UPPER BOUNDS ON THE SLOPE OF CERTAIN FIBERED SURFACES

MAKOTO ENOKIZONO

Abstract. We give an upper bound of the slope for finite cyclic covering fibrations of an elliptic surface which includes bielliptic fibrations. We also give an upper bound of the slope for triple cyclic covering fibrations of a ruled surface and hyperelliptic fibrations, which provides a new proof of Xiao’s upper bound.

Introduction

We work over the field of complex numbers $\mathbb{C}$. Let $S$ be a smooth projective surface, $B$ a smooth projective curve and $f : S \to B$ a relatively minimal fibration of curves of genus $g \geq 2$. Let $K_f$ denote the relative canonical bundle $K_S - f^*K_B$. Consider the following relative invariants:

\[ K_f^2 = K_S^2 - \deg K_F \deg K_B, \]
\[ \chi_f := \chi(O_S) - \chi(O_B)\chi(O_F), \]
\[ e_f := e(S) - e(B)e(F), \]

where $F$ denotes a general fiber of $f$ and $e(M)$ the topological Euler characteristic of $M$. Then the following facts are well known:

- (Noether) $12\chi_f = K_f^2 + e_f$.
- (Arakelov) $K_f$ is nef.
- (Ueno) $\chi_f \geq 0$, and $\chi_f = 0$ if and only if $f$ is locally trivial (i.e., a holomorphic fiber bundle).
- (Zeuthen–Segre) $e_f \geq 0$, and $e_f = 0$ if and only if $f$ is smooth.

Assume that $f$ is not locally trivial. We put

\[ \lambda_f := \frac{K_f^2}{\chi_f} \]

and call it the slope of $f$. Then one sees $0 < \lambda_f \leq 12$ from the above results. In 1987, Xiao showed in [18] the inequality

\[ \lambda_f \geq 4 - \frac{4}{g}. \]

This inequality is sharp and it is well known that the slope attains the lower bound only if the fibration $f$ is hyperelliptic, i.e., a general fiber $F$ is a hyperelliptic curve ([13] and [18]).
As to the upper bound, Kodaira [11] constructed examples of fibrations with slope 12, which are nowadays called Kodaira fibrations. Thus the inequality $\lambda_f \leq 12$ is sharp among all fibered surfaces. On the other hand, Matsusaka [16] obtained an upper bound smaller than 12 for hyperelliptic case and Xiao [20] improved this bound. In [5], upper bounds for genus 3 fibrations are studied from another point of view.

We studied in [7] primitive cyclic covering fibrations of type $(g, h, n)$. Roughly speaking, it is a fibration of genus $g$ over a curve obtained as the relatively minimal model of an $n$-sheeted cyclic branched covering of another fibration of genus $h$ (for the precise definition, see Definition 1.1). Here the genus of a fibration means that of general fibers and the term “primitive” means that the branch locus of the cyclic covering of degree $n$ is divisible by $n$ in the Picard group. Note that hyperelliptic fibrations are nothing more than such fibrations of type $(g, 0, 2)$ and that bielliptic fibrations of genus $g \geq 6$ are those of type $(g, 1, 2)$ (cf. [3], [6]). Here, a fibration is called bielliptic if a general fiber is a bielliptic curve, i.e., a non-singular projective curve obtained as a double covering of an elliptic curve. In [7], we established the lower bound of the slope for such fibrations of type $(g, h, n)$ extending former results for $n = 2$ in [2] and [6]. Furthermore, when $h = 0$ and $n \geq 4$, we obtained even the upper bound (expressed as a function in $g$ and $n$) which is strictly smaller than 12. Recall that known examples of Kodaira fibrations, including Kodaira’s original ones [11], are presented as primitive cyclic covering fibrations with $h \geq 2$ and, in fact, there exist such fibrations for any $h \geq 2$ (see, [4] and [10]). Hence, as far as the upper bound of the slope strictly smaller than 12 concerns, the remaining cases to be examined are $h = 0$, $n = 3$ and $h = 1$, $n \geq 2$.

The purpose of the present article is to give an affirmative answer to the above mentioned upper bound problem by introducing numerical invariants attached to the singularities of the branch locus of the cyclic covering, and improving a coarser estimates in [7]. When $h = 0$, a vertical component of the branch locus on a relatively minimal model is always a non-singular rational curve and this fact makes it much easier to handle singularities on the branch locus. On the other hand, when $h \geq 1$, we must pay attention to all subcurves of fibers and their singularities in a fibration of genus $h$, which seems quite terrible. Fortunately enough, when $h = 1$, we have Kodaira’s classification of singular fibers [12] from which we know that major components are rational curves and singularities are mild. This gives us a hope to extend results for $h = 0$ to fibrations of type $(g, 1, n)$. In fact, we can show the following:

**Theorem 0.1** (Theorem 2.3). Assume that $g \geq (2n - 1)(3n - 1)/(n + 1)$. Then, there exists a function $\text{Ind} : \mathcal{A}_{g, 1, n} \to \mathbb{Q}_{\geq 0}$ from the set $\mathcal{A}_{g, 1, n}$ of all fiber germs of primitive cyclic covering fibrations of type $(g, 1, n)$ such that $\text{Ind}(F_p) = 0$ for a general $p \in B$ and

$$K_f^2 = \frac{12(n - 1)}{2n - 1} \chi_f + \sum_{p \in B} \text{Ind}(F_p)$$

for any primitive cyclic covering fibration $f : S \to B$ of type $(g, 1, n)$.  

---

2
Theorem 0.2 (Theorem 3.1). Let $f : S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$. Then

$$\lambda_f \leq 12 - \begin{cases} 
\frac{6n^2}{(n+1)(g-1)}, & \text{if } n \geq 4, \text{ or } n = 3 \text{ and } g = 4 \\
\frac{24}{4g-17}, & \text{if } n = 3 \text{ and } g > 4, \\
\frac{2}{g-2}, & \text{if } n = 2 \text{ and } g \geq 3.
\end{cases}$$

In particular, we have the slope equality and the upper bound of the slope for bielliptic fibrations. We remark that the upper bounds in Theorem 0.2 are “fiberwise” sharp as we shall see in Example 3.4. We do not know, however, whether there exist primitive cyclic covering fibrations of type $(g, 1, n)$ whose slopes attain the bounds.

It remains to investigate the case that $h = 0$ and $n \leq 3$. The above method also applies to this case and we obtain the following:

Theorem 0.3 (Theorem 4.1). Let $f : S \to B$ be a primitive cyclic covering fibration of type $(g, 0, n)$.

1. If $g = 4$ and $n = 3$, then $\lambda_f \leq 129/17$ with the equality sign holding if and only if any singular fiber of $f$ is a triple fiber.
2. Assume that $n = 3$. If either $g > 4$, or $g = 4$ and $f$ has no triple fibers, then

$$\lambda_f \leq 12 - \frac{72(g+1)}{4g^2 + g + 13 - 36\delta},$$

where $\delta = 0$ if $g + 2 \in 6\mathbb{Z}$ and $\delta = 1$ otherwise.
3. If $n = 2$, then

$$\lambda_f \leq 12 - \frac{4(2g+1)}{g^2 - 1 + \delta} = 12 - \frac{4(2g+1)}{g^2 + \frac{(-1)^{g-1} - 1}{2}},$$

where $\delta = 0$ if $g$ is odd and $\delta = 1$ if $g$ is even, i.e., $2\delta = 1 + (-1)^g$.

We obtained the complete classification of singular fibers of primitive cyclic covering fibrations of type $(4, 0, 3)$ in [8] and computed some local invariants for each singular fiber. One may use it to reprove Theorem 0.3 (1). Note also that Theorem 0.3 (3) provides a new proof of Xiao’s upper bound for hyperelliptic fibrations in [20] referred above. It is shown in [15] that the inequality is optimal for given $g$.

The rough strategy of the proof of Theorems 0.2 and 0.3 is as follows: Let $f : S \to B$ be a primitive cyclic covering fibrations of type $(g, 1, n)$ and assume $h = 1$ or 0. By definition, there exist an elliptic surface ($h = 1$ case) or a ruled surface ($h = 0$ case) $\tilde{W} \to B$ and a cyclic covering $\tilde{S} \to \tilde{W}$ of degree $n$ with the branch locus $\tilde{R}$ such that $f$ is the relatively minimal model of the composition $\tilde{S} \to \tilde{W} \to B$. We take a birational morphism $\tilde{\psi} : \tilde{W} \to W$ over $B$ such that the fibered surface $\varphi : W \to B$ is relatively minimal and put $R := \tilde{\psi}_*\tilde{R}$. Then the induced morphism $\tilde{R} \to R$ can be viewed as the resolution of singularities. The
singularities of $R$ (and the fiber of $\varphi$ when $h = 1$) over a point $p \in B$ determine the fiber germ $F_p = f^{-1}(p)$ of $f$. Moreover, we can see (because $h = 1$ or 0) that the invariants $K_f^2$, $\chi_f$ and $e_f$ are localized at the singular fiber germs $F_p$ as

$$K_f^2 = \sum_{p \in B} K_f^2(F_p), \quad \chi_f = \sum_{p \in B} \chi_f(F_p), \quad e_f = \sum_{p \in B} e_f(F_p),$$

where the local invariants $K_f^2(F_p)$, $\chi_f(F_p)$ and $e_f(F_p)$ are defined by the singularities of $R$ and the fiber of $\varphi$ over $p \in B$ and zero for all but finite points $p \in B$. Thus in order to find an upper bound of the slope of $f$, it is sufficient to find the upper bound of the “local slope” $K_f^2(F_p)/\chi_f(F_p)$ for arbitrary fiber germ $F_p$ of primitive cyclic covering fibrations of type $(g,h,n)$. The main part of the proof is to find a maximal number $\mu > 0$ such that $(12 - \mu)\chi_f(F_p) - K_f^2(F_p)$ is non-negative for any fiber germ $F_p$ by analyzing all possible singularities appearing in the branch locus $R$.

The organization of the paper is as follows. In §1, we recall basic results from [7] and [8] on primitive cyclic covering fibrations and introduce some notation for the later use. In §2, we observe the local concentration of relative invariants of primitive cyclic covering fibrations of type $(g,1,n)$ on a finite number of fiber germs and show Theorem 0.1. §3 will be devoted to the proof of Theorem 0.2. In the course of the study, we freely use Kodaira’s table of singular fibers of elliptic surfaces. Finally in §4, we show Theorem 0.3.

Acknowledgment. The author expresses his sincere gratitude to Professor Kazuhiro Konno for many suggestions and warm encouragement.

1. Preliminaries

In this section, we recall and state basic results for primitive cyclic covering fibrations in [7].

1.1. Definition and the setting.

Definition 1.1. A relatively minimal fibration $f: S \to B$ of genus $g \geq 2$ is called a primitive cyclic covering fibration of type $(g,h,n)$, if there exist a (not necessarily relatively minimal) fibration $\tilde{\varphi}: \tilde{W} \to B$ of genus $h \geq 0$, and an $n$ sheeted cyclic covering

$$\tilde{\theta}: \tilde{S} = \text{Spec}_{\tilde{W}} \left( \bigoplus_{j=0}^{n-1} \mathcal{O}_{\tilde{W}}(-j\tilde{d}) \right) \to \tilde{W}$$

branched over a smooth (possibly reducible) curve $\tilde{R} \in \lfloor n\tilde{d} \rfloor$ and a line bundle $\tilde{d}$ on $\tilde{W}$ such that $f$ is the relatively minimal model of $\tilde{f} := \tilde{\varphi} \circ \tilde{\theta}$.

Let $f: S \to B$ be a primitive cyclic covering fibration of type $(g,h,n)$. Throughout the paper, we will continue to use the notation given in Definition 1.1. Let $\tilde{F}$ and $\tilde{R}$ be general fibers of $\tilde{f}$ and $\tilde{\varphi}$, respectively. Then the restriction map $\tilde{\theta}|_{\tilde{F}}: \tilde{F} \to \tilde{R}$ is an $n$ sheeted cyclic
covering branched over \( \tilde{R} \cap \tilde{\Gamma} \). Since the genera of \( \tilde{F} \) and \( \tilde{\Gamma} \) are \( g \) and \( h \), respectively, the Hurwitz formula gives us

\[
(1.1) \quad r := \frac{2(g - 1 - n(h - 1))}{n - 1}.
\]

Since \( \tilde{R} \) is divisible by \( n \), the number \( r \) is a multiple of \( n \). Let \( \bar{\sigma} \) be a generator of \( \text{Aut}(\tilde{S}/\tilde{W}) \simeq \mathbb{Z}/n\mathbb{Z} \) and \( \rho: \tilde{S} \to S \) the natural birational morphism. By assumption, the fixed point locus \( \text{Fix}(\bar{\sigma}) \) is a disjoint union of smooth curves and \( \theta(\text{Fix}(\bar{\sigma})) = \tilde{R} \). Let \( \varphi: W \to B \) be a relatively minimal model of \( \bar{\sigma} \) and \( \tilde{\psi}: \tilde{W} \to W \) the natural birational morphism. Since \( \tilde{\psi} \) is a succession of blow-ups, we can write \( \tilde{\psi} = \psi_1 \circ \cdots \circ \psi_N \), where \( \psi_i: W_i \to W_{i-1} \) denotes the blow-up at \( x_i \in W_{i-1} \) (\( i = 1, \ldots, N \)) with \( W_0 = W \) and \( W_N = \tilde{W} \). We define reduced curves \( R_i \) on \( W_i \) inductively as \( R_{i-1} = (\psi_i)_* R_i \) starting from \( R_N = \tilde{R} \) down to \( R_0 =: R \). We also put \( E_i = \psi_i^{-1}(x_i) \) and \( m_i = \text{mult}_{x_i}(R_{i-1}) \) for \( i = 1, 2, \ldots, N \).

1.2. Resolution process and invariants.

Lemma 1.2 ([7] Lemma 1.5). With the above notation, the following hold for any \( i = 1, \ldots, N \).

(1) Either \( m_i \in n\mathbb{Z} \) or \( m_i \in n\mathbb{Z} + 1 \). Moreover, \( m_i \in n\mathbb{Z} \) holds if and only if \( E_i \) is not contained in \( R_i \).

(2) \( R_i = \psi_i^* R_{i-1} - n\left[ \frac{m_i}{n} \right] E_i \), where \( [t] \) is the greatest integer not exceeding \( t \).

(3) There exists \( \mathfrak{d}_i \in \text{Pic}(W_i) \) such that \( \mathfrak{d}_i = \psi_i^* \mathfrak{d}_{i-1} - \left[ \frac{m_i}{n} \right] E_i \) and \( R_i \sim n\mathfrak{d}_i \), \( \mathfrak{d}_N = \mathfrak{d} \).

Let \( E \) be a \((-1)\)-curve on a fiber of \( \tilde{f} \). If \( E \) is not contained in \( \text{Fix}(\bar{\sigma}) \), then \( L := \tilde{\theta}(E) \) is a \((-1)\)-curve and \( \tilde{\theta}^* L \) is the sum of \( n \) disjoint \((-1)\)-curves containing \( E \). Contracting them and \( L \), we may assume that any \((-1)\)-curve on a fiber of \( \tilde{f} \) is contained in \( \text{Fix}(\bar{\sigma}) \). Then \( \bar{\sigma} \) induces an automorphism \( \sigma \) of \( S \) over \( B \) and \( \rho \) is the blow-up of all isolated fixed points of \( \sigma \) ([7] Lemma 1.9). One sees easily that there is a one-to-one correspondence between
\((-k\)-curves contained in \(\text{Fix}(\tilde{\sigma})\) and \((-kn\)-curves contained in \(\tilde{R}\) via \(\tilde{\theta}\). Hence, the number of blow-ups in \(\rho\) is that of vertical \((-n\)-curves contained in \(\tilde{R}\).

From Lemma 1.2, we have

\[
K_{\tilde{R}} = \tilde{\psi}^* K_{\varphi} + \sum_{i=1}^{N} E_i, \tag{1.2}
\]

\[
\tilde{d} = \tilde{\psi}^* d - \sum_{i=1}^{N} \left[ \frac{m_i}{n} \right] E_i, \tag{1.3}
\]

where \(E_i\) denotes the total transform of \(E_i\). Since

\[
K_{\tilde{S}} = \tilde{\theta}^* (K_{\tilde{W}} + (n-1)\tilde{d})
\]

and

\[
\chi(O_{\tilde{S}}) = n\chi(O_{\tilde{W}}) + \frac{1}{2} \sum_{j=1}^{n-1} j\tilde{d}(j\tilde{d} + K_{\tilde{W}}),
\]

we get

\[
K_f^2 = n(K_{\varphi}^2 + 2(n-1)K_{\tilde{\varphi}}\tilde{d} + (n-1)^2\tilde{d}^2), \tag{1.4}
\]

\[
\chi_f = n\chi_{\tilde{\varphi}} + \frac{1}{2} \sum_{j=1}^{n-1} j\tilde{d}(j\tilde{d} + K_{\tilde{\varphi}}). \tag{1.5}
\]

1.3. Singularity indices. Now we introduce some numerical invariants for fiber germs \(F_p = f^{-1}(p)\) by using the data of the resolution process of \(\tilde{R} \to R\) over \(p \in B\):

**Definition 1.3 (Singularity indices \(\alpha\) and \(\varepsilon\)).** (1) Let \(k\) be a positive integer. For \(p \in B\), we consider all the singular points (including infinitely near ones) of \(R\) over \(p\). We let \(\alpha_k(F_p)\) be the number of singular points of multiplicity either \(kn\) or \(kn + 1\) among them, and call it the \(k\)-th singularity index of \(F_p\), the fiber of \(f : S \to B\) over \(p\). Clearly, we have \(\alpha_k(F_p) = 0\) except for a finite number of \(p \in B\). We put \(\alpha_k := \sum_{p \in B} \alpha(F_p)\) and call it the \(k\)-th singularity index of \(f\).

(2) Let \(D_1\) be the sum of all \(\tilde{\varphi}\)-vertical \((-n\)-curves contained in \(\tilde{R}\) and put \(\tilde{R}_0 := \tilde{R} - D_1\). We denote by \(\alpha_0(F_p)\) the ramification index of \(\tilde{\varphi}|_{\tilde{R}_0} : \tilde{R}_0 \to B\) over \(p\), that is, the ramification index of \(\tilde{\varphi}|_{(\tilde{R}_0)_h} : (\tilde{R}_0)_h \to B\) over \(p\) minus the sum of the topological Euler number of irreducible components of \((\tilde{R}_0)_v\) over \(p\), where \(\tilde{R}_0 = (\tilde{R}_0)_h + (\tilde{R}_0)_v\) is the unique decomposition to the horizontal part and vertical part over \(B\). Then \(\alpha_0(F_p) = 0\) except for a finite number of \(p \in B\), and we have

\[
\sum_{p \in B} \alpha_0(F_p) = (K_{\tilde{\varphi}} + \tilde{R}_0)\tilde{R}_0
\]

by definition. We put \(\alpha_0 := \sum_{p \in B} \alpha(F_p)\) and call it the 0-th singularity index of \(f\).
(3) Let $\varepsilon(F_p)$ be the number of $(-1)$-curves contained in $\widetilde{F}_p$, and put $\varepsilon := \sum_{p \in B} \varepsilon(F_p)$. This is no more than the number of blowing-ups appearing in $\rho: \widetilde{S} \to S$.

From (1.2) and (1.3), we have
\[
(K_\varphi + R)\widetilde{R} = \left(\tilde{\psi}^*(K_\varphi + R) + \sum_{i=1}^{N} \left(1 - n \left\lceil \frac{m_i}{n} \right\rceil \right) E_i \right) \left(\tilde{\psi}^* R - \sum_{i=1}^{N} n \left\lfloor \frac{m_i}{n} \right\rfloor E_i \right)
\]
\[
= (K_\varphi + R)R - \sum_{i=1}^{N} n \left\lfloor \frac{m_i}{n} \right\rfloor \left(n \left\lfloor \frac{m_i}{n} \right\rfloor - 1\right)
\]
\[
= (K_\varphi + R)R - n \sum_{k \geq 1} k(nk - 1)\alpha_k.
\]
(1.6)

On the other hand, it follows from $\tilde{R}_0 D_1 = 0$ and the definition of $\alpha_0$ and $\varepsilon$ that
\[
(K_\varphi + \tilde{R})\tilde{R} = (K_\varphi + \tilde{R}_0)\tilde{R}_0 + D_1(K_\varphi + D_1) = \alpha_0 - 2\varepsilon.
\]
Hence we have
\[
(K_\varphi + R)R = n \sum_{k \geq 1} k(nk - 1)\alpha_k + \alpha_0 - 2\varepsilon
\]
by (1.6) and (1.7). Since $K_\varphi^2 = K_\varphi^2 + \varepsilon$, $\chi_\varphi = \chi_f$, (1.2), (1.3), (1.4) and (1.5), we get
\[
K_\varphi^2 = nK_\varphi^2 + 2(n - 1)K_\varphi R + \frac{(n - 1)^2}{n} R^2 - \sum_{k \geq 1} ((n - 1)k - 1)^2 \alpha_k + \varepsilon
\]
and
\[
\chi_f = n\chi_\varphi + \frac{(n - 1)(2n - 1)}{12n} R^2 + \frac{n - 1}{4} K_\varphi R - \frac{n(n - 1)}{12} \sum_{k \geq 1} ((2n - 1)k^2 - 3k) \alpha_k.
\]
(1.9) (1.10)

From (1.8), (1.9), (1.10) and Noether’s formula, we have
\[
\chi_f = n\chi_\varphi + \frac{(n - 1)(2n - 1)}{12n} R^2 + \frac{n - 1}{4} K_\varphi R - \frac{n(n - 1)}{12} \sum_{k \geq 1} ((2n - 1)k^2 - 3k) \alpha_k.
\]
(1.11)

1.4. Decomposition of the branch locus and more indices. We define some notation for the later use. For an effective vertical divisor $T$ on a fibered surface over $B$ and a point $p \in B$, we denote by $T(p)$ the greatest subdivisor of $T$ consisting of components of the fiber over $p$. Then we can write $T = \sum_{p \in B} T(p)$.

