delta$ of $\Sigma^1_{n-1}$ (resp. $\Sigma^1_{2h+n-1}$).

Let $\phi$ be a mapping class in $\Gamma^1_{n-1}$ such that

$$\begin{array}{llll} \phi(c_1') = v_1', & \phi(c_1'') = v_1'', & \phi(c_2) = c_6, & \phi(c_3) = c_5, \\ \phi(c_4) = c_4, & \phi(c_5) = c_3, & \phi(c_6) = c_2, & \phi(c_7) = w, \\ \phi(c_i) = c_i & & & \end{array}$$

for $i = 8, 9, \ldots, 2n-1$. If we consider $\phi$ as a mapping class in $\Gamma^1_{2h+n-1}$, then we add the condition

$$\phi(D_j) = D_j$$

for $j = 0, 1, \ldots, 2h$, where $D_j$ are the simple closed curves as in Figure 3.

4.5. **Elementary lemmas.** We construct some relations by applying elementary transformations. These relations will be used to construct new relations obtained by lantern substitutions in Section 5.2. We recall that for two positive factorizations $\rho_1$ and $\rho_2$, we write $\rho_1 \equiv \rho_2$ when $\rho_2$ is obtained from $\rho_1$ by applying a finite series of elementary transformations, simultaneous conjugations and cyclic permutations (see Section 4.1).

Let $\gamma_1, \ldots, \gamma_k$ be a sequence of simple closed curves on an oriented surface such that $\gamma_i$ and $\gamma_j$ are disjoint if $|i-j| \geq 2$ and $\gamma_i$ intersects $\gamma_{i+1}$ at exactly one point. We recall the following relations:

$$t_{\gamma_i} \cdot t_{\gamma_{i+1}} \equiv t_{t_{\gamma_i}(\gamma_{i+1})} \cdot t_{\gamma_i},$$
$$t_{\gamma_{i+1}} \cdot t_{\gamma_i} \equiv t_{\gamma_i} \cdot t_{t^{-1}_{\gamma_i}(\gamma_{i+1})}.$$

Note that for $|i-j|>1$, we have

$$t_{\gamma_i}\cdot t_{\gamma_j}\equiv t_{\gamma_j}\cdot t_{\gamma_i}.$$

Using the braid relation $t_{\gamma_i}t_{\gamma_{i+1}}t_{\gamma_i}=t_{\gamma_{i+1}}t_{\gamma_i}t_{\gamma_{i+1}}$, we obtain

$$t_{\gamma_k}t_{\gamma_{k-1}}\cdots t_{\gamma_{m+1}}t_{\gamma_m}\cdot t_{\gamma_{i+1}}\equiv t_{\gamma_i}\cdot t_{\gamma_k}t_{\gamma_{k-1}}\cdots t_{\gamma_{m+1}}t_{\gamma_m},\tag{1}$$

$$t_{\gamma_m}t_{\gamma_{m+1}}\cdots t_{\gamma_{k-1}}t_{\gamma_k}\cdot t_{\gamma_i}\equiv t_{\gamma_{i+1}}\cdot t_{\gamma_m}t_{\gamma_{m+1}}\cdots t_{\gamma_{k-1}}t_{\gamma_k}\tag{2}$$

for $m\leq i\leq k-1$.

**Lemma 27.** *For $2\leq k$, we have the following relations:*

(a) $t_{\gamma_k}\cdots t_{\gamma_2}t_{\gamma_1}\cdot t_{\gamma_k}\cdots t_{\gamma_3}t_{\gamma_2}\equiv t_{t_{\gamma_{k-1}}(\gamma_k)}\cdots t_{t_{\gamma_2}(\gamma_3)}t_{t_{\gamma_1}(\gamma_2)}\cdot t_{\gamma_1}^k$,

(b) $t_{\gamma_{k-1}}\cdots t_{\gamma_2}t_{\gamma_1}\cdot t_{\gamma_k}\cdots t_{\gamma_3}t_{\gamma_2}\equiv t_{t_{\gamma_{k-1}}(\gamma_k)}\cdots t_{t_{\gamma_2}(\gamma_3)}t_{t_{\gamma_1}(\gamma_2)}\cdot t_{\gamma_1}^{k-1}$,

(c) $t_{\gamma_2}t_{\gamma_3}\cdots t_{\gamma_k}\cdot t_{\gamma_1}t_{\gamma_2}\cdots t_{\gamma_k}\equiv t_{\gamma_1}^k\cdot t_{t_{\gamma_1}^{-1}(\gamma_2)}t_{t_{\gamma_2}^{-1}(\gamma_3)}\cdots t_{t_{\gamma_{k-1}}^{-1}(\gamma_k)}$,

(d) $t_{\gamma_2}t_{\gamma_3}\cdots t_{\gamma_k}\cdot t_{\gamma_1}t_{\gamma_2}\cdots t_{\gamma_{k-1}}\equiv t_{\gamma_1}^{k-1}\cdot t_{t_{\gamma_1}^{-1}(\gamma_2)}t_{t_{\gamma_2}^{-1}(\gamma_3)}\cdots t_{t_{\gamma_{k-1}}^{-1}(\gamma_k)}$.

*Proof.* Below we denote the arrangement using the relation (i) by $\xrightarrow{\text{(i)}}$. The proof will be given by induction on $k$. Suppose that $k=2$. Then, we have

$$t_{\gamma_2}t_{\gamma_1}\cdot t_{\gamma_2}\xrightarrow{(1)}t_{\gamma_1}\cdot t_{\gamma_2}t_{\gamma_1}\equiv t_{t_{\gamma_1}(\gamma_2)}\cdot t_{\gamma_1}^2.$$

Hence, the conclusion of (a) holds for $k=2$.

Let us assume inductively that the relation (a) holds for $k=i$. Hence, we have

$$t_{\gamma_i}\cdots t_{\gamma_2}t_{\gamma_1}\cdot t_{\gamma_i}\cdots t_{\gamma_3}t_{\gamma_2}\equiv t_{t_{\gamma_{i-1}}(\gamma_i)}\cdots t_{t_{\gamma_2}(\gamma_3)}t_{t_{\gamma_1}(\gamma_2)}\cdot t_{\gamma_1}^i.$$

Then,

$$\begin{aligned}t_{\gamma_{i+1}}\cdots t_{\gamma_2}t_{\gamma_1}\cdot t_{\gamma_{i+1}}\cdots t_{\gamma_3}t_{\gamma_2}&\xrightarrow{(1)}t_{\gamma_i}t_{\gamma_{i-1}}\cdots t_{\gamma_2}t_{\gamma_1}\cdot t_{\gamma_{i+1}}t_{\gamma_i}\cdots t_{\gamma_2}t_{\gamma_1}\\&\equiv t_{\gamma_i}\cdot t_{\gamma_{i+1}}\cdot t_{\gamma_{i-1}}\cdots t_{\gamma_2}t_{\gamma_1}\cdot t_{\gamma_i}\cdots t_{\gamma_2}t_{\gamma_1}\\&\equiv t_{t_{\gamma_i}(\gamma_{i+1})}\cdot t_{\gamma_i}\cdot t_{\gamma_{i-1}}\cdots t_{\gamma_2}t_{\gamma_1}\cdot t_{\gamma_i}\cdots t_{\gamma_2}t_{\gamma_1}\\&\equiv t_{t_{\gamma_i}(\gamma_{i+1})}\cdot t_{t_{\gamma_{i-1}}(\gamma_i)}\cdots t_{t_{\gamma_2}(\gamma_3)}t_{t_{\gamma_1}(\gamma_2)}\cdot t_{\gamma_1}^i\cdot t_{\gamma_1}.\end{aligned}$$

This proves part (a). The proofs of (b), (c) and (d) are similar, and therefore omitted. ∎

## 5. Proofs of Theorems A and B

5.1. **Hurwitz equivalent relations.** The purpose of this section is to prove Proposition 28 below. The Lefschetz fibrations corresponding to the relations in Proposition 28 are used to construct Lefschetz fibrations in Theorems A and B.

Let us consider the curves $t_{c_1'}(c_2)$, $t_{c_1'}^{-1}(c_2)$, $t_{c_i}(c_{i+1})$ and $t_{c_i}^{-1}(c_{i+1})$ on $\Sigma_{n-1}^1$ (resp. $\Sigma_{2h+n-1}^1$) for $2\leq i\leq 2n-2$, and let $\delta$ be the boundary curve of $\Sigma_{n-1}^1$

(resp. $\Sigma^1_{2h+n-1}$). These curves are shown in Figure 5. For abbreviation, we set

$$\begin{aligned}
&\Psi_{n-1} := t_{c_{2n-1}} t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c_1'} t_{c_1''} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}} t_{c_{2n-1}},\\
&\Omega_{2h} := t_{D_0} t_{D_1} t_{D_2} \cdots t_{D_{2h}},\\
&\mathbb{D} := t_{t_{c_{2n-2}}(c_{2n-1})} \cdots t_{t_{c_3}(c_4)} t_{t_{c_2}(c_3)} t_{t_{c_1'}(c_2)},\\
&\mathbb{D}' := t_{t_{c_{2n-2}}(c_{2n-1})} \cdots t_{t_{c_3}(c_4)} \cdot t_{t_{c_3}^{2n-1}(t_{c_2}(c_3))} t_{t_{c_3}^{2n-1}(t_{c_1'}(c_2))},\\
&\mathbb{D}'' := t_{t_{c_{2n-2}}(c_{2n-1})} \cdots t_{t_{c_3}(c_4)} \cdot t_{t_{c_3}^{2n}(t_{c_2}(c_3))} t_{t_{c_3}^{2n}(t_{c_1'}(c_2))},\\
&\mathbb{E} := t_{t^{-1}_{c_1'}(c_2)} t_{t^{-1}_{c_2}(c_3)} t_{t^{-1}_{c_3}(c_4)} \cdots t_{t^{-1}_{c_{2n-2}}(c_{2n-1})}.
\end{aligned}$$

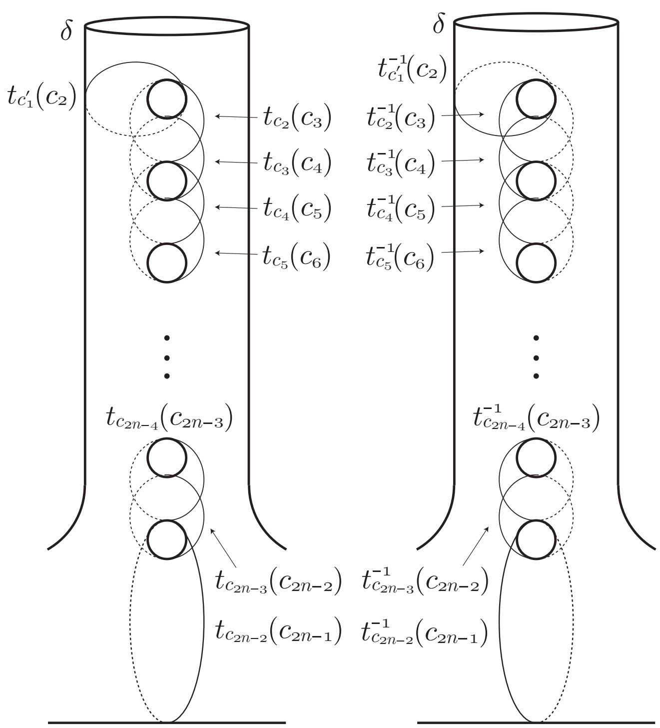


FIGURE 5. The curves $t_{c_1'}(c_2)$, $t^{-1}_{c_1'}(c_2)$, $t_{c_i}(c_{i+1})$ and $t^{-1}_{c_i}(c_{i+1})$ on $\Sigma^1_{n-1}$ (resp. $\Sigma^1_{2h+n-1}$) and the boundary curve $\delta$ of $\Sigma^1_{n-1}$ (resp. $\Sigma^1_{2h+n-1}$).

**Proposition 28.** *The following relation holds in* $\Gamma^1_{n-1}$*:*

$$t_\delta = (\Psi_{n-1})^2 \equiv t_{c_3}^{2n} t_{c_1'}^{2n-1} t_{c_1''} \cdot \mathbb{E} \cdot \mathbb{D}'',$$

*and the following relation holds in* $\Gamma^1_{2h+n-1}$*:*

$$\begin{aligned} t_\delta = \overline{\theta}^2 \ &\equiv \ \Psi_{n-1} \cdot t_{c_1''} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}} \cdot (\Omega_{2h})^2 \cdot t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c_1''} \\ &\equiv t_{c_3}^{2n-1} t_{c_1'}^{2n-2} t_{c_1''} \cdot \mathbb{E} \cdot (\Omega_{2h})^2 \cdot \mathbb{D}'. \end{aligned}$$

Note that the equality $t_\delta = (\Psi_{n-1})^2$ (resp. $t_\delta = \overline{\theta}^2$) in Proposition 28 follows from Proposition 25 (resp. 26) and $\overline{\eta} = \Psi_{n-1}$. Moreover, by an argument similar to the proof of Proposition 6.5 in [36] and using $t_{c_1'} \cdot t_{c_1''} \equiv t_{c_1''} \cdot t_{c_1'}$, we see that $\overline{\theta}^2 \equiv \Psi_{n-1} \cdot t_{c_1''} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}} \cdot (\Omega_{2h})^2 \cdot t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c_1''}$. We can also prove this fact using Proposition 26 (and therefore $T \cdot \overline{\theta} \equiv \overline{\theta} \cdot T$), $c_{2n-1} = \overline{\theta}(c_{2n-1}) = T\Omega_{2h} t_{c_{2n-1}}(c_{2n-1}) = T\Omega_{2h}(c_{2n-1})$ (and therefore $T\Omega_{2h} \cdot t_{c_{2n-1}} \equiv t_{c_{2n-1}} \cdot T\Omega_{2h}$) and the relations (3)–(5) below, where $T = t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c_1'} t_{c_1''} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}}$. Proposition 28 immediately follows from Lemmas 29 and 30. Below we denote the arrangement using cyclic permutations by $\xrightarrow{\text{c.p.}}$.

**Lemma 29.** *The following holds in* $\Gamma^1_{n-1}$*:*

$$(\Psi_{n-1})^2 \equiv \mathbb{D} \cdot t_{c_1'}^{2n} \cdot t_{c_1''} \cdot t_{c_1'}^{2n-1} \cdot \mathbb{E},$$

*and the following holds in* $\Gamma^1_{2h+n-1}$*:*

$$\begin{aligned} &\Psi_{n-1} \cdot t_{c_1''} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}} \cdot (\Omega_{2h})^2 \cdot t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c_1''} \\ &\equiv \mathbb{D} \cdot t_{c_1'}^{2n-1} \cdot t_{c_1''} \cdot t_{c_1'}^{2n-2} \cdot \mathbb{E} \cdot (\Omega_{2h})^2. \end{aligned}$$

*Proof.* Note that $t_{c_1'} \cdot t_{c_1''} \equiv t_{c_1''} \cdot t_{c_1'}$ by $c_1' \cap c_1'' = \emptyset$. From Figures 6 and 7, we have $\Psi_{n-1}(c_1') = c_1''$ and $\Psi_{n-1}(c_i) = c_i$ for $2 \le i \le 2n-2$. Similarly, we obtain $\Psi_{n-1}(c_1'') = c_1'$. This gives

$$\Psi_{n-1} \cdot t_{c_1'} \equiv t_{c_1''} \cdot \Psi_{n-1}, \tag{3}$$

$$\Psi_{n-1} \cdot t_{c_1''} \equiv t_{c_1'} \cdot \Psi_{n-1}, \tag{4}$$

$$\Psi_{n-1} \cdot t_{c_i} \equiv t_{c_i} \cdot \Psi_{n-1} \tag{5}$$

for $2 \le i \le 2n-2$.

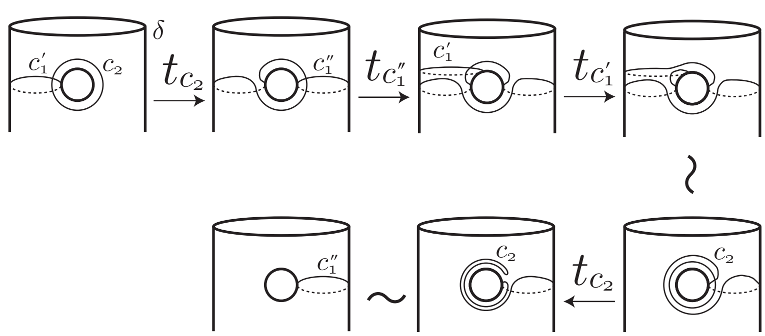


FIGURE 6.

