

PRIMES OF THE FORM $X^3 + NY^3$ AND A FAMILY OF NON-SINGULAR PLANE CURVES WHICH VIOLATE THE LOCAL-GLOBAL PRINCIPLE

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ABSTRACT. Let n be an integer such that $n = 5$ or $n \geq 7$. In this article, we introduce a recipe for a certain infinite family of non-singular plane curves of degree n which violate the local-global principle. Moreover, each family contains infinitely many members which are not geometrically isomorphic to each other. Our construction is based on two arithmetic objects; that is, prime numbers of the form $X^3 + NY^3$ due to Heath-Brown and Moroz and the Fermat type equation of the form $x^3 + Ny^3 = Lz^n$, where N and L are suitably chosen integers. In this sense, our construction is an extension of the family of odd degree n which was previously found by Shimizu and the author. The previous construction works only if the given degree n has a prime divisor p for which the pure cubic fields $\mathbb{Q}(p^{1/3})$ or $\mathbb{Q}((2p)^{1/3})$ satisfy a certain indivisibility conjecture of Ankeny-Artin-Chowla-Mordell type. In this time, we focus on the complementary cases, namely the cases of even degrees and exceptional odd degrees. Consequently, our recipe works well as a whole. This means that we can unconditionally produce infinitely many explicit non-singular plane curves of every degree $n = 5$ or $n \geq 7$ which violate the local-global principle. This gives a conclusion of the classical story of searching explicit ternary forms violating the local-global principle, which was initiated by Selmer (1951) and extended by Fujiwara (1972) and others.

1. INTRODUCTION

In the theory of Diophantine equations, the local-global principle for quadratic forms established by Minkowski and Hasse is one of the major culminations (cf. [27, Theorem 8, Ch. IV]). It states that every quadratic hypersurface of the projective space \mathbb{P}^n ($n \geq 1$) defined over \mathbb{Q} has a rational point over \mathbb{Q} if (and only if) it has a rational point over \mathbb{R} and \mathbb{Q}_p for every prime number p . Here and after, $\mathbb{Q}, \mathbb{R}, \mathbb{Q}_p$ denote as usual the field of rational, real, and p -adic numbers respectively.

In contrast to the quadratic case, there exist many homogeneous forms of higher degrees which violate the local-global principle. For example, Selmer [26] found that a non-singular plane cubic curve defined by

$$(1) \quad 3X^3 + 4Y^3 = 5Z^3, \quad \text{or equivalently} \quad X^3 + 6Y^3 = 10Z^3$$

has rational points \mathbb{R} and \mathbb{Q}_p for every prime number p but not over \mathbb{Q} . In this situation, we say that (the curve defined by) eq. (1) violates the local-global principle or it is a counterexample to the local-global principle. From eq. (1), we can easily construct reducible (especially singular) counterexamples of higher degrees. Note also that in the case of weighted homogeneous forms,

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Lind [15] and Reichardt [24] independently found the following first counterexample

$$X^4 - 17Y^4 = 2Z^2.$$

For more information on this topic, we refer the reader to a nice introductory article [1] by Aitken and Lemmermeyer.

More recently, some people have studied infinite families of non-singular plane curves which violate the local-global principle. The existence of such a family of cubic curves was proven by Colliot-Thélène and Poonen in [6], and an explicit example was constructed first by Poonen in [22] as follows.

Theorem 1.1 ([22]). *For any $t \in \mathbb{Q}$, the equation*

$$5X^3 + 9Y^3 + 10Z^3 + 12 \left(\frac{t^2 + 82}{t^2 + 22} \right)^3 (X + Y + Z)^3 = 0$$

defines a non-singular plane cubic curve $C = C_t$ defined over \mathbb{Q} which violates the local-global principle. Moreover, there exists a set of $t \in \mathbb{Q}$ which gives infinitely many geometrically non-isomorphic classes of such curves.

In contrast to the cubic case, it seems that there is only a little number of explicit non-singular plane curves of a higher odd degree which violates the local-global principle. After the above examples by Lind, Reichardt, and Selmer, Fujiwara [8] found that a non-singular plane quintic curve defined by

$$(2) \quad (X^3 + 5Z^3)(X^2 + XY + Y^2) = 17Z^5$$

violates the local-global principle. His idea is to reduce the proof of the insolubility over \mathbb{Q} to the determination of primitive solutions of the Fermat type equation $x^3 + 5y^3 = 17z^5$. Here and after, a triple of integers $(x, y, z) \in \mathbb{Z}^{\oplus 3}$ is called primitive if $\gcd(x, y, z) = 1$. After Fujiwara, Cohen [5, Corollary 6.4.11] gave several counterexamples of the form $x^p + by^p + cz^p = 0$ of degree $p = 3, 5, 7, 11$ with $b, c \in \mathbb{Z}$, but his construction is still so restricted. More recently, Shimizu and the author conditionally succeeded in generalizing Fujiwara's idea in general higher odd degrees. Our construction is based on the generation of prime numbers by cubic polynomials established by Heath-Brown and Moroz [9, 10].

Theorem 1.2 ([10, Theorem 1]). *Let $f_0 \in \mathbb{Z}[X, Y]$ be an irreducible binary cubic form, $\rho \in \mathbb{Z}$, $(\gamma_1, \gamma_2) \in \mathbb{Z}^{\oplus 2}$, and γ_0 be the greatest common divisor of the coefficients of $f_0(\rho x + \gamma_1, \rho y + \gamma_2)$. Set $f(x, y) := \gamma_0^{-1} f_0(\rho x + \gamma_1, \rho y + \gamma_2)$. Suppose that $\gcd(f(\mathbb{Z}^{\oplus 2})) = 1$. Then, the set $f(\mathbb{Z}^{\oplus 2})$ contains infinitely many prime numbers.*

By using the above result, Shimizu and the author proved the following theorem.

Theorem 1.3 (cf. [11, Theorem 1.1] and Remark 4.3). *Let p be an odd prime number and $P = 2p$ or p so that $P \not\equiv \pm 1 \pmod{9}$. Let $\epsilon = \alpha + \beta P^{1/3} + \gamma P^{2/3} \in \mathbb{R}_{>1}$ be the fundamental unit of $\mathbb{Q}(P^{1/3})$ with $\alpha, \beta, \gamma \in \mathbb{Z}$. Set*

$$\iota = \begin{cases} 1 & \text{if } \beta \not\equiv 0 \pmod{p} \text{ or } \beta \equiv \gamma \equiv 0 \pmod{p} \\ 2 & \text{if } \beta \equiv 0 \pmod{p} \text{ and } \gamma \not\equiv 0 \pmod{p} \end{cases}.$$

Let $n \geq 5$ be an odd integer divisible by p^ι . Then, there exist infinitely many $(n-3)/2$ -tuples of pairs of integers (b_j, c_j) ($1 \leq j \leq (n-3)/2$) satisfying the following condition:

There exist infinitely many integers L such that the equation

$$(3) \quad (X^3 + P^\iota Y^3) \prod_{j=1}^{\frac{n-3}{2}} (b_j^2 X^2 + b_j c_j XY + c_j^2 Y^2) = LZ^n$$

define non-singular plane curves of degree n which violate the local-global principle.

Moreover, for each n , there exists a set of such $(n-3)/2$ -tuples $((b_j, c_j))_{1 \leq j \leq (n-3)/2}$ which gives infinitely many geometrically non-isomorphic classes of such curves of degree n .

Here, note that for $p = 3$, we can take $P = 6$ so that $\beta \equiv \gamma \equiv 0 \pmod{3}$, hence $\iota = 1$ (cf. Remark 4.3). On the other hand, Shimizu and the author [11] formulated the following conjecture, which implies again $\iota = 1$ for all $p \neq 3$.