Let $\tilde{R} = \tilde{R}_h + \tilde{R}_v$ be the decomposition to the horizontal and vertical parts over $B$. We consider subdivisors $D = \sum_i L^i$ of $\tilde{R}_v(p)$ which satisfy the following three conditions:

(i) $L^1$ is the proper transform of an irreducible component $\Gamma^1$ of the fiber $\Gamma_p = \varphi^{-1}(p)$ or that of an exceptional $(-1)$-curve $E^1$ appearing in $\tilde{\psi}$. We define $C^1$ to be $E^1$ or $\Gamma^1$ according to whether $L^1$ is the proper transform of which curve.
(ii) For $i \geq 2$, $L^i$ is the proper transform of an irreducible component $\Gamma^i$ of $\Gamma_p$ intersecting $C^k$ for some $k < i$ or that of an exceptional $(-1)$-curve $E^j$ appearing in $\psi$ the center of which is on $(\text{the proper transform of})$ $C^k$ for some $k < i$, where we inductively define $\Gamma^i$ to be $E^j$ or $\Gamma^i$ according to whether $L^i$ is the proper transform of which curve.

(iii) $D = \sum_i L^i$ is maximal among subdivisors satisfying (i) and (ii).

Then $\widetilde{R_v}(p)$ can be decomposed into the sum of such subdivisors uniquely. We denote it as

$$\widetilde{R_v}(p) = D^1(p) + \cdots + D^{\eta_p}(p), \quad D^{r}(p) = \sum_{k \geq 1} L^{r,k}$$

where each $D^{r}(p) = \sum_{k \geq 1} L^{r,k}$ satisfies (i), (ii), (iii). Let $C^{t,k}$ be the exceptional curve or the component of the fiber $\Gamma_p$ the proper transform of which is $L^{r,k}$. We moreover decompose $D^{r}(p)$ as

$$D^{r}(p) = D^{m}(p) + D^{n}(p),$$

where $D^{m}(p)$ (resp. $D^{n}(p)$) is the subdivisor consisting of components of $D^{r}(p)$ which are not $\tilde{\psi}$-exceptional (resp. are $\tilde{\psi}$-exceptional). Let $\eta_p^m$ be the number of $t = 1, \ldots, \eta_p$ such that $D^{m}(p)$ is $\psi$-exceptional (i.e., $D^{m}(p) = 0$) and put $\eta_p^r := \eta_p - \eta_p^m$.

Let us introduce more numerical invariants of fiber germs $F_p$ by counting specific curves in $\widetilde{R_v}(p)$:

**Definition 1.4 (Index $j$).** (1) For integers $m \geq 0$ and $a > 0$, let $j_{m,a}(F_p)$ (resp. $j^t_{m,a}(F_p)$, $j^m_{m,a}(F_p)$, $j^m_{m,a}(F_p)$) be the number of irreducible curves of genus $m$ with self-intersection number $-an$ contained in $\widetilde{R_v}(p)$ (resp. $D^{r}(p)$, $D^{m}(p)$, $D^{n}(p)$). We put

$$j^a_{m,a}(F_p) := \sum_{m \geq 0} j^a_{m,a}(F_p), \quad j^a_{m,\bullet}(F_p) := \sum_{a \geq 0} j^a_{m,a}(F_p), \quad j_{m,a}(F_p) := \sum_{t=1}^{\eta_p^r} j^{t}_{m,a}(F_p).$$

Similarly, we define $j^i(F_p) = j^i_{\bullet,\bullet}(F_p)$, $j^m_{\bullet,\bullet}(F_p)$, $j^m_{\bullet,a}(F_p)$, etc. Note that $j^m_{\bullet,\bullet}(F_p) = 0$ holds for any $m \geq 1$ since all irreducible $\tilde{\psi}$-exceptional curves are rational. Rearranging the index if necessary, we may assume that

$$D^{n}(p) = \sum_{k=1}^{j^m(F_p)} L^{t,k}, \quad D^{m}(p) = \sum_{k=j^m(F_p)+1}^{j^r(F_p)} L^{t,k}.$$

Then we put

$$L^{t,k} := L^{t,k}, \quad L^{t,k} := L^{t,j(F_p)+k}, \quad C^{m,k} := C^{t,k}, \quad C^{m,k} := C^{t,j(F_p)+k}.$$

(2) Let $\alpha^+(F_p)$ be the ramification index of $\tilde{\varphi} : \tilde{R}_h \to B$ over $p$ and put

$$\alpha^-_0(F_p) := \alpha_0(F_p) - \alpha^+_0(F_p).$$

By definition, we have $\varepsilon(F_p) = j_{0,1}(F_p)$ and

$$\alpha^-_0(F_p) = \sum_{m \geq 0} (2m - 2) j_{m,\bullet}(F_p) + 2\varepsilon(F_p).$$
(3) Let $\eta_p$ be the number of $t=1, \ldots, \eta_p$ such that $D^t(p)$ consists of $\tilde{\psi}$-exceptional $(-n)$-curves, that is, $j^t(F_p) = j^\eta_{p,1}(F_p)$. Clearly we have $\eta_p \leq \eta''_p$. We put $\bar{\eta}_p := \eta''_p - \eta_p$.

**Definition 1.5** (Vertical type singularity and indices $t$ and $\kappa$). (1) Let $x$ be a (possibly infinitely near) singular point of $R$. For $t=1, \ldots, \eta_p$ and $u \geq 1$, $x$ is said to be a $(t, u)$-vertical type singularity or simply a $u$-vertical type singularity if the number of $C^{t, k}$'s whose proper transforms pass through $x$ is $u$. If $x$ is a $(t, u)$-vertical type singularity and the multiplicity of it belongs to $n \mathbb{Z}$ (resp. $n \mathbb{Z} + 1$), we call it a $(t, u)$-vertical $n \mathbb{Z}$ type singularity (resp. $(t, u)$-vertical $n \mathbb{Z} + 1$ type singularity).

(2) Let $\iota^{t, (u)}(F_p)$, $\kappa^{t, (u)}(F_p)$ respectively be the number of $(t, u)$-vertical $n \mathbb{Z}$, $n \mathbb{Z} + 1$ type singularities over $p$ and put

$$
\iota^t(F_p) := \sum_{u \geq 1} (u-1) \iota^{t, (u)}(F_p), \quad \iota(F_p) := \sum_{t=1}^{\eta_p} \iota^t(F_p),
$$

$$
\kappa^t(F_p) := \sum_{u \geq 1} (u-1) \kappa^{t, (u)}(F_p), \quad \kappa(F_p) := \sum_{t=1}^{\eta_p} \kappa^t(F_p).
$$

Let $\iota^{t, (u)}_k(F_p)$, $\kappa^{t, (u)}_k(F_p)$ respectively be the number of $(t, u)$-vertical type singularities with multiplicity $kn$, $kn + 1$ and we define $\iota^t_k(F_p)$, $\kappa^t_k(F_p)$ and $\kappa_k(F_p)$ similarly.

**Definition 1.6** (Indices $\alpha'$ and $\alpha''$). (1) We say that a singular point $x$ of $R$ is involved in $D^t(p)$ if there exists a component $L^{t, k}$ of $D^t(p)$ such that its birational image (that is, the proper transform of $C^{t, k}$) passes through $x$ or $C^{t, k}$ is the exceptional $(-1)$-curve by the blow-up at $x$. A singular point $x$ of $R$ is involved in $\tilde{R}_v(p)$ if it is involved in $D^t(p)$ for some $t$.

(2) Let $\alpha'_k(F_p)$ (resp. $\alpha''_k(F_p)$) denotes the number of singularities with multiplicity $kn$ or $kn + 1$ over $p$ not involved in $\tilde{R}_v(p)$ (resp. involved in $\tilde{R}_v(p)$). Let $\alpha''_k(F_p)$ denote the number of singularities with multiplicity $kn$ or $kn + 1$ over $p$ involved in $D^t(p)$. Then we have

$$
\alpha_k(F_p) = \alpha'_k(F_p) + \alpha''_k(F_p), \quad \alpha''_k(F_p) = \sum_{t=1}^{\eta_p} \alpha''_k(F_p)
$$

by the definition of the decomposition $\tilde{R}_v(p) = D^1(p) + \cdots + D^{\eta_p}(p)$.

(3) Let $\alpha^{n \mathbb{Z}}_k(F_p)$, $\alpha^{n \mathbb{Z}+1}_k(F_p)$ respectively denote the number of singularities with multiplicity $kn$, $kn + 1$ over $p$. Clearly we have

$$
\alpha_k(F_p) = \alpha^{n \mathbb{Z}}_k(F_p) + \alpha^{n \mathbb{Z}+1}_k(F_p).
$$

Similarly, we define $\alpha^{m \mathbb{Z}}_k(F_p)$, $\alpha^{m \mathbb{Z}+1}_k(F_p)$, etc.

For $n = 2$ case, we recall the notion of singularities of type $(2k + 1 \to 2k + 1)$ introduced in [19].

**Definition 1.7** (Singularity of type $(2k + 1 \to 2k + 1)$). (1) Suppose that $n = 2$. If the exceptional curve $E_x$ of the blow-up at $x \in R$ with odd multiplicity $2k + 1$ has only one singularity $y$ of $R$, then the multiplicity of $R$ at $y$ equals $2k + 2$ and $E_x$ contributes to $j^{\eta}_{1,1}(F_p)$.
Conversely, the exceptional curve $E$ contributing to $j''_{0,1}(F_p)$ has such a pair $(x,y)$ since $E$ is blown up only once. Then we call the pair $(x,y)$ a singularity of type $(2k+1 \to 2k+1)$ (cf. [19], p.605).

(2) Let $\alpha_{(2k+1\to 2k+1)}(F_p)$ be the number of singularities of type $(2k+1 \to 2k+1)$ over $p$ (i.e., $s_{2k+1}(F_p)$ in the notation of [19] Definition 5). Then we have

$$j''_{0,1}(F_p) = \sum_{k \geq 1} \alpha_{(2k+1\to 2k+1)}(F_p).$$

(3) We decompose

$$\alpha_{(2k+1\to 2k+1)}(F_p) = \alpha_{(2k+1\to 2k+1)}^{\text{tr}}(F_p) + \alpha_{(2k+1\to 2k+1)}^{\text{co}}(F_p)$$

and

$$\alpha_{(2k+1\to 2k+1)}^{\text{co}}(F_p) = \alpha_{(2k+1\to 2k+1)}^{\text{co},0}(F_p) + \alpha_{(2k+1\to 2k+1)}^{\text{co},1}(F_p)$$

as follows: Let $\alpha_{(2k+1\to 2k+1)}^{\text{tr}}(F_p)$ be the number of singularities of type $(2k+1 \to 2k+1)$ over $p$ at which any local branch of the horizontal part $R_i$ intersects the fiber over $p$ transversely (that is, the proper transform of the vertical component passing through $x$ does not pass through $y$). Let $\alpha_{(2k+1\to 2k+1)}^{\text{co},0}(F_p)$ (resp. $\alpha_{(2k+1\to 2k+1)}^{\text{co},1}(F_p)$) be the number of singularities $(x,y)$ of type $(2k+1 \to 2k+1)$ over $p$ such that the proper transform of the vertical component passing through $x$ also passes through $y$ and is not contained in $R$ (resp. is contained in $R$).

1.5. Singularity diagrams. Let $C$ be a smooth component of some $R_i$ which is a component of $\Gamma_p$ or the exceptional $(-1)$-curve over $p$ appearing in $\psi$ (that is $C = C^{t,k}$ for some $t, k$). Now we focus on the resolution of $R$ occurring over $C$. If $C$ is on $W_i$, we drop the index and set $R = R_i$ for simplicity. Let $R' := R - C$ as a divisor. Let $x_1, \ldots, x_t$ be all the points of $C \cap R'$. We put $x_{i,1} := x_i$ and $m_{i,1} := m_i = \text{mult}_{x_i}(R)$. We define $\psi_{i,1} : W_{i,1} \to W$ to be the blow-up at $x_{i,1}$ and $E_{i,1} := \psi_{i,1}^{-1}(x_{i,1})$ and $R_{i,1} := \psi_{i,1}^* R - n[m_{i,1}/n]E_{i,1}$. Inductively, we define $x_{i,j}$, $m_{i,j}$ to be the intersection point of the proper transform of $C$ and $E_{i,j-1}$, the multiplicity of $R_{i,j-1}$ at $x_{i,j}$, and if $m_{i,j} > 1$, we define $\psi_{i,j} : W_{i,j} \to W_{i,j-1}$, $E_{i,j}$ and $R_{i,j}$ to be the blow-up at $x_{i,j}$, the exceptional curve for $\psi_{i,j}$ and $R_{i,j} := \psi_{i,j}^* R_{i,j-1} - n[m_{i,j}/n]E_{i,j}$, respectively. Put

$$i_{\text{hm}} := \max\{j \mid m_{i,j} > 1\},$$

that is, the number of blowing-ups occurring over $x_i$. We may assume that $i_{\text{hm}} \geq (i + 1)_{\text{hm}}$ for $i = 1, \ldots, l - 1$ after rearranging the index if necessary. Put $r' := R'C$ and $c := \sum_{i=1}^t i_{\text{hm}}$. If $C$ is a fiber $\Gamma$ of $\varphi$, $r'$ is the number of branch points $r$. If $C$ is an exceptional curve, $r'$ is the multiplicity of $R$ at the point to which $C$ is contracted. Clearly, $c$ is the number of blow-ups occurring over $C$. Set $d_{i,j} := [m_{i,j}/n]$. Then the following lemmas hold:

**Lemma 1.8 ([7] Proposition 3.6).** We have

$$\frac{r' + c}{n} = \sum_{i=1}^t \sum_{j=1}^{i_{\text{hm}}} d_{i,j}.$$ 

This is a special case of the following lemma:
Lemma 1.9. Let \( f : S \to B \) be a primitive cyclic covering fibration of type \((g,h,n)\). Let \( C \) be a curve contained in \( R \), \( L \) the proper transform of \( C \) on \( \widetilde{W} \) and \( x_1, x_2, \ldots, x_c \) all the singularities of \( R \) on \( C \) (including infinitely near ones). We put \( m_i := \text{mult}_{x_i}(R) \), \( k_i := \text{mult}_{x_i}(C) \) and \( d_i := \lfloor m_i/n \rfloor \). Then, we have
\[
\frac{RC - L^2}{n} = \sum_{i=1}^c k_i d_i.
\]

Proof. We may assume that \( \psi_i \) is the blow-up at \( x_i \) for \( i = 1, \ldots, c \). Then, we can write \( L = e \psi^* C - \sum_{i=1}^c k_i E_i \) and \( \tilde{R} = \psi^* R - \sum_{i=1}^N nd_i E_i \). Thus, we have \( \tilde{R} L = RC - \sum_{i=1}^c nk_i d_i \).

On the other hand, since \( \tilde{R} - L \) and \( L \) are disjoint, we have \( L^2 = \tilde{R}L \). From these equalities, the assertion follows.

The following is easy to prove:

Lemma 1.10 ([7] Lemma 3.7). (1) When \( n \geq 3 \), then \( m_{i,j} \geq m_{i,j+1} \). When \( n = 2 \), then \( m_{i,j} + 1 \geq m_{i,j+1} \) with equality holds only if \( m_{i,j-1} \in 2\mathbb{Z} \) (if \( j > 1 \)) and \( m_{i,j} \in 2\mathbb{Z} + 1 \).

(2) If \( m_{i,j-1} \in n\mathbb{Z} + 1 \) and \( m_{i,j} \in n\mathbb{Z} \), then \( m_{i,j} > m_{i,j+1} \).

(3) \( m_{i,i} \in n\mathbb{Z} \).

We recall the singularity diagrams introduced in [8]:

Definition 1.11 (Singularity diagram). By using the datum \( \{m_{i,j}\} \), one can construct a diagram as in Table 1, where we insert the pair \((x_{i,j}, m_{i,j})\) into the square at the \( i \)-th column from the left and the \( j \)-th row from the bottom. We call it the singularity diagram of \( C \).

<table>
<thead>
<tr>
<th>( x_{1,1} ), ( m_{1,1} )</th>
<th>( \cdots )</th>
<th>( x_{1,\eta} ), ( m_{1,\eta} )</th>
<th>( \cdots )</th>
<th>( x_{1,1,l} ), ( m_{1,1,l} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
</tr>
</tbody>
</table>

Remark 1.12. When \( C \) is not contained in \( R \), the corresponding singularity diagram of \( C \) can be defined similarly, see Definition 2.7 in [8].

Definition 1.13. For \( t = 1, \ldots, \eta \) and \( k = 1, \ldots, j^t(F_p) \), let \( D^{t,k} \) denote the singularity diagram of \( C^{t,k} \). We call the sequence of diagrams \( D^{t,1}, D^{t,2}, \ldots, D^{t,j^t(F_p)} \) the sequence of singularity diagrams associated with \( D^t(p) \).

Then the following lemma is clear.
Lemma 1.14 (cf. [8] Lemma 2.9). Let $D^{t,1}, D^{t,2}, \ldots, D^{t,j} (F_p)$ be the sequence of singularity diagrams associated with $D^t(p)$. Let $l^{i,k}$ be the number of points in $R' \cap C^{t,k}$ and $(x_{i,j}^{t,k}, m_{i,j}^{t,k})$, $i = 1, \ldots, l^{i,k}$, $j = 1, \ldots, i_{bm}$ denote the entries of $D^{t,k}$. Let $(x_{i,j}^{t,p}, m_{i,j}^{t,p})$ be a singularity on $C^t$ such that $m_{i,j}^{t,p} \in \mathbb{N} + 1$, and $m_{i,j-1}^{t,p} \in \mathbb{N}$ when $j > 1$. Let $q > p$ be the integer such that $C^t_q$ is the exceptional curve for the blow-up at $x_{i,j}^{t,p}$. Then, for every $1 \leq p' \leq p$, $i'$, $j'$ satisfying $(x_{i',j'}^{t,p'}, m_{i',j'}^{t,p'}) = (x_{i,j}^{t,p}, m_{i,j}^{t,p})$, the diagram $D^{t,q}$ has $(x_{i',j'+1}^{t,p'}, m_{i',j'+1}^{t,p'})$ as an entry in the bottom row.

Example 1.15. Let $t = 1, \ldots, \eta_p$ and suppose that $D^t(p)$ consists of $\tilde{\eta}$-exceptional $(-n)$-curves, that is, $t$ contributes to $\eta_p$. Then $C^{t,k} = C^{m,k}$ is a $(-1)$-curve and blown up $n - 1$ times for any $k$.

1. If $n = 2$, then the point to which $C^{t,1}$ contracts is a singularity of type $(m \to m)$ for some odd integer $m$. Indeed, $R'C^{t,1} = m$ and from Lemma 1.8, the singularity diagram of $C^{t,1}$ is the following:

$$
\begin{array}{c}
\eta_p \\
\hline
m + 1 \\
\hline
D^{t,1}
\end{array}
$$

where we drop the symbol indicating the singular point on $C^{t,1}$ for simplicity. Since $m + 1$ is even, we have $j^t(F_p) = 1$. This observation gives us

$$(1.14) \quad \bar{\eta}_p = \sum_{k \geq 1} \left( \alpha_{(2k+1 \to 2k+1)}^{tr}(F_p) + \alpha_{(2k+1 \to 2k+1)}^{co,0}(F_p) \right).$$

2. Suppose that $n = 3$. Let $m$ be the multiplicity of $R$ at the center of $C^{t,1}$. Then $R'C^{t,1} = m$ and from Lemma 1.8, all possible singularity diagram of $C^{t,1}$ are the following:

$$(i) \quad \begin{array}{c}
n_1 \\
\hline
n_2 \\
\hline
D^{t,1}
\end{array} \quad (ii) \quad \begin{array}{c}
n_2 \\
\hline
n_1 \\
\hline
D^{t,1}
\end{array} \quad (iii) \quad \begin{array}{c}
n_1 \\
\hline
m_1 \\
\hline
D^{t,1}
\end{array}$$

where the integers $n_i \in 3\mathbb{Z}$ and $m_i \in 3\mathbb{Z} + 1$ satisfy that $m + 2 = n_1 + n_2$ in the case (i), $m + 2 = n_1 + n_2$ and $n_2 \leq n_1$ in the case (ii), $m + 3 = m_1 + n_1$ and $n_1 < m_1$ in the case (iii). If the diagram $D^{t,1}$ is (i) or (ii), then $j^t(F_p) = 1$ since there are no $3\mathbb{Z} + 1$ type singularities on $C^{t,1}$. If the diagram $D^{t,1}$ is (iii), then $j^t(F_p) > 1$ and the singularity diagram $D^{t,2}$ of $C^{t,2}$ which is obtained by the blow-up at the singularity with multiplicity $m_1$ is (i) or (ii) from Lemma 1.14. Thus we have $j^t(F_p) = 2$.

3. When $n \geq 4$, then the number $j^t(F_p)$ is not bounded. For example, we can consider the following sequence of singularity diagrams associated with $D^t(p)$ when $n = 4$:
where $n_k \in 4\mathbb{Z}$, $m_k \in 4\mathbb{Z} + 1$ and $C^{t,k}$, $k \geq 2$ is the exceptional curve obtained by the blow-up at the multiplicity on $C^{t,k-1}$ with multiplicity $m_{k-1}$. From Lemma 1.8, we have $m_k + 4 = m_{k+1} + n_{2k+1} + n_{2k+2}$ for any $k \geq 1$.

1.6. Gonality of primitive cyclic covering fibrations. Recall that the gonality $\text{gon}(C)$ of a non-singular projective curve $C$ is the minimum of the degree of morphisms onto $\mathbb{P}^1$. The gonality $\text{gon}(f)$ of a fibered surface $f: S \to B$ is defined to be that of a general fiber (cf. [14]).