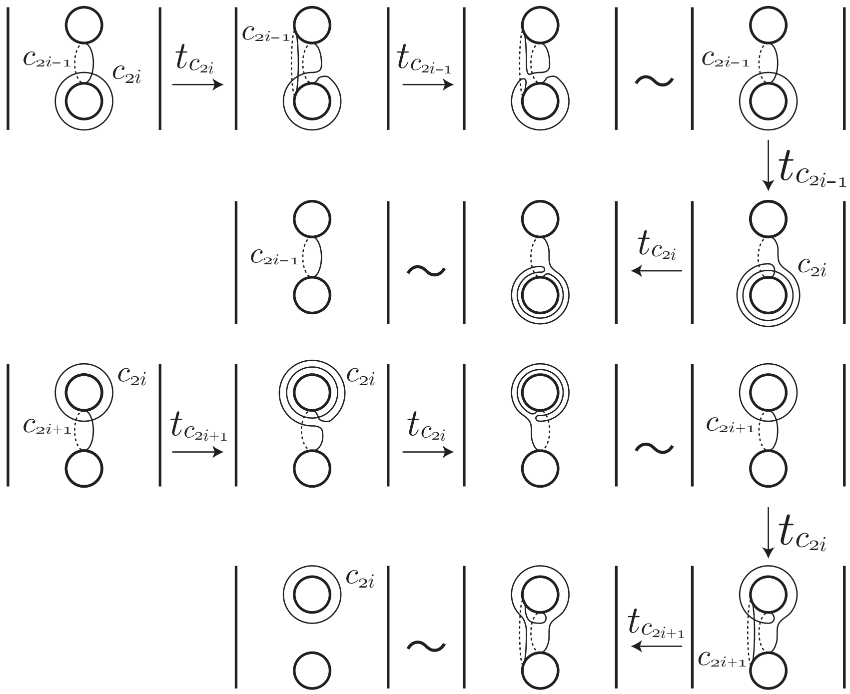


FIGURE 7.

First, we show the former relation in Lemma 29. From the relations above, we have

$$\begin{aligned}
&(\Psi_{n-1})^2 \equiv \Psi_{n-1} \cdot t_{c_{2n-1}} \cdots t_{c_2} t_{c_1''} t_{c_1'} t_{c_2} \cdots t_{c_{2n-1}} \\
&\equiv t_{c_{2n-1}} \cdots t_{c_2} t_{c_1'} \cdot \Psi_{n-1} \cdot t_{c_1'} t_{c_2} \cdots t_{c_{2n-1}} \\
&= t_{c_{2n-1}} \cdots t_{c_2} t_{c_1'} \cdot t_{c_{2n-1}} \cdots t_{c_2} t_{c_1'} \cdot t_{c_1''} t_{c_2} \cdots t_{c_{2n-1}} \cdot t_{c_1'} t_{c_2} \cdots t_{c_{2n-1}}.
\end{aligned}$$

By applying Lemma 27 (a) and (d) to the former and the latter parts of this word, respectively, we obtain

$$\begin{aligned}
&t_{c_{2n-1}} \cdots t_{c_2} t_{c_1'} \cdot t_{c_{2n-1}} \cdots t_{c_2} t_{c_1'} \cdot t_{c_1''} t_{c_2} \cdots t_{c_{2n-1}} \cdot t_{c_1'} t_{c_2} \cdots t_{c_{2n-1}} \\
&\equiv t_{t_{c_{2n-2}}(c_{2n-1})} \cdots t_{t_{c_2}(c_3)} t_{t_{c_1'}(c_2)} \cdot t_{c_1'}^{2n} \cdot t_{c_1''} \cdot t_{c_1'}^{2n-1} t_{t_{c_1'}^{-1}(c_2)} t_{t_{c_2}^{-1}(c_3)} \cdots t_{t_{c_{2n-2}}^{-1}(c_{2n-1})} \\
&= \mathbb{D} \cdot t_{c_1'}^{2n} \cdot t_{c_1''} \cdot t_{c_1'}^{2n-1} \cdot \mathbb{E},
\end{aligned}$$

which gives the former relation in Lemma 29.

Next, we give the proof of the latter relation in Lemma 29. An argument similar to the former relation in Lemma 29 gives

$$\begin{aligned}
&\Psi_{n-1} \cdot t_{c_1''} t_{c_2} \cdots t_{c_{2n-2}} \cdot (\Omega_{2h})^2 \cdot t_{c_{2n-2}} \cdots t_{c_2} t_{c_1''} \\
&\xrightarrow{\text{c.p.}} t_{c_{2n-2}} \cdots t_{c_2} t_{c_1''} \cdot \Psi_{n-1} \cdot t_{c_1''} t_{c_2} \cdots t_{c_{2n-2}} \cdot (\Omega_{2h})^2 \\
&\equiv t_{c_{2n-2}} \cdots t_{c_2} t_{c_1'} \cdot \Psi_{n-1} \cdot t_{c_1'} t_{c_2} \cdots t_{c_{2n-2}} \cdot (\Omega_{2h})^2. \qquad (6)
\end{aligned}$$

Here, we consider $t_{c_{2n-2}}\cdots t_{c_2}t_{c_1'}\cdot\Psi_{n-1}\cdot t_{c_1'}t_{c_2}\cdots t_{c_{2n-2}}$ in the relation (6), which becomes

$$t_{c_{2n-2}}\cdots t_{c_2}t_{c_1'}\cdot t_{c_{2n-1}}\cdots t_{c_2}t_{c_1'}t_{c_1''}t_{c_2}\cdots t_{c_{2n-1}}\cdot t_{c_1'}t_{c_2}\cdots t_{c_{2n-2}}.$$

By applying Lemma 27 (b) and (d) to the former and the latter parts of this word, respectively, we obtain

$$\begin{aligned}&t_{c_{2n-2}}\cdots t_{c_2}t_{c_1'}\cdot t_{c_{2n-1}}\cdots t_{c_2}t_{c_1'}t_{c_1''}t_{c_2}\cdots t_{c_{2n-1}}\cdot t_{c_1'}t_{c_2}\cdots t_{c_{2n-2}}\\ &\equiv t_{t_{c_{2n-2}}(c_{2n-1})}\cdots t_{t_{c_2}(c_3)}t_{t_{c_1'}(c_2)}\cdot t_{c_1'}^{2n-1}\cdot t_{c_1''}\cdot t_{c_1'}^{2n-2}\\ &\quad\cdot t_{t_{c_1'}^{-1}(c_2)}t_{t_{c_2}^{-1}(c_3)}\cdots t_{t_{c_{2n-2}}^{-1}(c_{2n-1})}\\ &=\mathbb{D}\cdot t_{c_1'}^{2n-1}\cdot t_{c_1''}\cdot t_{c_1'}^{2n-2}\cdot\mathbb{E},\end{aligned}$$

and the lemma follows. □

**Lemma 30.** *The following holds in $\Gamma_{2h+n-1}$:*

$$\mathbb{D}\cdot t_{c_1'}^{2n-1}t_{c_1''}t_{c_1'}^{2n-2}\cdot\mathbb{E}\cdot(\Omega_{2h})^2\equiv t_{c_3}^{2n-1}t_{c_1'}^{2n-2}t_{c_1''}\cdot\mathbb{E}\cdot(\Omega_{2h})^2\cdot\mathbb{D}',$$

*and the following holds in $\Gamma_{n-1}$:*

$$\mathbb{D}\cdot t_{c_1'}^{2n}\cdot t_{c_1''}\cdot t_{c_1'}^{2n-1}\cdot\mathbb{E}\equiv t_{c_3}^{2n}\cdot t_{c_1'}^{2n-1}\cdot t_{c_1''}\cdot\mathbb{E}\cdot\mathbb{D}''.$$

*Proof.* It follows easily from Figure 8 that $t_{t_{c_2}(c_3)}t_{t_{c_1'}(c_2)}(c_1')=c_3$ , and therefore $t_{t_{c_2}(c_3)}t_{t_{c_1'}(c_2)}\cdot t_{c_1'}\equiv t_{c_3}\cdot t_{t_{c_2}(c_3)}t_{t_{c_1'}(c_2)}$. Therefore, without applying cyclic

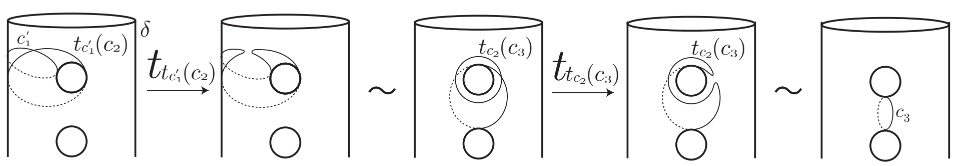


FIGURE 8.

permutations, we obtain

$$\begin{aligned}\mathbb{D}\cdot t_{c_1'}^{2n-1}&=t_{t_{c_{2n-2}}(c_{2n-1})}\cdots t_{t_{c_3}(c_4)}t_{t_{c_2}(c_3)}t_{t_{c_1'}(c_2)}\cdot t_{c_1'}^{2n-1}\\ &\equiv t_{t_{c_{2n-2}}(c_{2n-1})}\cdots t_{t_{c_4}(c_5)}t_{t_{c_3}(c_4)}\cdot t_{c_3}^{2n-1}\cdot t_{t_{c_2}(c_3)}t_{t_{c_1'}(c_2)}\\ &\equiv t_{t_{c_{2n-2}}(c_{2n-1})}\cdots t_{t_{c_4}(c_5)}t_{t_{c_3}(c_4)}\cdot t_{t_{c_3}^{2n-1}(t_{c_2}(c_3))}t_{t_{c_3}^{2n-1}(t_{c_1'}(c_2))}\cdot t_{c_3}^{2n-1}\\ &=\mathbb{D}'\cdot t_{c_3}^{2n-1}.\end{aligned}$$

Similarly, we obtain

$$\mathbb{D}\cdot t_{c_1'}^{2n}\equiv\mathbb{D}''\cdot t_{c_3}^{2n}$$

without applying cyclic permutations. Using this, we have

$$\begin{aligned}\mathbb{D}\cdot t_{c_1'}^{2n-1}t_{c_1''}t_{c_1'}^{2n-2}\cdot\mathbb{E}\cdot(\Omega_{2h})^2&\equiv\mathbb{D}'\cdot t_{c_3}^{2n-1}t_{c_1''}t_{c_1'}^{2n-2}\cdot\mathbb{E}\cdot(\Omega_{2h})^2\\ &\xrightarrow{\text{c.p.}} t_{c_3}^{2n-1}t_{c_1''}t_{c_1'}^{2n-2}\cdot\mathbb{E}\cdot(\Omega_{2h})^2\cdot\mathbb{D}',\end{aligned}$$

and similarly

$$\mathbb{D} \cdot t_{c_1'}^{2n} t_{c_1''} t_{c_1'}^{2n-1} \cdot \mathbb{E} \equiv \mathbb{D}'' \cdot t_{c_3}^{2n} t_{c_1''} t_{c_1'}^{2n-1} \cdot \mathbb{E} \xrightarrow{\text{c.p.}} t_{c_3}^{2n} t_{c_1''} t_{c_1'}^{2n-1} \cdot \mathbb{E} \cdot \mathbb{D}''.$$

By $t_{c_1'} \cdot t_{c_1''} \equiv t_{c_1''} \cdot t_{c_1'}$, we obtain the required formula. □

### 5.2. **Proofs of Theorems A and B.** We now prove Theorem A.

*Proof of Theorem A.* First, we construct a genus-$(n-1)$ Lefschetz fibration $f_i : X_i \to \mathbb{S}^2$. From the former relation of Proposition 28, we have the following lift in $\Gamma_{n-1}^1$ of the positive factorization $\eta^2$ of id in $\Gamma_{n-1}$ defined in Section 4.3:

$$t_\delta = t_{c_3}^{2n} t_{c_1'}^{2n-1} t_{c_1''} \cdot \mathbb{E} \cdot \mathbb{D}''. \tag{7}$$

Since $\phi(c_1') = v_1'$, $\phi(c_1'') = v_1''$ and $\phi(c_3) = c_5$ (see Section 4.4), by applying simultaneous conjugations by $\phi$ to the equation (7), we get

$$t_\delta = t_{c_5}^{2n} t_{v_1'}^{2n-1} t_{v_1''} \cdot {}_\phi(\mathbb{E}) \cdot {}_\phi(\mathbb{D}''). \tag{8}$$

Here, we set ${}_\phi(\mathbb{F}) = t_{\phi(a_1)} \cdots t_{\phi(a_k)}$ for a product $\mathbb{F} = t_{a_1} \cdots t_{a_k}$ of Dehn twists. Moreover, by applying cyclic permutations to the equation (7) we have

$$t_\delta = \mathbb{E} \cdot \mathbb{D}'' t_{c_3}^{2n} t_{c_1'}^{2n-1} t_{c_1''}. \tag{9}$$

Since $c_1', c_1'', c_3, c_5, v_1'$ are disjoint from each other, by combining the equations (8), (9), we obtain

$$\begin{aligned} t_\delta^2 &= \mathbb{E} \cdot \mathbb{D}'' t_{c_3}^{2n} t_{c_1'}^{2n-1} t_{c_1''} \cdot t_{c_5}^{2n} t_{v_1'}^{2n-1} t_{v_1''} \cdot {}_\phi(\mathbb{E}) \cdot {}_\phi(\mathbb{D}'') \\ &\equiv \mathbb{E} \cdot \mathbb{D}'' (t_{c_1'} t_{c_3} t_{c_5} t_{v_1'})^{2n-1} t_{c_1''} t_{c_3} t_{c_5} t_{v_1''} \cdot {}_\phi(\mathbb{E}) \cdot {}_\phi(\mathbb{D}''). \end{aligned} \tag{10}$$

From the relation (10) and the lantern relation $t_{c_1'} t_{c_3} t_{c_5} t_{v_1'} = t_x t_y t_z$, we obtain

$$t_\delta^2 = \mathbb{E} \cdot \mathbb{D}'' (t_x t_y t_z)^i (t_{c_1'} t_{c_3} t_{c_5} t_{v_1'})^{2n-1-i} t_{c_1''} t_{c_3} t_{c_5} t_{v_1''} \cdot {}_\phi(\mathbb{E}) \cdot {}_\phi(\mathbb{D}'') \tag{11}$$

for $0 \leq i \leq 2n-1$. By letting $f_i : X_i \to \mathbb{S}^2$ be the genus-$g$ Lefschetz fibration corresponding to the positive factorization (11) of $t_\delta^2$, we see that $f_i$ admits a $(-2)$-section for $0 \leq i \leq 2n-1$.

Second, we show that $X_i$ is minimal for $0 \leq i \leq 2n-1$. We see that $X_0$ is obtained by fiber summing two copies of $Y(n, 0)$ in Section 4.3 since both (8) and (9) are lifts of the positive factorization $\eta^2$ of id in $\Gamma_{n-1}$. Therefore, the minimality of $X_0$ follows from Corollary 17. For $1 \leq i \leq 2n-1$, $X_i$ was obtained from $X_0$ via $i$-times lantern substitutions (i.e., by $i$-times rational blowdown surgeries along spheres of self-intersection $-4$). By Lemma 20, we see that $X_i$ is minimal for any $i$.

Third, we compute $\chi(X_i)$ and $c_1^2(X_i)$ for $0 \leq i \leq 2n-1$. By Theorem 14, Proposition 28 and Section 4.3, $Y(n, 0)$ is diffeomorphic to $\mathbb{CP}^2 \sharp (4n+1)\overline{\mathbb{CP}^2}$.

Since $X_i$ is obtained by $i$-times rational blowdown along spheres of self-intersection $-4$ to the fiber sum $f_0 : X_0 \to \mathbb{S}^2$ of two copies of the genus-$(n-1)$ Lefschetz fibration $Y(n,0) \to \mathbb{S}^2$ by $\phi$, by Lemma 12 and Theorem 19, we have

$$\sigma(X_i) = 2\sigma(Y(n,0)) + i = -8n + i,$$
$$e(X_i) = 2e(Y(n,0)) + 4\{(n-1)-1\} - i = 12n - i,$$

and hence, we obtain

$$\chi(X_i) = n, \qquad c_1^2(X_i) = i.$$

Finally, we compute the fundamental group $\pi_1(X_i)$ of $X_i$ for $0 \leq i \leq 2n-1$. Let us consider the genus-$(n-1)$ Lefschetz fibration $f' : X' \to \mathbb{S}^2$ corresponding to the positive factorization (7) of $t_\delta$ in $\Gamma_{n-1}^1$. Since this fibration is isomorphic to the Lefschetz fibration $f : Y(n,0) \to \mathbb{S}^2$ by Theorem 14 and Proposition 28, we have $\pi_1(X') = \pi_1(Y(n,0)) = 1$ by Lemma 15. Here, it is easy to check that the normal subgroup generated by the vanishing cycles of $f_i$ contains that of $f'$ for $i = 0,1,2,\ldots,2n-1$. This and Lemma 15 give $\pi_1(X_i) = 1$ for $i = 1,2,\ldots,2n-1$.