Conjecture 1.4. *Let $p \neq 3$ be a prime number, $P = p$ or $2p$, and $\epsilon = \alpha + \beta p^{1/3} + \gamma p^{2/3} \in \mathbb{R}_{>1}$ be the fundamental unit of $\mathbb{Q}(P^{1/3})$ with $\alpha, \beta, \gamma \in (1/3)\mathbb{Z}$. Then, we have $\beta \not\equiv 0 \pmod{p}$.*

The above Conjecture 1.4 for pure cubic fields $\mathbb{Q}(P^{1/3})$ may be regarded as an analogue of the classical Ankeny-Artin-Chowla-Mordell conjecture for real quadratic fields $\mathbb{Q}(p^{1/2})$ (cf. [2, 17]). In fact, the authors verified in [11] that Conjecture 1.4 for all $p < 10^5$ by numerical examination using Magma [3]. Anyway, Theorem 1.3 gives, in a certain uniform manner, an explicit construction of infinitely many non-singular plane curves which violate the local-global principle, but it works only conditionally for higher odd degrees. As far as the author knows, this is the best result on explicit construction of non-singular plane curves of a general odd degree which violate the local-global principle.

On the other hand, in even degree case, after Fujiwara's counterexamples of degree 5, the first counterexamples of degree 4 were found by Bremner-Lewis-Morton [4] and Schinzel [25] independently, whose equations are given as follows respectively:

$$3X^4 + 4Y^4 = 19Z^4 \quad \text{and} \quad X^4 - 2Y^4 - 16Y^2 Z^2 - 49Z^4 = 0.$$

Moreover, after Poonen's family of counterexamples of degree 3, Nguyen [18–20] gave infinitely many counterexamples of almost all even degrees.¹ Although it is difficult to describe his result in complete detail, we recall here a part of them as follows:

Theorem 1.5 ([19, Theorem 1.4]). *Let $n = 2k$ be an even integer with $k \geq 1$. Take p, d, m , and α as follows:*

- (1) p is a prime number such that $p \equiv 1 \pmod{8}$.
- (2) d is an integer which is a quadratic non-residue in \mathbb{F}_p^\times and prime to n .
- (3) m is an even integer such that $q := d^2 + pm^2$ is a prime number.
- (4) α is a rational number such that $\alpha \in \mathbb{Z}_l$ for every prime divisor l of dp and $\alpha \neq 0, qp^{-k}, qd^{-k}, (m(d+p) - 2q)((dp)^k - d^k - p^k)^{-1}$.

Set $s = pq^2$, $t = d(d+p)$, $u = -q$, $v = pm^2(d+p)$, $w = -dq$, $A = q - \alpha p^k$, $B = q - \alpha d^k$, and $C = m(d+p) - 2q - \alpha((dp)^k - d^k - p^k)$. Then, the equation

$$sX^{4k+2} + Y^{4k-2}(tX^2 + uY^2)(vX^2 + wY^2) - Z^2(AX^{2k} + BY^{2k} + CX^k Y^k + \alpha Z^{2k})^2 = 0$$

defines a plane curve $C_{p,d,m,\alpha}$ of degree $4k+2$ over \mathbb{Q} which violates the local-global principle.

¹In fact, the recipe in Theorem 1.5 actually gives explicit counterexamples of degree $4k+2$ for every $k \geq 1$. On the other hand, Nguyen [20] asserted only for $k \neq 1, 2, 4$ that the recipe in Theorem 1.6 actually gives explicit counterexamples of degree $4k$. See also [18] for an explicit infinite family of counterexamples of degree 4.

Theorem 1.6 (a specialized form at $k = 0$ of [20, Theorem 3.1]). *Let $m, n \geq 1$ be integers such that $m < n$. Let p be a prime number and $\alpha, \beta \in \mathbb{Z}$. Define a homogeneous polynomial $d(X, Y) \in \mathbb{Z}[X, Y]$ by*

$$d(X, Y) = X^m(p(X + Y))^n - (-Y)^m(p(X + Y) + Y)^n.$$

Suppose that

- (1) $p \equiv 1 \pmod{8}$.
- (2) $\gcd(\alpha\beta, p) = 1$.
- (3) *Every odd prime divisor of $\alpha d(\alpha, \beta)$ is a quadratic residue in \mathbb{F}_p^\times .*
- (4) β *is a quadratic non-residue in \mathbb{F}_p^\times .*
- (5) *Every odd prime divisor of $\beta^2 + p(\alpha + \beta)^2$ is prime to $n - m$.*

Then, there exists a family of explicit homogeneous polynomials $Q_\zeta = Q_\zeta(X, Y, Z) \in \mathbb{Q}[X, Y, Z]$ of degree $2n - 1$ parametrized by a rational number $\zeta \in \mathbb{Q}$ such that the equation

$$Q_\zeta(X, Y, Z)^2 Z^2 = p(\alpha X^{2n} + \beta Y^{2n})^2 + Y^{4n-4m}((p\alpha + (p+1)\beta)X^{2m} - p(\alpha + \beta)Y^{2m})^2$$

defines a plane curve $C_{p,\alpha,\beta,\zeta}$ of degree $4n$ over \mathbb{Q} which violates the local-global principle.

A remarkable character which the families obtained by Nguyen share is that each member of these families covers a hyperelliptic curve which violates the local-global principle, and the latter violation of the local-global principle is explained by the Brauer-Manin obstruction by a certain Brauer class of degree 2.

As the first main theorem of this article, we introduce another family of non-singular plane curves of even degrees $n \geq 8$ which violate the local-global principle.

Theorem 1.7 (First main theorem). *Let u be an odd square-free integer such that $u \not\equiv \pm 4 \pmod{9}$. Set $P = 2u$. Let $\epsilon = \alpha + \beta P^{1/3} + \gamma P^{2/3} \in \mathbb{R}_{>1}$ be the fundamental unit of $\mathbb{Q}(P^{1/3})$ with $\alpha, \beta, \gamma \in \mathbb{Z}$. Suppose that β is even and the class number of $\mathbb{Q}(P^{1/3})$ is odd. Let $n \geq 8$ be an even integer, and $m \geq 3$ be an odd integer such that $m < n$. Then, there exist infinitely many $(n-6)/2$ -tuples of pairs of integers (b_j, c_j) ($1 \leq j \leq (n-6)/2$) satisfying the following condition:*

There exist infinitely many prime numbers l and infinitely many pairs of integers (b_0, c_0) such that the equation

$$(4) \quad (X^3 + P^2 Y^3)(b_0 X^3 + l c_0 Y^3) \prod_{j=1}^{(n-6)/2} (b_j^2 X^2 + b_j c_j X Y + c_j^2 Y^2) = l^m Z^n$$

define non-singular plane curves of degree n which violate the local-global principle.

Moreover, for each n , there exists a set of such $(n-4)/2$ -tuples $((b_j, c_j))_{0 \leq j \leq (n-6)/2}$ which gives infinitely many geometrically non-isomorphic classes of such curves of degree n .

Here, note that the plane curve defined by eq. (4) covers a hyperelliptic curve defined by

$$(5) \quad (X^3 + P^2 Y^3)(b_0 X^3 + l c_0 Y^3) \prod_{j=1}^{(n-6)/2} (b_j^2 X^2 + b_j c_j X Y + c_j^2 Y^2) = l^m Z^2.$$

This geometric situation is similar to Nguyen's examples in Theorems 1.5 and 1.6. However, we demonstrate in §6 that the above hyperelliptic curve may has a \mathbb{Q} -rational point. In particular, the violation of the local-global principle for the plane curve defined by eq. (4) cannot be explained by the Brauer-Manin obstruction on the hyperelliptic curve defined by eq. (5) in general. This arithmetic situation is in contrast to Nguyen's examples.

On the other hand, as its visual suggested, the family in Theorem 1.7 is a variant of the family of odd degrees in Theorem 1.3. In fact, their proofs are quite similar. For example, the key ingredients of the proof of Theorem 1.7 are the generation of prime numbers of the form $X^3 + P^2Y^3$ (cf. Theorem 1.2) and a property of the primitive solutions of the Fermat type equation $x^3 + P^2y^3 = l^mz^n$. Moreover, the proof of the infinitude of geometric isomorphy is essentially given in the previous work [11, Lemma 4.1].