**Proposition 1.16.** Let $\theta: F \to \Gamma$ be a totally ramified covering of degree $n$ between smooth projective curves branched over $r$ points. If $r \geq 2n \text{gon}(\Gamma)$, then $\text{gon}(F) = n \text{gon}(\Gamma)$. In particular, the gonality of a primitive cyclic covering fibration of type $(g, h, n)$ is $n \text{gon}(\varphi)$, when $r \geq 2n \text{gon}(\varphi)$.

**Proof.** Assume contrary that $F$ has a morphism onto $\mathbb{P}^1$ of degree $k < n \text{gon}(\Gamma)$. This together with the covering $\theta: F \to \Gamma$ defines a morphism $\Phi : F \to \mathbb{P}^1 \times \Gamma$. If $\Phi$ is of degree $m$ onto the image $\Phi(F)$, then $m$ is a common divisor of $n$ and $k$. By the genus formula, the arithmetic genus of $\Phi(F)$ is $(n/m - 1)k/m + (h - 1)n/m + 1$. Now, let $F'$ be the normalization of $\Phi(F)$. Since the covering $F \to \Gamma$ factors through $F'$, we see that the induced covering $F' \to \Gamma$ is a totally ramified covering of degree $n/m$ branched over $r$ points. Then, by the Hurwitz formula, we have $2g(F') - 2 = (2h - 2)n/m + (n/m - 1)r$. Since the genus $g(F')$ of $F'$ is not bigger than the arithmetic genus of $\Phi(F)$, we get $r \leq 2k/m$ when $n > m$, which is impossible, since $r \geq 2n \text{gon}(\Gamma)$ and $k < n \text{gon}(\Gamma)$. Thus, we get $n = m$. Then $F'$ is isomorphic to $\Gamma$ and therefore the morphism $F \to \mathbb{P}^1$ factors through $\Gamma$. Hence we have $k \geq n \text{gon}(\Gamma)$ by the definition of the gonality of $\Gamma$, which contradicts $k < n \text{gon}(\Gamma)$. A more careful study shows that any gonality pencil of $F$ is the pull-back of a gonality pencil of $\Gamma$ when $r > 2n \text{gon}(\Gamma)$.

2. **Primitive cyclic covering fibrations of type $(g, 1, n)$**

Let $f: S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$. Since $\varphi: W \to B$ is a relatively minimal elliptic surface, $K_{\varphi}$ is numerically equivalent to $\left(\chi_{\varphi} + \sum_{p \in B} \left(1 - \frac{1}{m_p}\right)\right)\Gamma$ by the canonical bundle formula, where $m_p$ denotes the multiplicity of the fiber $\Gamma_p$ of $\varphi$ over $p$. In particular, we have $K^2_{\varphi} = 0$. For $p \in B$, we define

$$\nu(F_p) := 1 - \frac{1}{m_p}, \quad \nu := \sum_{p \in B} \nu(F_p).$$

Then, we have $K_{\varphi}R = (\chi_{\varphi} + \nu)r$, where $r$ is the number defined in in (1.1). Combining these equalities with (1.8), (1.9), (1.10) and (1.11), we get the following lemma:
Lemma 2.1. The following equalities hold.

\[
\begin{align*}
K^2_f & = \sum_{k \geq 1} ((n+1)(n-1)k - n) \alpha_k + \frac{(n-1)^2}{n}(\alpha_0 - 2\varepsilon) \\
& \quad + \frac{(n+1)(n-1)r}{n}(\chi_\phi + \nu) + \varepsilon. \\
\chi_f & = \frac{1}{12} (n-1)(n+1) \sum_{k \geq 1} k\alpha_k + \frac{(n-1)(2n-1)}{12n}(\alpha_0 - 2\varepsilon) \\
& \quad + \frac{(n+1)(n-1)r}{12n}(\chi_\phi + \nu) + n\chi_\phi. \\
e_f & = (n-1)\alpha_0 + n \sum_{k \geq 1} \alpha_k - (2n-1)\varepsilon + 12n\chi_\phi.
\end{align*}
\]

Thus we can localize the invariants \(K^2_f\), \(\chi_f\) and \(e_f\):

Definition 2.2 (Localization of \(K^2_f\), \(\chi_f\) and \(e_f\)). For \(p \in B\), we put \(\chi_\phi(F_p) := e_\phi(\Gamma_p)/12\) and

\[
\begin{align*}
K^2_f(F_p) & := \sum_{k \geq 1} ((n+1)(n-1)k - n) \alpha_k(F_p) + \frac{(n-1)^2}{n}(\alpha_0(F_p) - 2\varepsilon(F_p)) \\
& \quad + \frac{(n+1)(n-1)r}{n}(\chi_\phi(F_p) + \nu(F_p)) + \varepsilon(F_p), \\
\chi_f(F_p) & := \frac{1}{12} (n-1)(n+1) \sum_{k \geq 1} k\alpha_k(F_p) + \frac{(n-1)(2n-1)}{12n}(\alpha_0(F_p) - 2\varepsilon(F_p)) \\
& \quad + \frac{(n+1)(n-1)r}{12n}(\chi_\phi(F_p) + \nu(F_p)) + n\chi_\phi(F_p), \\
e_f(F_p) & := (n-1)\alpha_0(F_p) + n \sum_{k \geq 1} \alpha_k(F_p) - (2n-1)\varepsilon(F_p) + 12n\chi_\phi(F_p).
\end{align*}
\]

Then by Lemma 2.1, we have

\[
\begin{align*}
K^2_f = \sum_{p \in B} K^2_f(F_p), \quad \chi_f = \sum_{p \in B} \chi_f(F_p), \quad e_f = \sum_{p \in B} e_f(F_p).
\end{align*}
\]

Then the following slope equality holds:

Theorem 2.3. Let \(f : S \to B\) be a primitive cyclic covering fibration of type \((g, 1, n)\). Then

\[
K^2_f = \lambda_{g,1,n} \chi_f + \sum_{p \in B} \text{Ind}(F_p),
\]
where \( \lambda_{g,1,n} := \frac{12(n-1)}{2n-1} \) and \( \text{Ind}(F_p) \) is defined by

\[
\text{Ind}(F_p) = n \sum_{k \geq 1} \left( \frac{(n+1)(n-1)}{2n-1} k - 1 \right) \alpha_k(F_p) + \frac{n-1}{2n-1} \left( (n+1)r - 12n \right) \chi_{\varphi}(F_p)
\]
\[
+ \frac{(n+1)(n-1)r}{2n-1} \nu(F_p) + \varepsilon(F_p).
\]

Moreover, if \( r \geq \frac{12n}{n+1} \), then \( \text{Ind}(F_p) \) is non-negative for any \( p \in B \).

**Proof.** Since \( K^2_f = \sum_{p \in B} K^2_f(F_p), \chi_f = \sum_{p \in B} \chi_f(F_p) \) and \( K^2_f(F_p) - \lambda_{g,1,n} \chi_{\varphi}(F_p) \) is defined by \( \text{Ind}(F_p) \), the claim follows. \( \square \)

For an oriented compact real 4-dimensional manifold \( X \), the signature \( \text{Sign}(X) \) of \( X \) is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form on \( H^2(X) \). From Lemma 2.1, we observe the local concentration of \( \text{Sign}(S) \) to a finite number of fiber germs.

**Corollary 2.4** (cf. [1]). Let \( f : S \to B \) be a primitive cyclic covering fibration of type \((g,1,n)\). Then

\[
\text{Sign}(S) = \sum_{p \in B} \sigma(F_p),
\]

where \( \sigma(F_p) \) is defined by

\[
\sigma(F_p) = n \sum_{k \geq 1} \left( \frac{(n+1)(n-1)}{3} k - 1 \right) \alpha_k(F_p) + \left( \frac{(n-1)(n+1)r}{3n} - 8n \right) \chi_{\varphi}(F_p)
\]
\[
+ \frac{(n+1)(2n-1)}{3n} \varepsilon(F_p) + \frac{(n+1)(n-1)r}{3n} \nu(F_p) - \frac{(n+1)(n-1)}{3n} \alpha_0(F_p).
\]

**Proof.** By the index theorem (cf. [9, p. 126]), we have

\[
\text{Sign}(S) = \sum_{p+q \equiv 0 \mod 2} h^{p,q}(S) = K^2_f - 8 \chi_f.
\]

On the other hand, we can see that

\[
\sigma(F_p) = K^2_f(F_p) - 8 \chi_f(F_p)
\]

by a computation. \( \square \)

**3. Upper bound of the slope: the case of type \((g,1,n)\)**

**3.1. Main theorem.** In this section, we prove the following theorem:
Theorem 3.1. Let $f: S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$.

1. If $n \geq 4$ or $n = 3$ and $g = 4$, then we have
   \[ K_f^2 \leq \left( 12 - \frac{12n^2}{r(n-1)(n+1)} \right) \chi_f. \]

2. If $n = 3$ and $g \geq 7$, then we have
   \[ K_f^2 \leq \left( 12 - \frac{24}{4g-17} \right) \chi_f. \]

3. If $n = 2$ and $g \geq 3$, then we have
   \[ K_f^2 \leq \left( 12 - \frac{2}{g-2} \right) \chi_f. \]

Corollary 3.2. Let $f: S \to B$ be a relatively minimal bielliptic fibered surface of genus $g \geq 3$. Then, we have
   \[ K_f^2 \leq \left( 12 - \frac{2}{g-2} \right) \chi_f. \]

Proof. Let $f: S \to B$ be a relatively minimal fibered surface of genus $g$ whose general fiber $F$ is a double cover of a smooth curve $\Gamma$ of genus $h$. If $g > 4h + 1$, an involution of the general fiber $F$ of $f$ over $\Gamma$ is unique. Then, the fibration $f$ has a global involution since it is relatively minimal (cf. [6] p.553). Hence $f$ is a primitive cyclic covering fibration of type $(g, h, 2)$. In particular, a relatively minimal bielliptic fibered surface of genus $g \geq 6$ is a primitive cyclic covering fibration of type $(g, 1, 2)$. In the case of $g \leq 5$, we use the semi-stable reduction. We may assume that the slope $\lambda_f$ is greater than 8. Taking a suitable base change $B' \to B$, we get the base change fibration $f': S' \to B'$ which is semi-stable and the bielliptic involution on $F$ extends to a global involution, that is, primitive cyclic covering fibration of type $(g, 1, 2)$. From Theorem 3.1 (3), we have $\lambda_{f'} \leq 12 - 2/(g-2)$. On the other hand, we have $\lambda_f \leq \lambda_{f'}$ from Corollary 4.3 in [17]. Thus the claim holds.

In particular, any bielliptic fibration is not a Kodaira fibration. Namely, the following holds.

Corollary 3.3. Let $B_g \subset M_g$ be the bielliptic locus on the moduli space $M_g$ of smooth curves of genus $g$. Then $B_g$ contains no complete subvarieties of positive dimension.

3.2. Strategy of the proof and examples. Before proving the main theorem, we explain the strategy of the proof. Let $f: S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$. As explained in the introduction, in order to find an upper bound of the slope of $f$, it suffices to find a positive number $\mu$ such that $(12 - \mu)\chi_f(F_p) \geq K_f^2(F_p)$ for any fiber germ $F_p$ of $f$. By the definition of $\chi_f(F_p)$ and $K_f^2(F_p)$, the number $(12 - \mu)\chi_f(F_p) - K_f^2(F_p)$ is expressed as singularity indices $\alpha_0^+(F_p)$, $\alpha_0^-(F_p)$, $\alpha_k(F_p)$, $\eta_{0,k}(F_p)$, etc. Since all indices are non-negative by definition, if we can find $\mu$ such that the coefficient of each index is non-negative, the proof is finished. But these coefficients are sometimes negative for any $\mu \geq 0$. Fortunately, the coefficient of $\alpha_0^+(F_p)$ is positive for small $\mu > 0$. The key point
of the proof is to estimate $\alpha^+_0(F_p)$ from below by using other singularity indices. Indeed, the number $\alpha^+_0(F_p)$ can be estimated by the number of vertical type singularities, which is the reason that the indices $\iota(F_p)$ and $\kappa(F_p)$ were introduced in §1. Then the next step is to establish some relations of $\iota(F_p), \kappa(F_p)$ and other indices such as $\alpha_k(F_p), j_{0,a}(F_p)$. For $n = 2, 3$ cases, more detailed estimations for $j_{0,1}^r(F_p), j_{0,1}^{tr}(F_p)$ are needed. For each case, we estimate $(12 - \mu)\chi_f(F_p) - K_f^2(F_p)$ from below by using the remaining indices $\alpha'_r(F_p), \eta'_p, \eta''_p, j_{k,a}(F_p)$, etc, and find the maximal $\mu > 0$ such that all the coefficients of these indices are non-negative. Indeed, the number $\mu$ is defined by

$$
\mu := \begin{cases} 
\frac{12n^2}{r(n-1)(n+1)}, & \text{if } n \geq 4, \\
\frac{24}{4r-13}, & \text{if } n = 3, \\
\frac{4}{r-2}, & \text{if } n = 2,
\end{cases}
$$

and determined as the coefficient of $\alpha'_{r/n}(F_p)$ (resp. $\eta''_p, \alpha'^{tr}_{(r-1\rightarrow r-1)}(F_p)$) vanishes when $n \geq 4$ (resp. $n = 3, n = 2$).

There exist singular fiber germs $F_p$ attaining the upper bound of $K_f(F_p)/\chi_f(F_p)$:

**Example 3.4.** The following are all the singular fiber germs $F_p$ such that $K_f^2(F_p) = (12 - \mu)\chi_f(F_p)$ (we can prove this fact easily by the proof of Theorem 3.1).

(i) Assume that $n \geq 4$, or $n = 3$ and $g = 4$. Consider the situation that $\Gamma_p$ is smooth and $F_p$ is obtained by the following sequence of singularity diagrams associated with $\Gamma_p$ (here we use the singularity diagrams for curves not contained in $R$, see Definitions 2.7, 2.8 in [8]):

\[
\begin{array}{cccc}
(x_1, r) & (x_2, r) & \ldots & (x_k, r) & (1, 1, \ldots, 1) \\
\Gamma^0_p & E^0_1 & & E^0_{l-1} & E^0_l
\end{array}
\]
where in the figure of $F_p$, the symbol $\bullet$ is the intersection of the 1-dimensional fixed locus of the cyclic covering automorphism and $A_{k,i}$, $i = 1, \ldots, n$ are $n$ copies of the proper transform of $E_k$ obtained by the $n$-cyclic covering, $A_0$ is the $n$-cyclic covering of the elliptic curve $\Gamma_p$ (may be reducible), $A_l$ is the $n$-cyclic covering of $E_l$ branched at $r$ points. Then $g(A_{k,i}) = 0$ for $k = 1, \ldots, l - 1$, $g(A_l) = (r/2 - 1)(n - 1)$ and we can write

$$F_p = A_0 + \sum_{i=1}^{n} (A_{1,i} + \cdots + A_{l-1,i}) + A_l.$$ 

$F_p$ satisfies $\alpha_{k}(F_p) = 0$ for $k = 0, 1, \ldots, r/n - 1$, $\alpha_{r/n}(F_p) = l$, $\varepsilon(F_p) = 0$, $\nu(F_p) = 0$ and $\chi_{\varphi}(F_p) = 0$. Thus we have

$$K_f^2(F_p) = \left( \frac{r(n-1)(n+1)}{n} - n \right) l, \quad \chi_f(F_p) = \frac{r(n-1)(n+1)l}{12n}, \quad e_f(F_p) = nl,$$

and then $e_f(F_p)/\chi_f(F_p) = 12n^2/r(n-1)(n+1) = \mu$.

(ii) Assume that $n = 3$ and $g > 4$. Consider the situation that $\Gamma_p$ is smooth and $F_p$ is obtained by the following sequence of singularity diagrams associated with $\Gamma_p$:

\begin{align*}
\begin{array}{c}
(x, r-2) \\
\gamma_p
\end{array} & \xrightarrow{(1,1)} \\
\begin{array}{c}
\begin{array}{c}
(y, r-2) \\
E_x^1
\end{array}
\end{array} & \xrightarrow{(w, 3)} \\
\begin{array}{c}
\begin{array}{c}
(z, r-3) \\
E_y^1
\end{array}
\end{array} & \xrightarrow{(w, 3)} \\
\begin{array}{c}
\begin{array}{c}
(1, 1, \ldots, 1) \\
E_z^0
\end{array}
\end{array} & \xrightarrow{(1, 1, 1)} \\
\begin{array}{c}
\begin{array}{c}
(1, 1, 1) \\
E_w^0
\end{array}
\end{array}
\end{align*}
where the symbol $\circ$ is the isolated fixed point of the cyclic covering automorphism. Then we can write $F_p = A_0 + 2A_1 + A_3$, $p_\alpha(A_0) = 4$, $g(A_1) = 1$ and $g(A_2) = r - 5$. $F_p$ satisfies $\alpha_0(F_p) = 1$, $\alpha_1(F_p) = l$, $\alpha_{r/3-1}(F_p) = 3$, $\alpha_k(F_p) = 0$ for $k \neq 1, r/3 - 1$, $\varepsilon(F_p) = 2$, $\nu(F_p) = 0$ and $\chi_p(F_p) = 0$. Thus we have

$$K^2_f(F_p) = 8r - 30, \quad \chi_f(F_p) = \frac{4r - 13}{6}, \quad e_f(F_p) = 4$$

and then $e_f(F_p)/\chi_f(F_p) = 24/(4r - 13) = \mu$.

(iii) Assume that $n = 2$ and $g \geq 3$. Consider the situation that $\Gamma_p$ is smooth and $F_p$ is obtained by the following sequence of singularity diagrams associated with $\Gamma_p$:

$$\left(\begin{array}{c}
(x_1, r - 1) \\
\Gamma^0_p
\end{array}\right) (1) \left(\begin{array}{c}
E^1_1 \\
(x_2, r)
\end{array}\right) \left(\begin{array}{c}
\vdots
\end{array}\right) \left(\begin{array}{c}
E^{0}_{2l - 1}
\end{array}\right) \left(\begin{array}{c}
(x_{2l-1}, r - 1) \\
(1, 1, \ldots, 1)
\end{array}\right) \left(\begin{array}{c}
E^1_{2l - 1} \\
(x_{2l}, r)
\end{array}\right) \left(\begin{array}{c}
\vdots
\end{array}\right) \left(\begin{array}{c}
E^0_{2l - 1}
\end{array}\right)$$

where the symbol $\circ$ is the isolated fixed point of the cyclic covering automorphism.
Then we can write \( F_p = A_0 + A_1 + \cdots + A_l \), \( p_a(A_0) = 2 \), \( g(A_k) = 0 \) for \( k = 1, \ldots, l - 1 \) and \( g(A_l) = r/2 - 1 \). \( F_p \) satisfies \( \alpha_{r/2-1}(F_p) = \alpha_{r/2}(F_p) = l \), \( \alpha_k(F_p) = 0 \) for \( k \neq r/2 - 1, r/2 \), \( \varepsilon(F_p) = l \), \( \nu(F_p) = 0 \) and \( \chi_{\varphi}(F_p) = 0 \). Thus
\[
K^2_f(F_p) = (3r - 7)l, \quad \chi_f(F_p) = \frac{(r - 2)l}{4}, \quad e_f(F_p) = l
\]
and then \( e_f(F_p)/\chi_f(F_p) = 4/(r - 2) = \mu \).

**Notation 3.5.** In the rest of the section, we freely use the notation and terminology defined in §1 for primitive cyclic covering fibrations.

For a condition or a Roman numeral \( \mathcal{P} \), we put \( \delta_{\mathcal{P}} = 1 \) if the condition \( \mathcal{P} \) holds or \( \Gamma_p \) is a singular fiber of type \( \mathcal{P} \), and \( \delta_{\mathcal{P}} = 0 \) otherwise.

### 3.3. Classification of vertical type singularities.

Let \( f : S \to B \) be a primitive cyclic covering fibration of type \((g, 1, n)\). We fix a point \( p \in B \). Let \( m = m_p \) be the multiplicity of the fiber \( \Gamma_p \) of \( \varphi \) over \( p \). Since \( \varphi \) is of genus 1, we have \( j_m \cdot (F_p) = 0 \) for any \( m \geq 2 \). From the classification of singular fibers of relatively minimal elliptic surfaces ([12]), we have the following lemma for \( u \)-vertical type singularities:

**Lemma 3.6.** There exist no \( u \)-vertical type singularities of \( R \) for \( u \geq 4 \). All possible 3-vertical type singularities are as follows.

**Type (II) :** \( \Gamma_p \) is a singular fiber of type (II) in the Kodaira’s table ([12]) (i.e., it is a singular rational curve with one cusp) and it is contained in \( R \). The cusp on \( \Gamma_p \) is a singularity of type \( n\mathbb{Z} + 1 \) and the singularity at which the proper transform of \( \Gamma_p \) and the exceptional curve \( E_1 \) for the blow-up at the cusp intersect is also of type \( n\mathbb{Z} + 1 \). Then, the proper transforms of \( \Gamma_p \) and \( E_1 \) and the exceptional curve \( E_2 \) for the blow-up at this singularity form a 3-vertical type singularity.

![Diagram](https://via.placeholder.com/150)

**Type (III) :** \( \Gamma_p \) is a singular fiber of type (III) in the Kodaira’s table (i.e., it consists of two nonsingular rational curves intersecting each other at one point of order two) and it is contained in \( R \). The singularity on \( \Gamma_p \) is a singularity of type \( n\mathbb{Z} + 1 \). Then, the proper transforms of \( \Gamma_p \) and the exceptional curve \( E_1 \) for the blow-up at this singularity form a 3-vertical type singularity.

![Diagram](https://via.placeholder.com/150)
Type (IV) : $\Gamma_p$ is a singular fiber of type (IV) in the Kodaira’s table (i.e., it consists of three nonsingular rational curves intersecting one another at one point transversely) and it is contained in $R$. The singularity on $\Gamma_p$ is a 3-vertical type singularity.