This is the desired conclusion. $\square$

**Remark 31.** We can apply one more lantern substitution to the resulting relation obtained by applying $(2n-1)$-times lantern substitutions to the relation (11) as follows. It is easy to see that $c_1''$ and $v_1''$ correspond to $c_1'$ and $v_1'$ under the inclusion $\Sigma_{n-1}^1 \to \Sigma_{n-1}$, respectively. Consequently, we get a genus-$(n-1)$ Lefschetz fibration $f_{2n} : X_{2n} \to \mathbb{S}^2$ (however, the existence of a section of $f_{2n}$ is not guaranteed).

We next prove Theorem B.

*Proof of Theorem B.* For a given finitely presented group $G$, we give a mapping class $\rho_G$ in $\Gamma_{2h+n-1}^1$ defined in Section 6.2 such that $\rho_G(v_1') = v_1'$, $\rho_G(v_1'') = v_1''$ and $\rho_G(c_5) = c_5$.

We construct a genus-$(2h+n-1)$ Lefschetz fibration $f_{G,i} : Y_i \to \mathbb{S}^2$. From Propositions 26 and 28, we have the following lift in $\Gamma_{2h+n-1}^1$ of Gurtas' relation in $\Gamma_{2h+n-1}$:

$$t_\delta = t_{c_3}^{2n-1} t_{c_1'}^{2n-2} t_{c_1''} \cdot \mathbb{E} \cdot (\Omega_{2h})^2 \cdot \mathbb{D}'. \tag{12}$$

Moreover, by applying cyclic permutations to this equation, we have

$$t_\delta = \mathbb{E} \cdot (\Omega_{2h})^2 \cdot \mathbb{D}' \cdot t_{c_3}^{2n-1} t_{c_1'}^{2n-2} t_{c_1''}.$$

From $\phi(c_1') = v_1'$, $\phi(c_1'') = v_1''$ and $\phi(c_3) = c_5$ (see Section 4.4), we get the following lift of Gurtas' relation by applying simultaneous conjugations by $\rho_G\phi$ to the equation (12):

$$t_\delta = t_{c_5}^{2n-1} t_{v_1'}^{2n-2} t_{v_1''} \cdot {}_{\rho_G\phi}(\mathbb{E}) \cdot ({}_{\rho_G\phi}(\Omega_{2h}))^2 \cdot {}_{\rho_G\phi}(\mathbb{D}'), \tag{13}$$

where

$$\begin{aligned}
{}_{\rho_G\phi}(\Omega_{2h}) &= t_{\rho_G\phi(D_0)} t_{\rho_G\phi(D_1)} \cdots t_{\rho_G\phi(D_{2h})},\\
{}_{\rho_G\phi}(\mathbb{E}) &= t_{\rho_G\phi t_{c_1'}^{-1}(c_2)} t_{\rho_G\phi t_{c_2}^{-1}(c_3)} t_{\rho_G\phi t_{c_3}^{-1}(c_4)} \cdots t_{\rho_G\phi t_{c_{2n-2}}^{-1}(c_{2n-1})},\\
{}_{\rho_G\phi}(\mathbb{D}') &= t_{\rho_G\phi t_{c_{2n-2}}(c_{2n-1})} \cdots t_{\rho_G\phi t_{c_4}(c_5)} t_{\rho_G\phi t_{c_3}(c_4)}\\
&\quad \cdot t_{\rho_G\phi t_{c_3}^{2n-1}(t_{c_2}(c_3))} t_{\rho_G\phi t_{c_3}^{2n-1}(t_{c_1'}(c_2))}.
\end{aligned}$$

Here, since $c_1', c_1'', c_3, c_5, v_1'$ are disjoint from each other, we have

$$t_{c_3}^{2n-1} t_{c_1'}^{2n-2} t_{c_1''} t_{c_5}^{2n-1} t_{v_1'}^{2n-2} t_{v_1''} \equiv (t_{c_1'} t_{c_3} t_{c_5} t_{v_1'})^{2n-2} t_{c_1''} t_{c_3} t_{c_5} t_{v_1''}.$$

Therefore, by combining the equations (12) and (13), we obtain

$$t_\delta^2 = \mathbb{E}(\Omega_{2h})^2 \mathbb{D}' (t_{c_1'} t_{c_3} t_{c_5} t_{v_1'})^{2n-2} t_{c_1''} t_{c_3} t_{c_5} t_{v_1''} \cdot {}_{\rho_G\phi}(\mathbb{E}) ({}_{\rho_G\phi}(\Omega_{2h}))^2 {}_{\rho_G\phi}(\mathbb{D}').$$

From this relation and the lantern relation $t_{c_1'} t_{c_3} t_{c_5} t_{v_1'} = t_x t_y t_z$, we obtain

$$\begin{aligned}
t_\delta^2 =& \mathbb{E}(\Omega_{2h})^2 \mathbb{D}' (t_x t_y t_z)^i (t_{c_1'} t_{c_3} t_{c_5} t_{v_1'})^{2n-2-i}\\
&\cdot t_{c_1''} t_{c_3} t_{c_5} t_{v_1''} \cdot {}_{\rho_G\phi}(\mathbb{E}) ({}_{\rho_G\phi}(\Omega_{2h}))^2 {}_{\rho_G\phi}(\mathbb{D}')
\end{aligned} \tag{14}$$

for $0 \leq i \leq 2n-2$. By letting $f_{G,i} : Y_i \to \mathbb{S}^2$ be the genus-$g$ Lefschetz fibration corresponding to the positive factorization (14) of $t_\delta^2$ in $\Gamma^1_{2h+n-1}$, we see that $f_{G,i}$ admits a $(-2)$-section for $0 \leq i \leq 2n-2$.

By Theorem 14, Proposition 28 and Section 4.3, $Y(n,h)$ is diffeomorphic to $(\Sigma_h \times \mathbb{S}^2) \sharp 4n\overline{\mathbb{CP}}^2$. We see that $f_{G,0}$ is obtained by fiber summing two copies of the genus-$(2h+n-1)$ Lefschetz fibration $Y(n,h) \to \mathbb{S}^2$ since both (12) and (13) are lifts of Gurtas' relation. Moreover, the positive factorization corresponding to $f_{G,i}$ is obtained by applying $i$-times lantern substitutions to the positive factorization corresponding to $f_{G,0}$. Therefore, the proofs of $\chi(Y_i) = n$, $c_1^2(Y_i) = i$ and the minimality of $Y_i$ are similar to the proof of Theorem A, and so we omit them.

If $h \geq N + \ell_k - 1$, then Theorem B follows from Proposition 34 in Section 6 that $\pi_1(Y_i) \cong G$ for $i = 0, 1, \ldots, 2n-2$. We postpone the proof until Section 6.

This is the desired conclusion. □

**Remark 32.** It is easy to construct a mapping class $\rho$ in $\Gamma^1_g$ such that $\rho(v_1') = v_1'$, $\rho(v_1'') = v_1''$ and $\rho(c_5) = c_5$, for example, $\rho = \mathrm{id}$. Therefore, from the proof of Theorem B, we see that there is a genus-$(2h+n-1)$ Lefschetz fibration on a minimal symplectic 4-manifold with $(\chi, c_1^2) = (n, i)$ for $0 \leq i \leq 2n-2$, where $h$ and $n$ are integers in Theorem B. From Proposition 13, there are $2n-2$ indecomposable genus-$(2h+n-1)$ Lefschetz fibrations with minimal total spaces. Hence, we obtain Theorem C except that the fundamental group $\pi_1(Y_i)$ is isomorphic to $G$.

**Remark 33.** We can apply one more lantern substitution to the relation (14) in "$\Gamma_{2h+n-1}$", but the fibration corresponding to the resulting positive factorization does not guarantee the existence of a section. Therefore, we do not consider such a fibration since it is difficult to determine the fundamental group.

*Proof of Corollary 2.* Since for $1 \leq i \leq 2n-7$, the total spaces of Lefschetz fibrations in Theorems A and B do not satisfy the Noether inequality, they cannot admit any complex structure with either orientation. Therefore, the fibrations are nonholomorphic. □

*Proof of Corollary 3.* In the notation of Theorem B, if $G = 1$, then $N = 0$ and $k = 0$ (therefore $\ell_k = 1$). This gives $h \geq 0$. Notice that $k = 0$ satisfies the condition $2n - 8 \geq k$ in Theorem B for $n - 1 \geq 3$. Moreover, we have $h \geq 9$ from the inequality $h > (5n-3)/2$ in Theorem C. This proves Corollary 3. □

*Proof of Corollary 4.* In the notation of Theorem B, let us consider $G = \langle x_1 \mid x_1^M \rangle \cong \mathbb{Z}_M$. Then, we have $N = 1$, $k = 1$ and $\ell_k = 1$, and therefore $h \geq 1$ in Theorem B. If $n-1 \geq 4$, then $k = 1$ satisfies the condition $2n-8 \geq k$. Therefore, by Theorem B, there exists a genus-$g$ Lefschetz fibration on $Y_i$ such that $Y_i$ is minimal and $\pi_1(Y_i) \cong \mathbb{Z}_M$ for $g = 2h+n-1 \geq 2h+4$. Moreover, we have $h \geq 12$ from the inequality $h > (5n-3)/2$ in Theorem C. This proves Corollary 4. □

## 6. Computation of fundamental groups

In this section, we prove Proposition 34 stating that $\pi_1(Y_i)$ is isomorphic to $G$, which was postponed in the proof of Theorem B in Section 5.2.

**Proposition 34.** *Let $f_{G,i} : Y_i \to \mathbb{S}^2$ be the genus-$(2h+n-1)$ Lefschetz fibration corresponding to the following positive factorization of $t_\delta^2$ in $\Gamma^1_{2h+n-1}$:*

$$t_\delta^2 = t_{c_1''}\mathbb{E}(\Omega_{2h})^2\mathbb{D}'t_{c_3}(t_xt_yt_z)^i(t_{c_1'}t_{c_3}t_{c_5}t_{v_1'})^{2n-2-i} \cdot t_{c_5}t_{v_1''}{}_{\rho_G\phi}(\mathbb{E})({}_{\rho_G\phi}(\Omega_{2h}))^2{}_{\rho_G\phi}(\mathbb{D}'),$$

*where*

$$\begin{aligned}
\Omega_{2h} &:= t_{D_0}t_{D_1}t_{D_2}\cdots t_{D_{2h}},\\
\mathbb{D}' &:= t_{t_{c_{2n-2}}(c_{2n-1})}\cdots t_{t_{c_3}(c_4)}\cdot t_{t_{c_3}^{2n-1}(t_{c_2}(c_3))}t_{t_{c_3}^{2n-1}(t_{c_1'}(c_2))},\\
\mathbb{E} &:= t_{t_{c_1'}^{-1}(c_2)}t_{t_{c_2}^{-1}(c_3)}t_{t_{c_3}^{-1}(c_4)}\cdots t_{t_{c_{2n-2}}^{-1}(c_{2n-1})},\\
{}_{\rho_G\phi}(\Omega_{2h}) &= t_{\rho_G\phi(D_0)}t_{\rho_G\phi(D_1)}\cdots t_{\rho_G\phi(D_{2h})},\\
{}_{\rho_G\phi}(\mathbb{E}) &= t_{\rho_G\phi t_{c_1'}^{-1}(c_2)}t_{\rho_G\phi t_{c_2}^{-1}(c_3)}t_{\rho_G\phi t_{c_3}^{-1}(c_4)}\cdots t_{\rho_G\phi t_{c_{2n-2}}^{-1}(c_{2n-1})},\\
{}_{\rho_G\phi}(\mathbb{D}') &= t_{\rho_G\phi t_{c_{2n-2}}(c_{2n-1})}\cdots t_{\rho_G\phi t_{c_4}(c_5)}t_{\rho_G\phi t_{c_3}(c_4)}\\
&\quad\cdot t_{\rho_G\phi t_{c_3}^{2n-1}(t_{c_2}(c_3))}t_{\rho_G\phi t_{c_3}^{2n-1}(t_{c_1'}(c_2))}.
\end{aligned}$$

*Then, $\pi_1(Y_i) \cong G$ for $0 \leq i \leq 2n-2$.*

6.1. **The fundamental group of $\Sigma_{2h+n-1}$.** In this subsection, we give a presentation of the fundamental group of $\Sigma_{2h+n-1}$. This presentation is used to compute the fundamental group of $Y_i$ in Proposition 34.

Fix a base point $\bullet \in \Sigma_{2h+n-1}$, and let $\pi_1(\Sigma_{2h+n-1})$ be the fundamental group of $\Sigma_{2h+n-1}$ at the base point $\bullet$. For a simple closed curve $c$ on $\Sigma_{2h+n-1}$, we identify $c$ with an element of $\pi_1(\Sigma_{2h+n-1})$ by choosing a path from $\bullet$ to some point on $c$. We use the same symbol for a loop based at $\bullet$ and its based homotopy class. Similarly, we use the same symbol for a diffeomorphism and its isotopy class, or a simple closed curve and its isotopy class. A simple loop based at $\bullet$ and a simple closed curve will even be denoted by the same symbol. It will cause no confusion as it will be clear from the context which one we mean. For $a$ and $b$ in $\pi_1(\Sigma_{2h+n-1})$, the notation $ab$ means that we first traverse $a$ and then $b$.

Let $a_1, b_1, \ldots, a_{2h}, b_{2h}, \alpha_1, \beta_1, \ldots, \alpha_{n-1}, \beta_{n-1}, s_1, \ldots, s_{2h+n-1}$ be the "simple closed curves" on $\Sigma_{2h+n-1}$ as shown in Figure 9, and we use the same symbols for the "loops" obtained from the simple closed curves by choosing the straight path from $\bullet$ as shown in the figure. Then, the fundamental group $\pi_1(\Sigma_{2h+n-1})$ has the following presentation:

$$\pi_1(\Sigma_{2h+n-1}) = \langle a_1, b_1, \ldots, a_{2h}, b_{2h}, \alpha_1, \beta_1, \ldots, \alpha_{n-1}, \beta_{n-1} \mid s_{2h+n-1} = 1\rangle.$$

Note that

$$s_k = b_k^{-1} \cdots b_1^{-1}(a_1 b_1 a_1^{-1}) \cdots (a_k b_k a_k^{-1})$$

for $k \leq 2h$ and that

$$s_{2h+k} = \beta_{n-k}^{-1} \cdots \beta_{n-1}^{-1} s_{2h}(\alpha_{n-1}\beta_{n-1}\alpha_{n-1}^{-1}) \cdots (\alpha_{n-k}\beta_{n-k}\alpha_{n-k}^{-1})$$

for $1 \leq k \leq n-1$.

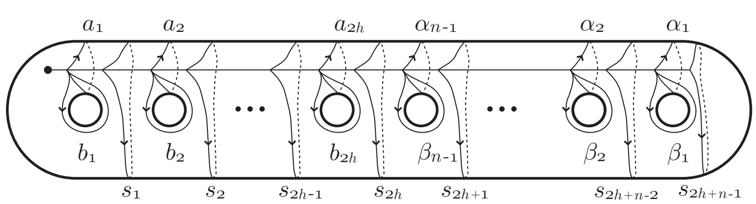


FIGURE 9. The base point $\bullet$ in $\Sigma_{2h+n-1}$, the simple closed curves $a_i, b_i, \alpha_j, \beta_j, s_k, s_{2h+k}$ on $\Sigma_{2h+n-1}$ and the loops $a_i, b_i, \alpha_j, \beta_j, s_k, s_{2h+k}$ in $\Sigma_{2h+n-1}$ based at $\bullet$ obtained from the simple closed curves.