It should be noted, however, that if we replace the cubic form $X^3 + P^2Y^3$ in Theorem 1.7 to the original cubic forms $X^3 + p^\iota Y^3$ or $X^3 + (2p)^\iota Y^3$ in Theorem 1.3 for “ $p = 2$ ”, then the proof of Theorem 1.7 is broken down. In fact, we need an important but implicit condition under which the proofs of Theorems 1.3 and 1.7 work, that is, for the fundamental unit ϵ of $\mathbb{Q}(P^{1/3})$, the modulo p^ι class of ϵ^{p-1} has order at most p , which is automatic whenever $p \geq 3$. However, the modulo 2^ι class of the fundamental unit $\epsilon = 1 + 2^{1/3} + 2^{2/3}$ of $\mathbb{Q}(2^{1/3})$ has order $2^{\iota+1}$ for every $\iota \geq 1$. The condition on the parity of β in Theorem 1.7 ensures that the above argument works well. Therefore, the existence of an auxiliary integer u satisfying the conditions in Theorem 1.7, for example $u = 3, 7, 17, 21, 35, 39, \dots$, is essential in order to ensure that the recipe in Theorem 1.7 works well for all even degrees $n \geq 8$ unconditionally. For the arithmetic background of this subtlety, see Proposition 4.1 and Remark 4.5.

In order to explain the second main theorem, we mention another interesting aspect of Theorem 1.7. Note that although the cubic form $X^3 + P^2Y^3$ in Theorem 1.7 suggests the analogy with the case of $\iota = 2$ in Theorem 1.3, we do not need to assume that n is divisible by $4 = 2^2$. This is justified by a key ingredient Proposition 4.1 (see also Remark 4.2). In fact, by combining Proposition 4.1 with the existing result Theorem 1.3, we obtain the following second main theorem in odd degree case. Although the appearance of Theorem 1.8 is almost the same as that of Theorem 1.3, there is a crucial difference between them. In Theorem 1.8, we assume only that n is divisible by p but not necessarily divisible by p^ι . This releases us from the highly mysterious Conjecture 1.4.

Theorem 1.8 (Second main theorem). *Let p be an odd prime number and $P = 2p$ or p so that $P \not\equiv \pm 1 \pmod{9}$. Let $\epsilon = \alpha + \beta P^{1/3} + \gamma P^{2/3} \in \mathbb{R}_{>1}$ be the fundamental unit of $\mathbb{Q}(P^{1/3})$ with $\alpha, \beta, \gamma \in \mathbb{Z}$. Set*

$$\iota = \begin{cases} 1 & \text{if } \beta \not\equiv 0 \pmod{p} \text{ or } \beta \equiv \gamma \equiv 0 \pmod{p} \\ 2 & \text{if } \beta \equiv 0 \pmod{p} \text{ and } \gamma \not\equiv 0 \pmod{p} \end{cases}.$$

Let $n \geq 5$ be an odd integer divisible by p . Then, there exist infinitely many $(n-3)/2$ -tuples of pairs of integers (b_j, c_j) ($1 \leq j \leq (n-3)/2$) satisfying the following condition:

There exist infinitely many integers $L \in \mathbb{Z}$ such that the equation

$$(6) \quad (X^3 + P^\iota Y^3) \prod_{j=1}^{(n-3)/2} (b_j^2 X^2 + b_j c_j XY + c_j^2 Y^2) = LZ^n$$

define non-singular plane curves of degree n which violate the local-global principle.

Moreover, for each n , there exists a set of such $(n-3)/2$ -tuples $((b_j, c_j))_{1 \leq j \leq (n-3)/2}$ which gives infinitely many geometrically non-isomorphic classes of such curves of degree n .

As a consequence of Theorems 1.7 and 1.8, we obtain the following conclusion:

Corollary 1.9. *For every integer n such that $n = 5$ or $n \geq 7$, there exist infinitely many non-singular plane curves of the form eq. (4) or eq. (6) according to the parity of n which violate the local-global principle.*

Here, we should emphasize that although it is unclear from the above statements, the proofs of Theorems 1.3 and 1.8 (resp. Theorem 1.7) show that for every odd (resp. even) integer n such that $n = 5$ or $n \geq 7$, we have an algorithm to produce arbitrarily many explicit parameters (b_j, c_j) and L (resp. l) for which eq. (6) (resp. eq. (4)) define non-singular plane curves of degree n which violate the local-global principle. For numerical examples, see [11, §5] and §6 of the present article respectively. Note also that for exceptionally small degrees $n = 3, 4, 6$, we already have an algorithm due to Poonen [22] and Nguyen [18, 19]. (For non-singularity of Nguyen's curves of degree 6 in [19], see also §7.) Therefore, for every arbitrarily given degree $n \geq 3$, we now obtain an algorithm to produce arbitrarily many explicit non-singular plane curves of degree n which violate the local-global principle. This gives a conclusion of the classical story of searching explicit ternary forms violating the local-global principle, which was initiated by Selmer and extended by Fujiwara and others.

We conclude the introduction by presenting the organization of this article. In §2, we give a recipe which exhibits how to construct counterexamples to the local-global principle defined by eq. (4) from prime numbers of the form $X^3 + P^2Y^3$ and the Fermat type equations $x^3 + P^2y^3 = Lz^n$. In §3, we reduce the proof of Theorem 1.7 to the key Theorem 3.1 by using Theorem 1.2. By an exactly similar manner, we reduce the proof of Theorem 1.8 to the key Theorem 3.1. The proof of the key Theorem 3.1 itself is given in §4, where we again use Theorem 1.2. In §5, we give a variant Theorem 5.2 of Theorem 1.7 focusing on degrees divisible by 4. In §6, we demonstrate how our construction in Theorem 1.7 works for each given degree. We give two numerical examples of degree 8 following the proofs of Theorems 1.7 and 5.2 respectively. It should be emphasized that both of these examples are $\mathbb{Z}/4\mathbb{Z}$ -coverings of hyperelliptic curves with \mathbb{Q} -rational points. This means that the violation of the local-global principle for the former plane curves cannot be explained by the Brauer-Manin obstruction on the latter hyperelliptic curves, which is in contrast to Nguyen's examples in Theorems 1.5 and 1.6.

2. CONSTRUCTION FROM PRIME NUMBERS AND FERMAT TYPE EQUATIONS

Let $p = 2$, u be an odd integer, $P = 2u$, and $\iota = 1$ or 2 . We fix these integers throughout this section. In this section, we prove the following proposition, which gives explicit counterexamples to the local-global principle of even degree n under the assumption that we have

- sufficiently many prime numbers of the form $P^\iota b^3 + c^3$ with $b, c \in \mathbb{Z}$ and
- integers L such that the equation $x^3 + P^\iota y^3 = Lz^n$ has a specific property.

In what follows, for each prime number l , $v_l(n)$ denotes the additive l -adic valuation of $n \in \mathbb{Z}$. Furthermore, for every non-zero integer L , we denote the radical of L by $\text{rad } L := \prod_l l$, where l runs over the prime numbers such that $v_l(L) \geq 1$.

Proposition 2.1. *Let n be an even integer such that $n \geq 8$. Let b_j, c_j ($0 \leq j \leq (n-6)/2$), and L be integers satisfying the following conditions:*

- (1) *For every $j \geq 1$, $P^\iota b_j^3 + c_j^3$ is a prime number $\equiv 2 \pmod{3}$ and prime to P .*
- (2) *$L \not\equiv \pm 1$ and $\gcd(L, b_j c_j) = 1$ for every $j \geq 0$. Moreover, for every prime divisor l of L , we have $l \not\equiv 2$, $l \equiv 2 \pmod{3}$, and $2 \leq v_l(L) < n$.*
- (3) *If $P \equiv \pm 2, \pm 4 \pmod{9}$ and $b_0 \not\equiv 0 \pmod{3}$, then $P^\iota b_0 - \text{rad } L \cdot c_0 = \pm 3^k$ with some $k \neq 1$. If $P \not\equiv \pm 2, \pm 4 \pmod{9}$ or $b_0 \equiv 0 \pmod{3}$, then $P^\iota b_0 - \text{rad } L \cdot c_0 = \pm 1$.*
- (4) *For every prime divisor q of P such that $q \equiv 2 \pmod{3}$, $\gcd(q, b_0 c_0) = 1$.²*

²In fact, $\gcd(q, c_0) = 1$ follows automatically from the condition (3).