In particular, we have $\iota^{(u)}(F_p) = \kappa^{(u)}(F_p) = 0$ for $u \geq 4$ and $0 \leq \iota^{(3)}(F_p) + \kappa^{(3)}(F_p) \leq 1$.

3.4. Lower bound of $\alpha_0^+(F_p)$. Next, we give a lower bound of $\alpha_0^+(F_p)$ by using $\iota(F_p)$ and $\kappa(F_p)$.

Lemma 3.7. We have

$$\alpha_0^+(F_p) \geq \left(1 - \frac{1}{m}\right) r + (n - 2)(\iota(F_p) + 2\kappa(F_p)) + \beta_p.$$  

where $\beta_p := \delta_{n \neq 2} \left( (n - 7)\delta_{I} - (n + 1)\delta_{III} - 2\delta_{IV} \right) \iota^{(3)}(F_p) + \delta_{n=2} \sum_{k \geq 1} 2k\alpha_{(2k+1 \to 2k+1)}(F_p)$, and $\delta_p$ is as in Notation 3.5.

Proof. Let $\widetilde{\Gamma}_p = m\widetilde{D}_p$ and $\widetilde{D}_p = \sum_i m_i G_i$ the irreducible decomposition. Then we have

$$\alpha_0^+(F_p) = r - \#(\text{Supp}(\widetilde{R}_h) \cap \text{Supp}(\widetilde{\Gamma}_p))$$  

$$\geq r - \sum_i \widetilde{R}_h G_i$$  

$$= \left(1 - \frac{1}{m}\right) r + \sum_i (m_i - 1)\widetilde{R}_h G_i.$$  

For a $(t, 2)$ or $(t, 3)$-vertical $n\mathbb{Z}$ type singularity $x$, we denote by $E_x^t$ the exceptional curve for the blow-up at $x$. Let $m_x^t$ be the multiplicity of $\widetilde{D}_p$ along $\widetilde{E}_x^t$, the proper transform of $E_x^t$ on $\widetilde{W}$. Then, we have

$$\sum_i (m_i - 1)\widetilde{R}_h G_i \geq \sum_{t=1}^{n_p} \sum_{x: (t, u) \ n\mathbb{Z}, \ u \geq 2} (m_x^t - 1)\widetilde{R}_h \widetilde{E}_x^t,$$

where the second summation in the right hand side is taken over all $(t, u)$-vertical $n\mathbb{Z}$ type singularities with $u \geq 2$. If there exists a singular point of type $n\mathbb{Z}$ on $E_x^t$, we replace $E_x^t$ with the exceptional curve $E$ obtained by blowing up at this point. Repeating this procedure, we may assume that there exist no singular points of type $n\mathbb{Z}$ on $E_x^t$. If there exists a singular point of type $n\mathbb{Z} + 1$ on $E_x^t$, the proper transform of the exceptional curve obtained
by blowing up at this point belongs to other $D_u^v(p)$. Since the multiplicity of $\widetilde{\Gamma}_p$ along it is not less than $m^t_x > 1$, we do not have to consider this situation. Thus, we may assume that there exist no singular points on $E^t_x$ and we have $\bar{R}_h E^t_x \geq n - 2$ if $x$ is a 2-vertical type singularity, and $\bar{R}_h E^t_x \geq n - 3$ if $x$ is a 3-vertical type singularity. We can see that

$$\sum_{x: t, z \in \mathbb{Z}} m^t_x \geq 2t^1(F_p) + 2\kappa^1(F_p) \text{ for any } t \text{ with } t^{(3)}(F_p) = 0.$$

Thus, if $t^{(3)}(F_p) = 0$, we have

$$\alpha^+_0(F_p) \geq \left(1 - \frac{1}{m}\right) r + (n - 2)(\iota(F_p) + 2\kappa(F_p)).$$

If $t^{(3)}(F_p) = 1$, then $m = 1$ and $\Gamma_p$ is of type (II), (III) or (IV) from Lemma 3.6. We may assume that $D^1_u(p) \neq 0$. Let $x_0$ be the 3-vertical $n\mathbb{Z}$ type singularity over $p$. Suppose that $\Gamma_p$ is of type (II). Then, we can see that $m^1_{x_0} = 6$ and $\sum_{x: (1, 2) \in \mathbb{Z}} m^1_x \geq 2t^1(2)(F_p) + 2(\kappa^1(F_p) - 1)$. Then we have

$$\alpha^+_0(F_p) \geq \left(1 - \frac{1}{m}\right) r + 5(n - 3) + (n - 2)(t^1(2)(F_p) + 2(\kappa^1(F_p) - 1))$$

$$+ (n - 2) \sum_{t=2}^{m} \left( t^1(F_p) + 2\kappa^1(F_p) \right)$$

$$= \left(1 - \frac{1}{m}\right) r + n - 7 + (n - 2)(\iota(F_p) + 2\kappa(F_p)).$$

Suppose that $\Gamma_p$ is of type (III). Then, we can see that $m^1_{x_0} = 4$ and $\sum_{x: (1, 2) \in \mathbb{Z}} m^1_x \geq 2t^1(2)(F_p) + 2(\kappa^1(F_p) - 1)$. Then we have

$$\alpha^+_0(F_p) \geq \left(1 - \frac{1}{m}\right) r + 3(n - 3) + (n - 2)(t^1(2)(F_p) + 2(\kappa^1(F_p) - 1))$$

$$+ (n - 2) \sum_{t=2}^{m} \left( t^1(F_p) + 2\kappa^1(F_p) \right)$$

$$= \left(1 - \frac{1}{m}\right) r - n - 1 + (n - 2)(\iota(F_p) + 2\kappa(F_p)).$$

Suppose that $\Gamma_p$ is of type (IV). Then, we can see that $m^1_{x_0} = 3$ and $\sum_{x: (1, 2) \in \mathbb{Z}} m^1_x \geq 2t^1(2)(F_p) + 2\kappa^1(F_p)$. Then we have

$$\alpha^+_0(F_p) \geq \left(1 - \frac{1}{m}\right) r + 2(n - 3) + (n - 2)(t^1(2)(F_p) + 2\kappa^1(F_p))$$

$$+ (n - 2) \sum_{t=2}^{m} \left( t^1(F_p) + 2\kappa^1(F_p) \right)$$

$$= \left(1 - \frac{1}{m}\right) r - 2 + (n - 2)(\iota(F_p) + 2\kappa(F_p)).$$
Suppose that \( n = 2 \). For a \((2k+1 \to 2k+1)\) singularity \((x, y)\), let \( E_y \) denotes the exceptional curve for the blow-up at \( y \) and \( m_y \) the multiplicity of \( \Gamma_p \) along \( \widetilde{E}_y \), the proper transform of \( E_y \). Then we have
\[
\sum_i (m_i - 1) \tilde{R}_h G_i \geq \sum_{k \geq 1} \sum_{(x,y):(2k+1 \to 2k+1)} (m_y - 1) \tilde{R}_h \tilde{E}_y.
\]
By an argument similar to the above, we may assume that there are no singular points on \( E_y \). Then we have \( \tilde{R}_h \tilde{E}_y = 2k \) for any \((2k+1 \to 2k+1)\) singularity \((x, y)\). On the other hand, we have \( m_y \geq 2 \) for any \((2k+1 \to 2k+1)\) singularity \((x, y)\) involved in \( \alpha^{(2k+1 \to 2k+1)}_p \). Thus, we obtain
\[
\sum_{(x,y):(2k+1 \to 2k+1)} (m_y - 1) \tilde{R}_h \tilde{E}_y \geq 2k \alpha^{(2k+1 \to 2k+1)}_p (F_p).
\]

3.5. Estimation of indices. First we translate the index \( \alpha'' \) into other indices as follows.

**Lemma 3.8.** The following equalities hold.

\[
\text{(3.1)} \quad \sum_{k \geq 1} \alpha''_k (F_p) = \eta''_p + \sum_{a \geq 1} an'_{j,a} (F_p) + \sum_{a \geq 1} (an - 2 - \delta_{a,1,\|I\|}) j'_{0,a} (F_p) + \sum_{a \geq 1} (an - 1) j''_{0,a} (F_p) - \iota (F_p) - \kappa (F_p).
\]

\[
\text{(3.2)} \quad \sum_{k \geq 1} k \alpha'''_k (F_p) = \gamma_p + \sum_{a \geq 1} a j_{\ast,a} (F_p) + \sum_{k \geq 1} k \left( \alpha''_{k+1} (F_p) - \iota_k (F_p) - \kappa_k (F_p) \right).
\]

where \( \gamma_p = \sum_{r=1}^{\eta''_p} \gamma''_p \) and \( \gamma''_p \) is defined to be the following (i), (ii), (iii):

(i) \( \gamma''_p := d^{1,1}_r \) if \( D^{m_1} (p) = 0 \), where \( m^{1,1}_r \) is the multiplicity of the singularity to which \( C^{m_1} \) contracts and \( d^{1,1}_r := [m^{1,1}_r/n] \).

(ii) \( \gamma''_p := \sum_{k=1}^{\gamma^{(F_p)}} RC^{m_1,k} / n \) if \( D^{m_1} (p) \neq 0 \) and any \( C^{m_1,k} \) is smooth.

(iii) \( \gamma''_p := r/mn - d^{1,1}_r \) if \( C^{m_1,1} = (\Gamma_p)_{\text{red}} \) is singular, where \( m^{1,1}_r \) is the multiplicity of the singular point of \( R \) which is singular for \( C^{m_1,1} \) and \( d^{1,1}_r := [m^{1,1}_r/n] \).

**Proof.** Let \( a^{m_1,k} \), \( a^{m_1,k} \) be the integers such that \( (L^{m_1,k})^2 = -a^{m_1,k} \), \( (L^{m_1,k})^2 = -a^{m_1,k} \). If \( C^{m_1,k} \) is smooth, then \( C^{m_1,k} \) is blown up \( a^{m_1,k} + (C^{m_1,k})^2 \) times. If \( C^{m_1,k} \) is a singular rational curve, then \( C^{m_1,k} \) is blown up \( a^{m_1,k} - 3 \) times. Since \( C^{m_1,k} \) is a \((-1)\)-curve, \( C^{m_1,k} \) is blown up \( a^{m_1,k} - 1 \) times. Hence, if \( D^{m_1} (p) \neq 0 \) and every \( C^{m_1,k} \) is smooth (resp. \( C^{m_1,1} \) is singular rational), the number of singular points associated with \( D^{i}(p) \) is \( \sum_k \left( a^{m_1,k} + (C^{m_1,k})^2 \right) \) (resp. \( \sum_k \left( a^{m_1,k} - 3 \right) \) )
\[
\sum_k \left( a^{m_1,k} - 3 \right) - \iota (F_p) - \kappa (F_p).
\]

Therefore, we have
\[
\sum_{k \geq 1} \alpha''_k (F_p) = \sum_{a \geq 1} an'_{1,a} (F_p) + \sum_{a \geq 1} (an - 2 - \delta_{a,1,\|I\|}) j'_{0,a} (F_p) + \sum_{a \geq 1} (an - 1) j''_{0,a} (F_p) - \iota (F_p) - \kappa (F_p).
\]
If $D''(p) = 0$, we have
\[ \sum_{k \geq 1} \alpha_k^{m_t}(F_p) = 1 + \sum_{a \geq 1} (an - 1)j_{0,a}^{m_t}(F_p) - \iota^t(F_p) = \kappa^t(F_p). \]

Summing up for $t = 1, \ldots, \eta_p$, we have (3.1).

Let $r_{t,k} := RC^{m,k}$, $m_{t,k}$ the multiplicity of $R$ at the center of $C_{m,k}$ and $d_{t,k} := [m_{t,k}/n]$. Let $x_{1}^{m,k}, \ldots, x_{\gamma}^{m,k}$ (resp. $x_{\gamma}^{m,k}, \ldots, x_{\gamma}^{m,k}$) be all the singular points on $C_{m,k}$ (resp. on $C_{m,k}'$), including infinitely near ones. Put $m_{i}^{m,k} := \text{mult}_{x_{i}}^{m,k}(R)$, $d_{i}^{m,k} := [m_{i}^{m,k}/n]$, $m_{i}^{m,k} := \text{mult}_{x_{i}}^{m,k}(R)$ and $d_{i}^{m,k} := [m_{i}^{m,k}/n]$. Applying Lemma 1.9 to $C_{m,k}$ and $C_{m,k}'$, we get that $r_{t,k}/n + a_{t,k} = \sum_{i} d_{i}^{t,k}$ if $C_{m,k}$ is smooth, $r_{t,k}/n + a_{t,k} = \sum_{i} d_{i}^{t,k}$ if $C_{m,k}'$ is singular rational, and $a_{t,k} + a_{m,k} = \sum_{i} d_{i}^{m,k}$. If $D''(p) \neq 0$ and every $C_{m,k}$ is smooth, then
\[
\sum_{k} \frac{r_{t,k}}{n} + \sum_{a \geq 1} a_{k,a}^{t}(F_p) + \sum_{k \geq 1} k\alpha_{k}^{t,nm+1}(F_p) = \sum_{k} \left( \frac{r_{t,k}}{n} + a_{t,k}^{n} \right) + \sum_{k} (d_{t,k}^{m,k} + a_{m,k})
= \sum_{k} \sum_{i} d_{i}^{m,k} + \sum_{k} \sum_{i} d_{i}^{m,k}
= \sum_{k \geq 1} k \left( \alpha_{k}^{m_t}(F_p) + \iota_{k}(F_p) + \kappa_{k}(F_p) \right).
\]

Similarly, if $D''(p) \neq 0$ and $C_{m,1}$ is singular rational, we have
\[
\frac{r}{nm} - d_{1,1} + \sum_{a \geq 1} a_{k,a}^{t}(F_p) + \sum_{k \geq 1} k\alpha_{k}^{t,nm+1}(F_p) = \sum_{k \geq 1} k \left( \alpha_{k}^{m_t}(F_p) + \iota_{k}(F_p) + \kappa_{k}(F_p) \right).
\]

If $D''(p) = 0$, then
\[
\sum_{a \geq 1} a_{k,a}^{t}(F_p) + \sum_{k \geq 1} k\alpha_{k}^{t,nm+1}(F_p) = \sum_{k} (d_{t,k} + a_{m,k})
= \sum_{k} \sum_{i} d_{i}^{m,k}
= \sum_{k \geq 1} k \left( \alpha_{k}^{m_t}(F_p) + \iota_{k}(F_p) + \kappa_{k}(F_p) \right) - d_{1,1}.
\]

Summing up for $t = 1, \ldots, \eta_p$, we get (3.2).

Next we estimate $\gamma_p$ as in (3.2) from above and translate $\iota$ into other indices.

**Lemma 3.9.** The following hold.
\[
\gamma_p \leq \left( \frac{r}{n} - j_{0,1}(F_p) \delta_{n=2} - \delta_{m,1,n} \right) \delta_{\eta_p \neq 0} + \left( \frac{r}{n} - 1 \right) \eta_p'',
\]
\[
\iota(F_p) = j(F_p) - \eta_p + \delta_{\text{cyc}},
\]
where $\delta_{\text{cyc}}$ is defined to be 1 if the following (i), (ii), (iii) and (iv) hold and $\delta_{\text{cyc}} = 0$ otherwise.

(i) $\Gamma_p$ is a singular fiber of type $(m_{1,k})_{k \geq 1}$, (II), (III) or (IV).
(ii) Any irreducible component of $\Gamma_p$ is contained in $R$. 

(iii) $\nu^{(3)}(F_p) = \kappa^{(3)}(F_p) = 0$. 

(iv) The multiplicity of the singular point of $R$ which is singular for $(\Gamma_p)_{\text{red}}$ belongs to $n\mathbb{Z} + 1$ if $\Gamma_p$ is a singular fiber of type $(m \mathbb{I}_1)$ or $(\Pi)$. 

Proof. By the definition of $\gamma_p$, the first inequality is clear.

We consider the following graph $G^t$: The vertex set $V(G^t)$ is defined by the symbol set \{ $t$, $k$ \}. The edge set $E(G^t)$ is defined by the symbol set \{ $x$ \} $\cup$ \{ $y$ \} $\cup$ \{ $y'$ \}, where $x$, $y$ respectively move among $(t, 2)$, $(t, 3)$-vertical $n\mathbb{Z}$ type singularities. If the proper transform of $C^{t, k}$ meets that of $C^{t, k'}$ at a $(t, 2)$-vertical $n\mathbb{Z}$ type singularity $x$, the edge $e_x$ connects $v^{t, k}$ and $v^{t, k'}$. If the proper transforms of $C^{t, k}$, $C^{t, k'}$ and $C^{t, k''}$ ($k < k' < k''$) intersects in a $(t, 3)$-vertical $n\mathbb{Z}$ type singularity $y$, the edge $e_y$ connects $v^{t, k}$ and $v^{t, k'}$, and $e_y'$ connects $v^{t, k'}$ and $v^{t, k''}$. By the definition of the decomposition $\tilde{R}_e(p) = D^1(p) + \cdots + D^{n_p}(p)$, the graph $G^t$ is defined for any $t = 1, \ldots, n_p$. Clearly, $\nu(F_p)$ is the cardinality of $E(G^t)$. Thus, the number of cycles in $G^t$ equals $\nu(F_p) - j^{(3)}(F_p) + 1$. One sees that $G^t$ has at most one cycle, and it has one cycle only if \{ $C^{t, k}$ \} $\cup$ $\{ C^{t, k'} \}$ contains all irreducible components of $\Gamma_p$. Hence at most one $G^t$ has one cycle. We can see that $G^t$ has one cycle for some $t$ if and only if $\delta_{\text{cyc}} = 1$. Thus, we get $\nu(F_p) = j(F_p) - n_p + \delta_{\text{cyc}}$. 

For any singular point $x$ of $R$, the multiplicity $\text{mult}_x(R)$ at $x$ does not exceed $r/m + 1$ since $R(\Gamma_p)_{\text{red}} = r/m$. Thus we have $\alpha_k = 0$ for $k \geq r/nm + 1$. Moreover, the following lemma holds.

Lemma 3.10. If $n \geq 3$, then we have $\alpha^{n\mathbb{Z} + 1}_{\frac{r}{n}}(F_p) = 0$. If $n = 2$, then we have $\kappa^{n\mathbb{Z} + 1}_{\frac{r}{nm}}(F_p) = 0$.

Proof. If $\alpha^{n\mathbb{Z} + 1}_{\frac{r}{nm}}(F_p) \neq 0$, then there exists an irreducible component $C$ of $\Gamma_p$ contained in $R$ and a singular point $x$ of $R$ on $C$ with multiplicity $r/m + 1$ such that any local horizontal branch of $R$ around $x$ is not tangential to $C$ since $RD_p = r/m$. Then, the exceptional curve $E$ for the blow-up at $x$ and the proper transform of $C$ form a singular point of multiplicity 2. Hence we have $n = 2$ from Lemma 1.2. It is clear that all singular points with multiplicity $r/m + 1$ are infinitely near to $x$ and the exceptional curves for blow-ups of these singularities form a chain. In particular, any singular point with multiplicity $r/m + 1$ is a 1-vertical type singularity. 

To prove Theorem 3.1, more inequalities among several indices are needed:

Lemma 3.11. (1) The following inequality holds.

$$\sum_{k \geq 1} k \left( \alpha^{n\mathbb{Z} + 1}_{\frac{r}{n}}(F_p) - \kappa_k(F_p) \right) \leq \left( \frac{r}{n} - 1 \right) \left( j^{(n)}(F_p) - \kappa(F_p) \right)$$

$$+ \left( \frac{r}{n} - 2 \right) \kappa^{(3)}(F_p) + \alpha^{n\mathbb{Z} + 1}_{\frac{r}{nm}}(F_p).$$

(2) If $n = 2$, then the following stronger inequality holds.
Similarly as in (1), the assertion (2) follows.

\[
\sum_{k \geq 1} k \left( \alpha_k^{2Z+1}(F_p) - \kappa_k(F_p) \right) \leq \sum_{k \geq 1} k\alpha_{(2k+1\rightarrow 2k+1)}(F_p) + \left( \frac{r}{2} - 1 \right) \left( \sum_{a \geq 2} j_{0,a}''(F_p) - \kappa(F_p) \right) + \left( \frac{r}{2} - 2 \right) \kappa^{(3)}(F_p) + \alpha^{2Z+1}_{2/m}(F_p).
\]

**Proof.** From Lemma 3.10, we have

\[
\sum_{k \geq 1} k \left( \alpha_k^{nZ+1}(F_p) - \kappa_k(F_p) \right) = \sum_{k=1}^{\frac{r}{nm}-1} k \left( \alpha_k^{nZ+1}(F_p) - \kappa_k(F_p) \right) + \frac{r}{nm} \alpha^{nZ+1}_{r/n}(F_p)
\]

\[
= \sum_{k=1}^{\frac{r}{nm}-1} k \left( \alpha_k^{nZ+1}(F_p) - \kappa_k^{(2)}(F_p) - \kappa_k^{(3)}(F_p) \right) - \sum_{k=1}^{\frac{r}{nm}-1} k\kappa_k^{(3)}(F_p) + \frac{r}{nm} \alpha^{nZ+1}_{r/n}(F_p).
\]

Since \( \alpha_k^{nZ+1}(F_p) - \kappa_k^{(2)}(F_p) - \kappa_k^{(3)}(F_p) \geq 0 \) and \( \sum_{k=1}^{\frac{r}{nm}-1} k\kappa_k^{(3)}(F_p) = k_0\kappa_k^{(3)}(F_p) \) for some \( 1 \leq k_0 \leq r/n - 1 \) from Lemma 3.6, we have

\[
\sum_{k=1}^{\frac{r}{nm}-1} k \left( \alpha_k^{nZ+1}(F_p) - \kappa_k^{(2)}(F_p) - \kappa_k^{(3)}(F_p) \right) \leq \left( \frac{r}{n} - 1 \right) \left( \sum_{k=1}^{\frac{r}{nm}-1} \alpha_k^{nZ+1}(F_p) - \kappa_k^{(2)}(F_p) - \kappa_k^{(3)}(F_p) \right) - k_0\kappa^{(3)}(F_p) + \frac{r}{n} \alpha^{nZ+1}_{r/n}(F_p)
\]

\[
= \left( \frac{r}{n} - 1 \right) \left( \sum_{k=1}^{\frac{r}{nm}-1} \alpha_k^{nZ+1}(F_p) - \kappa(F_p) \right) + \left( \frac{r}{n} - 1 - k_0 \right) \kappa^{(3)}(F_p) + \frac{r}{n} \alpha^{nZ+1}_{r/n}(F_p).
\]

Combining the above inequality with \( j''(F_p) = \sum_{k=1}^{\frac{r}{nm}} \alpha_k^{nZ+1}(F_p) \) and \( r/n - 1 - k_0 \leq r/n - 2 \), the assertion (1) follows.