For simplicity of notation, we use the same symbols $D_0, D_1, \ldots, D_{2h}$ and $c_2, c_3, \ldots, c_{2n-1}$ for the images in $\Sigma_{2h+n-1}$ of the simple closed curves on $\Sigma^1_{2h+n-1}$ as in Figure 3 under the inclusion $\Sigma^1_{2h+n-1} \to \Sigma_{2h+n-1}$. Note that the images of $c_1'$ and $c_1''$ as in Figure 3 under the inclusion are isotopic. Let $c_1$ be the image of $c_1'$ under the inclusion. Recall that for a simple closed curve $c$ on $\Sigma_{2h+n-1}$, we identify $c$ with an element of $\pi_1(\Sigma_{2h+n-1})$

by choosing a path from the base point $\bullet$ to some point on $c$. Then, it is immediate from Figure 10 that, up to conjugation, the following equalities hold in $\pi_1(\Sigma_{2h+n-1})$:

(15) $$D_0 = (b_1 b_2 \cdots b_{2h})\alpha_{n-1}^{-1};$$

(16) $$D_{2k-1} = (a_k b_k b_{k+1} \cdots b_{2h+1-k} s_{2h+1-k} a_{2h+1-k})\alpha_{n-1}^{-1} \quad \text{for} \quad 1 \leq k \leq h;$$

(17) $$D_{2k} = (a_k b_{k+1} b_{k+2} \cdots b_{2h-k} s_{2h-k} a_{2h+1-k})\alpha_{n-1}^{-1} \text{ for } 1 \leq k \leq h-1;$$

(18) $$D_{2h} = (a_h s_h a_{h+1})\alpha_{n-1}^{-1};$$

(19) $$c_1 = \alpha_1;$$

(20) $$c_{2l} = \beta_l \quad \text{for} \quad 1 \leq l \leq n-1;$$

(21) $$c_{2l+1} = \alpha_l \alpha_{l+1}^{-1} \quad \text{for} \quad 2 \leq l \leq n-2;$$

(22) $$c_{2n-1} = s_h \alpha_{n-1}.$$

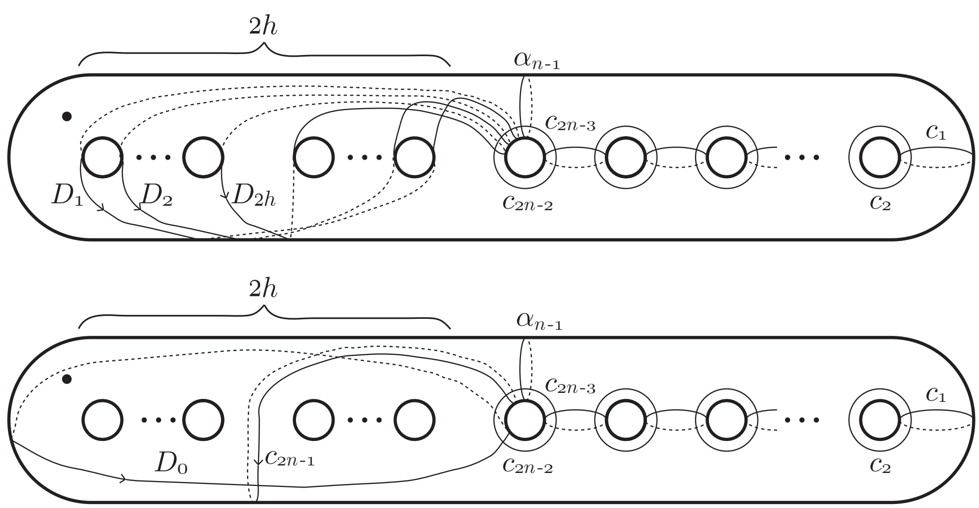


FIGURE 10. The simple closed curves $D_0, D_1, \ldots, D_{2h}$, $c_1, c_2, \ldots, c_{2n-1}$ on $\Sigma_{2h+n-1}$ and the base point $\bullet$.

6.2. **A mapping class $\rho_G$.** In this subsection, we define a mapping class $\rho_G$ in $\Gamma^1_{2h+n-1}$ such that $\rho_G\phi$ is used in Proposition 34, where $\phi$ is defined in Section 4.4. In order to define $\rho_G$, we present Proposition 35 below proved in [36], which is based on the works [38, 6].

Let $h'$ be a positive integer with $h' \leq h$. Let $a_1, b_1, \ldots, a_{2h}, b_{2h}, \alpha_1, \beta_1, \ldots, \alpha_{n-1}, \beta_{n-1}$ (resp. $a_1, b_1, \ldots, a_{h'}, b_{h'}$) be the "simple closed curves" on $\Sigma^1_{2h+n-1}$ (resp. $\Sigma^1_{h'}$) as shown in Figure 11, and we use the same symbols for the "loops" obtained from the simple closed curves by choosing the straight path from $\bullet$ as shown in the figure. The based homotopy classes of the loops are generators of the fundamental group $\pi_1(\Sigma^1_{2h+n-1})$ (resp.

$\pi_1(\Sigma^1_{h'})$) of $\Sigma^1_{2h+n-1}$ (resp. $\Sigma^1_{h'}$) at the base point $\bullet$. The loops in $\Sigma^1_{2h+n-1}$ (resp. $\Sigma^1_{h'}$) are contained in $\Sigma_{2h+n-1}$ (resp. $\Sigma_{h'}$), which is obtained by gluing a disk to the boundary component of $\Sigma^1_{2h+n-1}$ (resp. $\Sigma^1_{h'}$).

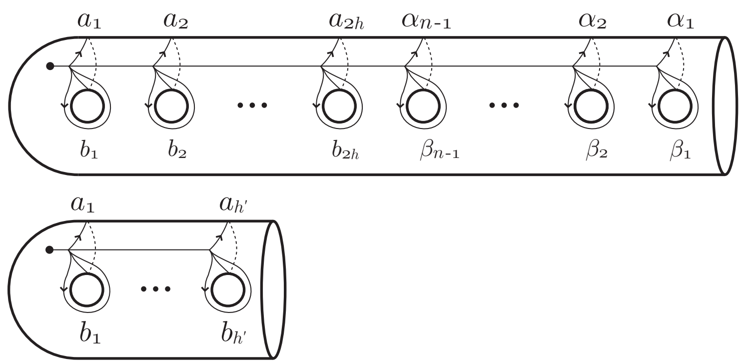


FIGURE 11. The simple closed curves $a_i, b_i, \alpha_j, \beta_j$ on $\Sigma^1_{2h+n-1}$ (resp. $a_i, b_i$ on $\Sigma^1_{h'}$) and the loops $a_i, b_i, \alpha_j, \beta_j$ in $\Sigma^1_{2h+n-1}$ (resp. $a_i, b_i$ in $\Sigma^1_{h'}$) based at $\bullet$.

For simplicity of exposition, we assume that the nonnegative integer $N$ (resp. $N + \ell_k - 1$) that appears in Proposition 35 satisfies $N \leq h$ (resp. $N + \ell_k - 1 \leq h$).

**Proposition 35** (Proposition 7.1 in [36])**.** *Let $F_N$ be a free subgroup of $\pi_1(\Sigma_N)$ of rank $N$ generated by the generators $a_1, \dots, a_N$, and let $r_1, \dots, r_k$ be arbitrary $k$ elements in $F_N$ represented as words in $a_1, \dots, a_N$. We write $\ell_k = \max_{1 \leq i \leq k}\{\ell(r_i)\}$, where $\ell(r_i)$ is the syllable length of $r_i$. Then, there are simple loops $R_1, \dots, R_k$ in $\Sigma_{N+\ell_k-1}$ based at $\bullet$ with the following properties: For each $1 \leq i \leq k$,*

- (i) *$R_i$ is freely homotopic to a simple closed curve which intersects $a_{N+\ell_k-1}$ transversely at exactly one point, and*
- (ii) *For the homomorphism $\lambda : \pi_1(\Sigma_{N+\ell_k-1}) \to \pi_1(\Sigma_N)$ defined by $\lambda(a_j) = a_j$ for $1 \leq j \leq N$ and $\lambda(c) = 1$ for $c \in \{a_{N+1}, a_{N+2}, \dots, a_{N+\ell_k-1}, b_1, b_2, \dots, b_{N+\ell_k-1}\}$, we have $\lambda(R_i) = r_i$, which is really an equality.*

Using Proposition 35, we can obtain the following proposition.

**Proposition 36.** *In the notation of Proposition 35, suppose that $h \geq N + \ell_k - 1$ and $2n - 8 \geq k$. Let $w$ be the simple closed curve on $\Sigma^1_{2h+n-1}$ as in Figure 4. Then, there are simple loops $R_1, R_2, \dots, R_k$ in $\Sigma^1_{2h+n-1}$ based at $\bullet$ with the following properties (see Figure 12):*

- (i) *$R_1, \dots, R_k$ are disjoint from $s_{2h+n-4}$,*
- (ii) *$R_1$ intersects $t^{-1}_{c_2}(w)$ transversely at exactly one point and $R_2, \dots, R_k$ do not intersect $t^{-1}_{c_2}(w)$,*
- (iii) *$R_2$ intersects $t^{-1}_{w}(c_8)$ transversely at exactly one point and $R_3, \dots, R_k$ do not intersect $t^{-1}_{w}(c_8)$,*

(iv) *For each* $3 \leq i \leq k$, $R_i$ *intersects* $t_{c_{i+5}}^{-1}(c_{i+6})$ *transversely at exactly one point and* $R_{i+1}, R_{i+2}, \ldots, R_k$ *are disjoint from* $t_{c_{i+5}}^{-1}(c_{i+6})$, *and*

(v) *Let* $\lambda : \pi_1(\Sigma_{2h+n-1}^1) \to \pi_1(\Sigma_N)$ *be the homomorphism defined by* $\lambda(a_j) = a_j$ *for* $1 \leq j \leq N$ *and* $\lambda(c) = 1$ *for* $c \in \{a_{N+1}, a_{N+2}, \ldots, a_h, b_1, b_2, \ldots, b_h, \alpha_1, \beta_1, \ldots, \alpha_{n-1}, \beta_{n-1}\}$. *Then, for each* $1 \leq i \leq k$, *we have* $\lambda(R_i) = r_i$, *which is really an equality.*

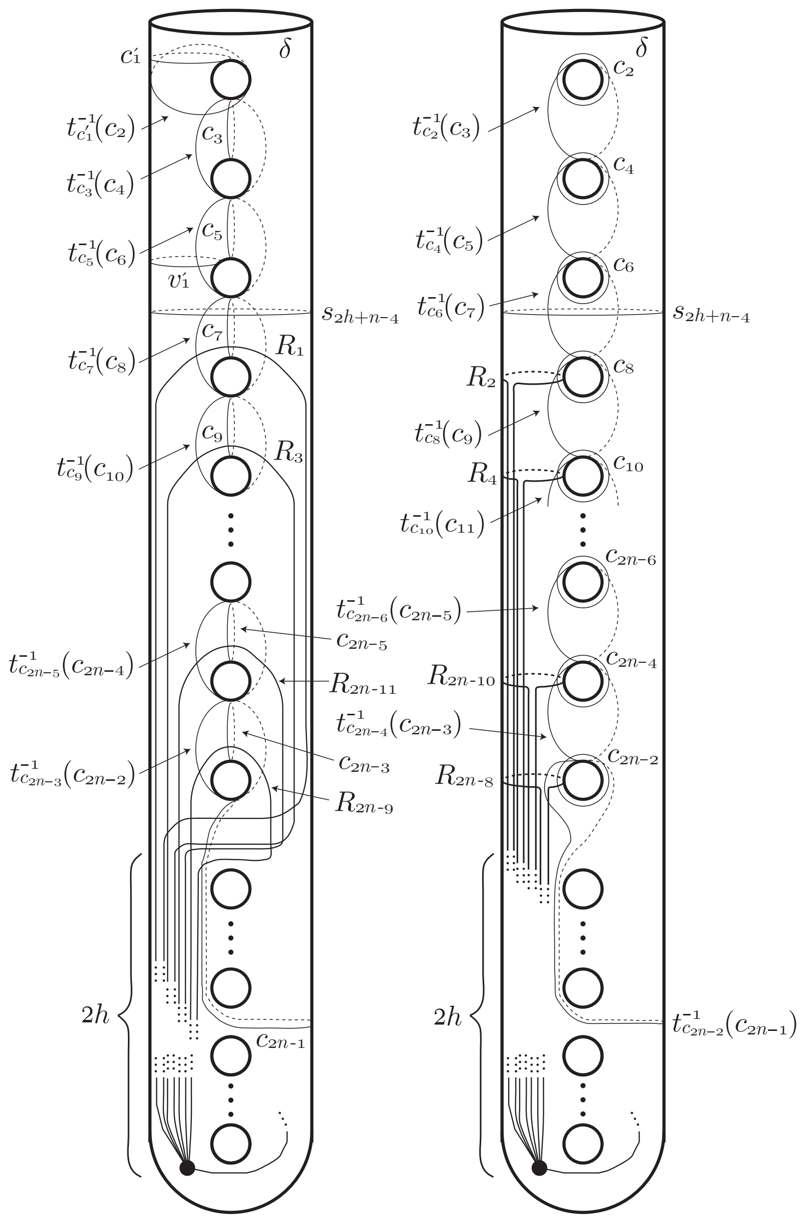


FIGURE 12. Simple loops $R_1, \ldots, R_k$ in $\Sigma_{2h+n-1}^1$ based at $\bullet$.

*Proof.* Let us consider the simple loops $R_1, \ldots, R_k$ in $\Sigma_{N+\ell_k-1}$ constructed in Proposition 35. By removing an open disk from $\Sigma_{N+\ell_k-1}$ near the simple closed curve $a_{N+\ell_k-1}$ and disjoint from all $R_i$, we obtain $\Sigma^1_{N+\ell_k-1}$ (cf. Figure 13 (a)). Moreover, we embed $\Sigma^1_{N+\ell_k-1}$ into $\Sigma^1_{2h+n-1}$ in such a way that for each $1 \leq t \leq N+\ell_k-1$, the loops $a_t, b_t$ in $\Sigma^1_{N+\ell_k-1}$ correspond to the loops $a_t, b_t$ in $\Sigma^1_{2h+n-1}$ (cf. Figure 13 (b)). Then, we can modify $R_1, \ldots, R_k$ so that each $R_i$ $(i = 1, \ldots, k)$ satisfies the properties of Proposition 36 by replacing $R_i$ with a simple representative of $R_{2p-1}(\beta_{n-1} \cdots \beta_{p+4}\beta_{p+3})^{\epsilon}$ if $i = 2p-1$, and $R_{2p}a^{\epsilon}_{p+3}$ if $i = 2p$, where $\epsilon = \pm 1$ (cf. Figure 12). This finishes the proof. $\square$

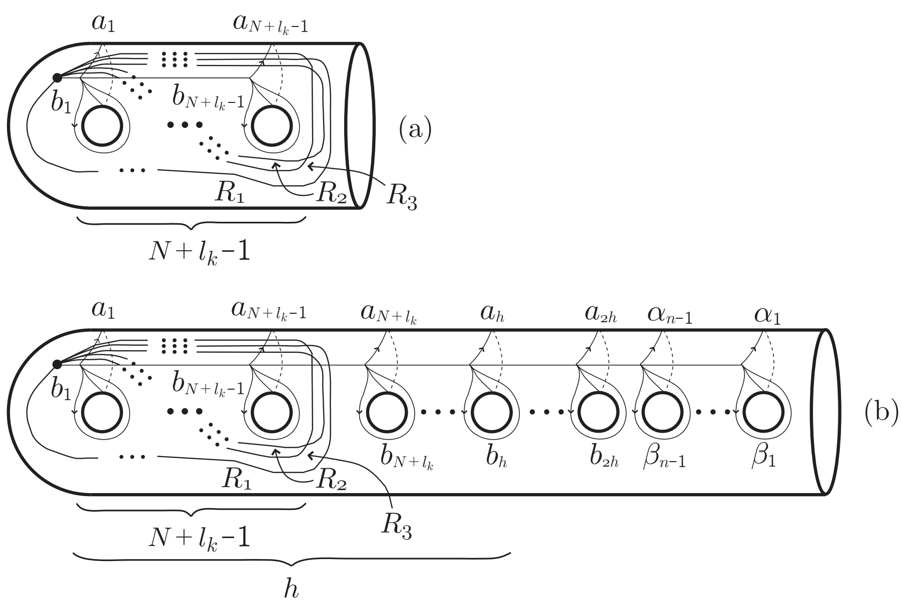


FIGURE 13. Modified loops $R_1, \ldots, R_k$ in $\Sigma^1_{2h+n-1}$ based at $\bullet$.

The first property in Proposition 36 is used to define a mapping class $\rho_G$ in $\Gamma^1_{2h+n-1}$ in the next paragraph. The second, third and fourth properties are used in the proof of Lemma 39 in the next subsection. Together with this, we use the fifth property in the proof of Proposition 34.

We define the mapping class $\rho_G$ in $\Gamma^1_{2h+n-1}$ to be

$$\rho_G = t_{R_1}t_{R_2}\cdots t_{R_k}t_{a_{N+1}}t_{a_{N+2}}\cdots t_{a_h}t_{b_{h+1}}t_{b_{h+2}}\cdots t_{b_{2h}}.$$

From the first property of $R_i$ for $1 \leq i \leq k$, $v_1' = \phi(c_1')$, $v_1'' = \phi(c_1'')$ and $c_5 = \phi(c_3)$ are disjoint from $R_1, R_2, \ldots, R_k$. Therefore, we see that

$$\rho_G(v_1') = v_1', \qquad \rho_G(v_1'') = v_1'' \qquad \text{and} \qquad \rho_G(c_5) = c_5,$$

which were used in the proof of Theorem B in Section 5.2. Moreover, we have

$$v_1' = \rho_G\phi(c_1'), \qquad v_1'' = \rho_G\phi(c_1'') \qquad \text{and} \qquad c_5 = \rho_G\phi(c_3).$$

6.3. **Proof of Proposition 34.** We prove Proposition 34 in this subsection.