- (5) If $P \not\equiv \pm 1 \pmod{9}$, then $L \equiv b_0 \prod_{j \geq 1} b_j^2 \not\equiv 0 \pmod{3}$ and $\sum_{j \geq 1} b_j^{-1} c_j \not\equiv 0 \pmod{3}$.
(6) For every primitive triple $(x, y, z) \in \mathbb{Z}^{\oplus 3}$ satisfying $x^3 + P^\iota y^3 = Lz^n$, we have $x \equiv y \equiv 0 \pmod{l}$ for some prime divisor l of L .

Then, the equation

$$(X^3 + P^\iota Y^3)(b_0 X^3 + \text{rad } L \cdot c_0 Y^3) \prod_{j=1}^{\frac{n-6}{2}} (b_j^2 X^2 + b_j c_j XY + c_j^2 Y^2) = LZ^n$$

violates the local-global principle.

We will prove Proposition 2.1 by dividing its proof into the local solubility part (Lemma 2.2) and the global insolubility part (Lemma 2.3).

Lemma 2.2 (local solubility). *Let n be an even integer such that $n \geq 8$. Let b_j, c_j, L ($0 \leq j \leq (n-6)/2$) be integers satisfying the following conditions:*

- (1) *For every prime divisor q of P such that $q \equiv 2 \pmod{3}$, $\gcd(q, b_0 c_0) = 1$.
(2) *If $P \not\equiv \pm 1 \pmod{9}$, then $L \equiv b_0 \prod_{j \geq 1} b_j^2 \not\equiv 0 \pmod{3}$ and $\sum_{j \geq 1} b_j^{-1} c_j \not\equiv 0 \pmod{3}$.*³*

Then, the equation

$$F(X, Y, Z) := (X^3 + P^\iota Y^3)(b_0 X^3 + c_0 Y^3) \prod_{j=1}^{\frac{n-6}{2}} (b_j^2 X^2 + b_j c_j XY + c_j^2 Y^2) - LZ^n = 0$$

has non-trivial solutions over \mathbb{R} and \mathbb{Q}_l for every prime number l .

Proof. We prove this statement along Fujiwara's argument in [8]. It is sufficient to consider the case $b_1 c_1 \neq 0$, because we can find a rational point $[1 : 0 : 0]$ for $b_1 = 0$ and $[0 : 1 : 0]$ for $c_1 = 0$. Then, since $\iota = 1$ or 2 , and $b_1, c_1 \neq 0$, the minimal splitting field of $(X^3 + P^\iota Y^3)(b_1^2 X^2 + b_1 c_1 XY + c_1^2 Y^2)$ is a Galois extension over \mathbb{Q} whose Galois group is isomorphic to the symmetric group of degree 3. In particular, the residual degree at every prime number is 1, 2, or 3. Moreover, since this extension is unramified at every prime number prime to $3P$ (and ∞), the polynomial $(X^3 + P^\iota Y^3)(b_1^2 X^2 + b_1 c_1 XY + c_1^2 Y^2)$ has a linear factor over \mathbb{R} and over \mathbb{Q}_v for every prime number v such that $\gcd(v, 3P) = 1$ or $v \equiv 1 \pmod{3}$. Here, note that the assumption (1) ensures that we obtain a rational point $[X : Y : Z] = [1 : -(c_0/b_0)^{1/3} : 0]$ over \mathbb{Q}_q for any prime divisor q of P such that $q \equiv 2 \pmod{3}$, and the assumption (2) ensures that we obtain a 3-adic lift of $[X : Y : Z] = [1 : 0 : 1]$ (cf. [11, Lemma 2.2]). \square

Lemma 2.3 (global insolubility). *Let n be an even integer such that $n \geq 8$. Let a, b_j, c_j, L ($0 \leq j \leq (n-6)/2$) be integers satisfying the following conditions:*

- (1) *For every $j \geq 1$, $\gcd(ab_j, c_j) = 1$ and each prime divisor q of $ab_j^3 + c_j^3$ satisfies $q \equiv 2 \pmod{3}$.
(2) *For every $j \geq 0$, $\gcd(L, b_j c_j) = 1$. Moreover, for every prime divisor l of L , $l \equiv 2 \pmod{3}$ and $2 \leq v_l(L) < n$.
(3) *If $a \equiv \pm 2, \pm 4 \pmod{9}$ and $b_0 \not\equiv 0 \pmod{3}$, then $ab_0 - \text{rad } L \cdot c_0 = \pm 3^k$ with some $k \neq 1$. If $a \not\equiv \pm 2, \pm 4 \pmod{9}$ or $b_0 \equiv 0 \pmod{3}$, then $ab_0 - \text{rad } L \cdot c_0 = \pm 1$.
(4) *For every primitive triple $(x, y, z) \in \mathbb{Z}^{\oplus 3}$ satisfying $x^3 + ay^3 = Lz^n$, we have $x \equiv y \equiv 0 \pmod{l}$ for some prime divisor l of L .****

³Obviously, another condition $b_0^2 c_0 \equiv \pm 1 \pmod{9}$ (or more generally $c_0/b_0 \in \mathbb{Q}_3^{\times 3}$) is sufficient to the 3-adic solubility. However, this condition does not fit to our proof of Theorem 1.7.

Then, there exist no triples $(X, Y, Z) \in \mathbb{Z}^{\oplus 3} \setminus \{(0, 0, 0)\}$ satisfying

$$(7) \quad (X^3 + aY^3)(b_0X^3 + \text{rad } L \cdot c_0Y^3) \prod_{j=1}^{\frac{n-6}{2}} (b_j^2X^2 + b_jc_jXY + c_j^2Y^2) = LZ^n.$$

Proof. We prove the assertion by contradiction. Let $(X, Y, Z) \in \mathbb{Z}^{\oplus 3}$ be a triple satisfying eq. (7). We may assume that $\gcd(X, Y, Z) = 1$. It is sufficient to prove that

$$(\star) \quad \gcd((X^3 + aY^3)L, (b_0X^3 + \text{rad } L \cdot c_0Y^3)(b_j^2X^2 + b_jc_jXY + c_j^2Y^2)) = 1 \quad \text{for every } j \geq 1.$$

Indeed, if eq. (\star) holds, then we have some divisor z of Z satisfying $X^3 + aY^3 = Lz^n$. Hence, by the assumption, we have $X \equiv Y \equiv 0 \pmod{l}$ for some prime divisor l of L . However, since $v_l(L) < n$, we also have $Z \equiv 0 \pmod{l}$, which contradicts that $\gcd(X, Y, Z) = 1$. In what follows, we prove eq. (\star) by contradiction.

First, suppose that a prime divisor q of $X^3 + aY^3$ divides $b_j^2X^2 + b_jc_jXY + c_j^2Y^2$ for some $j \geq 1$. Then, since $\gcd(X, Y, Z) = 1$ and $v_q(L) < n$, we see that $Y \not\equiv 0 \pmod{q}$. On the other hand, since q divides

$$b_j^3(X^3 + aY^3) - (b_jX - c_jY)(b_j^2X^2 + b_jc_jXY + c_j^2Y^2) = (ab_j^3 + c_j^3)Y^3,$$

we have $ab_j^3 + c_j^3 \equiv 0 \pmod{q}$, hence $q \equiv 2 \pmod{3}$. In particular, the polynomial $b_j^2T^2 + b_jc_jT + c_j^2$ is irreducible in $\mathbb{Z}_q[T]$. Moreover, since $b_j^2X^2 + b_jc_jXY + c_j^2Y^2 \equiv 0 \pmod{q}$ and $Y \not\equiv 0 \pmod{q}$, we have $c_j \equiv 0 \pmod{q}$, hence $ab_j \equiv 0 \pmod{q}$. However, it contradicts the assumption that $\gcd(ab_j, c_j) = 1$.