Assume \( n = 2 \). Note that any \((2k + 1 \rightarrow 2k + 1)\) singularity is not involved in \( \kappa(F_p) \). Then we have

\[
\sum_{k \geq 1} k \left( \alpha_k^{2Z+1}(F_p) - \kappa_k(F_p) \right) = \sum_{k=1}^{\frac{r}{nm}-1} k \left( \alpha_k^{2Z+1}(F_p) - \alpha_{(2k+1\rightarrow 2k+1)}(F_p) - \kappa_k(F_p) \right) + \sum_{k \geq 1} k\alpha_{(2k+1\rightarrow 2k+1)}(F_p) + \frac{r}{2m} \alpha^{2Z+1}_{2/m}(F_p).
\]

Similarly as in (1), the assertion (2) follows. \( \square \)
3.6. The cases of $n = 2$ and $n = 3$. When $n = 2, 3$, more delicate inequalities between several indices are needed. First, we estimate the index $\kappa$ from above:

**Lemma 3.12.** If $n = 2$, then we have

$$\kappa(F_p) \leq \frac{2}{3} \sum_{a \geq 2} (a - 1) j_{0,a}(F_p) - \frac{2}{3} \alpha \frac{t^{2z+1}}{2m}(F_p).$$

**Proof.** It is sufficient to show that

$$(3.3) \quad \kappa^t(F_p) \leq \frac{2}{3} \sum_{a \geq 2} (a - 1) j_{0,a}(F_p) - \frac{2}{3} \alpha \frac{t^{2z+1}}{2m}(F_p)$$

for any $t$. If $\kappa^t(F_p) = 0$, then it is clear. Thus, we may assume $\kappa^t(F_p) > 0$. By the definition of $\kappa$ and Lemma 3.10, we have

$$(3.4) \quad j^t(F_p) \geq \kappa^{t,(2)}(F_p) + \kappa^{t,(3)}(F_p) + \alpha \frac{t^{2z+1}}{2m}(F_p) + 2.$$

Since any blow-up at a $(t, u)$-vertical type singularity contributes $-u$ to the number

$$\sum_{k \geq 1} (L_{t,k})^2 = - \sum_{a \geq 1} 2a j_{*,a}^t(F_p)$$

and $\Gamma_p$ contains no $u$-vertical type singularity with $u \geq 2$ if $\Gamma_p$ is of type $(m I_0)$, we get

$$\sum_{a \geq 1} 2a j_{*,a}^t(F_p) \geq j_{1,*}^t(F_p) + (2 + \delta_{m1,1}) j_{0,*}^t(F_p) + j_{0,*}^m(F_p)$$

$$+ \alpha \frac{t^{2z+1}}{2m}(F_p) + \sum_{u=2,3} u \left( t^{t,(u)}(F_p) + \kappa^{t,(u)}(F_p) \right).$$

Combining this inequality with $j^t(F_p) \geq j^t(F_p) - 1$ and (3.4), we have

$$\sum_{a \geq 1} 2a j_{*,a}^t(F_p) \geq 3 j^t(F_p) + (1 + \delta_{m1,1}) j_{0,*}^m(F_p) + \alpha \frac{t^{2z+1}}{2m}(F_p)$$

$$+ 2 \kappa^t(F_p) - 2 - (t^{t,(3)}(F_p) + \kappa^{t,(3)}(F_p))$$

$$\geq 2 j^t(F_p) + (1 + \delta_{m1,1}) j_{0,*}^m(F_p) + 2 \alpha \frac{t^{2z+1}}{2m}(F_p)$$

$$+ 3 \kappa^t(F_p) - t^{t,(3)}(F_p) - 2 \kappa^{t,(3)}(F_p).$$

On the other hand, it is easily seen from Lemma 3.6 that

$$(1 + \delta_{m1,1}) j_{0,*}^m(F_p) - t^{t,(3)}(F_p) - 2 \kappa^{t,(3)}(F_p) \geq 0.$$

Hence we get (3.3), as desired. \[\square\]

We need the following upper bounds of $j_{0,1}^t(F_p)$, $j_{0,1}^m(F_p)$:

**Lemma 3.13.** (1) If $n = 3$, then the following hold.

1. If $j_{0,1}^m(F_p) \leq 2$ for any $t$, then

$$\frac{1}{2} j_{0,1}^t(F_p) \leq \eta_p - \delta_{\text{cyc}}.$$
(1, ii) If \( j_{0,1}^n(F_p) = 3 \) for some \( t \), then \( \Gamma_p \) is a singular fiber of type (IV), (I\(_k^*\)), (II\(^*\)), (III\(^*\)) or (IV\(^*\)) and
\[
\frac{1}{3} j_{0,1}^t(F_p) \leq \eta_p', \quad \delta_{cyc} = 0.
\]

(1, iii) If \( j_{0,1}^n(F_p) = 4 \) for some \( t \), then \( \Gamma_p \) is a singular fiber of type (I\(_k^*\)) and any component of \( \Gamma_p \) is contained in \( R \). Moreover, we have \( \eta_p' = 1, \ j_{0,1}^t(F_p) = 4 \) and \( \delta_{cyc} = 0 \).

(2) Assume \( n = 2 \). Let \( j_{0,2,odd}(F_p) \) denote the number of irreducible components \( C \) of \( \Gamma_p \) involved in \( j_{0,2}(F_p) \) which has a singular point of \( R \) of odd multiplicity. Then the following hold.

(2, i) If \( j_{0,2,odd}(F_p) \leq 2 \) for any \( t \), then
\[
j_{0,1}^t(F_p) + \frac{1}{2} j_{0,2,odd}(F_p) \leq \eta_p' - \delta_{cyc}.
\]

(2, ii) If \( j_{0,2,odd}(F_p) = 3 \) for some \( t \), then \( \Gamma_p \) is a singular fiber of type (IV), (I\(_k^*\)), (II\(^*\)), (III\(^*\)) or (IV\(^*\)) and
\[
j_{0,1}^t(F_p) + \frac{1}{3} j_{0,2,odd}(F_p) \leq \eta_p', \quad \delta_{cyc} = 0.
\]

(2, iii) If \( j_{0,2,odd}(F_p) = 4 \) for some \( t \), then \( \Gamma_p \) is a singular fiber of type (I\(_k^*\)) and any component of \( \Gamma_p \) is contained in \( R \). Moreover, we have \( \eta_p' = 1, \ j_{0,1}^t(F_p) = 0, \ j_{0,2,odd}(F_p) = 4 \) and \( \delta_{cyc} = 0 \).

**Proof.** If \( n = 3 \), then any curve \( C \) in \( \Gamma_p \) contributing to \( j_{0,1}^t(F_p) \) intersects at most one component of \( \Gamma_p \) contained in \( R \), since \( C \) is blown up just once. Thus, considering the classification of singular fibers of elliptic surfaces, we can show easily the assertion (1).

Suppose that \( n = 2 \). Any curve in \( \Gamma_p \) contributing to \( j_{0,1}^t(F_p) \) is not blown up and any curve in \( \Gamma_p \) contributing to \( j_{0,2,odd}(F_p) \) intersects at most one component of \( \Gamma_p \) contained in \( R \). Hence we can show the assertion (2) similarly. \( \square \)

**Lemma 3.14.** (1) If \( n = 3 \), then we have
\[
j_{0,1}^m(F_p) \leq 2\eta_p' + \sum_{a \geq 1} 2a j_{1,a}^t(F_p) + \sum_{a \geq 2} (2a - 2) j_{0,a}^t(F_p) + \sum_{a \geq 2} (2a - 1) j_{0,0}^t(F_p).
\]

(2) If \( n = 2 \), then we have
\[
\sum_{k \geq 1} \alpha_{(2k+1 \rightarrow 2k+1)}^a(F_p) \leq j_{0,2,odd}(F_p) + \sum_{a \geq 3} (a - 1) j_{0,a}^t(F_p) + \sum_{a \geq 2} (a - 1) j_{0,a}^t(F_p)
\]
\[
+ \sum_{a \geq 3} (a - 2) j_{0,a}^t(F_p) + \tilde{\eta}_p.
\]

**Proof.** Suppose that \( n = 3 \). Let \( C_1, \ldots, C_{j_{0,1}^m(F_p)} \) be all \((-1\)-curves in \( \{C_{m,k}\}_k \) contributing to \( j_{0,1}^m(F_p) \) and \( x_i \) the point to which \( C_i \) contracts for \( i = 1, \ldots, j_{0,1}^m(F_p) \). If \( C_i \neq C^t_{i,k} \), then \( x_i \) is contained in some \( C^t_{i,k} \). If \( C^t_{i,k} \) contributes to \( j_{0,1}^m(F_p) \) and \( k = 1 \), then \( j_{0,1}^m(F_p) = j_{0,1}^m(F_p) = 2 \) from Example 1.15 (2). If \( C^t_{i,k} \) contributes to \( j_{0,1}^m(F_p) \) and \( k \neq 1 \), then the point \( x_{i,k} \) to which
$C^{t,k}$ contracts is contained in another $C^{t,k'}$ which does not contribute to $j_{0,1}^{m}(F_p)$ from the argument of Example 1.15 (2). Moreover, $x_i$ is also contained in $C^{t,k'}$ since the singularity diagram of $C^{t,k}$ is of type (iii) in Example 1.15 (2) and Lemma 1.8. For a curve $C^{t,k}$ which does not contribute to $j_{0,1}^{m}(F_p)$, we consider how many points among $x_1, \ldots, x_{j_0^{m}}(F_p)$ it contains.

(i) Assume that $C^{t,k}$ contributes to $j_{0,a}^{m}(F_p)$ for some $a \geq 2$. Then $C^{t,k}$ is blown up $3a-1$ times. Let $(x_{i,j}, m_{i,j}), i = 1, \ldots, l, j = 1, \ldots, i_{bm}$ be the entries of the singularity diagram of $C^{t,k}$. We consider a subset of entries of the $i$-th column of its diagram $\{(x_{i,j}, m_{i,j})\}_{j=1}^{\infty}$ satisfying that

(*) $m_{i,j_0} \in 3\mathbb{Z}$ if $j_0 > 0, m_{i,j} \in 3\mathbb{Z} + 1$ for $j_0 < j < j_0 + N$ and $m_{i,j_0+N} \in 3\mathbb{Z}$.

Note that the set of all entries of the singularity diagram is the union of these subsets. Then we can see that the exceptional curve $C^{t,k'}$ obtained by the blow-up at $x^{t,k}, j_0 + 1 < j < j_0 + N$ does not contribute to $j_{0,1}^{m}(F_p)$ from Lemma 1.14. Hence it contains at most $2a - 1$ points among $x_1, \ldots, x_{j_0^{m}}(F_p)$.

(ii) Assume that $C^{t,k}$ contributes to $j_{0,a}^{m}(F_p)$. Then $C^{t,k}$ is blown up $3a - 2$ times when it is a $(-2)$-curve or $3a - 3$ times when it is a singular rational curve. Hence it contains at most $2a - 2$ points among $x_1, \ldots, x_{j_0^{m}}(F_p)$ by the same argument as in (i).

(iii) Assume that $C^{t,k}$ contributes to $j_{1,a}^{m}(F_p)$. Then $C^{t,k}$ is blown up $3a$ times. Hence it contains at most $2a$ points among $x_1, \ldots, x_{j_0^{m}}(F_p)$ by the same argument as in (i).

We estimate $j_{0,1}^{m}(F_p)$ from (i), (ii), (iii) as follows.

(a) If $D'(p) = 0$ and $j_{0,a}^{m}(F_p) = 0$ for any $a \geq 2$, then we have shown that $j_{0,1}^{m}(F_p) \leq 2$ in Example 1.15 (2).

(b) If $D'(p) = 0$ and $j_{0,a}^{m}(F_p) > 0$ for some $a \geq 2$, then $x_i$ is the point to which $C^{t,1}$ contracts or contained in some $C^{t,k}$ which contributes to $j_{0,a}^{m}(F_p)$ for some $a \geq 2$. Hence we have

$$j_{0,1}^{m}(F_p) \leq 1 + \sum_{a \geq 2} (2a - 1)j_{0,a}^{m}(F_p).$$

(c) If $D'(p) \neq 0$, then $x_i$ is contained in some $C^{t,k}$ which does not contribute to $j_{0,1}^{m}(F_p)$. Hence we have

$$j_{0,1}^{m}(F_p) \leq \sum_{a \geq 2} (2a - 2)j_{0,a}^{m}(F_p) + \sum_{a \geq 1} 2a j_{1,a}^{m}(F_p) + \sum_{a \geq 2} (2a - 1)j_{0,a}^{m}(F_p).$$

From (a), (b) and (c), we have

$$j_{0,1}^{m}(F_p) \leq \eta_p + \eta_p' + \sum_{a \geq 2} (2a - 2)j_{0,a}^{m}(F_p) + \sum_{a \geq 1} 2a j_{1,a}^{m}(F_p) + \sum_{a \geq 2} (2a - 1)j_{0,a}^{m}(F_p)$$

by summing up for $t = 1, \ldots, \eta_p$. Combining this with $\eta_p \leq \eta_p''$, the claim (1) follows.

Suppose $n = 2$. Let $x^{t,k}$ be the point to which $C^{m,k}$ is contracted and $m^{t,k}$ the multiplicity of $R$ at $x^{t,k}$. If $D'(p) = 0$, then $x^{t,k}, k \geq 2$ is contained in $C^{t,k'}$ for some $k' < k$. Otherwise, $x^{t,1}$ is also contained in $C^{t,k'}$ for some $k'$. If $C^{t,k}$ is smooth, a singularity with odd multiplicity which is not contained in $C^{t,k'}$ for any $k' > k$ corresponds to an entry $(x_{i,j}, m_{i,j})$ of the
singularity diagram $D_{t,k}$ of $C_{t,k}$ satisfying that $m_{i,j-1}$ is even if $j > 1$, and $m_{i,j}$ is odd and then corresponds to a subset of entries of the diagram satisfying $(\ast)$. For a curve $C_{t,k}$, we consider how many such subsets of entries of its singularity diagram there are.

(iv) If $C_{t,k}$ contributes to $j_{0,a}^m(F_p)$, then $C_{t,k}$ is blown up $2a - 1$ times. Then the singularity diagram of $C_{t,k}$ has at most $a - 1$ subsets satisfying $(\ast)$.

(v) If $C_{t,k}$ contributes to $j_{0,a}^\prime(F_p)$ and it is a $(-2)$-curve, then $C_{t,k}$ is blown up $2a - 2$ times. Then the singularity diagram of $C_{t,k}$ has at most $a - 1$ subsets satisfying $(\ast)$.

(vi) If $C_{t,k}$ contributes to $j_{0,a}^\prime(F_p)$ and it is a singular rational curve, then $C_{t,k}$ is blown up $2a - 3$ times. Considering the singularity diagram of the proper transform of $C_{t,k}$ by the blow-up at its singular point, $C_{t,k}$ has at most $a - 1$ singularities with odd multiplicity which is not contained in $C_{t,k}'$ for any $k' > k$.

(vii) If $C_{t,k}$ contributes to $j_{0,a}^\prime(F_p)$, then $C_{t,k}$ is blown up $2a$ times. Then the singularity diagram of $C_{t,k}$ has at most $a$ subsets satisfying $(\ast)$.

We estimate $j^m(F_p)$ using (iv), (v), (vi) and (vii) as follows.

(d) If $D^m(p) = 0$, then the number of singularities with odd multiplicity appearing in $\{C_{t,k}\}_k$ is $j^m(F_p) - 1$. Hence we have

$$j^m(F_p) - 1 \leq \sum_{a \geq 2} (a - 1)j_{0,a}^m(F_p).$$

(e) If $D^m(p) \neq 0$, then the number of singularities with odd multiplicity appearing in $\{C_{t,k}\}_k$ is $j^m(F_p)$. Hence we have

$$j^m(F_p) \leq j_{0,1,\text{odd}}^m(F_p) + \sum_{a \geq 3} (a - 1)j_{0,a}^m(F_p) + \sum_{a \geq 1} aj_{1,a}^m(F_p) + \sum_{a \geq 2} (a - 1)j_{0,a}^\prime(F_p).$$

From (d) and (e), we have

$$j^m(F_p) \leq \eta^m_p + j_{0,1,\text{odd}}^m(F_p) + \sum_{a \geq 2} (a - 1)j_{0,a}^\prime(F_p) + \sum_{a \geq 1} aj_{1,a}^\prime(F_p) + \sum_{a \geq 2} (a - 1)j_{0,a}^\prime(F_p)$$

by summing up for $t = 1, \ldots, \eta_p$. Combining this with (1.12) and (1.14), the claim (2) follows.

\[\Box\]

**Lemma 3.15.** (1) If $n = 3$ and $j_{0,1}^\prime(F_p) \neq 0$, then we have

$$\chi_{\varphi}(F_p) \geq \frac{1}{12}(j_{0,1}^\prime(F_p) + 1).$$

(2) If $n = 2$, then the following hold.

(2,1) If $\Gamma_p$ is a singular fiber not of type $(m_I_k), (\Pi_k^*), (\Pi^*), (\Pi^*)$, then we have

$$\chi_{\varphi}(F_p) \geq \frac{1}{12}(2j_{0,1}^\prime(F_p) + j_{0,2}^\prime(F_p) + j_{0,3}^\prime(F_p) + 1).$$

Moreover, all the cases where $\Gamma_p$ is a singular fiber of type $(m_I_k), (\Pi^*), (\Pi^*), (\Pi^*)$ and

$$\chi_{\varphi}(F_p) < \frac{1}{12}(2j_{0,1}^\prime(F_p) + j_{0,2}^\prime(F_p) + j_{0,3}^\prime(F_p) + 1)$$

30
are as follows.

where, in the dual graphs of $\Gamma_p$, the symbols $\circ$, $\bullet$, $\star$ respectively denote a $(-2)$-curve not contained in $R$, contributing to $j^{\prime}_{0,1}(F_p)$, contributing to $j^{\prime}_{0,2}(F_p)$ or $j^{\prime}_{0,3}(F_p)$. In these cases, we have

$$\chi_{\varphi}(F_p) = \frac{1}{12}(2j^{\prime}_{0,1}(F_p) + j^{\prime}_{0,2}(F_p) + j^{\prime}_{0,3}(F_p) - 1)$$

when $\Gamma_p$ is of type $(\text{III}^*)$ and $j^{\prime}_{0,2}(F_p) = j^{\prime}_{0,3}(F_p) = 0$, and

$$\chi_{\varphi}(F_p) = \frac{1}{12}(2j^{\prime}_{0,1}(F_p) + j^{\prime}_{0,2}(F_p) + j^{\prime}_{0,3}(F_p))$$

otherwise.

(2, ii) If $\Gamma_p$ is a singular fiber of type $(I^*_k)$, then

$$\chi_{\varphi}(F_p) \geq \frac{1}{12}(2j^{\prime}_{0,1}(F_p) + j^{\prime}_{0,2}(F_p) + j^{\prime}_{0,3}(F_p) - 2)$$

with equality holding if and only if $\Gamma_p$ and $R$ satisfies the condition indicated in the following figure.
Proof. (1) Suppose that \( n = 3 \) and \( j_{0,1}'(F_p) \neq 0 \). If \( \Gamma_p \) is not of type \((mI_k)\), the claim is clear. Thus we may assume that \( \Gamma_p \) is of type \((mI_k)\). If \( \chi_\varphi(F_p) = j_{0,1}'(F_p)/12 \), then any component of \( \Gamma_p \) contributes to \( j_{0,1}'(F_p) \) and contains at least 2 singular points of \( R \), which is a contradiction.

(2) Suppose that \( n = 2 \). Any irreducible component \( C \) of \( \Gamma_p \) contributing to \( j_{0,1}'(F_p) \) has no singular points of \( R \). Thus any component of \( \Gamma_p \) intersecting with the curve \( C \) is not contained in \( R \). From this observation and the classification of singular fibers of elliptic surfaces, the claims (2,i) and (2,ii) follow by an easy combinatorial argument. \( \square \)

3.7. Proof of the main theorem. Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let \( f : S \to B \) be a primitive cyclic covering fibration of type \((g,1,n)\). Let \( \mu \) be an arbitrary non-negative number and \( K_f^2(F_p), \chi_f(F_p) \) and \( e_f(F_p) \) as in Definition 2.2. Let us estimate \( (12 - \mu)K_f\chi_f(F_p) - K_f^2(F_p) = e_f(F_p) - \mu\chi_f(F_p) \) by using the (in)equalities established above.