We note that $c_1', c_1''$ (resp. $v_1', v_1''$) are mapped to $c_1$ (resp. $\alpha_3$) under the inclusion $\Sigma_{2h+n-1}^1 \to \Sigma_{2h+n-1}$. Similarly, $t_{c_1'}(c_2)$, $t_{c_1'}^{-1}(c_2)$ and $t_{c_3}^{2n-1}(t_{c_1'}(c_2))$ are mapped to $t_{c_1}(c_2)$, $t_{c_1}^{-1}(c_2)$ and $t_{c_3}^{2n-1}(t_{c_1}(c_2))$, respectively.

Let us consider the presentation of the fundamental group $\pi_1(\Sigma_{2h+n-1})$ of $\Sigma_{2h+n-1}$ at the base point $\bullet$ in Section 6.1. From the positive factorization of $t_\delta^2$ in Proposition 34 and Lemma 15, we see that $\pi_1(Y_i)$ has a presentation with generators

$$a_1, b_1, a_2, b_2, \ldots, a_{2h}, b_{2h}, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{n-1}, \beta_{n-1}$$

and with relations

$$s_{2h+n-1} = 1; \tag{23}$$
$$c_1 = x = y = z = \alpha_3 = 1; \tag{24}$$
$$t_{c_{j-1}}(c_j) = 1 \qquad \text{for } \ 4 \le j \le 2n-1; \tag{25}$$
$$t_{c_{j-1}}^{-1}(c_j) = 1 \qquad \text{for } \ 2 \le j \le 2n-1; \tag{26}$$
$$t_{c_3}^{2n-1}(t_{c_1}(c_2)) = t_{c_3}^{2n-1}(t_{c_2}(c_3)) = 1; \tag{27}$$
$$D_k = 1 \qquad \text{for } \ 0 \le k \le 2h, \tag{28}$$
$$\rho_G\phi(t_{c_{j-1}}(c_j)) = 1 \qquad \text{for } \ 4 \le j \le 2n-1; \tag{29}$$
$$\rho_G\phi(t_{c_{j-1}}^{-1}(c_j)) = 1 \qquad \text{for } \ 2 \le j \le 2n-1; \tag{30}$$
$$\rho_G\phi(t_{c_3}^{2n-1}(t_{c_1}(c_2))) = \rho_G\phi(t_{c_3}^{2n-1}(t_{c_2}(c_3))) = 1; \tag{31}$$
$$\rho_G\phi(D_k) = 1 \qquad \text{for } \ 0 \le k \le 2h. \tag{32}$$

We prepare some lemmas to show Proposition 34.

**Lemma 37.** *The following holds in $\pi_1(Y_i)$:*

$$\alpha_j = 1 \qquad \text{for } \ 1 \le j \le n-1; \tag{33}$$
$$\beta_j = 1 \qquad \text{for } \ 1 \le j \le n-1; \tag{34}$$
$$s_h = 1. \tag{35}$$

*Moreover, we can replace the relations (24)–(27) by the relations (33) and (34).*

*Proof.* It is easy to check that, up to conjugation, the following relations hold in $\pi_1(\Sigma_{2h+n-1})$:

$$\begin{aligned}
&c_1 = \alpha_1;\\
&t_{c_1}^{-1}(c_2) = \beta_1\alpha_1^{-1};\\
&t_{c_{2j}}^{-1}(c_{2j+1}) = \beta_j\alpha_{j+1}^{-1}\alpha_j;\\
&t_{c_{2j+1}}^{-1}(c_{2j+2}) = \beta_{j+1}\alpha_{j+1}^{-1}\alpha_j;\\
&t_{c_{2n-2}}^{-1}(c_{2n-1}) = s_h\alpha_{n-1}\beta_{n-1}.
\end{aligned}$$

These and the relations (23)–(25) give the relations (33)–(35).

We suppose that the relations (33) and (34) hold. Then, since the curves in the relations (24)–(27) are generated by $\alpha_j$ and $\beta_j$, we obtain the latter part. □

The following lemma is used to show Lemmas 39 and 40. For simplicity, take a base point in a regular fiber $\Sigma_g$ of a genus-$g$ Lefschetz fibration $f : X \to \mathbb{S}^2$. Let $\pi_1(X)$ be the fundamental group of $X$ at the base point. We identify a simple closed curve $c$ on $\Sigma_g$ with an element of $\pi_1(X)$ by choosing a path in $\Sigma_g$ from the base point to some point on $c$.

**Lemma 38** ([38])**.** *Let $c, x_1, x_2, \ldots, x_k$ be simple closed curves on a regular fiber $\Sigma_g$ of a genus-$g$ Lefschetz fibration $f : X \to \mathbb{S}^2$. If $x_i = 1$ in $\pi_1(X)$ for $i = 1, 2 \ldots, k$, then the following holds in $\pi_1(X)$:*

$$t_{x_k} \cdots t_{x_2} t_{x_1}(c) = c.$$

The following proof was suggested by the editor.

*Proof.* If $x_1$ is disjoint from $c$, then we have $t_{x_1}(c) = c$ on $\Sigma_g (\subset X)$. Suppose that $x_1$ intersects $c$. By the assumption, $x_1$ bounds an immersed disk $D$ in $X$. Then, $c$ and $t_{x_1}(c)$ are homotopic in $\Sigma_g \cup D (\subset X)$ relative to the base point $\bullet$. This gives $t_{x_1}(c) = c$ in $\pi_1(X)$. By repeating this argument, we obtain the claim. □

**Lemma 39.** *The following holds in $\pi_1(Y_i)$:*

$$R_j = 1 \qquad \text{for} \quad 1 \leq j \leq k. \tag{36}$$

*Moreover, we can replace the relations (29)–(31) by the relation (36).*

*Proof.* We first show that the relation (36) holds. Let us consider the relation (30).

Since each $\phi(t_{c_{j'-1}}^{-1}(c_{j'}))$ is generated by $\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2, \ldots, \alpha_{n-1}$, $\beta_{n-1}$ from the definition of $\phi$ (see Section 4.4), by Lemma 37 we have

$$\phi(t_{c_{j'-1}}^{-1}(c_{j'})) = 1 \tag{37}$$

in $\pi_1(Y_i)$ for $j' = 2, 3, \ldots, 2n-1$. By the definition of $\phi$, we have

$$\phi(t_{c_6}^{-1}(c_7)) = t_{c_2}^{-1}(w); \tag{38}$$

$$\phi(t_{c_7}^{-1}(c_8)) = t_w^{-1}(c_8); \tag{39}$$

$$\phi(t_{c_{j'-1}}^{-1}(c_{j'})) = t_{c_{j'-1}}^{-1}(c_{j'}) \tag{40}$$

for $j' = 9, 10, \ldots, 2n-1$.

Here, from the definition of $\phi$ (see Section 4.4) and Figures 4 and 9–12, we see that the simple closed curve $\phi(t_{c_{j'-1}}^{-1}(c_{j'}))$ is also disjoint from the simple closed curves $a_m$ and $b_m$ for all $m$ and for $j' = 2, 3, \ldots, 2n-1$. This gives

$$\begin{aligned}\rho_G\phi(t_{c_{j'-1}}^{-1}(c_{j'})) &= t_{R_1}\cdots t_{R_k}t_{a_{N+1}}t_{a_{N+2}}\cdots t_{a_h}t_{b_{h+1}}t_{b_{h+2}}\cdots t_{b_{2h}}\phi(t_{c_{j'-1}}^{-1}(c_{j'}))\\ &= t_{R_1}t_{R_2}\cdots t_{R_k}\phi(t_{c_{j'-1}}^{-1}(c_{j'})) = 1.\end{aligned}$$

Hence,

$$\rho_G\phi(t_{c_{j'-1}}^{-1}(c_{j'})) = t_{R_1}t_{R_2}\cdots t_{R_k}\phi(t_{c_{j'-1}}^{-1}(c_{j'})) = 1 \tag{41}$$

for $j' = 2, 3, \ldots, 2n-1$. Since $R_1$ intersects $t_{c_2}^{-1}(w)$ transversely at exactly one point and $R_2, \ldots, R_k$ do not intersect $t_{c_2}^{-1}(w)$ from Proposition 36, by the relation (38), up to conjugation, we obtain

$$\begin{aligned}1 &= t_{R_1}t_{R_2}\cdots t_{R_k}\phi(t_{c_6}^{-1}(c_7))\\ &= t_{R_1}t_{R_2}\cdots t_{R_k}(t_{c_2}^{-1}(w))\\ &= t_{R_1}(t_{c_2}^{-1}(w))\\ &= t_{c_2}^{-1}(w)R_1^{\varepsilon_1},\end{aligned}$$

where $\varepsilon_1 = \pm 1$. By the relations (37) and (38), we have

$$R_1 = 1.$$

Similarly, since $R_3, \ldots, R_k$ do not intersect $t_w^{-1}(c_8)$ from Proposition 36, by the relation (39) we obtain

$$\begin{aligned}1 &= t_{R_1}t_{R_2}\cdots t_{R_k}\phi(t_{c_7}^{-1}(c_8))\\ &= t_{R_1}t_{R_2}\cdots t_{R_k}(t_w^{-1}(c_8))\\ &= t_{R_1}t_{R_2}(t_w^{-1}(c_8)).\end{aligned}$$

Since $R_2$ intersects $t_w^{-1}(c_8)$ transversely at exactly one point from Proposition 36, by Lemma 38, up to conjugation, we get

$$\begin{aligned}1 &= t_{R_1}t_{R_2}(t_w^{-1}(c_8))\\ &= t_{R_2}(t_w^{-1}(c_8))\\ &= t_w^{-1}(c_8)R_2^{\varepsilon_2},\end{aligned}$$

where $\varepsilon_2 = \pm 1$. By the relation (37), we have

$$R_2 = 1.$$

Moreover, since $R_i$ intersects $\phi(t_{c_{i+5}}^{-1}(c_{i+6}))$ transversely at exactly one point and $R_{i+1}, R_{i+2}, \dots, R_k$ are disjoint from $t_{c_{i+5}}^{-1}(c_{i+6})$ for $3 \leq i \leq k$ from Proposition 36, by the relations (40) and (41) we see that for $j' = 9, 10, \dots, k$,

$$\begin{aligned} 1 &= t_{R_1} t_{R_2} \cdots t_{R_k} \phi(t_{c_{j'-1}}^{-1}(c_{j'})) \\ &= t_{R_1} t_{R_2} \cdots t_{R_k} (t_{c_{j'-1}}^{-1}(c_{j'})) \\ &= t_{R_1} t_{R_2} \cdots t_{R_{j'-6}} (t_{c_{j'-1}}^{-1}(c_{j'})) \end{aligned}$$

in $\pi_1(Y_i)$. Hence, we obtain

$$t_{R_1} t_{R_2} \cdots t_{R_{j'-6}} (t_{c_{j'-1}}^{-1}(c_{j'})) = 1$$

in $\pi_1(Y_i)$ for $j' = 9, 10, \dots, k$. Since $R_3$ intersects $t_{c_8}^{-1}(c_9)$ transversely at exactly one point and $R_4, \dots, R_k$ are disjoint from $t_{c_8}^{-1}(c_9)$ (see Proposition 36), by Lemma 38, up to conjugation, we have

$$\begin{aligned} 1 &= t_{R_1} t_{R_2} t_{R_3} (t_{c_8}^{-1}(c_9)) \\ &= t_{R_3} (t_{c_8}^{-1}(c_9)) \\ &= t_{c_8}^{-1}(c_9) R_3^{\varepsilon_3}, \end{aligned}$$

where $\varepsilon_3 = \pm 1$. From the relation (37), we have

$$R_3 = 1.$$

By repeating this argument, we obtain the former part of the claim.

We next show the latter part. Suppose that the relation (36) holds. From Figures 4, 5 and 9–12, each of the simple closed curves $\phi(t_{c_{j'-1}}^{-1}(c_{j'}))$, $\phi(t_{c_{j'-1}}(c_{j'}))$, $\phi(t_{c_3}^{2n-1}(t_{c_1}(c_2)))$ and $\phi(t_{c_3}^{2n-1}(t_{c_2}(c_3)))$ is disjoint from the simple closed curves $a_m$ and $b_m$ for all $m$. By Lemma 38, we can replace the relations (29)–(31) by the relations

$$\begin{aligned} &1 = \rho_G \phi(t_{c_{j-1}}(c_j)) = t_{R_1} \cdots t_{R_k} (\phi(t_{c_{j'-1}}(c_{j'}))) = \phi(t_{c_{j'-1}}(c_{j'})); \\ &1 = \rho_G \phi(t_{c_{j-1}}^{-1}(c_j)) = t_{R_1} \cdots t_{R_k} (\phi(t_{c_{j'-1}}^{-1}(c_{j'}))) = \phi(t_{c_{j'-1}}^{-1}(c_{j'})); \\ &1 = \rho_G \phi(t_{c_3}^{2n-1}(t_{c_1}(c_2))) = t_{R_1} \cdots t_{R_k} (\phi(t_{c_3}^{2n-1}(t_{c_1}(c_2)))) = \phi(t_{c_3}^{2n-1}(t_{c_1}(c_2))); \\ &1 = \rho_G \phi(t_{c_3}^{2n-1}(t_{c_2}(c_3))) = t_{R_1} \cdots t_{R_k} (\phi(t_{c_3}^{2n-1}(t_{c_2}(c_3)))) = \phi(t_{c_3}^{2n-1}(t_{c_2}(c_3))). \end{aligned}$$

Since $\phi(t_{c_{j-1}}^{-1}(c_j))$, $\phi(t_{c_{j-1}}(c_j))$, $\phi(t_{c_3}^{2n-1}(t_{c_1}(c_2)))$ and $\phi(t_{c_3}^{2n-1}(t_{c_2}(c_3)))$ are generated by $\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2, \dots, \alpha_{n-1}$, $\beta_{n-1}$, from Lemma 37 we obtain the relations (29)–(31). ☐

**Lemma 40.** *The following holds in $\pi_1(Y_i)$:*

$$a_m a_{2h+1-m} = 1 \quad \text{for} \quad 1 \leq m \leq h; \tag{42}$$

$$a_m = 1 \quad \text{for} \quad N+1 \leq m \leq h; \tag{43}$$

$$b_l = 1 \quad \text{for} \quad 1 \leq l \leq 2h. \tag{44}$$

*Moreover, we can replace the relations (23), (28), (32) and (35) by the relations (42)–(44).*

*Proof.* We first show the relation (42) by induction on $m$. By the relations (28), $\alpha_{n-1}=1$ (the relation (33)), (35) and (18), we see that the relation (42) holds for $m=h$. Suppose that we have $a_m a_{2h+1-m}=1$ for $m<h$. Then, from the relations (28), $\alpha_{n-1}=1$ (the relation (33)), (35), $D_{2m-1}=1$ (the relation (16)) and $D_{2m-2}=1$ (the relation (17)), we obtain

$$\begin{aligned}
&a_{m-1}b_m b_{m+1}\cdots b_{2h-(m-1)}s_{2h-(m-1)}a_{2h+1-(m-1)}=1;\\
&a_m b_m b_{m+1}\cdots b_{2h+1-m}s_{2h+1-m}a_{2h+1-m}=1
\end{aligned}$$

for $2\leq m\leq h$, which give $a_{m-1}a_{2h+1-(m-1)}=1$. Therefore, we obtain the relation (42) for $1\leq m\leq h$.