Secondly, suppose that a prime divisor l of L divides $b_j^2X^2 + b_jc_jXY + c_j^2Y^2$ for some $j \geq 1$. Then, since $l \equiv 2 \pmod{3}$ and $\gcd(L, b_jc_j) = 1$, we have $X \equiv Y \equiv 0 \pmod{l}$. However, since $v_l(L) < n$, we see that $Z \equiv 0 \pmod{l}$, which contradicts that $\gcd(X, Y, Z) = 1$.

Thirdly, suppose that a prime divisor q of $X^3 + aY^3$ divides $b_0X^3 + \text{rad } L \cdot c_0Y^3$. Then, since $\gcd(X, Y, Z) = 1$ and $v_q(L) < n$, we see that $Y \not\equiv 0 \pmod{q}$. On the other hand, since q divides

$$b_0(X^3 + aY^3) - (b_0X^3 + \text{rad } L \cdot c_0Y^3) = (ab_0 - \text{rad } L \cdot c_0)Y^3 = \pm 3^k Y^3,$$

we have $q = 3$, $k \geq 2$, $a \equiv \pm 2, \pm 4 \pmod{9}$, and $b_0 \not\equiv 0 \pmod{3}$. Since $Y \not\equiv 0 \pmod{3}$, we have $X^3 + aY^3 \equiv \pm 3 \pmod{9}$ and $X \not\equiv 0 \pmod{3}$. Moreover, we have $b_0X^3 + \text{rad } L \cdot c_0Y^3 \equiv (1+a)b_0X^3 \not\equiv 0 \pmod{9}$. Further, since we assume that $ab_j^3 + c_j^3$ ($1 \leq j \leq (n-6)/2$) is prime to 3, either b_j or c_j is prime to 3, hence $b_j^2X^2 + b_jc_jXY + c_j^2Y^2$ is not divisible by 9. Since L is prime to 3, we conclude that Z^n is divisible by 3 but not divisible by 3^n , a contradiction.

Finally, suppose that a prime divisor l of L divides $b_0X^3 + \text{rad } L \cdot c_0Y^3$. Then, since $\gcd(L, b_0) = 1$, we have $X \equiv 0 \pmod{l}$. Since $v_l(L) \geq 2$, we have $(a \cdot \text{rad } L \cdot c_0 \prod_{j \geq 1} c_j^2)Y^n \equiv 0 \pmod{l^2}$. On the other hand, since $ab_0 - \text{rad } L \cdot c_0 = \pm 3^k$, we have $\gcd(l, a) = 1$. Moreover, since $\gcd(L, c_j) = 1$ for every $j \geq 0$, we have $\gcd(l, ac_0 \prod_{j \geq 1} c_j^2) = 1$ for every $j \geq 0$, hence $Y \equiv 0 \pmod{l}$. However, since $v_l(L) < n$, we have $Z \equiv 0 \pmod{l}$, which contradicts that $\gcd(X, Y, Z) = 1$. This completes the proof. \square

3. REDUCTION TO THE FERMAT TYPE EQUATIONS $X^3 + P^u Y^3 = l^m Z^n$

Let p be a prime number and u be a square-free integer prime to p . Set $P = pu$. Let $\pi = P^{1/3} \in \mathbb{R}$ be the real cubic root of P , $K = \mathbb{Q}(\pi)$ be the pure cubic field generated by π , \mathcal{O}_K be the ring of integers in K , and $\epsilon = \alpha + \beta\pi + \gamma\pi^2 > 1$ be the fundamental unit of K with $\alpha, \beta, \gamma \in (1/3)\mathbb{Z}$. Note that the Galois closure of K in the field \mathbb{C} of complex numbers is $K(\zeta_3)$,

where $\zeta_3 \in \mathbb{C}$ is a fixed primitive cubic root of unity. For basic properties of these objects, see e.g. [7].

In what follows, we assume that $P \not\equiv \pm 1 \pmod{9}$. Then, since we assume that u is square-free, we have $\mathcal{O}_K = \mathbb{Z}[\pi]$ and so $\alpha, \beta, \gamma \in \mathbb{Z}$.

Theorem 3.1. *In the above setting, further suppose that $\beta \equiv 0 \pmod{p}$ and the class number of K is prime to p . Let n and m be positive integers such that $n \geq 3$, $n \equiv 0 \pmod{p}$, and $m \not\equiv 0 \pmod{p}$. Then, there exist infinitely many odd prime numbers l such that $l \equiv 2 \pmod{3}$ and for every primitive triple $(x, y, z) \in \mathbb{Z}^{\oplus 3}$ satisfying $x^3 + P^2 y^3 = l^m z^n$, we have $x \equiv y \equiv 0 \pmod{l}$.*

We will prove Theorem 3.1 in the next section. Here, we prove Theorems 1.7 and 1.8 by using Theorem 3.1. We start from the former.

Proof of Theorem 1.7 under Theorem 3.1. We prove the assertion by taking desired parameters so that the conditions in Proposition 2.1 hold. Note that since every step is constructive, we obtain an algorithm to produce arbitrarily many non-singular plane curves violating the local-global principle up to generation of the prime numbers l claimed in Theorem 3.1. An algorithm to produce these prime numbers l is explained in the top of the next section.

Let u be an odd square-free integer such that $P = 2u \not\equiv \pm 1 \pmod{9}$, $\beta \equiv 0 \pmod{2}$, and the class number of $K = \mathbb{Q}(P^{1/3})$ is odd. First, we consider the case $u \not\equiv 0 \pmod{3}$, i.e., $P \equiv \pm 2, \pm 4 \pmod{9}$.

Let $f(X, Y) = P^2(3X + 1)^3 + (3Y + 1)^3$ and $g(X, Y) = P^2(3X - 1)^3 + (3Y)^3$. Then, since $\gcd(f(0, 0), f(0, -1)) = 1$ and $\gcd(g(0, 0), g(0, 1)) = 1$, we have $\gcd(f(\mathbb{Z}^{\oplus 2})) = 1$ and $\gcd(g(\mathbb{Z}^{\oplus 2})) = 1$ respectively. By Theorem 1.2, there exist infinitely many distinct prime numbers of the form $q = f(B, C)$ (resp. $q = g(B, C)$) with $(B, C) \in \mathbb{Z}^{\oplus 2}$ and prime to P . Note that all of them satisfy $q \equiv 2 \pmod{3}$. Among such prime numbers q , take distinct $(n - 6)/2$ prime numbers $q_j = f(B_j, C_j)$ or $g(B_j, C_j)$ with $(B_j, C_j) \in \mathbb{Z}^{\oplus 2}$ ($1 \leq j \leq (n - 6)/2$) so that $\gcd(3, \sum_j b_j^{-1} c_j) = 1$, where $(b_j, c_j) := (3B_j + 1, 3C_j + 1)$ or $(3B_j - 1, 3C_j)$ according to whether $q_j = f(B_j, C_j)$ or $g(B_j, C_j)$.

For each $(n - 6)/2$ -tuple $((b_j, c_j))_{1 \leq j \leq (n-6)/2}$ taken as above, we have infinitely many prime numbers l satisfying the properties claimed in Theorem 3.1 with an additional condition that $\gcd(l, P b_j c_j) = 1$ for every $j \geq 1$. We fix such l arbitrarily.

Let Q be the product of the prime divisors q of P such that $q \equiv 2 \pmod{3}$ and $k \in \mathbb{Z}_{\geq 0}$. Then, since $\gcd(l, 3P) = 1$, there exist infinitely many pairs $(b, c_0) \in \mathbb{Z}^{\oplus 2}$ such that $\gcd(l, c_0) = 1$ and

$$3P^2 Q b - l c_0 = \pm 3^k + P^2, \quad \text{i.e.,} \quad P^2(-1 + 3Qb) - l c_0 = \pm 3^k.$$

Take such a pair (b, c_0) arbitrarily and set $b_0 = -1 + 3Qb$. Then, since $l \equiv 2 \pmod{3}$ and m is odd, we have $b_0 \prod_{j \geq 1} b_j^2 \equiv 2 \equiv l^m \pmod{3}$ and $\gcd(Q, b_0) = 1$. Moreover, we see that $l \neq 2$ and $\gcd(l, b_0 c_0) = 1$. Therefore, Proposition 2.1 implies that the equation

$$(X^3 + P^2 Y^3)(b_0 X^3 + l c_0 Y^3) \prod_{j=1}^{\frac{n-6}{2}} (b_j^2 X^2 + b_j c_j X Y + c_j^2 Y^2) = l^m Z^n$$

violates the local-global principle.