From Lemma 3.8, we have

\[
e_f(F_p) - \mu\chi_f(F_p) = A_n\alpha_0^+(F_p) + \sum_{k \geq 1} (n - \mu'k) \alpha_k(F_p) + n \sum_{k \geq 1} \alpha_k''(F_p) - \mu' \sum_{k \geq 1} k\alpha_k''(F_p) - \sum_{a \geq 1} (2A_n + \delta_{a=1})j_{0,a}(F_p) + C_n\chi_\varphi(F_p) - \frac{r\mu'}{n}\nu(F_p) = A_n\alpha_0^+(F_p) + \sum_{k \geq 1} (n - \mu'k) \alpha_k(F_p) + C_n\chi_\varphi(F_p) - \frac{r\mu'}{n} \left(1 - \frac{1}{m}\right) - \mu'\gamma_p + mn\nu' + \sum_{a \geq 1} (an^2 - a\mu') j_{1,a}'(F_p) + \sum_{a \geq 1} (n(an - 2 - \delta_{a=1}) - a\mu' - 2A_n - \delta_{a=1}) j_{0,a}'(F_p) + \sum_{a \geq 1} (n(an - 1) - a\mu' - 2A_n - \delta_{a=1}) j_{0,a}''(F_p) - \mu' \sum_{k \geq 1} k\alpha_k^{nZ+1}(F_p) - \sum_{k \geq 1} (n - \mu'k) \kappa_k(F_p) - \sum_{k \geq 1} (n - \mu'k) \kappa_k(F_p),\]

where we put

\[
A_n := n - 1 - \frac{(n-1)(2n-1)}{12n}\mu, \quad C_n := 12n - \left(\frac{r}{12n}(n-1)(n+1) + n\right)\mu
\]
and $\mu' := (n - 1)(n + 1)\mu/12$. Combining Lemma 3.9 with the above equality, we have

$$e_f(F_p) - \mu \chi_f(F_p) \geq A_n\alpha_0^+ (F_p) + \sum_{k \geq 1} (n - \mu'k) \alpha_k^+ (F_p) + C_n \chi (F_p) - \frac{r\mu'}{n} \left(1 - \frac{1}{m}\right) + \mu' \left(\frac{r}{n} - j_{0,1}(F_p)\delta_{n=2} - \delta_{m1,\mathbb{II}}\right) \delta_{\eta_p' \neq 0} + (n - \mu') \left(\eta_p' - \delta_{\text{cyc}}\right) + \left(2n - \frac{r\mu'}{n}\right) \eta_p''$$

(3.5) $\sum_{a \geq 1} n (an - 3 - \delta_{m1,\mathbb{II}}) - (a - 1)\mu' - 2A_n - \delta_{a=1}) j_{0,a}^0 (F_p)$

$\sum_{a \geq 1} (n(an - 2) - (a - 1)\mu' - 2A_n - \delta_{a=1}) j_{0,a}'' (F_p)$

$\sum_{a \geq 1} (n(an - 1) - (a - 1)\mu') j_{1,a}^0 (F_p) - \mu' \sum_{k \geq 1} k \alpha_k^{n\mathbb{Z}+1}(F_p) - \sum_{k \geq 1} (n - \mu'k) \kappa_k(F_p)$.

Assume that $A_n \geq 0$ and $C_n \geq 0$. We obtain by using Lemmas 3.7 and 3.11 (1) that

$$e_f(F_p) - \mu \chi_f(F_p) \geq \sum_{k \geq 1} (n - \mu'k) \alpha_k^+ (F_p) + \frac{r(m-1)(n-1)(4-\mu)}{4m} + C_n \chi (F_p) - \mu' \left(\frac{r}{n} - \delta_{m1,\mathbb{II}}\right) \delta_{\eta_p' \neq 0} + A_n \beta_p + (n - \mu' - (n - 2)A_n) \left(\eta_p' - \delta_{\text{cyc}}\right) + \left(2n - \frac{r\mu'}{n} - (n - 2)A_n\right) \eta_p''$$

(3.6) $\sum_{a \geq 1} ((n - 4)A_n + n(an - 3 - \delta_{m1,\mathbb{II}}) - (a - 1)\mu' - (1 - \delta_{n=2\mu'}\delta_{a=1}) j_{0,a}^0 (F_p)$

$\sum_{a \geq 1} ((n - 4)A_n + n(an - 2) - \mu' (a + \frac{r}{n} - 2) - \delta_{a=1}) j_{0,a}'' (F_p)$

$\sum_{a \geq 1} ((n - 2)A_n + n(an - 1) - (a - 1)\mu') j_{1,a}^0 (F_p)$

$\left(2n - 2\right)A_n - n + \left(\frac{r}{n - 1}\right) \mu' \kappa(F_p) - \mu' \left(\frac{r}{n - 2}\right) \kappa(3)(F_p) - \mu' \alpha^{n\mathbb{Z}+1}(F_p)$.

We put

$$\mu := \begin{cases} 
\frac{12n^2}{r(n-1)(n+1)}, & \text{if } n \geq 4, \\
\frac{24}{4r - 13}, & \text{if } n = 3, \\
\frac{4}{r - 2}, & \text{if } n = 2.
\end{cases}$$

Then it follows that $A_n > 0$ and $C_n > 0$. 33
We take cases. In each case, we will estimate the right hand side of (3.6) from below and check that all the coefficients of the indices are non-negative.

**Case (i):** We assume \( n \geq 4 \). We write \( r = kn \). The coefficient of \( \eta'_p \) is

\[
- \frac{1}{12} (n-1)(n+1) \mu - (n-2) A_n = -\frac{n^2 - 6n + 2}{k(n+1)} < 0.
\]

The coefficient of \( \eta''_p \) is

\[
2n - \frac{(n-1)(n+1)r \mu}{12n} - (n-2) A_n = -\frac{(n-2)(2n-1)}{k(n+1)}.
\]

It is negative if \( n \geq 5 \) or \( k \geq 2 \). Note that \( \eta''_p = 0 \) if \( k < n - 1 \) since the multiplicity \( m' \) of a singular point of type \( n \mathbb{Z} + 1 \) satisfies \( (n-1)^2 \leq m' \leq r - n + 1 \). Thus, we may not consider the case where \( n = 4 \) and \( k = 1 \). Using \( \eta'_p < j'(F_p) \) and \( \eta''_p < j''(F_p) \), the right hand side of (3.6) is greater than or equal to

\[
\sum_{k \geq 1} (n - \mu' k) \alpha'_k(F_p) + \frac{r(m-1)(n-1)(4-\mu)}{4m} + C_n X \varphi(F_p) + A_n \beta_p
\]

\[
- \mu' \left( \frac{r}{n} - \delta_{m_1, \#} \right) \delta_{\eta'_p \neq 0} + ((n-2) A_n - n + \mu') \delta_{\text{cyc}} + \sum_{a \geq 1} (a n^2 - a \mu') j'_{0,a}(F_p)
\]

\[
\sum_{a \geq 1} (n(n - 2 - \delta_{m_1, \#}) - a \mu' - 2 A_n - \delta_{a=1}) j'_{0,a}(F_p)
\]

\[
+ \sum_{a \geq 1} \left( a n^2 - \mu' \left( a + \frac{2r}{n} - 2 \right) - 2 A_n - \delta_{a=1} \right) j''_{0,a}(F_p)
\]

\[
+ \left( 2(n-2) A_n - n + \left( \frac{r}{n} - 1 \right) \mu' \right) \kappa(F_p) - \mu' \left( \frac{r}{n} - 2 \right) \kappa^{(3)}(F_p).
\]

Since \( 2(n-2) A_n - n + \mu' > 0 \), we have

\[
\left( 2(n-2) A_n - n + \left( \frac{r}{n} - 1 \right) \mu' \right) \kappa(F_p) - \mu' \left( \frac{r}{n} - 2 \right) \kappa^{(3)}(F_p)
\]

\[
\geq (2(n-2) A_n - n + \mu') \kappa(F_p)
\]

\[
\geq 0.
\]

The coefficient of \( j'_{0,1}(F_p) \) in (3.7) is

\[
n(n - 2 - \delta_{m_1, \#}) - \mu' - 2 A_n - \delta_{a=1}
\]

\[
= n^2 - (4 + \delta_{m_1, \#}) n + 1 - \frac{(n-1)(n-2)}{k(n+1)}.
\]

If \( \delta_{m_1, \#} = 0 \), (3.8) is negative if and only if \( n = 4 \) and \( k = 1 \). If \( \delta_{m_1, \#} = 1 \) and \( n \geq 5 \), (3.8) is non-negative. If \( \delta_{m_1, \#} = 1 \) and \( n = 4 \), (3.8) is \(-3 - 6/5k < 0\). Note that \( j'(F_p) = 0 \) if \( k = 1 \) since \( r = n \) and any singularity of \( R \) has the multiplicity \( n \). Thus, we may not consider the case where \( \delta_{m_1, \#} = 0 \), \( n = 4 \) and \( k = 1 \). We can check that the coefficient of \( j'_{0,a}(F_p) \) in (3.7)
is positive for \( a \geq 2 \). Moreover, we can also check that the coefficient of \( j^{''}_{0,a}(F_p) \) in (3.7) is positive for \( a \geq 1 \).

**Case (i,1):** We assume that \( \eta'_p = 0 \). Then, clearly (3.7) is non-negative.

**Case (i,2):** We assume that \( \eta'_p \neq 0, \chi_p(F_p) \geq 1/6 \) (i.e., \( \Gamma_p \) is a singular fiber not of type \((mI_1)\)) and \( \iota^{(3)}(F_p) = 0 \). Then we have

\[
C_n \chi_p(F_p) \geq \frac{1}{6} C_n = \frac{n}{6} \left( 11 - \frac{12n}{k(n-1)(n+1)} \right).
\]

The coefficient of \( \delta_{\eta',\neq0} \) is

\[
\mu' \left( \frac{r}{n} - \delta_{m_{1,\Pi}} \right) = -n + \frac{n}{k} \delta_{m_{1,\Pi}}.
\]

If \( n \neq 4 \) or \( \delta_{m_{1,\Pi}} = 0 \) or \( j^{''}_{0,1}(F_p) = 0 \), then (3.7) is positive since

\[
\frac{n}{6} \left( 11 - \frac{12n}{k(n-1)(n+1)} \right) - n = \frac{n}{6} \left( 5 - \frac{12n}{k(n-1)(n+1)} \right) > 0.
\]

Assume that \( n = 4 \) and \( \delta_{m_{1,\Pi}} = 1 \) and \( j^{''}_{0,1}(F_p) \neq 0 \). We denote this condition by \((\#)\). Then we have

\[
j^{''}_{0,1}(F_p) = 1 \quad \text{and then (3.7) is positive since}
\]

\[
\frac{n}{6} \left( 11 - \frac{12n}{k(n-1)(n+1)} \right) - n + \frac{n}{k} - 3 - \frac{6}{5k} = \frac{1}{3} + \frac{26}{15} > 0.
\]

**Case (i,3):** We assume that \( \eta'_p \neq 0 \) and \( \chi_p(F_p) = 1/12 \) (i.e., \( \Gamma_p \) is a singular fiber of type \((mI_1)\)). Let \( m_1 \) be the multiplicity of the singular point \( x_1 \) of \( R \) which is singular for \( \Gamma_p \). If \( m_1 \in n\mathbb{Z} \), then \( x_1 \) contributes \( m_1 - 2 \) to \( \alpha_0^+(F_p) \) (Note that \( x_1 \) is a 1-vertical type singularity). In particular, \( x_1 \) contributes at least \( n - 2 \) to \( \alpha_0^+(F_p) \). If the condition \((\#)\) does not hold, (3.7) is positive since

\[
\frac{n}{12} \left( 11 - \frac{12n}{k(n-1)(n+1)} \right) - n + (n - 2)A_n = n^2 - \frac{37}{12}n + 2 - \frac{n^2 + (n - 2)(n - 1)(2n - 1)}{k(n-1)(n+1)}
\]

increases monotonically with respect to \( n \) and the value at \( n = 4 \) is

\[
\frac{17}{3} - \frac{58}{15k} > 0.
\]

If the condition \((\#)\) holds, (3.7) is also positive since

\[
\frac{17}{3} - \frac{58}{15k} + \frac{4}{k} - \frac{6}{5k} = \frac{8}{3} - \frac{16}{15k} > 0.
\]

**Case (i,4):** We assume that \( \eta'_p \neq 0 \) and \( \chi_p(F_p) = 0 \) (i.e., \( \Gamma_p \) is a smooth elliptic curve). Then (3.7) is positive since \( j^{''}_{1,\cdot}(F_p) = 1 \).

**Case (i,5):** We assume that \( \iota^{(3)}(F_p) = 1 \). From Lemma 3.6, we may consider the following 3 cases.
Case (i,5,II): If $\Gamma_p$ is a singular fiber of type (II), then $\beta_p = n - 7$, $\chi_\varphi(F_p) = 1/6$ and $\kappa(F_p) = \kappa^{(2)}(F_p) \geq 1$. From the argument in the case (i,2), it is sufficient to show that

$$A_n\beta_p + \left(2(n - 2)A_n - n + \left(\frac{r}{n} - 1\right)\mu'\right)\kappa(F_p) > 0.$$ 

This inequality is true, since

$$(n - 7)A_n + 2(n - 2)A_n - n + \left(\frac{r}{n} - 1\right)\mu' = (3n - 11)\left(n - 1 - \frac{2n - 1}{k(n + 1)}\right) - \frac{n}{k} > 0.$$ 

Note that $k \geq n - 1$ since $j''(F_p) \neq 0$.

Case (i,5,III): If $\Gamma_p$ is a singular fiber of type (III), then $\beta_p = -n - 1$, $\chi_\varphi(F_p) = 1/4$ and $\kappa(F_p) = \kappa^{(2)}(F_p) \geq 1$. Since $j''(F_p) \geq 1$ and the coefficient of $j''_0,F_p$ is greater than 1 for any $a \geq 1$, it is sufficient to show that

$$1 + A_n\beta_p + \frac{1}{12}C_n + \left(2(n - 2)A_n - n + \left(\frac{r}{n} - 1\right)\mu'\right)\kappa(F_p) \geq 0.$$ 

The left hand side of it is greater than or equal to

$$1 - (n + 1)A_n + \frac{1}{12}C_n + 2(n - 2)A_n - n + \left(\frac{r}{n} - 1\right)\mu' = \frac{n^2 - 61}{12}n + 6 - \left(\substack{3n^3 - 12n^2 + 15n - 5 \\ k(n - 1)(n + 1)}\right),$$

and (3.9) increases monotonically with respect to $n$. If $n = 4$, (3.9) is equal to $5/3 - 11/3k$ and it is positive. Note that $k \geq n - 1$ since $j''(F_p) \neq 0$.

Case (i,5,IV): If $\Gamma_p$ is a singular fiber of type (IV), then $\beta_p = -2$, $\chi_\varphi(F_p) = 1/3$ and $j''_0,F_p \geq 3$. Thus, it is sufficient to show that $A_n\beta_p + C_n/6 + 3 \cdot (3.8)$ is positive. By a computation, this is equal to

$$3n^2 - \frac{73}{6}n + 5 - \frac{3n^3 - 14n^2 + 21n - 8}{k(n - 1)(n + 1)}$$

and we can check that it is positive.

From (i,1) through (i,5), we have $e_f(F_p) - \mu\chi_\varphi(F_p) \geq 0$ for $n \geq 4$. On the other hand, if $n = 3$ and $g = 4$, one can easily classify all singular fibers of primitive cyclic covering fibrations of type $(4,1,3)$ because $R$ has no singularities of multiplicity greater than 3, and check $e_f(F_p) \geq (9/2)\chi_\varphi(F_p)$ for any fiber germ $F_p$. Hence Theorem 3.1 (1) follows.

Case (ii): We assume $n = 3$ and $g > 4$. The coefficient of $j''_0,F_p$ in the right hand side of (3.6) is

$$-\left(\frac{2}{9}r + \frac{17}{18}\right)\mu < 0.$$
Applying Lemma 3.14 (1) to the term of $j_0''(F_p)$, the right hand side of (3.6) is greater than or equal to

$$
\sum_{k \geq 1} \left(3 - \frac{2}{3}\mu k\right) a_k'(F_p) + \frac{r(m - 1)(4 - \mu)}{2m} + \left(36 - \frac{2}{9}\mu + 3\right) \chi(F_p)
$$

$$
+ \left(2 - \frac{5}{18}\mu\right) \beta_p - \frac{2}{3}\mu \left(\frac{r}{3} - \delta_{m_1, h}\right) \delta_{\eta_p'} \neq 0 + \left(1 - \frac{7}{18}\mu\right) (\eta_p' - \delta_{\text{cyc}})
$$

$$
+ \sum_{\alpha \geq 2} \left(9a - 11 - 3\delta_{m_1, h} - \left(\frac{4}{9}(a - 1)r - \frac{11}{9}a + \frac{17}{18}\mu\right) \right) j_{0,\alpha}(F_p)
$$

$$
+ \left(-3 + \frac{5}{18}\mu\right) j_{0,1}(F_p) + \sum_{\alpha \geq 2} \left(9a - 8 - \left(\frac{4}{9}ar - \frac{11}{9}a + \frac{2}{3}\mu\right) \right) j_{0,\alpha}(F_p)
$$

$$
+ \sum_{\alpha \geq 1} \left(9a - 1 - \left(\frac{4}{9}ar - \frac{11}{9}a + \frac{7}{18}\mu\right) \right) j_{1,\alpha}(F_p)
$$

$$
+ \left(1 + \left(\frac{2}{9}r - \frac{11}{9}\right) \mu\right) \kappa(F_p) - \frac{2}{3}\mu \left(\frac{r}{3} - 2\right) \kappa^{(3)}(F_p).
$$

We remark that the term of $\eta_p''$ vanishes by the definition of $\mu$ and

$$
\left(1 + \left(\frac{2}{9}r - \frac{11}{9}\right) \mu\right) \kappa(F_p) - \frac{2}{3}\mu \left(\frac{r}{3} - 2\right) \kappa^{(3)}(F_p)
$$

$$
\geq \left(1 + \frac{1}{9}\mu\right) \kappa(F_p)
$$

$$
\geq 0.
$$

We can check that the coefficient of $j_{0,\alpha}(F_p)$ (resp. $j_{0,\alpha}(F_p)$, $j_{1,\alpha}(F_p)$) in (3.10) are positive for $a \geq 2$ (resp. $a \geq 2$, $a \geq 1$).

**Case (ii,1):** Assume that $\eta_p' = 0$. Then (3.10) is non-negative.

**Case (ii,2):** Assume that $\eta_p' \neq 0$ and $j_{0,1}''(F_p) = 4$ for some $t$. From Lemma 3.13 (1,iii), it follows that $\Gamma_p$ is a singular fiber of type $(I_k^*)$ for some $k$; $\chi(F_p) = (k + 6)/12$, $r \geq 9$, $\Gamma_p \subset R$, $\eta_p' = 1$, $j_{0,1}(F_p) = 4$ and $\delta_{\text{cyc}} = 0$. Considering the terms of $\chi(F_p)$, $\delta_{\eta_p' \neq 0}$, $\eta_p'$ and $j_{0,1}(F_p)$, (3.10) is greater than or equal to

$$
\left(36 - \frac{2}{9}\mu + 3\right) \mu K + \frac{r + 6}{12} - \frac{2}{9}r\mu + \left(1 - \frac{7}{18}\mu\right) + 4 \left(-3 + \frac{5}{18}\mu\right)
$$

$$
= 7 - \frac{8(3r + 2)}{3(4r - 13)}.
$$

This is positive since $r \geq 9$. 37
Case (ii,3): Assume that \( \eta'_p \neq 0 \), \( \iota^3(F_p) = 0 \), \( r \geq 9 \) and \( j''_0,1(F_p) \geq 3 \) for any \( t \). Then

\[
\frac{1}{3}j''_0,1(F_p) \leq \eta'_p - \delta_{cyc}
\]

from Lemma 3.13 (1,i), (1,ii). From this and Lemma 3.15 (1), (3.10) is greater than or equal to

\[
\left( 36 - \left( \frac{2}{9}r + 3 \right) \mu \right) \frac{j''_0,1(F_p) + 1}{12} - \frac{2}{9}r \mu + \frac{1}{3} \left( 1 - \frac{7}{18} \mu \right) j'_0,1(F_p) + \left( -3 + \frac{5}{18} \mu \right) j''_0,1(F_p).
\]

One can check by a computation that this is positive since \( r \geq 9 \).

Case (ii,4): Assume that \( \eta'_p \neq 0 \), \( \iota^3(F_p) = 0 \), \( r = 6 \) and \( \chi_\varphi(F_p) \geq (j''_0,1(F_p) + 2)/12 \). By the same argument as in the case (ii,3), (3.10) is greater than or equal to

\[
\left( 36 - \left( \frac{2}{9}r + 3 \right) \mu \right) \frac{j''_0,1(F_p) + 2}{12} - \frac{2}{9}r \mu + \frac{1}{3} \left( 1 - \frac{7}{18} \mu \right) j'_0,1(F_p) + \left( -3 + \frac{5}{18} \mu \right) j''_0,1(F_p)
= -\frac{13}{99}j''_0,1(F_p) + \frac{50}{33}.
\]

On the other hand, one sees that \( j''_0,1(F_p) \leq 6 \) since \( r = 6 \). Then

\[
-\frac{13}{99}j''_0,1(F_p) + \frac{50}{33} > 0.
\]

Case (ii,5): Assume that \( \eta'_p \neq 0 \), \( \iota^3(F_p) = 0 \), \( r = 6 \) and \( \chi_\varphi(F_p) < (j''_0,1(F_p) + 2)/12 \). If \( j''_0,1(F_p) = 0 \), then (3.10) is positive since \( j'(F_p) \neq 0 \). Then there are the following two cases only.

Case (ii,5,1): \( \Gamma_p \) is a singular fiber of type \((n, I_2)\) and only one component of \( \Gamma_p \) is contained in \( R \) and brown up just once.

Case (ii,5,2): \( \Gamma_p \) is a singular fiber of type \((n, I_3)\) and only two component of \( \Gamma_p \) are contained in \( R \) and brown up just once at the intersection point of these.

In both cases, we can see that

\[
\gamma_p \leq \left( \frac{r}{n} - 1 \right) \delta_{\eta'_p \neq 0} + \left( \frac{r}{n} - 1 \right) \eta''_p
\]

from the proof of Lemma 3.9, since the component of \( \Gamma_p \) not contained in \( R \) intersects \( R_h \). Thus, it is sufficient to show that

\[
\left( 36 - \left( \frac{2}{9}r + 3 \right) \mu \right) \frac{j''_0,1(F_p) + 1}{12} - \frac{2}{3} \mu \left( \frac{r}{3} - 1 \right) + \left( 1 - \frac{7}{18} \mu \right) + \left( -3 + \frac{5}{18} \mu \right) j''_0,1(F_p)
\]

is positive. This is equal to \((-2j''_0,1(F_p) + 10)/11 > 0\) by a computation.