We next show the relation (43). From the definition of $\phi$ (see Section 4.4) and Lemma 38, we see that

$$\rho_G\phi(D_j)=\rho_G(D_j)=t_{a_{N+1}}t_{a_{N+2}}\cdots t_{a_h}t_{b_{h+1}}t_{b_{h+2}}\cdots t_{b_{2h}}(D_j)$$

for $j=0,1,\ldots,2h$. It is easy to check that from this, up to conjugation, the following equalities hold in $\pi_1(\Sigma_{2h+n-1})$:

$$\begin{aligned}
\rho_G\phi(D_{2m-1})&=(a_m b_m b_{m+1}\cdots b_{2h+1-m}s_{2h+1-m}a_{2h+1-m})\\
&\qquad\cdot b_{2h+1-m}^{-1}\cdot\alpha_{n-1}^{-1}\cdot a_h\cdots a_{m+1}a_m;\\
\rho_G\phi(D_{2m})&=(a_m b_{m+1}b_{m+2}\cdots b_{2h-m}s_{2h-m}a_{2h+1-m})\\
&\qquad\cdot b_{2h+1-m}^{-1}\cdot\alpha_{n-1}^{-1}\cdot a_h\cdots a_{m+2}a_{m+1};\\
\rho_G\phi(D_{2h-1})&=(a_h b_h b_{h+1}s_{h+1}a_{h+1})\cdot b_{h+1}^{-1}\cdot\alpha_{n-1}^{-1}\cdot a_h;\\
\rho_G\phi(D_{2h})&=(a_h s_h a_{h+1})\cdot b_{h+1}^{-1}\cdot\alpha_{n-1}^{-1}
\end{aligned}$$

for $N+1\leq m\leq h-1$. Therefore, by the relations (28), $\alpha_{n-1}=1$ (the relation (33)) and $\rho_G\phi(D_{2m-1})=\rho_G\phi(D_{2m})=1$, we get

$$\begin{aligned}
&b_{2h+1-m}^{-1}\cdot a_h\cdots a_{m+1}a_m=1;\\
&b_{2h+1-m}^{-1}\cdot a_h\cdots a_{m+2}a_{m+1}=1;\\
&b_{h+1}^{-1}\cdot a_h=1;\\
&b_{h+1}^{-1}=1
\end{aligned}$$

for $N+1\leq m\leq h-1$. This gives

$$a_m=1$$

for $N+1\leq m\leq h$.

Here, we show the relation (44). From the definition of $\phi$ (see Section 4.4), Lemma 38 and the relation (43), we see that

$$\begin{aligned}
\rho_G\phi(D_j)=\rho_G(D_j)&=t_{a_{N+1}}t_{a_{N+2}}\cdots t_{a_h}t_{b_{h+1}}t_{b_{h+2}}\cdots t_{b_{2h}}(D_j)\\
&=t_{b_{h+1}}t_{b_{h+2}}\cdots t_{b_{2h}}(D_j)
\end{aligned}$$

for $j = 0, 1, \ldots, 2h$. Once again, it is easy to check that from this, up to conjugation, the following equalities hold in $\pi_1(\Sigma_{2h+n-1})$:

$$\begin{aligned}
\rho_G\phi(D_0) &= (b_1 b_2 \cdots b_{2h}) \cdot \alpha_{n-1}^{-1};\\
\rho_G\phi(D_{2m-1}) &= (a_m b_m b_{m+1} \cdots b_{2h+1-m} s_{2h+1-m} a_{2h+1-m}) \cdot b_{2h+1-m}^{-1} \cdot \alpha_{n-1}^{-1};\\
\rho_G\phi(D_{2m}) &= (a_m b_{m+1} b_{m+2} \cdots b_{2h-m} s_{2h-m} a_{2h+1-m}) \cdot b_{2h+1-m}^{-1} \cdot \alpha_{n-1}^{-1}
\end{aligned}$$

for $1 \leq k \leq h$. By the relations $\alpha_{n-1} = 1$ (the relation (33)), (28) and (32), we obtain

$$b_{2h-m+1} = 1$$

for $1 \leq m \leq h$. From the relation (42), $D_{2m-1} = 1$ (the relation (16)), $D_{2m} = 1$ (the relation (17)) and its proof, we get

$$\begin{aligned}
b_m b_{m+1} \cdots b_{2h+1-m} s_{2h+1-m} &= 1;\\
b_{m+1} b_{m+2} \cdots b_{2h-m} s_{2h-m} &= 1.
\end{aligned}$$

These give $b_m s_{2h-m}^{-1} b_{2h+1-m} s_{2h+1-m} = 1$ for $1 \leq m \leq h$. Therefore, by the definition of $s_m$ (see Section 6.1),

$$\begin{aligned}
1 &= b_m s_{2h-m}^{-1} b_{2h+1-m} (b_{2h+1-m}^{-1} s_{2h-m} (a_{2h+1-m} b_{2h+1-m} a_{2h+1-m}^{-1}))\\
&= b_m a_{2h+1-m} b_{2h+1-m} a_{2h+1-m}^{-1}
\end{aligned}$$

for $1 \leq m \leq h$. Hence, by $b_{2h-m+1} = 1$ for $1 \leq m \leq h$, we have

$$b_l = 1$$

for $1 \leq l \leq 2h$.

Finally, we show the latter part. Suppose that the relations (42)–(44) hold. From the arguments above, $D_j, \rho_G\phi(D_j)$ and $s_k$ are generated by $a_m$, $b_m$, $\alpha_l$, and $\beta_l$. By the relations (33) and (34), we obtain the relations (23), (28), (32) and (35).

This finishes the proof. $\square$

We now prove Proposition 34.

*Proof of Proposition 34.* By Lemmas 37, 39 and 40, $\pi_1(Y_i)$ has a presentation with generators

$$a_1, b_1, a_2, b_2, \ldots, a_{2h}, b_{2h}, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{n-1}, \beta_{n-1}$$

and with relations

$$\begin{aligned}
&(45) \qquad a_m a_{2h+1-m} = 1 && \text{for } 1 \leq m \leq h;\\
&(46) \qquad a_m = 1 && \text{for } N+1 \leq m \leq h;\\
&(47) \qquad b_l = 1 && \text{for } 1 \leq l \leq 2h;\\
&(48) \qquad \alpha_s = \beta_s = 1 && \text{for } 1 \leq s \leq n-1;\\
&(49) \qquad R_j = 1 && \text{for } 1 \leq j \leq k
\end{aligned}$$

for $0 \leq i \leq 2n-2$. From the relations (45)–(48), we see that $\pi_1(Y_i)$ is generated by $a_1, \ldots, a_N$. By the relations (46)–(49), and using the map $\lambda$

appearing in Proposition 36, we get a word representing the element $r_j$ for $1 \leq j \leq k$. Moreover, the fifth relation gives $r_j = 1$ for $1 \leq j \leq k$. Therefore, we see that $\pi_1(Y_i)$ has a presentation with generators $a_1, a_2, \ldots, a_N$ and with relations

$$r_j = 1 \qquad\qquad \text{for} \quad 1 \leq j \leq k,$$

and hence $\pi_1(Y_i)$ is isomorphic to $G$ for $0 \leq i \leq 2n-2$.

This finishes the proof. □

## 7. Proof of Propositions 25 and 26

This section gives the proofs of Propositions 25 and 26. In the figures below, we denote the arrangement using the Dehn twist $t_c$ about a simple closed curve $c$ and the arrangement using an isotopy by $\xrightarrow{c}$ and $\sim$, respectively.

### 7.1. Proof of Proposition 25.

Let $S_1$ be the subsurface of $\Sigma^1_{2n+n-1}$ of genus $n-1$ with two boundary curves $\delta$ and $\delta'$ as in Figure 3, and let $\Gamma(S_1)$ be the mapping class group of $S_1$. We consider the simple closed curves $c'_1, c''_1, c_2, \ldots, c_{2n-2}, c'_{2n-1}, c''_{2n-1}$ and the arcs $\tau, \tau_{2n-1}$ on $S_1 \subset \Sigma^1_{2h+n-1}$ as in Figures 3 and 14. We denote by $H_\delta$ (resp. $H_{\delta'}$) the half twist about $\delta$ (resp. $\delta'$), hence $t_\delta = H^2_\delta$ (resp. $t_{\delta'} = H^2_{\delta'}$), satisfying $H_\delta(c'_1) = c''_1$, $H_\delta(c_i) = c_i$, $H_\delta(c'_{2n-1}) = c''_{2n-1}$ and $H_\delta(\tau_{2n-1}) = \tau_{2n-1}$ (resp. $H_{\delta'}(c'_1) = c'_1$, $H_{\delta'}(c_i) = c_i$, $H_{\delta'}(c'_{2n-1}) = c'_{2n-1}$ and $H_{\delta'}(\tau) = \tau$ ) for $i = 2, 3, \ldots, 2n-2$. Therefore, we have $H_\delta t_{c'_{2n-1}} t_{c''_{2n-1}} = t_{c'_{2n-1}} t_{c''_{2n-1}} H_\delta$ (resp. $H_{\delta'} t_{c'_{2n-1}} t_{c''_{2n-1}} = t_{c'_{2n-1}} t_{c''_{2n-1}} H_{\delta'}$) from the relation $t_{\phi(c)} = \phi t_c \phi^{-1}$ in Section 4.1. We first show the following proposition.

**Proposition 41.** *The following relation holds in $\Gamma(S_1)$:*

$$t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c'_1} t_{c''_1} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}} = H_\delta H_{\delta'} t^{-1}_{c''_{2n-1}} t^{-1}_{c'_{2n-1}}.$$

*Proof.* We prove the equation using the *Alexander method*. The collection $\{c_2, c_3, \ldots, c_{2n-2}, \tau, \tau_{2n-1}\}$ "fills" $S_1$ and satisfies the assumptions of the Alexander method (see for example [24]). Therefore, from the above-mentioned properties of $H_\delta$ and $H_{\delta'}$, it suffices to show that

(i) $t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c'_1} t_{c''_1} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}}(c_i) = c_i$ for $i = 2, 3, \ldots, 2n-3$,
(ii) $t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c'_1} t_{c''_1} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}}(c_{2n-2}) = t^{-1}_{c''_{2n-1}} t^{-1}_{c'_{2n-1}}(c_{2n-2})$,
(iii) $t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c'_1} t_{c''_1} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}}(\tau) = H_\delta(\tau)$, and
(iv) $t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c'_1} t_{c''_1} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}}(\tau_{2n-1}) = H_{\delta'}(\tau_{2n-1})$.

First, we show (i). Note that $c_j$ is disjoint from $c_k$ if $|j-k| \geq 2$ and intersects $c_k$ at exactly one point if $|j-k| = 1$. Moreover, $c_j$ is disjoint from $c'_1$ and $c''_1$ for $j = 3, 4, \ldots, 2n-2$. Therefore, it follows easily from Figure 7 that for $j = 2, 3, \ldots, 2n-3$, we have

$$t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c'_1} t_{c''_1} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}}(c_j) = c_j.$$

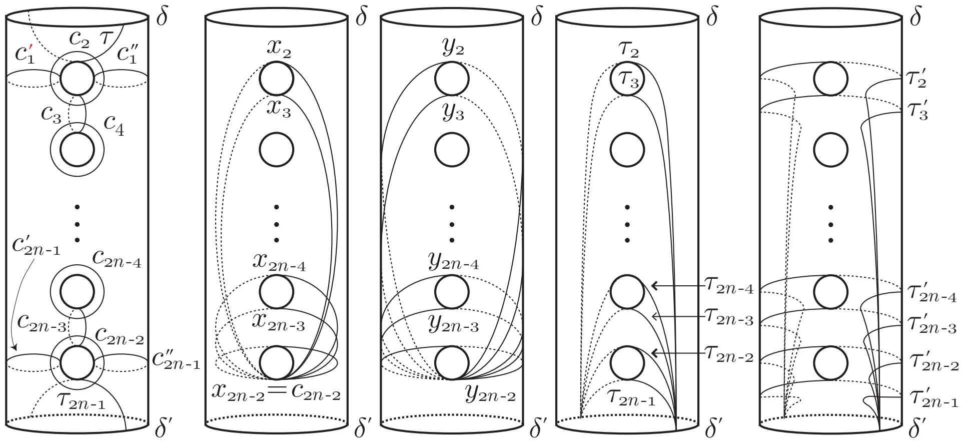


FIGURE 14.

Next, we show (ii). Let $x_2, x_3, \ldots, x_{2n-2}$ be the simple closed curves on $S_1$ as in Figure 14. Note that $c_{2n-2} = x_{2n-2}$. It is easy to check that

$$t_{c_{2i-1}}(x_{2i}) = x_{2i-1},$$
$$t_{c_{2i-2}}(x_{2i-1}) = x_{2i-2}$$

for $i = 2, \ldots, n-1$ from Figure 15. This gives

$$t_{c_2}t_{c_3}\cdots t_{c_{2n-2}}(c_{2n-2}) = t_{c_2}t_{c_3}\cdots t_{c_{2n-2}}(x_{2n-2}) = x_2.$$

Let $y_2, y_3, \ldots, y_{2n-2}$ be the simple closed curves on $S_1$ as in Figure 14. Notice that

$$y_2 = t_{c_1'}t_{c_1''}(x_2),$$
$$t_{c_{2n-2}}(y_{2n-2}) = t^{-1}_{c''_{2n-1}}t^{-1}_{c'_{2n-1}}(c_{2n-2})$$

(cf. Figure 16). It is immediate that

$$t_{c_{2i-2}}(y_{2i-2}) = y_{2i-1},$$
$$t_{c_{2i-1}}(y_{2i-1}) = y_{2i}$$

for $i = 2, 3, \ldots, n-1$ from Figure 17. This gives

$$t_{c_{2n-2}}\cdots t_{c_3}t_{c_2}t_{c_1'}t_{c_1''}(x_2) = t_{c_{2n-2}}(y_{2n-2}) = t^{-1}_{c''_{2n-1}}t^{-1}_{c'_{2n-1}}(c_{2n-2}).$$

Therefore, we have

$$t_{c_{2n-2}}\cdots t_{c_3}t_{c_2}t_{c_1'}t_{c_1''}t_{c_2}t_{c_3}\cdots t_{c_{2n-2}}(c_{2n-2}) = t^{-1}_{c''_{2n-1}}t^{-1}_{c'_{2n-1}}(c_{2n-2}).$$

The proof of (iii) is straightforward from Figure 18 since $\tau$ and $H_\delta(\tau)$ are disjoint from $c_3, c_4, \ldots, c_{2n-2}$.

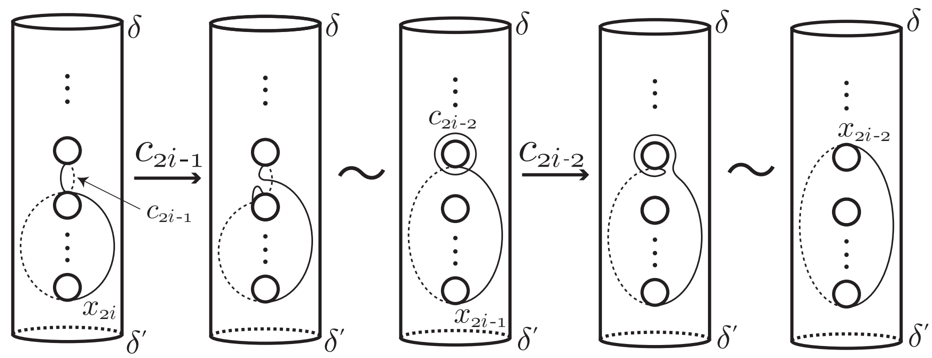


FIGURE 15.

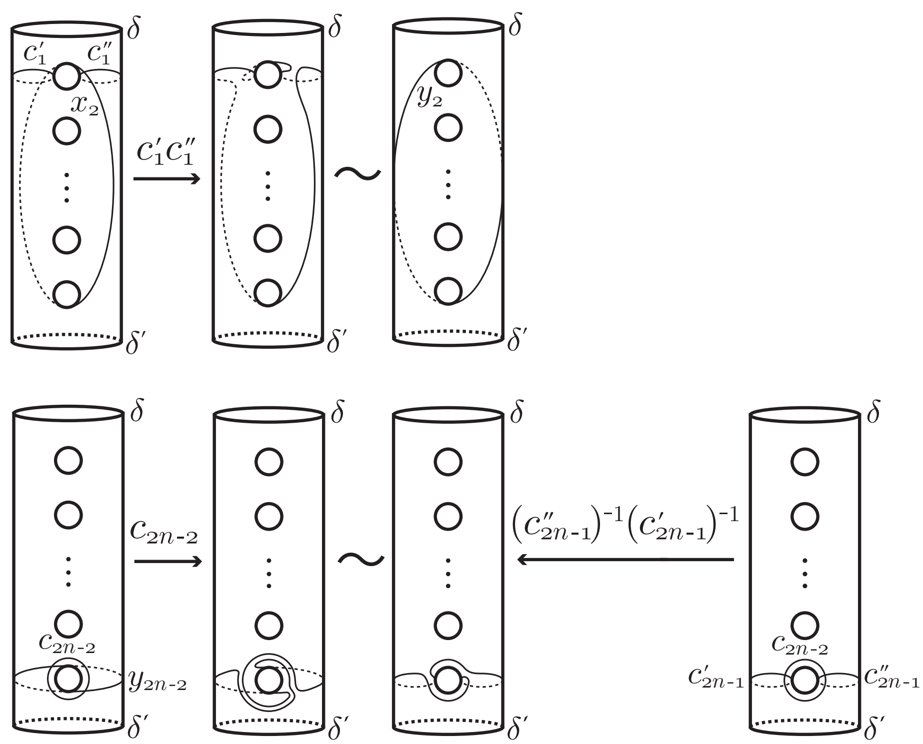


FIGURE 16.