The non-singularity is a consequence of the following two facts:

- (1) Since q_j and q_k are distinct prime numbers, $[b_j : c_j] \neq [b_k : c_k]$ for any distinct $j, k \geq 1$.
- (2) Since $\gcd(l, b_0 c_0) = 1$ and all of l, b_0, c_0 are odd, two polynomials $X^3 + P^2 Y^3$ and $b_0 X^3 + l c_0 Y^3$ are both irreducible in $\mathbb{Q}[X, Y]$ and cannot have a common root in \mathbb{C} .

The infinitude of the geometric isomorphy classes of plane curves follows from Schwarz's theorem on the finiteness of the automorphism group of a non-singular algebraic curve of genus ≥ 2 . For the detail, see [11, Lemma 4.1]. This completes the proof in the case $u \not\equiv 0 \pmod 3$.

Finally, if $u \equiv 0 \pmod 3$, then we take $f(X, Y) = P^2(3X + 1)^3 + (3Y + 1)^3$ and $g(X, Y) = P^2(3X - 1)^3 + (3Y + 1)^3$ so that we can take (b_j, c_j) such that $\sum_{j \geq 1} b_j^{-1} c_j \not\equiv 0 \pmod 3$. In this case, $b_0 = -1 + 3Qb$ and $c_0 \in \mathbb{Z}$ with $P^2 b_0 - l c_0 = \pm 1$ works well. This completes the proof. \square

Next, we prove Theorem 1.8 by using Theorem 3.1. Here, recall the following proposition, which is a counterpart of Proposition 2.1 and reduces Theorem 1.8 to the generation of prime numbers of the form $X^3 + P^2 Y^3$ and a certain Fermat type equation $x^3 + P^2 y^3 = Lz^n$.

Proposition 3.2 (cf. [11, Proposition 2.1]). *Let n be an odd integer such that $n \geq 5$, p be a prime number, and $P = p$ or $2p$, $\iota = 1$ or 2 . Let b_j, c_j, L ($1 \leq j \leq (n-3)/2$) be integers satisfying the following conditions:*

- (1) *For every j , $P^\iota b_j^3 + c_j^3$ is a prime number $\equiv 2 \pmod 3$ and prime to P .*
- (2) *$\gcd(L, b_j c_j) = 1$ for every j . Moreover, for every prime divisor l of L , $l \equiv 2 \pmod 3$ and $v_l(L) < n$.*
- (3) *For every prime divisor q of P such that $q \equiv 2 \pmod 3$ (only $q = 2, p$ are possible), we have $L \equiv \prod_j b_j^2 \not\equiv 0 \pmod q$ and $\sum_j b_j^{-1} c_j \not\equiv 0 \pmod q$.*
- (4) *If $P \not\equiv \pm 1 \pmod 9$, then $L \equiv \prod_j b_j^2 \not\equiv 0 \pmod 3$ and $\sum_j b_j^{-1} c_j \not\equiv 0 \pmod 3$.*
- (5) *For every primitive triple $(x, y, z) \in \mathbb{Z}^{\oplus 3}$ satisfying $x^3 + P^\iota y^3 = Lz^n$, there exists a prime divisor l of L such that $x \equiv y \equiv 0 \pmod l$.*

Then, the equation

$$(X^3 + P^\iota Y^3) \prod_{j=1}^{\frac{n-3}{2}} (b_j^2 X^2 + b_j c_j XY + c_j^2 Y^2) = LZ^n$$

violates the local-global principle.

Proof of Theorem 1.8 under Theorem 3.1. The proof is almost parallel to the proof of Theorem 1.7. We prove the assertion by taking desired parameters so that the conditions in Proposition 3.2 hold. In this time, we do not need to take extra parameters b_0 and c_0 .

Let p be an odd prime number. Take $u = 2$ or 1 , i.e., $P = 2p$ or p so that $P \not\equiv \pm 1 \pmod 9$. Thanks to Theorem 1.3 (see also Remark 4.3), it is sufficient to consider the case where $p \neq 3$ and $\beta \equiv 0 \pmod p$. Note that since the class numbers of $\mathbb{Q}(p^{1/3})$ and $\mathbb{Q}((2p)^{1/3})$ are prime to p (cf. [11, Lemma 3.4]), we can actually apply Theorem 3.1.

Let $f(X, Y) = P^2(3PX + 1)^3 + (3PY + 1)^3$ and $g(X, Y) = P^2(3PX - 1)^3 + (3PY + 3)^3$. Then, since $\gcd(f(0, 0), f(0, 1), f(0, -1)) = 1$ and $\gcd(g(0, 0), g(0, 1), g(0, -1)) = 1$, we have $\gcd(f(\mathbb{Z}^{\oplus 2})) = 1$ and $\gcd(g(\mathbb{Z}^{\oplus 2})) = 1$ respectively. By Theorem 1.2, there exist infinitely many distinct prime numbers of the form $q = f(B, C)$ (resp. $q = g(B, C)$) with $(B, C) \in \mathbb{Z}^{\oplus 2}$ and prime to P . Note that all of them satisfy $q \equiv 2 \pmod 3$. Among such prime numbers q , take distinct $(n-3)/2$ prime numbers $q_j = f(B_j, C_j)$ or $g(B_j, C_j)$ with $(B_j, C_j) \in \mathbb{Z}^{\oplus 2}$ ($1 \leq j \leq (n-6)/2$) so that $\gcd(3p, \sum_j b_j^{-1} c_j) = 1$, where $(b_j, c_j) := (3PB_j + 1, 3PC_j + 1)$ or $(3PB_j - 1, 3PC_j + 3)$ according to whether $q_j = f(B_j, C_j)$ or $g(B_j, C_j)$. Here, note that since n is odd, $\sum_j b_j^{-1} c_j$ is odd whenever P is even.

For each $(n-3)/2$ -tuple $((b_j, c_j))_{1 \leq j \leq (n-3)/2}$ taken as above, we have infinitely many prime numbers l satisfying the properties claimed in Theorem 3.1 with an additional condition that

$\gcd(l, 2Pb_jc_j) = 1$ for every j . We fix such l arbitrarily. Then, since both l and p are odd and $b_j \equiv \pm 1 \pmod{3P}$ for every j , we see that $l^{p-1} \equiv \prod_{j \geq 1} b_j^2 \equiv 1 \pmod{3P}$. Therefore, Proposition 3.2 implies that the equation

$$(X^3 + P^2Y^3) \prod_{j=1}^{\frac{n-6}{2}} (b_j^2X^2 + b_jc_jXY + c_j^2Y^2) = l^{p-1}Z^n$$

violates the local-global principle.

The non-singularity is a consequence of the following two facts:

- (1) Since q_j and q_k are distinct prime numbers, $[b_j : c_j] \neq [b_k : c_k]$ for any distinct j, k .
- (2) Since $X^3 + P^2Y^3$ is irreducible in $\mathbb{Q}[X, Y]$, it cannot have a multiple root nor a common root with $b_j^2X^2 + b_jc_jXY + c_j^2Y^2$ for any j .