Case (ii,6): Assume that \( \iota^3(F_p) = 1 \). From Lemma 3.6, we may consider the following 3 cases.

Case (ii,6,1): If \( \Gamma_p \) is a singular fiber of type (II), then we have \( \chi_\varphi(F_p) = 1/6 \), \( \beta_p = -4 \), \( \delta_{cyc} = 0 \), \( j'_0,1(F_p) = 0 \), \( j''_0,1(F_p) = 1 \), \( j''_0,1(F_p) \geq 2 \) and \( \kappa(F_p) = \kappa^3(F_p) \geq 1 \). One can see easily that (3.10) is positive by a computation.
**Case (ii,6,III):** If $\Gamma_p$ is a singular fiber of type (III), then we have $\chi_p(F_p) = 1/4$, $\beta_p = -4$, $\delta_{\text{cyc}} = 0$, $j_0'(F_p) = 0$, $j_0''(F_p) = 2$, $j_0''(F_p) \geq 1$ and $\kappa(F_p) = \kappa^{(2)}(F_p) \geq 1$. One can see easily that (3.10) is positive by a computation.

(ii,6,IV) If $\Gamma_p$ is a singular fiber of type (IV), then we have $\chi_p(F_p) = 1/3$, $\beta_p = -2$, $\delta_{\text{cyc}} = 0$, $j_0''(F_p) = 3$ and $\kappa^{(3)}(F_p) = 0$. If $j_0'(F_p) \leq 2$, then we can check that (3.10) is positive. Suppose that $j_0'(F_p) = 3$. Then any component of $\Gamma_p$ is brown up only once. Thus, the multiplicity of $R$ at the singularity on $\Gamma_p$ is $r/3 + 3 \geq 6$. In particular, $r \geq 9$. Since this singularity is a 3-vertical 3Z type singularity, we have

$$\alpha^+_0(F_p) \geq \iota(F_p) + 2\kappa(F_p) + \beta_p + 3$$

from the proof of Lemma 3.7. Then it suffices to show that

$$\left(2 - \frac{5}{18} \mu\right) + \left(36 - \left(\frac{2}{9} r + 3\right) \mu\right) \frac{1}{3} - \frac{2}{9} r \mu + \left(1 - \frac{7}{18} \mu\right) + 3 \left(-3 + \frac{5}{18} \mu\right)$$

is positive. This is equal to

$$6 - \frac{4(16r + 45)}{9(4r - 13)} > 0.$$  

From (ii,1) through (ii,6), we have $e_f(F_p) - \mu \chi_f(F_p) \geq 0$. Hence Theorem 3.1 (2) follows.

**Case (iii):** We assume $n = 2$ and $g \geq 3$. From Lemmas 3.7 and 3.11 (2), the right hand side of (3.5) is greater than or equal to

$$\sum_{k \geq 1} \left(2 - \frac{1}{4} \mu k\right) \alpha_k(F_p) + \frac{r(m - 1)(4 - \mu)}{4m} + \left(24 - \left(\frac{r}{8} + 2\right) \mu\right) \chi_p(F_p)$$

$$- \frac{1}{4} \mu \left(\frac{r}{2} - j_0'(F_p) - \delta_{m_{11}, II}\right) \delta_{\eta_0,' \neq 0} + \left(2 - \frac{1}{4} \mu\right) \left(\eta'_p - \delta_{\text{cyc}}\right) + \left(4 - \frac{1}{8} r \mu\right) \eta_p$$

$$+ \sum_{a \geq 1} \left(4a - 8 - 2\delta_{m_{11}, II} - \delta_{r = 1} - \frac{1}{4}(a - 2) \mu\right) j_{0, a}'(F_p)$$

$$+ \sum_{a \geq 2} \left(4a - 6 - \frac{1}{4}(a + \frac{r}{2} - 3) \mu\right) j_{0, a}''(F_p) + \sum_{a \geq 1} \left(4a - 2 - \frac{1}{4}(a - 1) \mu\right) j_{1, a}'(F_p)$$

$$+ \sum_{k \geq 1} \left(1 - \frac{1}{8}(r - 2) \mu - \frac{1}{4} \mu k\right) \alpha^{(2k+1-2k+1)}(F_p) - \frac{1}{4} \mu \alpha^{(2k+1)}(F_p)$$

$$+ \sum_{k \geq 1} \left(1 - \frac{1}{8}(r - 2) \mu + \left(2 - \frac{1}{2} \mu\right) k\right) \alpha^{(2k+1)}(F_p) - \left(2 - \frac{1}{8}(r - 2) \mu\right) \kappa(F_p)$$

$$+ \sum_{k \geq 1} \left(-3 + \frac{1}{4} \mu + \left(2 - \frac{1}{2} \mu\right) k\right) \alpha^{(2k+1)}(F_p) - \frac{1}{8}(r - 4) \mu \kappa^{(3)}(F_p).$$
The coefficients of $\alpha_{(2k+1 \rightarrow 2k+1)}^{tr}(F_p)$ and $\alpha_{(2k+1 \rightarrow 2k+1)}^{co,0}(F_p)$ in (3.11) are non-negative and that of $\alpha_{(r-1 \rightarrow r-1)}^{tr}(F_p)$ is 0 by the definition of $\mu$. The coefficient of $\alpha_{(2k+1 \rightarrow 2k+1)}^{co,1}(F_p)$ in (3.11) is positive except for $k = 1$ and that of $\alpha_{(3 \rightarrow 3)}^{co,1}(F_p)$ is $-1 - \mu/4 < 0$. The coefficient of $\kappa(F_p)$ in (3.11) is equal to $-3/2$. Applying Lemmas 3.12 and 3.14 (2) to (3/2)$\kappa(F_p)$ and $(1 + \mu/4) \sum_{k \geq 1} \alpha_{(2k+1 \rightarrow 2k+1)}^{co,1}(F_p)$, we see that (3.11) is greater than or equal to

$$
\sum_{k \geq 1} \left(2 - \frac{1}{4} \mu k\right) \alpha_k(F_p) + \frac{r(m - 1)(4 - \mu)}{4m} + \left(24 - \left(\frac{r}{8} + 2\right) \mu\right) \chi_{\varphi}(F_p)
-
\frac{1}{4} \mu \left(\frac{r}{2} - \delta_{m,1,\#}\right) \eta_p \left(\delta_{m,1,\#} \leq 0\right) + \left(2 - \frac{1}{4} \mu\right) \left(\eta_p - \delta_{cyc}\right) + \left(3 - \frac{1}{8} (r + 2) \mu\right) \tilde{\eta}_p
-
\left(5 - \frac{1}{2} \mu\right) \eta_p^0(F_p) - \left(5 - \frac{1}{2} \mu\right) \eta_p^0(F_p) - \left(2 + \frac{1}{4} \mu\right) \eta_p^0(F_p)
-
\left(2 - \frac{1}{2} \mu\right) \eta_p^0(F_p) + \sum_{a \geq 1} \left(2a - 6 - \frac{1}{2} (a - 3) \mu\right) \eta_p^0(F_p)
$$

(3.12)

where we put $\eta_p^0(F_p) := \eta_p(F_p) - \frac{1}{4} \mu \eta_p^0(F_p)$.

**Case (iii,1):** Assume that $\eta_p = 0$. Then (3.12) is clearly non-negative.

**Case (iii,2):** Assume that $\eta_p \neq 0$ and $\eta_p^t(F_p) = 4$ for some $t$. From Lemma 3.13 (2,iii), $\Gamma_p$ is a singular fiber of type $(I_k^m)$, $\Gamma_p \subset R$, $k^3(F_p) = 0$, $\eta_p = 1$, $\delta_{cyc} = 0$, $\eta_p^0(F_p) = 0$ and $\eta_p^0(F_p) = 4$. Clearly we have $\chi_{\varphi}(F_p) = (\eta_p^0(F_p) + \eta_p^0(F_p) + 1)/12$. Considering the terms of $\chi_{\varphi}(F_p)$, $\delta_{m,1,\#} \neq 0$, $\eta_p$, $\eta_p^0(F_p)$, $\eta_p^0(F_p)$ and $\eta_p^0(F_p)$, (3.12) is greater than or equal to

$$
(24 - \left(\frac{r}{8} + 2\right) \mu) \frac{1}{12} \left(\eta_p^0(F_p) + \eta_p^0(F_p) + 5\right) - \frac{1}{8} r \mu + \left(2 - \frac{1}{4} \mu\right)
$$

$$
- \frac{3}{4} \mu \eta_p^0(F_p)
$$

$$
= \left(1 - \left(\frac{r}{96} + \frac{1}{6}\right) \mu\right) \eta_p^0(F_p) + \left(2 - \left(\frac{r}{96} + \frac{11}{12}\right) \mu\right) \eta_p^0(F_p) + 4 - \left(\frac{17}{96} r + \frac{25}{12}\right) \mu,
$$

40
which is positive.

Case (iii,3): Assume that \( \eta_p' \neq 0 \), \( \kappa^{(3)}(F_p) = 0 \), \( \delta_{m_1, II} = 0 \), \( j_{0, 2, odd}(F_p) \leq 3 \) for any \( t \) and

\[
\chi_\varphi(F_p) \geq \frac{1}{12} (2j_0'(F_p) + j_0''(F_p) + 1).
\]

From Lemma 3.13 (2,i), (2,ii), we have \( j_0'(F_p) + j_0''(F_p)/3 \leq \eta' - \delta_{cyc} \). Then (3.12) is greater than or equal to

\[
(24 - (\frac{8}{r} + 2) \mu) \frac{1}{12} (2j_0'(F_p) + j_0''(F_p) + 1) - \frac{1}{8} r \mu
\]

\[
+ \left(2 - \frac{1}{4} \mu \right) j_0'(F_p) + \frac{1}{3} j_0''(F_p) - \left(5 - \frac{1}{2} \mu \right) j_0'(F_p) - j_0''(F_p)
\]

\[
- \left(2 + \frac{1}{4} \mu \right) j_0''(F_p) - \frac{3}{4} \mu j_0''(F_p)
\]

\[
= (1 - \frac{r}{48} + \frac{12}{12} \mu) j_0'(F_p) + \frac{2}{3} - \left(\frac{r}{96} + \frac{1}{2} \right) \mu j_0''(F_p)
\]

\[
+ \left(1 - \frac{r}{96} + \frac{1}{6} \mu \right) j_0''(F_p) + \frac{2}{3} - \left(\frac{r}{96} + \frac{11}{12} \mu \right) j_0''(F_p)
\]

\[
+ 2 - \left(\frac{13}{96} r + \frac{1}{6} \right) \mu.
\]

The coefficients of \( j_0'(F_p) \), \( j_0''(F_p) \), \( j_0'''(F_p) \) and the constant term are positive since \( r \geq 4 \), and \( j_0''(F_p) \) is also positive for \( r \geq 6 \). Thus the above equation is positive when \( r \geq 6 \). If \( r = 4 \), then one can check by an easy computation that (3.12) is also positive, since \( j_0''(F_p) \leq 2 \).

Case (iii,4): Assume that \( \eta_p' \neq 0 \), \( \kappa^{(3)}(F_p) = 0 \), \( \delta_{m_1, II} = 0 \), \( j_{0, 2, odd}(F_p) \leq 3 \) for any \( t \) and

\[
\chi_\varphi(F_p) < \frac{1}{12} (2j_0'(F_p) + j_0''(F_p) + 1).
\]

From Lemma 3.15 (2), \( \Gamma_p \) is of type \( (m_1 I_k) \), \( (I_0^*) \), \( (II^*) \), \( (III^*) \) or \( (IV^*) \). Considering the numbers \( j_0'(F_p) \), \( \chi_\varphi(F_p) \) and Lemma 3.15 (2), we can see that (3.12) is positive by the same argument as in (iii,3) except \( \Gamma_p \) is of type \( (I_0^*) \) and \( j_0''(F_p) = 4 \).

Suppose that \( \Gamma_p \) is of type \( (I_0^*) \) and \( j_0'(F_p) = 4 \). Then the component not contributing to \( j_0'(F_p) \) is a double component in \( \Gamma_p \) and intersects with \( R_h \). Thus we have

\[
\alpha_0^+(F_p) \geq \frac{r}{2} + \sum_{k \geq 1} 2k \alpha_{(2k+1 \rightarrow 2k+1)}(F_p)
\]

from the proof of Lemma 3.7. Then one can see that \( \epsilon_{\mu}(F_p) - \mu \chi_\varphi(F_p) \) is positive by a computation.

Case (iii,5): Assume that \( \eta_p' \neq 0 \), \( \kappa^{(3)}(F_p) = 0 \) and \( \delta_{m_1, II} = 1 \). Then \( j_0'(F_p) = 0 \), \( j'(F_p) = 1 \), \( \eta_p' = 1 \) and \( \chi_\varphi(F_p) = 1/12 \) or \( 1/6 \). If \( j_{0, 2, even}(F_p) = 1 \), then \( (\Gamma_p)_{red} \) is blown up just once. Thus, the multiplicity of the singularity of \( \bar{R} \) which is singular for \( (\Gamma_p)_{red} \) is even. Hence
\(\delta_{\text{cyc}} = 0\). Then we can check by a computation that (3.12) is positive. If \(j_{0,2,\text{even}}'(F_p) = 0\), then we can also check by a computation that (3.12) is positive.

**Case (iii,6):** Assume that \(\kappa^{(3)}(F_p) = 1\). From Lemma 3.6, we may consider the following 3 cases.

**Case (iii,6,II):** If \(\Gamma_p\) is a singular fiber of type (II), then \(n_p' = 1\), \(j'(F_p) = 1\), \(j_{0,a}(F_p) = 0\) for \(a \leq 3\), \(j''(F_p) - j'_{0,1}(F_p) \geq 3\) and \(\chi_\varphi(F_p) = 1/6\). Considering the terms of \(\delta_{n_p', \neq 0}, n_p', \kappa^{(3)}(F_p), j_{0,a}(F_p), j''_{0,a}(F_p)\) and \(\chi_\varphi(F_p)\) in (3.12), we can check that (3.12) is positive.

**Case (iii,6,III):** If \(\Gamma_p\) is a singular fiber of type (III), then \(n_p' = 1\), \(j'(F_p) = 2, j_{0,a}(F_p) = 0\) for \(a \leq 2\), \(j''(F_p) - j'_{0,1}(F_p) \geq 2\) and \(\chi_\varphi(F_p) = 1/4\). Then we can also check that (3.12) is positive.

**Case (iii,6,IV):** If \(\Gamma_p\) is a singular fiber of type (IV), then \(n_p' = 1\), \(j'(F_p) = 3, j'_{0,1}(F_p) = j_{0,2,\text{even}}'(F_p) = 0, j''(F_p) - j''_{0,1}(F_p) \geq 1\) and \(\chi_\varphi(F_p) = 1/3\). Similarly, we can check that (3.12) is positive.

From (iii,1) through (iii,6), we have \(e_f(F_p) - \mu\chi_f(F_p) \geq 0\). Hence Theorem 3.1 (3) follows.

\[\square\]

3.8. **Fibrations attaining the upper bound of the slope.** From Theorem 3.1 and Example 3.4, we can characterize primitive cyclic covering fibrations of type \((g, 1, n)\) whose slope attains the upper bound in Theorem 3.1.

**Corollary 3.16.** Let \(f : S \to B\) be a primitive cyclic covering fibration of type \((g, 1, n)\). Then the slope \(\lambda_f\) attains the upper bound in Theorem 3.1 if and only if any singular fiber of \(f\) is as in Example 3.4.

**Proof.** Let \(f : S \to B\) be a primitive cyclic covering fibration of type \((g, 1, n)\) attaining the upper bound in Theorem 3.1. Then any fiber \(F_p\) satisfies \(K_f^2(F_p) = (12 - \mu)\chi_f(F_p)\). Considering the estimates in the proof of Theorem 3.1, we can see that for singular fibers \(F_p\), the indices \(\alpha_k(F_p), \varepsilon(F_p), \nu(F_p)\) and \(\chi_\varphi(F_p)\) are the same as in Example 3.4 and \(F_p\) is obtained as in Example 3.4. \(\square\)

4. **Upper bound of the slope: the case of type \((g, 0, 3)\), \((g, 0, 2)\)**

4.1. **Main theorem.** In [7], we have established the slope equality for a primitive cyclic covering fibration of type \((g, 0, n)\) by using the above representations of \(K_f^2, \chi_f\) and \(e_f\), and an upper bound of its slope when \(n \geq 4\). In this section, we show that a more careful study as in the previous section gives us an upper bound of the slope for \(n = 2, 3\) cases. Namely, we show the following theorem:

**Theorem 4.1.** (1) Let \(f : S \to B\) be a primitive cyclic covering fibration of type \((4, 0, 3)\). Then we have

\[K_f^2 \leq \frac{129}{17} \chi_f\]

with the equality holding if and only if any singular fiber of \(f\) is a triple fiber.
(2) Let \( f : S \to B \) be a primitive cyclic covering fibration of type \((g, 0, 3)\). Assume that \( g > 4 \), or \( g = 4 \) and \( f \) has no triple fibers. Then we have

\[
K_f^2 \leq \left( 12 - \frac{72(r - 1)}{4r^2 - 15r + 27 - 36\delta} \right) \chi_f,
\]

where \( \delta := \begin{cases} 
0, & \text{if } r \in 6\mathbb{Z}, \\
1, & \text{if } r \not\in 6\mathbb{Z}.
\end{cases} \)

(3) Let \( f : S \to B \) be a primitive cyclic covering fibration of type \((g, 0, 2)\), i.e., a relatively minimal hyperelliptic fibration of genus \( g \). Then we have

\[
K_f^2 \leq \left( 12 - \frac{4(2g + 1)}{g^2 - 1 + \delta} \right) \chi_f,
\]

where \( \delta := \begin{cases} 
0, & \text{if } g \text{ is odd}, \\
1, & \text{if } g \text{ is even}.
\end{cases} \)

Remark 4.2. Theorem 4.1 (3) was shown by Xiao in [20] and the upper bound is known to be sharp [15].

4.2. Localization of invariants. Let \( f : S \to B \) be a primitive cyclic covering fibration of type \((g, 0, n)\). We freely use notations in §1 and §3. Since \( \widetilde{\varphi} : \widetilde{W} \to B \) is a ruled surface, its relatively minimal models are not unique. By performing elementary transformations, we can choose a standard one:

Lemma 4.3 (cf. [7] Lemma 3.1, [19] Lemma 6 for \( n = 2 \)). There exists a relatively minimal model \( \varphi : W \to B \) of \( \widetilde{\varphi} \) such that if \( n = 2 \) and \( g \) is even, then

\[
\text{mult}_x(R) \leq \frac{r}{2} = g + 1
\]

for all \( x \in R \), and otherwise,

\[
\text{mult}_x(R_h) \leq \frac{r}{2} = \frac{g}{n - 1} + 1
\]

for all \( x \in R_h \), where \( R_h \) denotes the horizontal part of \( R \), that is, the sum of all \( \varphi \)-horizontal components of \( R \).