Finally, we show (iv). Let $\tau_2, \tau_3, \ldots, \tau_{2n-1}$ be the arcs on $S_1$ as in Figure 14. It follows easily from Figure 19 that

$$\begin{aligned} t_{c_{2i}}(\tau_{2i+1}) &= \tau_{2i} && \text{for } i = 1, 2, \ldots, n-1, \\ t_{c_{2i-1}}(\tau_{2i}) &= \tau_{2i-1} && \text{for } i = 2, 3, \ldots, n-1. \end{aligned}$$

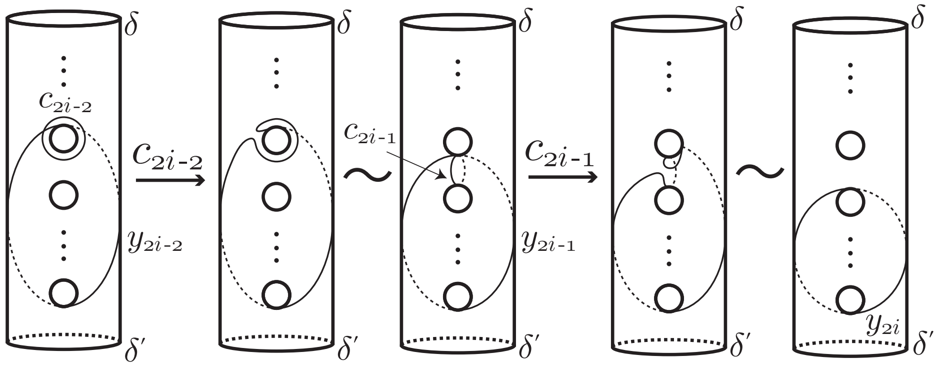


FIGURE 17.

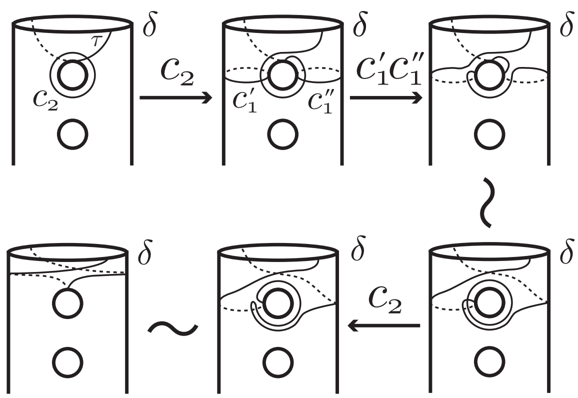


FIGURE 18.

This gives

$$t_{c_2}t_{c_3}\cdots t_{c_{2n-2}}(\tau_{2n-1}) = \tau_2.$$

Let $\tau_2', \tau_3', \ldots, \tau_{2n-1}'$ be the arcs on $S_1$ as in Figure 14. It is easily seen that

$$H_{\delta'}(\tau_{2n-1}) = \tau_{2n-1}'.$$

From Figure 20 we see that

$$t_{c_1'}t_{c_1''}(\tau_2) = \tau_2'.$$

It is immediate that we have

$$\begin{aligned} t_{c_{2i}}(\tau_{2i}') &= \tau_{2i+1}' && \text{for } i = 1, 2, \ldots, n-1, \\ t_{c_{2i+1}}(\tau_{2i+1}') &= \tau_{2i+2}' && \text{for } i = 1, 2, \ldots, n-2 \end{aligned}$$

from Figure 21. Therefore, we have

$$t_{c_{2n-2}}\cdots t_{c_3}t_{c_2}t_{c_1'}t_{c_1''}(\tau_2) = \tau_{2n-1}' = H_{\delta'}(\tau_{2n-1}).$$

From the equations above, we obtain

$$t_{c_{2n-2}}\cdots t_{c_3}t_{c_2}t_{c_1'}t_{c_1''}t_{c_2}t_{c_3}\cdots t_{c_{2n-2}}(\tau_{2n-1}) = H_{\delta'}(\tau_{2n-1}).$$

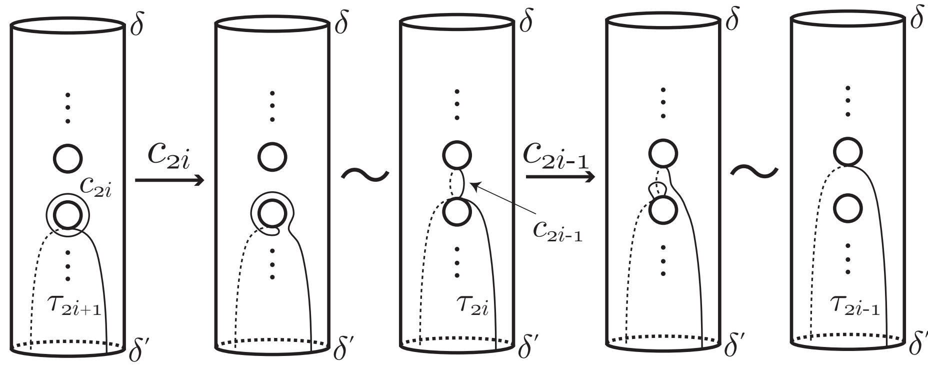


FIGURE 19.

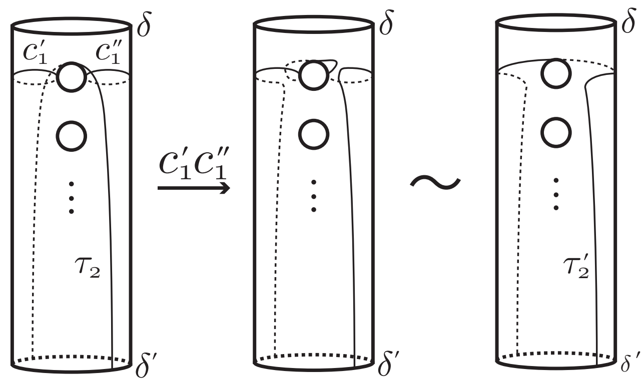


FIGURE 20.

This completes the proof. □

Next, we show Proposition 25.

*Proof of Proposition 25.* By gluing a disk to $S_1$ along $\delta'$, we obtain $\Sigma^1_{n-1}$ so that the curves $c'_{2n-1}$ and $c''_{2n-1}$ are isotopic to $c_{2n-1}$ in Figure 2. Therefore, since $h_\delta$ is the image of $H_\delta H_{\delta'}$ under the map $\Gamma^1_{n-1} \to \Gamma_{n-1}$ induced by the inclusion $\Sigma^1_{n-1} \to \Sigma_{n-1}$, from Proposition 41, we obtain

$$t_{c_{2n-2}}\cdots t_{c_3}t_{c_2}t_{c_1'}t_{c_1''}t_{c_2}t_{c_3}\cdots t_{c_{2n-2}} = h_\delta t^{-2}_{c_{2n-1}},$$

and thus

$$t_{c_{2n-1}}t_{c_{2n-2}}\cdots t_{c_3}t_{c_2}t_{c_1'}t_{c_1''}t_{c_2}t_{c_3}\cdots t_{c_{2n-2}}t_{c_{2n-1}} = h_\delta.$$

This finishes the proof. □

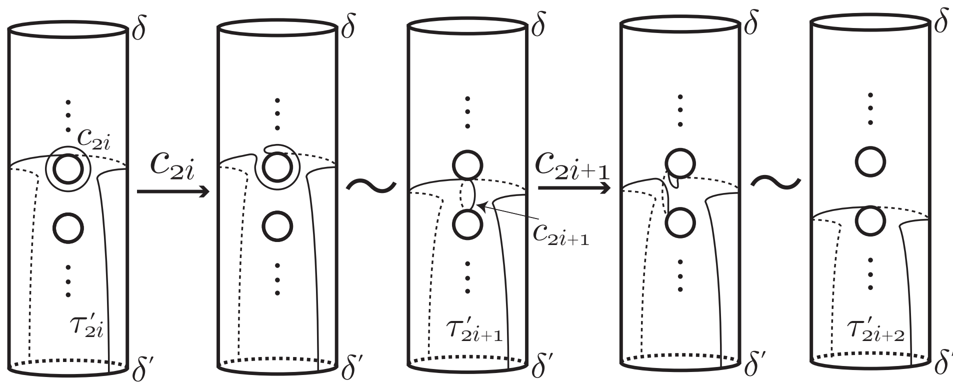


FIGURE 21.

7.2. **Proof of Proposition 26.** This subsection gives the proof of Proposition 26.

Let $S_2$ be the subsurface of $\Sigma^1_{2h+n-1}$ of genus $2h$ with two boundary curves $c'_{2n-1}$ and $c''_{2n-1}$ as in Figure 3, and let $\Gamma(S_2)$ be the mapping class group of $S_2$. We denote by $H'_{\delta'}$ the half twist about $\delta'$, hence $t_{\delta'} = (H'_{\delta'})^2$, satisfying $H'_{\delta'}(a_j) = a_{4h+2-j}$, where $a_j$ and $\delta'$ are the simple closed curves on $S_2(\subset \Sigma^1_{2h+n-1})$ as in Figure 22. We first show the following proposition.

**Proposition 42.** *The following relation holds in $\Gamma(S_2)$:*

$$t_{D_0}t_{D_1}t_{D_2}\cdots t_{D_{2h}}t_{c_{2n-1}} = t_{c'_{2n-1}}t_{c''_{2n-1}}(H'_{\delta'})^{-1},$$

*where $D_0, D_1, \ldots, D_{2h}, c_{2n-1}$ are the simple closed curves on $S_2(\subset \Sigma^1_{2h+n-1})$ as in Figure 3.*

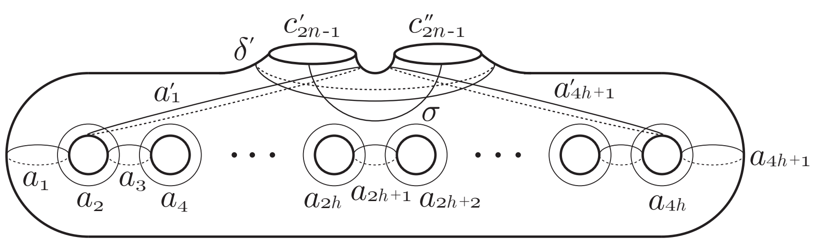


FIGURE 22.

*Proof.* We prove the equation using the *Alexander method*. Let $\sigma$ be the arc on $S_2$ as in Figure 22, and let $a_1, a_2, \ldots, a_{4h+1}, a'_1, a'_{4h+1}$ be the simple closed curves on $S_2$ as in the figure. Then, the collection $\{a_1, a_2, \ldots, a_{4h}, a'_{4h+1}, \sigma\}$

"fills" $S_2$ and satisfies the assumptions of the Alexander method (see for example [24]). Therefore, since $t_{c'_{2n-1}}$ and $t_{c''_{2n-1}}$ are in the center of $\Gamma(S_2)$, it suffices to show that

(i) $t_{D_0}t_{D_1}t_{D_2}\cdots t_{D_{2h}}t_{c_{2n-1}}(a_j) = a_{4h+2-j}$ for $j = 1, 2, \dots, 4h+1$,
(ii) $t_{D_0}t_{D_1}t_{D_2}\cdots t_{D_{2h}}t_{c_{2n-1}}(a'_{4h+1}) = (H'_{\delta'})^{-1}(a'_1)$, and
(iii) $t_{D_0}t_{D_1}t_{D_2}\cdots t_{D_{2h}}t_{c_{2n-1}}(\sigma) = t_{c'_{2n-1}}t_{c''_{2n-1}}(\sigma)$.

First, we show (i). It is easy to check that

$$t_{D_{2i-1}}t_{D_{2i}}(a_{2i}) = a_{4h+2-2i} \qquad (j = 2i),$$
$$t_{D_{2i-2}}t_{D_{2i-1}}(a_{2i-1}) = a_{4h+2-(2i-1)} \qquad (j = 2i-1, j \neq 2h+1)$$

for $i = 1, 2, \dots, h$ from Figures 23, 24, 25 and 26. Moreover, we also see at once that

$$t_{D_{2h}}t_{c_{2n-1}}(a_{2h+1}) = a_{2h+1}$$

from Figure 27. We notice that $a_j$ and $a_{4h+2-j}$ are disjoint from $D_k$ and $c_{2n-1}$ for $i \neq 2h+1$ and $k \neq j-1, j$ and that $a_{2h+1}$ is disjoint from $D_k$ for $k \neq 2h$. Therefore, for $j = 1, 2, \dots, 4h+1$, we have

$$t_{D_0}t_{D_1}t_{D_2}\cdots t_{D_{2h}}t_{c_{2n-1}}(a_j) = a_{4h+2-j}.$$

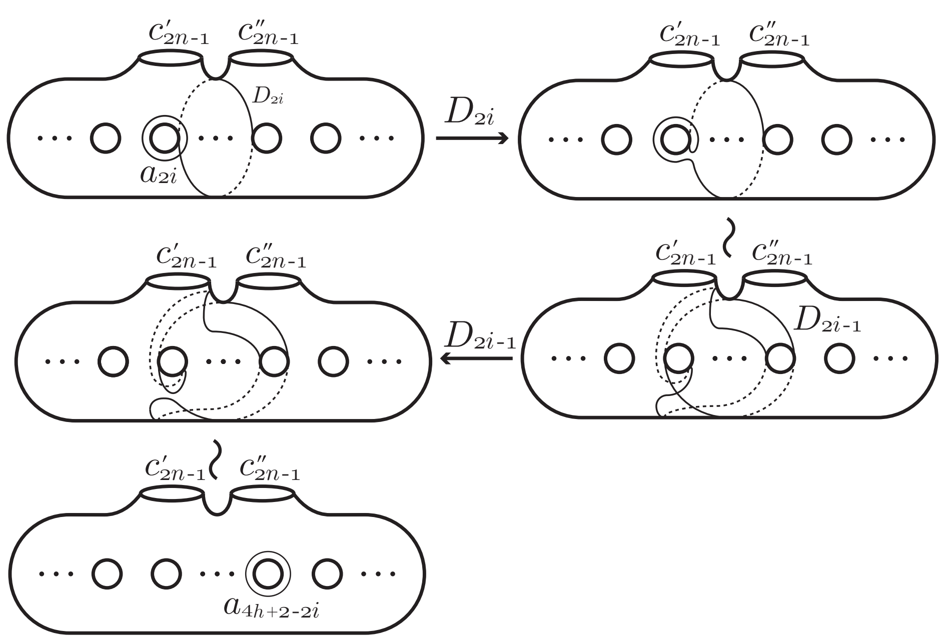


FIGURE 23.

Next, we show (ii). This immediately follows from Figures 28 and 29.

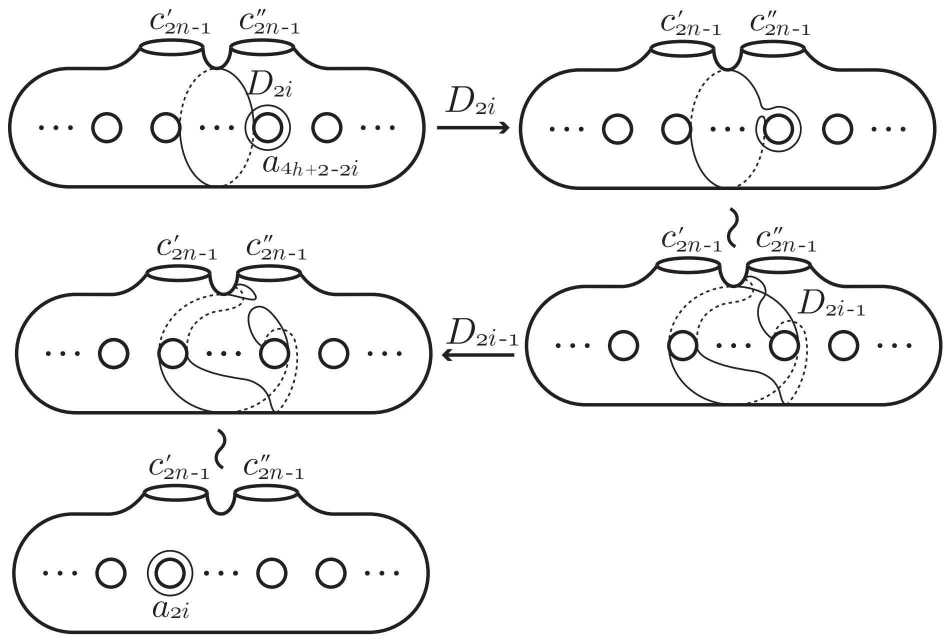


FIGURE 24.