The infinitude of the geometric isomorphy classes of plane curves follows from Schwarz's theorem on the finiteness of the automorphism group of a non-singular algebraic curve of genus ≥ 2 . For the detail, see [11, Lemma 4.1]. This completes the proof. \square

4. PROOF OF THEOREM 3.1

In order to prove Theorem 3.1, we again use Theorem 1.2. ⁴ Let p be a prime number satisfying the conditions in Theorem 3.1. Let $h(A, B) = (3A - 1)^3 + P(3B + 1)^3$ or $h(A, B) = (3pA - 1)^3 + P(3pB + 3)^3$ according to $p = 3$ or not. ⁵ Then, since $\gcd(h(0, 0), h(1, 0), h(-1, 0)) = 1$, we have $\gcd(h(\mathbb{Z}^{\oplus 2})) = 1$. Therefore, Theorem 1.2 implies that there exist infinitely many prime numbers l of the form

$$l = a^3 + Pb^3 \equiv 2 \pmod{3} \quad \text{with} \quad (a, b) = \begin{cases} (3A - 1, 3B + 1) & \text{if } p = 3 \\ (3pA - 1, 3pB + 3) & \text{if } p \neq 3 \end{cases}$$

Thus, Theorem 3.1 is obtained from the case of $(\iota, \nu) = (2, 1)$ in the following proposition.

Proposition 4.1. *Let $(\iota, \nu) = (1, 2), (2, 1)$. Let p be a prime number and u be a square-free positive integer prime to p . Set $P = pu$ and $\pi = P^{1/3} \in \mathbb{R}$. Suppose that $P \not\equiv \pm 1 \pmod{9}$, the fundamental unit $\epsilon = \alpha + \beta\pi + \gamma\pi^2$ of $K = \mathbb{Q}(\pi)$ with $\alpha, \beta, \gamma \in \mathbb{Z}$ satisfies $\beta \equiv 0 \pmod{p}$, and the class number of K is prime to p . If $(\iota, \nu) = (1, 2)$, then further assume that $\gamma \equiv 0 \pmod{p}$. Let l be a prime number such that $\gcd(l, P) = 1$ and $l \equiv 2 \pmod{3}$. Assume that there exist $a + b\pi + c\pi^2 \in \mathcal{O}_K$ with $a, b, c \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$ satisfying the following conditions:*

- (1) $l = N_{K/\mathbb{Q}}(a + b\pi + c\pi^2) (= a^3 + b^3P + c^3P^2 - 3abcP)$.
- (2) (a) If $(\iota, \nu) = (1, 2)$, then $\binom{m}{2}b^2 + mac$ is prime to p . ⁶
(b) If $(\iota, \nu) = (2, 1)$, then mb is prime to p .

⁴Here, the use of Theorem 1.2 is not essential unlike the situation in the proof of Theorem 1.7 (resp. Theorem 1.8) under Theorem 3.1. In fact, the constraint for $(a, b, c) \in \mathbb{Z}^{\oplus 3}$ in Proposition 4.1 is relatively weak. In particular, it is sufficient for the proof of Theorem 3.1 that there exist infinitely many prime numbers of the form $l = N_{K/\mathbb{Q}}(a + b\pi + c\pi^2)$ with $b \not\equiv 0 \pmod{p}$, which we can verify by the ring class field theory and the Chebotarev density theorem (cf. [13, Ch. V] and [16]).

⁵If $p = 2$ and u has no prime divisors of the form $q \equiv 2 \pmod{3}$, then $h(A, B) = (6A + 1)^3 + P(6B - 1)^3$ also works. More generally, if we want to obtain concrete examples with small coefficients, we can easily modify the polynomials generating the prime numbers with desired properties.

⁶In fact, the condition $\nu = 2$ is too strict for odd p , and $p^\nu \geq 3$ is sufficient.

Then, for every integer $n \geq 3$ divisible by p^ν and every primitive triple $(x, y, z) \in \mathbb{Z}^{\oplus 3}$ satisfying $x^3 + P^\iota y^3 = l^m z^n$, we have $x \equiv y \equiv 0 \pmod l$.

Proof. The proof is almost parallel to the proof of [11, Theorem 3.1] (cf. Remark 4.2). We prove the assertion by contradiction. Suppose that there exists a primitive triple $(x, y, z) \in \mathbb{Z}^{\oplus 3}$ such that $x^3 + P^\iota y^3 = l^m z^n$, and either x or y is prime to l .

First, note that since either x or y is prime to l and $\gcd(l, P) = 1$, $x^2 - xy\pi^\iota + y^2\pi^{2\iota}$ cannot be divisible by l . Moreover, $l \equiv 2 \pmod 3$ splits in K to the product of two prime ideals \mathfrak{p}_l and \mathfrak{p}_{l^2} of norms l and l^2 respectively. Suppose that $x + y\pi$ is divisible by \mathfrak{p}_{l^2} . Then, the product of its conjugates $(x + \zeta_3 y\pi^\iota)(x + \zeta_3^2 y\pi^\iota) = x^2 - xy\pi^\iota + y^2\pi^{2\iota}$ is divisible by l , a contradiction (cf. the following argument for $q \equiv 2 \pmod 3$). Therefore, $x^2 - xy\pi^\iota + y^2\pi^{2\iota}$ is divisible by $\mathfrak{p}_{l^2}^m$ but not divisible by \mathfrak{p}_l . Accordingly, $x + y\pi^\iota$ is divisible by \mathfrak{p}_l^m but not divisible by \mathfrak{p}_{l^2} .

Next, suppose that $x + y\pi^\iota$ is divisible by a prime ideal above a prime divisor q of z . Then, since P is square-free, $\iota < 3$, and (x, y, z) is primitive, we see that $\gcd(q, P) = 1$. Moreover, both of $x + y\pi^\iota$ and $x^2 - xy\pi^\iota + y^2\pi^{2\iota}$ are not divisible by q itself because if this is the case, then $x^3 + P^\iota y^3$ is divisible by q itself, hence the condition $P \not\equiv \pm 1 \pmod 9$ implies that $x \equiv y \equiv 0 \pmod q$, which contradicts that (x, y, z) is primitive. As a consequence, the possible decomposition types of q in K are as follows:

- (1) $(q) = \mathfrak{p}_{q,1}\mathfrak{p}_{q,2}\mathfrak{p}_{q,3}$, i.e., $q \equiv 1 \pmod 3$ and $P \pmod q \in \mathbb{F}_q^{\times 3}$.
- (2) $(q) = \mathfrak{p}_q\mathfrak{p}_{q^2}$, i.e., $q \equiv 2 \pmod 3$ and $\gcd(q, P) = 1$.
- (3) $(q) = \mathfrak{p}_q^3$, i.e., $q = 3$.

In each case, we have the following conclusion:

- (1) If $x + y\pi^\iota$ is divisible by distinct two prime ideals above q , say $\mathfrak{p}_{q,1}$ and $\mathfrak{p}_{q,2}$, then $x^2 - xy\pi^\iota + y^2\pi^{2\iota}$ is divisible by $(\mathfrak{p}_{q,1}\mathfrak{p}_{q,3}) \cdot (\mathfrak{p}_{q,2}\mathfrak{p}_{q,3})$, hence by q , a contradiction. Therefore, we may assume that $x + y\pi^\iota$ is divisible by $\mathfrak{p}_{q,1}^{nv_q(z)}$ but not by $\mathfrak{p}_{q,2}$ nor $\mathfrak{p}_{q,3}$ by replacing $\mathfrak{p}_{q,1}, \mathfrak{p}_{q,2}, \mathfrak{p}_{q,3}$ to each other if necessary.
- (2) In this case, q is decomposed in $K(\zeta_3)$ so that $\mathfrak{p}_q = \mathfrak{P}_{q^2,1}$ and $\mathfrak{p}_{q^2} = \mathfrak{P}_{q^2,2}\mathfrak{P}_{q^2,3}$. If $x + y\pi^\iota$ is divisible by \mathfrak{p}_{q^2} , then $x^2 - xy\pi^\iota + y^2\pi^{2\iota}$ is divisible by $(\mathfrak{P}_{q^2,1}\mathfrak{P}_{q^2,2}) \cdot (\mathfrak{P}_{q^2,1}\mathfrak{P}_{q^2,3})$, hence by q , a contradiction. Therefore, $x + y\pi^\iota$ is divisible by $\mathfrak{p}_q^{nv_q(z)}$ but not by \mathfrak{p}_{q^2} .
- (3) In this case, since $x^3 + P^\iota y^3$ is divisible by \mathfrak{p}_3^{3n} , $x + y\pi^\iota$ is divisible by \mathfrak{p}_3^n . Since $n \geq 3$, $x + \pi^\iota y$ is divisible by 3. Moreover, since P is square-free and $\iota < 3$, π^ι cannot be divisible by 3. Hence, both x and y are divisible by 3, which contradicts that (x, y, z) is primitive.