We take a relatively minimal model \( \varphi : W \to B \) satisfying the inequality in Lemma 4.3. It is easily seen that the value of \( \alpha_k(F_p) \) does not depend on the choice of the relatively minimal model of \( \widetilde{\varphi} \) satisfying Lemma 4.3. Since \( \varphi : W \to B \) is a relatively minimal ruled surface, we have \( K_{\varphi}^2 = 0 \) and \( R^2 = -rK_{\varphi}R \). Combining these equalities with (1.8), (1.9), (1.10) and (1.11), we get the following lemma similarly as in §2:

\[ K_f^2 = \frac{n - 1}{r - 1} \left( \frac{(n - 1)r - 2n}{n}(\alpha_0 - 2\varepsilon) + (n + 1) \sum_{k \geq 1} k(-nk + r)\alpha_k \right) - n \sum_{k \geq 1} \alpha_k + \varepsilon. \]

\[ \chi_f = \frac{n - 1}{12(r - 1)} \left( \frac{(2n - 1)r - 3n}{n}(\alpha_0 - 2\varepsilon) + (n + 1) \sum_{k \geq 1} k(-nk + r)\alpha_k \right). \]

\[ e_f = (n - 1)\alpha_0 + n \sum_{k \geq 1} \alpha_k - (2n - 1)\varepsilon. \]

Thus we can localize the invariants \( K_f^2, \chi_f \) and \( e_f \):

**Definition 4.5 (Localization of \( K_f^2, \chi_f \) and \( e_f \)).** For \( p \in B \), we put

\[ K_f^2(F_p) := \frac{n - 1}{r - 1} \left( \frac{(n - 1)r - 2n}{n}(\alpha_0(F_p) - 2\varepsilon(F_p)) + (n + 1) \sum_{k \geq 1} k(-nk + r)\alpha_k(F_p) \right) - n \sum_{k \geq 1} \alpha_k(F_p) + \varepsilon(F_p), \]

\[ \chi_f(F_p) := \frac{n - 1}{12(r - 1)} \left( \frac{(2n - 1)r - 3n}{n}(\alpha_0(F_p) - 2\varepsilon(F_p)) + (n + 1) \sum_{k \geq 1} k(-nk + r)\alpha_k(F_p) \right), \]

\[ e_f(F_p) := (n - 1)\alpha_0(F_p) + n \sum_{k \geq 1} \alpha_k(F_p) - (2n - 1)\varepsilon(F_p). \]

Then by Lemma 4.4, we have

\[ K_f^2 = \sum_{p \in B} K_f^2(F_p), \quad \chi_f = \sum_{p \in B} \chi_f(F_p), \quad e_f = \sum_{p \in B} e_f(F_p). \]

**4.3. Estimation of indices.** Recall the following lemma from [7] (cf. Lemmas 3.7, 3.8 and 3.9):

**Lemma 4.6 ([7] Lemma 5.2).** The following hold:

1. \( \iota(F_p) = j(F_p) - \eta_p \).
2. \( \alpha_0^+(F_p) \geq (n - 2)(j(F_p) - \eta_p + 2\kappa(F_p)) + \delta_{n=2} \sum_{k \geq 1} 2k\alpha_{(2k+1, 2k+1)}(F_p). \)
3. \( \sum_{k \geq 1} \alpha_k^+(F_p) = \sum_{a \geq 1} (an - 2)j_{0,a}(F_p) + 2\eta_p - \kappa(F_p). \)

For \( n = 2, 3 \) cases, we show the following inequalities:

**Lemma 4.7.** The following hold.

1. If \( n = 3 \), then we have \( j_{0,1}(F_p) \leq 2\eta_p + 1 + \sum_{a \geq 2}(2a - 1)j_{0,a}(F_p) \), with the equality holding only if \( F_p \) is a triple fiber, \( r \in 26 \mathbb{Z} + 6 \), \( j(F_p) = j_{0,1}(F_p) \), \( \kappa(F_p) = 1 \) and

\[ \alpha_0^+(F_p) \geq \frac{5(r - 6)}{9} + j(F_p) - \eta_p + 2\kappa(F_p). \]
(2) If $n = 2$, then

$$
\sum_{k \geq 1} \alpha_{(2k+1 \to 2k+1)}^{c,1}(F_p) \leq \hat{\eta}_p + 2\eta'_p + \sum_{a \geq 3} (a - 2)j_{0,a}(F_p) - j_{0,1}(F_p).
$$

Proof. Suppose that $n = 3$. By the same argument as in the proof of Lemma 3.13, we have

$$
j_{0,1}''(F_p) \leq \eta''_p + \sum_{a \geq 1} 2a j_{0,a}'(F_p) + \sum_{a \geq 2} (2a - 1)j_{0,a}''(F_p).
$$

Hence, if $\Gamma_p \not\subseteq R$, we have

$$
j_{0,1}(F_p) \leq 2\eta_p + \sum_{a \geq 2} (2a - 1)j_{0,a}(F_p).
$$

If $\Gamma_p \subseteq R$ and $\Gamma_p$ does not contribute to $j_{0,1}'(F_p)$, we have

$$
j_{0,1}(F_p) \leq 2\eta_p - 1 + \sum_{a \geq 2} (2a - 1)j_{0,a}(F_p),
$$

since $j_{0,1}''(F_p) = j_{0,1}(F_p), \sum_{a \geq 1} 2a j_{0,a}'(F_p) + \sum_{a \geq 2} (2a - 1)j_{0,a}''(F_p) = \sum_{a \geq 2} (2a - 1)j_{0,a}(F_p) + 1$ and $\eta''_p \leq \eta'_p \leq \eta_p - 1$. If $\Gamma_p \subseteq R$ and $\Gamma_p$ contributes to $j_{0,1}'(F_p)$, we have

$$
j_{0,1}(F_p) \leq 2\eta_p - 1 + \sum_{a \geq 2} (2a - 1)j_{0,a}(F_p),
$$

since $j_{0,1}''(F_p) = j_{0,1}(F_p) - 1, \sum_{a \geq 1} 2a j_{0,a}'(F_p) + \sum_{a \geq 2} (2a - 1)j_{0,a}''(F_p) = \sum_{a \geq 2} (2a - 1)j_{0,a}(F_p) + 2$ and $\eta''_p \leq \eta'_p \leq \eta_p - 1$. Thus the inequality in (1) holds.

Assume that the equality

$$j_{0,1}(F_p) = 2\eta_p + \sum_{a \geq 2} (2a - 1)j_{0,a}(F_p)
$$

holds. Then it follows that $\eta_p = \eta'_p$ and the sequence of singularity diagrams associated with $D^t(p)$ containing the proper transform of the fiber $\Gamma_p = C^{t,1}$ is the following:

$$
\begin{array}{c|c|c}
 n_1 & n_2 & n_1 \\
 m_2 & m_2 & n_2 \\
 m_1 & D^{t,3}
\end{array}
\begin{array}{c|c|c}
 n_1 & n_2 & n_1 \\
 m_2 & m_2 & n_2 \\
 m_1 & D^{t,2}
\end{array}
\begin{array}{c|c|c|c}
 n_1 & n_2 & n_1 \\
 m_2 & m_2 & n_2 \\
 m_1 & D^{t,1}
\end{array}
$$

where $n_i \in 3\mathbb{Z}, m_i \in 3\mathbb{Z} + 1$ and $C^{t,k}$ is the exceptional curve obtained by the blow-up at the singularity with multiplicity $m_{k-1}$ for $k = 2, 3$. From Lemma 1.8, we have $r + 5 = m_1 + m_2 + n_1, m_1 + 3 = m_2 + n_2$ and $m_2 + 2 = n_1 + n_2$. In particular, we get $r = 3(n_1 + n_2 - 4) \in 9\mathbb{Z} + 6$. Since $\hat{\eta}_p = \eta''_p$, the sequence of singularity diagrams associated with another $D^t(p)$ is the following from Example 1.15 (2):
at the singularity with multiplicity $m$, it is sufficient to show that

$$2^3$$

is the multiplicity of the fiber over $j$, where $n$ is the multiplicity of $R$ at the center of $C$. Clearly, we have $j(F_p) = j_0(F_p) = 2\eta_p + 1$, $\kappa(F_p) = 1$. Moreover, we see that $F_p$ is a triple fiber since the multiplicity of the fiber over $p$ along the exceptional curve obtained by the blow-up at a singularity of type $3Z$ is divisible by 3. Hence, we get $\alpha_0^C(F_p) \geq 2r/3$. To finish the proof of (1), it is sufficient to show that $2r/3 - (j(F_p) - \eta_p + 2\kappa(F_p)) \geq 5(r - 6)/9$, that is,

$$\eta_p \leq \frac{r + 3}{9}.$$ To prove this, we will show that for any $n_i \in 3Z \geq 0$ and a singularity $x$ of $R$ with multiplicity $x$, the number of singularities of type $3Z + 1$ over $x$ is less than or equal to $2n_1/3 - 2$ by induction on $n_1/3$. If $n_1 = 3$, the claim is obvious. We assume that $n_1 > 3$ and the claim holds true until $n_3 - 1$. Let $x_1, \ldots, x_l$ be all singularities of type $3Z + 1$ over $x$ which is not over any singularity of type $3Z + 1$ over $x$. Let $n_i$ denote the multiplicity of $R$ at $x_i$ for $i = 1 \ldots, l$. Then, the sequence of singularity diagrams associated with $D^t(p)$ obtained by the blow-up at $x_i$ is the following:

where $n_{i,j} \in 3Z$, $m_{i,j} \in 3Z + 1$ satisfy that $m_{i} + 3 = m_{i,1} + n_{i,1}$, $m_i + 2 = n_{i,1} + n_{i,2}$ and $C_{t,1}$, $C_{t,2}$ respectively denote the exceptional curves obtained by the blow-up at the singularity $x_i$, the singularity with multiplicity $m_{i,1}$. By the assumption of induction, the number of singularities of type $3Z + 1$ over $x_i$ is less than or equal to $2(n_{i,1} + n_{i,2})/3 - 3$. Hence the number of singularities of type $3Z + 1$ over $x$ is less than or equal to $\sum_{i=1}^l 2(n_{i,1} + n_{i,2})/3 - 2l$. If $l = 1$, then we have $n_1 \geq m_1 + 2 = 2n_{1,1} + n_{1,2} - 3$. Thus, we get

$$\frac{2}{3}n_1 - 2 \geq \frac{2}{3}(2n_{1,1} + n_{1,2}) - 4 \geq \frac{2}{3}(n_{1,1} + n_{1,2}) - 2.$$ If $l = 2$, then we have $n_1 \geq m_1 + m_2 + 1 = 2n_{1,1} + n_{1,2} + 2n_{2,1} + n_{2,2} - 9$. Thus, we get
\[
\frac{2}{3}n_1 - 2 \geq \frac{2}{3} (2n_{1,1} + n_{1,2} + 2n_{2,1} + n_{2,2}) - 8 \\
\geq \frac{2}{3} (n_{1,1} + n_{1,2} + n_{2,1} + n_{2,2}) - 4.
\]

If \(l \geq 3\), then we have \(n_1 \geq \sum_{i=1}^l m_i = \sum_{i=1}^l (2n_{i,1} + n_{i,2}) - 5l\). Thus, we get

\[
\frac{2}{3}n_1 - 2 \geq \frac{2}{3} \sum_{i=1}^l (2n_{i,1} + n_{i,2}) - \frac{10}{3}l - 2 \\
\geq \frac{2}{3} \sum_{i=1}^l (n_{i,1} + n_{i,2}) - 2l
\]

with equality holds only if \(l = 3\), \(n_1 = m_1 + m_2 + m_3\) and \(n_{i,1} = 3\) for any \(i\). Hence, the number of singularities of type \(3Z + 1\) over a singularity with multiplicity \(n_1\) is less than or equal to \(2n_1/3 - 2\). Now \(r\) is equal to \(3(n_1 + n_2 - 4)\) and the number of singularities of type \(3Z + 1\) over \(p\) is less than or equal to \(2(n_1 + n_2)/3 - 2\). Namely, we have

\[j(F_p) - 1 = j''_{0,1}(F_p) \leq \frac{2}{9}r + \frac{2}{3}.
\]

Hence, we have \(\eta_p = (j(F_p) - 1)/2 = (r + 3)/9\), which is the desired inequality.

Next suppose that \(n = 2\). By the same argument as in the proof of Lemma 3.13, we have

\[
\sum_{k \geq 1} \alpha_{(2k+1 \to 2k+1)}^\text{co}(F_p) \leq \widehat{\eta}_p + \sum_{a \geq 1} a j'_{0,a}(F_p) + \sum_{a \geq 3} (a - 2) j''_{0,a}(F_p).
\]

Since \(\sum_{a \geq 1} a j'_{0,a}(F_p) + \sum_{a \geq 3} (a - 2) j''_{0,a}(F_p) = 2\eta'_p + \sum_{a \geq 3} (a - 2) j'_{0,a}(F_p) - j'_{0,1}(F_p)\), the claim (2) follows.

\[\square\]

**Lemma 4.8** (cf. Lemma 3.12). If \(n = 2\), then we have

\[\kappa(F_p) \leq \frac{2}{3} \sum_{a \geq 2} (a - 1) j'_{0,a}(F_p).
\]

**Proof.** It is sufficient to show that

\[\kappa'(F_p) \leq \frac{2}{3} \sum_{a \geq 2} (a - 1) j'_{0,a}(F_p)
\]

for any \(t\). If \(\kappa'(F_p) = 0\), then it is clear. Thus, we may assume \(\kappa'(F_p) > 0\). Then clearly we have \(j'(F_p) \geq \kappa'(F_p) + 2\). Since any blow-up at a \((t,2)\)-vertical type singularity contributes \(-2\) to the number \(\sum_{k \geq 1} (L^t)^2 = -\sum_{a \geq 1} 2a j'_{0,a}(F_p)\) and \(\Gamma_p\) contains no \(2\)-vertical type singularities, we get

\[-\sum_{a \geq 1} 2a j'_{0,a}(F_p) \leq -j'(F_p) + (-2) (\nu'(F_p) + \kappa'(F_p)) = -(3j'(F_p) + 2\kappa'(F_p) - 2).
\]
Combining this equality with $j^t(F_p) \geq \kappa^t(F_p) + 2$, we have

$$\sum_{a \geq 1} 2aj^t_{0,a}(F_p) \geq 2j^t(F_p) + 3\kappa^t(F_p).$$

Hence we get $\kappa^t(F_p) \leq \frac{2}{3} \sum_{a \geq 2}(a - 1)j^t_{0,a}(F_p)$. \qed

### 4.4. Proof of main theorem.

Now, we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Let $f: S \to B$ be a primitive cyclic covering fibration of type $(g, 0, n)$. For any $p \in B$ and $\mu \geq 0$, we have from Definition 4.5 and Lemma 4.6 (3) that

$$e_f(F_p) - \mu \chi_f(F_p) = (n - 1)\alpha_0(F_p) + n \sum_{k \geq 1} \alpha_k(F_p) - (2n - 1)\varepsilon(F_p) - \mu' \left( \frac{r(2n - 1) - 3n}{n} \alpha_0(F_p) + (n + 1) \sum_{k \geq 1} (-nk^2 + rk)\alpha_k(F_p) - 2\frac{(r(2n - 1) - 3n)}{n} \varepsilon(F_p) \right)$$

$$= A_n \alpha_0(F_p) + \sum_{k \geq 1} (Q(k) + B_n) \alpha_k(F_p) - (2A_n + 1) \varepsilon(F_p)$$

$$= A_n \alpha^+_0(F_p) + \sum_{k \geq 1} (Q(k) + B_n) \alpha'_k(F_p) + \sum_{k \geq 1} (Q(k) + B_n) \alpha''_k(F_p) - 2A_n j(F_p) - j_{0,1}(F_p)$$

$$= A_n \alpha^+_0(F_p) + \sum_{k \geq 1} (Q(k) + B_n) \alpha'_k(F_p) + \sum_{k \geq 1} Q(k) \alpha''_k(F_p)$$

$$+ \sum_{a \geq 1} ((an - 2)B_n - 2A_n - \delta_{a=1}) j_{0,a}(F_p) + 2B_n \eta_p - B_n \kappa(F_p),$$

where we put

$$\mu' := \frac{n - 1}{12(r - 1)} \mu, \quad A_n := n - 1 - \frac{r(2n - 1) - 3n}{n} \mu', \quad B_n := n - \frac{(n + 1)(r^2 - \delta n^2)}{4n} \mu'$$

and

$$Q(k) := \mu' \left( n(n + 1) \left( k - \frac{r}{2n} \right)^2 - \frac{n(n + 1)\delta}{4} \right) \geq 0.$$
Assume that \( A_n \geq 0 \) and \( B_n \geq 0 \). From Lemma 4.6 (2), \( e_f(F_p) - \mu \chi_f(F_p) \) is greater than or equal to

\[
\left( (n - 2) (j(F_p) - \eta_p + 2\kappa(F_p)) + \delta_n = 2 \sum_{k \geq 1} 2k\alpha_{(2k+1 \rightarrow 2k+1)}(F_p) \right) A_n + \sum_{k \geq 1} Q(k)\alpha_k''(F_p) + \sum_{a \geq 1} ((an - 2)B_n - 2A_n - \delta_{a=1}) j_{0,a}(F_p) + 2B_n\eta_p - B_n\kappa(F_p)
\]

\[
= \delta_n = 2 \sum_{k \geq 1} 2k\alpha_{(2k+1 \rightarrow 2k+1)}(F_p) A_n + \sum_{a \geq 1} ((n - 4)A_n + (an - 2)B_n - \delta_{a=1}) j_{0,a}(F_p)
\]

(4.1) \( + \sum_{k \geq 1} Q(k)\alpha_k''(F_p) + (2B_n - (n - 2)A_n)\eta_p + (2(n - 2)A_n - B_n)\kappa(F_p). \)

We take cases.

**Case (i):** We first assume that \( n = 3 \).

**Case (i,1):** Assume that \( j_{0,1}(F_p) \leq 2\eta_p + \sum_{a \geq 2} (2a - 1)j_{0,a}(F_p) \). We define

\[
\mu := \frac{72(r - 1)}{4r^2 - 15r + 27 - 36\delta}.
\]

Then we have \( A_3 > 0 \) and \( B_3 > 0 \). Since the coefficient \(-A_3 + 2B_3\) of \( \eta_p \) in (4.1) is also positive, (4.1) is greater than or equal to

(4.2) \[ \sum_{a \geq 2} \left( a - \frac{3}{2} \right) A_3 + (a - 1)B_3 \] \( j_{0,a}(F_p) + \left( -\frac{3}{2}A_3 + 2B_3 - 1 \right) j_{0,1}(F_p) + (2A_3 - B_3)\kappa(F_p). \]

By the definition of \( \mu \), we have \(-\frac{3}{2}A_3 + 2B_3 - 1 = 0 \) and \( 2A_3 - B_3 > 0 \). Thus (4.2) is non-negative.

**Case (i,2):** Assume that \( j_{0,1}(F_p) = 2\eta_p + 1 + \sum_{a \geq 2} (2a - 1)j_{0,a}(F_p) \) and \( r > 6 \). We define

\[
\mu := \frac{72(r - 1)}{4r^2 - 15r + 27 - 36\delta}.
\]

From Lemma 4.7 (1), we have

\[
e_f(F_p) - \mu \chi_f(F_p)
\]

\[
\geq \frac{5(r - 6)}{9} A_3 + (-A_3 + B_3 - 1) j_{0,1}(F_p) + (-A_3 + 2B_3)\eta_p + (2A_3 - B_3)
\]

\[
= \frac{5(r - 6)}{9} A_3 + (-\frac{3}{2}A_3 + 2B_3 - 1) j_{0,1}(F_p) + \frac{5}{2}A_3 - 2B_3
\]

\[
\geq \frac{15}{2} A_3 - 2B_3
\]

\[
> 0.
\]

49
Case (i,3): Assume that $j_{0,1}(F_p) = 2\eta_p + 1 + \sum_{a \geq 2} (2a - 1)j_{0,a}(F_p)$ and $r = 6$. We define

$$\mu := \frac{129}{17}.$$ 

From Lemma 4.7 (1) and $j_{0,1}(F_p) = 3$, we have

$$e_j(F_p) - \mu \chi_j(F_p) \geq -2A_3 + 4B_3 - 3 = 0,$$

where the last equality follows by the definition of $\mu$.

From (i,1) through (i,3), we have $e_j(F_p) - \mu \chi_j(F_p) \geq 0$. Hence Theorem 4.1 (1), (2) follows.

Case (ii): We assume that $n = 2$. Put

$$\mu := \frac{4(2g+1)}{g^2 - 1 + \delta}.$$

Then we have $2B_2 - 2A_2 - 1 - 6(1 - \delta)\mu' = 0$, $A_2 > 0$ and $B_2 \geq 1$. From (1.12) and (1.14), (4.1) is equal to

$$\sum_{k \geq 1} Q(k)\alpha''_k(F_p) + \sum_{a \geq 2} (-2A_2 + (2a - 2)B_2)j_{0,a}(F_p) - B_2\kappa(F_p)$$

$$+ (2B_2 - 2A_2 - 1) \sum_{k \geq 1} \alpha'^{tr}_{(2k+1 \rightarrow 2k+1)}(F_p) + \sum_{k \geq 1} (2(k - 1)A_2 + 2B_2 - 1) \alpha^co_{(2k+1 \rightarrow 2k+1)}(F_p)$$

$$+ \sum_{k \geq 1} (2(k - 1)A_2 - 1) \alpha^co_{(2k+1 \rightarrow 2k+1)}(F_p) - (2A_2 + 1)j_{0,1}'(F_p) + 2B_2(\eta_p' + \gamma_p).$$

Using Lemma 4.7 (2) for $-\sum_{k \geq 1} \alpha^co_{(2k+1 \rightarrow 2k+1)}(F_p)$, (4.1) is greater than or equal to

(4.3)

$$\sum_{k \geq 1} Q(k)\alpha''_k(F_p) + \sum_{a \geq 2} (-2A_2 + (2a - 2)B_2 - (a - 2))j_{0,a}(F_p) - B_2\kappa(F_p)$$

$$+ (2B_2 - 2A_2 - 1) \sum_{k \geq 1} \alpha'^{tr}_{(2k+1 \rightarrow 2k+1)}(F_p) + \sum_{k \geq 1} (2(k - 1)A_2 + 2B_2 - 1) \alpha^co_{(2k+1 \rightarrow 2k+1)}(F_p)$$

$$+ \sum_{k \geq 1} 2(k - 1)A_2\alpha^co_{(2k+1 \rightarrow 2k+1)}(F_p) - 2A_2j_{0,1}'(F_p) + (2B_2 - 2)\eta_p' + (2B_2 - 1)\gamma_p.$$ 

By the definition of $\mu$, we see that

(4.4)

$$\sum_{k \geq 1} Q(k)\alpha''_k(F_p) + (2B_2 - 2A_2 - 1) \sum_{k \geq 1} \alpha'^{tr}_{(2k+1 \rightarrow 2k+1)}(F_p) \geq 0$$

with the equality holding if and only if $\alpha''_{(g-1)/2}(F_p) = \alpha'^{tr}_{(g \rightarrow g)}(F_p)$ and $\alpha''_k(F_p) = 0$ for $k \neq (g - 1)/2$ when $g$ is odd, or $\alpha''_{g/2}(F_p) = \alpha'^{tr}_{(g+1 \rightarrow g+1)}(F_p)$ and $\alpha''_k(F_p) = 0$ for $k \neq g/2$ when $g$ is even. If $j_{0,1}'(F_p) = 1$, then $g$ is odd and $\Gamma_p \subset R$ contains a singularity of type $(g+2 \rightarrow g+2)$
since $\Gamma_p$ is blown up just twice and Lemma 4.3. Thus, $\Gamma_p$ contributes to $\alpha_{(g+2\to g+2)}^{co}(F_p)$ and then we have
\[
\sum_{k \geq 1} 2(k - 1)A_2\alpha_{(2k+1\to 2k+1)}^{co}(F_p) - 2A_2j_{0,1}^j(F_p) \geq 0.
\]
From Lemma 4.8 and
\[-2A_2 + \frac{4}{3}(a - 1)B_2 - (a - 2) > 0,
\]
we have
\[
\sum_{a \geq 2} (-2A_2 + (2a - 2)B_2 - (a - 2))j_{0,a}(F_p) - B_2\kappa(F_p) \geq 0.
\]
From (4.4), (4.5), (4.6) and $B_2 \geq 1$, (4.3) is non-negative. Hence Theorem 4.1 (3) follows. □

REFERENCES


Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda, Chiba 278-8510, Japan

Email address: enokizono_makoto@ma.noda.tus.ac.jp