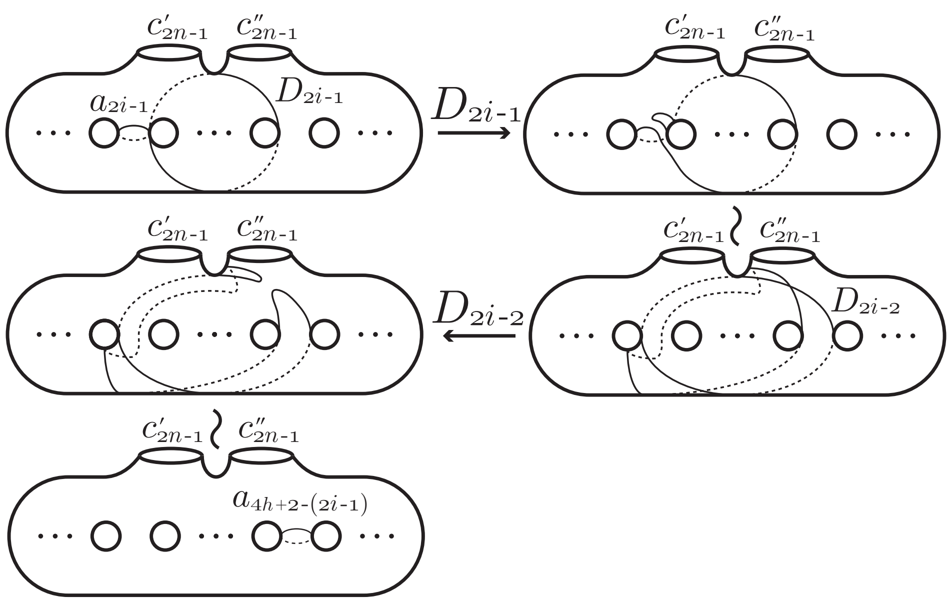


FIGURE 25.

Finally, we show (iii). Let $\sigma_1, \sigma_2, \dots, \sigma_h$ be the arcs on $S_2$ as in Figure 30. Note that

$$\sigma_h = t_{c_{2n-1}}(\sigma).$$

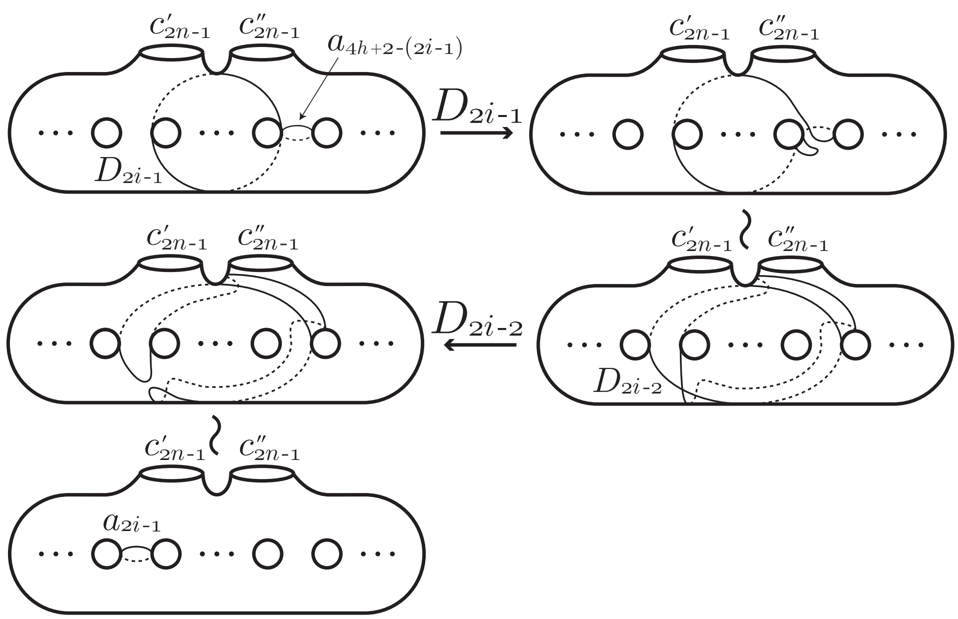


FIGURE 26.

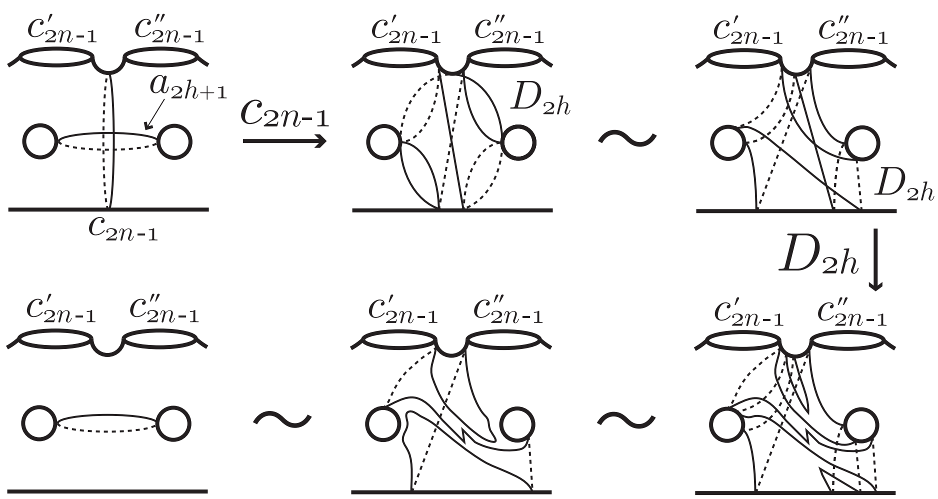


FIGURE 27.

It follows easily from Figures 31 and 32 that

$$t_{D_{2i-1}} t_{D_{2i}}(\sigma_i) = \sigma_{i-1},$$
$$t_{D_0}(\sigma_1) = t_{c'_{2n-1}} t_{c''_{2n-1}}(\sigma)$$

for $i = 2, \dots, h$. This gives

$$t_{D_0} t_{D_1} t_{D_2} \cdots t_{D_{2h}} t_{c_{2n-1}}(\sigma) = t_{c'_{2n-1}} t_{c''_{2n-1}}(\sigma).$$

This completes the proof. □

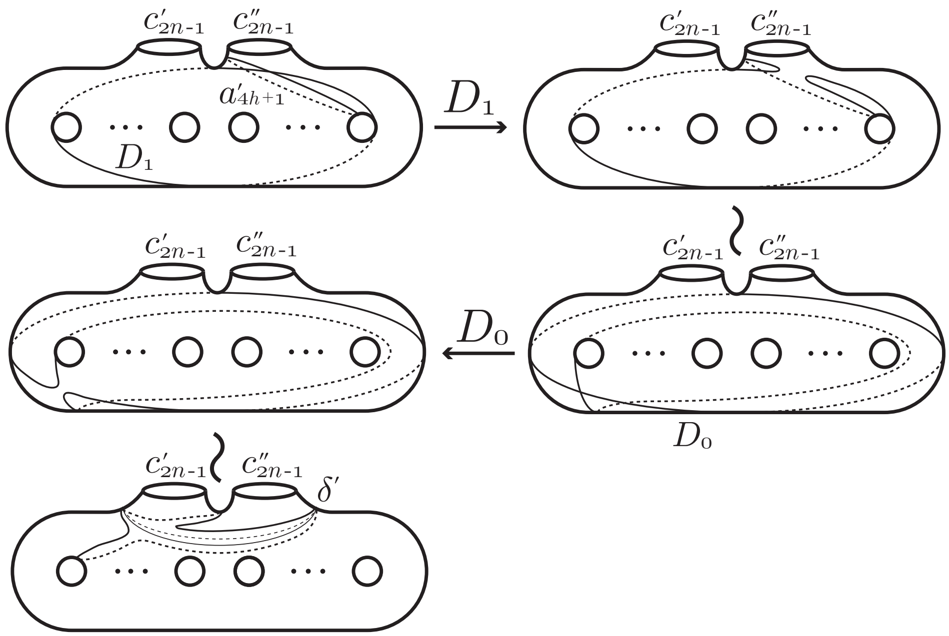


Figure 28.

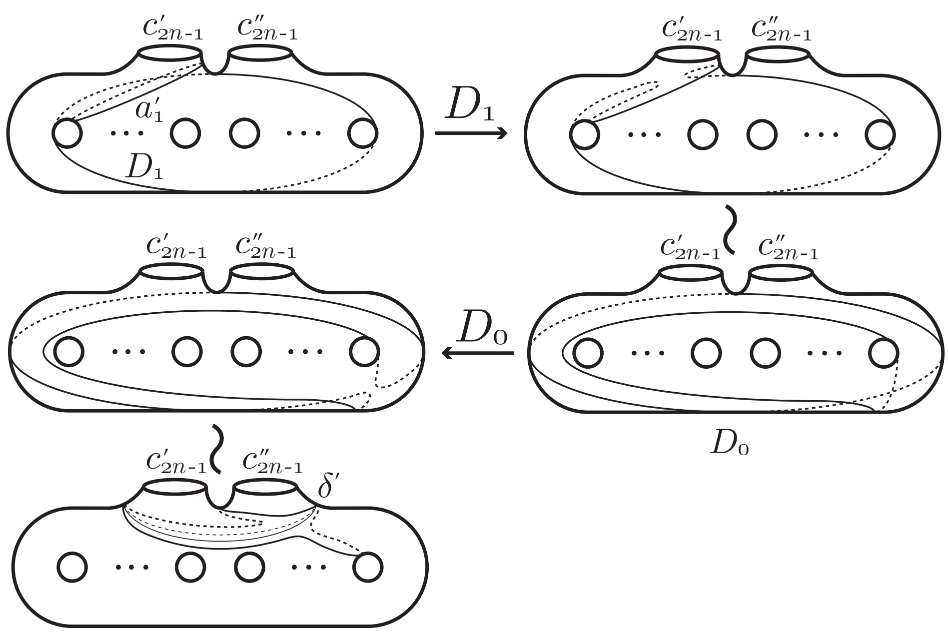


Figure 29.

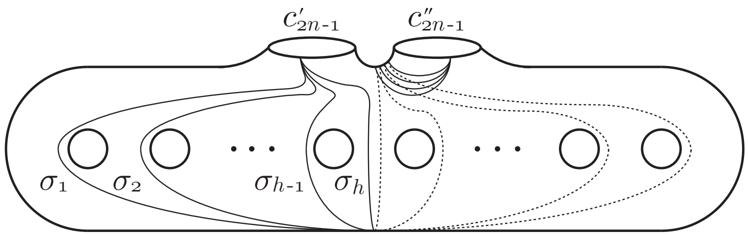


FIGURE 30.

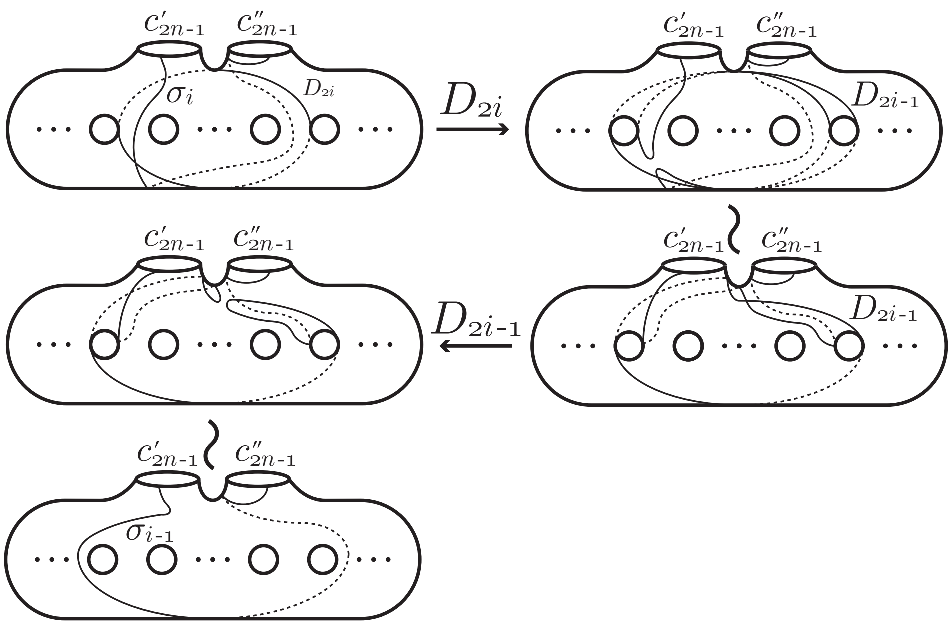


FIGURE 31.

Next, we show Proposition 26.

*Proof of Proposition 26.* It follows immediately from Propositions 41 and 42 that

$$t_{c_{2n-2}} \cdots t_{c_3} t_{c_2} t_{c'_1} t_{c''_1} t_{c_2} t_{c_3} \cdots t_{c_{2n-2}} t_{D_0} t_{D_1} t_{D_2} \cdots t_{D_{2h}} t_{c_{2n-1}} = h'_\delta,$$

and the proof is complete. □

## 8. Remarks on genus-2 Lefschetz fibrations

We give some remarks on genus-2 Lefschetz fibrations.

For abbreviation, a genus-2 Lefschetz fibration $f : X \to \mathbb{S}^2$ is said to be *of type* $(s_0, s)$ if $f$ has $s_0$ nonseparating and $s$ separating singular fibers. Then, we see that $e(X) = -4 + s_0 + s$ and $\sigma(X) = -\frac{3}{5}s_0 - \frac{1}{5}s$ using the signature formula for genus-2 Lefschetz fibrations given by Matsumoto [44] (which is generalized by Endo [22] to genus-$g$ hyperelliptic Lefschetz fibrations). Then, it is easy to check that $c_1^2(X) - 2\chi(X) + 6 = s \geq 0$, and hence every genus-2 Lefschetz fibration satisfies the Noether inequality $2\chi(X) - 6 \leq c_1^2(X)$.

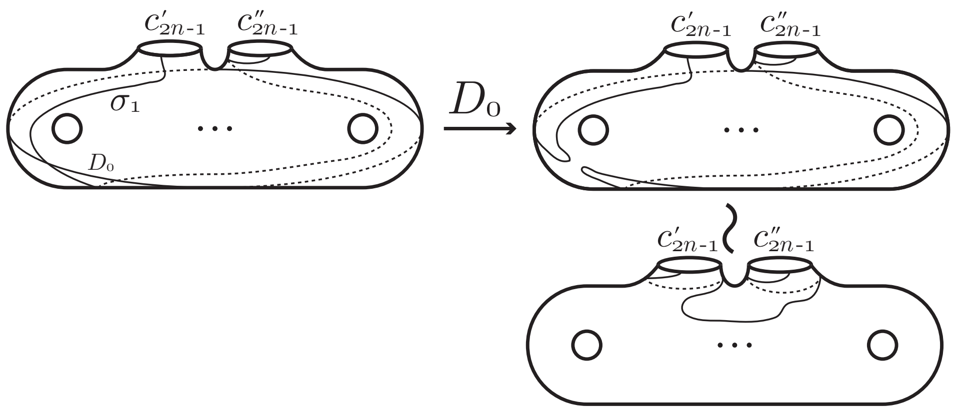


FIGURE 32.

All explicit examples of indecomposable genus-2 Lefschetz fibrations with minimal total spaces have been constructed by Xiao [57] as far as the authors know. Such examples are of types $(6, 7)$ and $(12, 19)$. In general, every genus-2 Lefschetz fibration of type $(s_0, 2s_0 - 5)$ is indecomposable (see [4]). From the arguments in [4], we find that the total space is minimal except for $s_0 = 4$. Moreover, we can show that the total space of a genus-2 Lefschetz fibration of type $(s_0, 2s_0 - 5)$ is not simply connected as follows (therefore the total spaces of the above-mentioned examples given by Xiao are not simply connected).

**Proposition 43.** *If $X$ admits a genus-2 Lefschetz fibration of type $(s_0, 2s_0 - 5)$, then $b_1(X) \geq 2$.*

*Proof.* From the assumption that $s = 2s_0 - 5$, we have

$$\begin{aligned} e(X) &= 3s_0 - 9 = 2 - 2b_1(X) + b_2^+(X) + b_2^-(X), \\ \sigma(X) &= -s_0 + 1 = b_2^+(X) - b_2^-(X). \end{aligned}$$

This gives $2 - 2b_1(X) + 2b_2^-(X) = 4s_0 - 10$. Here, by $b_2^-(Y) \geq s + 1$ for a genus-2 Lefschetz fibration $Y \to \mathbb{S}^2$ of type $(s_0, s)$ (see, for example, Lemma 2.4 in [40]) and the assumption that $s = 2s_0 - 5$, we have $b_1(X) \geq 2$. ☐

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