As a consequence, we see that there exists an integral ideal \mathfrak{w} of \mathcal{O}_K such that

$$(x + y\pi^\iota) = \mathfrak{p}_l^m \mathfrak{w}^n \quad \text{and} \quad (\mathfrak{w}, 3P) = 1.$$

Then, since the first assumption implies that \mathfrak{p}_l is generated by $a + b\pi + c\pi^2$, \mathfrak{w}^n is a principal ideal. Moreover, since we assume that the class number of K is prime to p , the ideal \mathfrak{w}^{n/p^ν} itself is also generated by a single element $w_0 + w_1\pi + w_2\pi^2 \in \mathcal{O}_K$ with $w_0, w_1, w_2 \in \mathbb{Z}$. Therefore, there exists $k \in \mathbb{Z}$ such that

$$x + y\pi^\iota = \epsilon^k (a + b\pi + c\pi^2)^m (w_0 + w_1\pi + w_2\pi^2)^{p^\nu}.$$

Since we assume that $\beta \equiv 0 \pmod p$, we have

$$x + y\pi^\iota \equiv \alpha^{k-1} (\alpha + k\gamma\pi^2) a^{m-2} \left(a^2 + mab\pi + \left(\binom{m}{2} b^2 + mac \right) \pi^2 \right) (w_0 + w_1\pi^{p^\nu}) \pmod p.$$

Here, note that since $\epsilon = \alpha + \beta\pi + \gamma\pi^2$ is a unit of K , $\alpha \not\equiv 0 \pmod{p}$. Moreover, since $\gcd(l, P) = 1$, $a \not\equiv 0 \pmod{p}$. Furthermore, since $\gcd(\mathfrak{w}, P) = 1$, $w_0 \not\equiv 0 \pmod{p}$.

If $(\iota, \nu) = (1, 2)$, then since we assume that $\gamma \equiv 0 \pmod{p}$, the above congruence between the coefficients of π^2 contradicts the assumption.

If $(\iota, \nu) = (2, 1)$, then the above congruence between the coefficients of π contradicts the assumption. This completes the proof. \square

Remark 4.2. It should be noted that a key ingredient in the previous work [11, Theorem 3.1] is a counterpart of Proposition 4.1 in the cases $(\iota, \nu) = (1, 1), (2, 2)$. This restriction on (ι, ν) binds Theorem 1.3 with the assumption that the degree n is divisible by p^ℓ . In this time, we are released from this strong assumption thanks to Proposition 4.1.

Remark 4.3 (Outline of the proof Theorem 1.3 in the case $\beta \not\equiv 0 \pmod{p}$). For the completeness, we explain an outline of the proof Theorem 1.3 in the case $\beta \not\equiv 0 \pmod{p}$, which was essentially given in [11]. The idea is similar to the case $\beta \equiv 0 \pmod{p}$ as given in the previous section. In the case $\beta \not\equiv 0 \pmod{p}$, we produce the parameters (b_j, c_j) ($1 \leq j \leq (n-3)/2$) by using appropriate polynomials e.g. $f(X, Y) = P(3PX \pm 1)^3 + (3PY + 1)^3$ and $g(X, Y) = P(3PX \mp 1)^3 + (3PY + 3)^3$ according to $P \equiv \pm 1 \pmod{3}$. The crucial point appears in the difference between the proofs of [11, Theorem 3.1] and Proposition 4.1: In the former, in order to deduce a contradiction from the modulo p comparison of the coefficients of π^2 , it is sufficient that the coefficient $l^m = N_{K/\mathbb{Q}}((a + b\pi + c\pi^2)^m)$ of z^n satisfies additional technical conditions that $b \equiv 0 \pmod{p}$ and a curious quantity

$$\left(\frac{\beta}{2\alpha} - \frac{\gamma}{\beta}\right)^2 - \frac{2c}{a} \cdot m$$

is not a quadratic residue modulo p (cf. [11, Lemma 3.5]). This condition is verified by generating prime numbers of the form e.g. $l = h(A, B) = (3PA + 1)^3 + P^2(3PC + 1)$ and considering sufficiently many even integers m in the range $1 \leq m \leq p-1$ (cf. effective Pólya-Vinogradov inequalities as obtained in [21]). Here, we take even m in order to ensure the 3-adic and p -adic solubility (cf. Proposition 3.2). Similarly, if $p = 3$, we can produce (b_j, c_j) ($1 \leq j \leq (n-3)/2$) by e.g. $f(X, Y) = P^\iota(PX + 1)^3 + (PY - 1)^3$ and $g(X, Y) = P^\iota(PX - 1)^3 + (PY - 1)^3$ and generate l by e.g. $h(A, C) = (6A - 1)^3 + P^{2\iota}(6C + 1)$.

Remark 4.4. If $p = 2$, then $\binom{m}{2}b^2 + mac \equiv \binom{m}{2}b^2 + mc \pmod{2}$, and it is prime to $p = 2$ if and only if one of the following conditions holds:

- (1) $m \equiv 1 \pmod{4}$ and c is odd.
- (2) $m \equiv 2 \pmod{4}$ and b is odd.
- (3) $m \equiv 3 \pmod{4}$ and $b + c$ is odd.

Remark 4.5. It seems plausible that there exist infinitely many odd integers u such that the class numbers of the cubic fields $\mathbb{Q}((2u)^{1/3})$ are odd: In fact, the class number of $\mathbb{Q}((2u)^{1/3})$ is odd if at least one of the 2-Selmer groups of the two elliptic curves $E^{(\pm u)}$ defined by $y^2 = x^3 \pm 2u$ is trivial (cf. [14]). Note also that the class number of $\mathbb{Q}((2u)^{1/3})$ is odd if and only if the class number of its Galois closure $\mathbb{Q}((2u)^{1/3}, \zeta_3)$ is odd (cf. [23]). For related topics, see e.g. [12, 28].

5. A VARIANT OF THEOREM 1.7

In §3, we proved Theorem 3.1 from the case $(\iota, \nu) = (2, 1)$ in Proposition 4.1 with $p = 2$. It is obvious that we can prove a counterpart of Theorem 3.1 from the other case $(\iota, \nu) = (1, 2)$. For

instance, we obtain the following corollary by applying the exactly same argument as the proof of Theorem 3.1 with $p = 2$ (with the same polynomial $h(A, B) = (6A - 1)^3 + P(6B + 3)^3$).

Corollary 5.1. *Let u be an odd square-free integer such that $u \not\equiv \pm 4 \pmod{9}$. Set $P = 2u$. Let $\pi = P^{1/3} \in \mathbb{R}$ be the real cubic root of P , $K = \mathbb{Q}(\pi)$ be the pure cubic field generated by π , and $\epsilon = \alpha + \beta\pi + \gamma\pi^2$ be the fundamental unit of K with $\alpha, \beta, \gamma \in \mathbb{Z}$. Suppose that both β and γ are even and the class number of K is odd. Then, there exist infinitely many odd prime numbers $l \equiv 2 \pmod{3}$ such that there exist no primitive solution of $x^3 + Py^3 = l^2z^4$.*

Moreover, by combining Corollary 5.1 and appropriate prime generating polynomials, we can prove a variant of Theorem 1.7. For instance, if $u \equiv \pm 1 \pmod{3}$, then by taking $f(X, Y) = P(3X \mp 1)^3 + (3Y + 1)^3$ and $g(X, Y) = P(3X \pm 1)^3 + (3Y)^3$ in place of f and g in the proof of Theorem 1.7 and $b_0 = 1 + 3Qb$ in place of $-1 + 3Qb$ so that $Pb_0 - lc_0 = \pm 3^k$ with some $k \geq 0$, we obtain the following.

Theorem 5.2. *Let n be an integer such that $n \geq 8$ and $n \equiv 0 \pmod{4}$, and P be as in Corollary 5.1. Then, there exist infinitely many $(n - 6)/2$ -tuples of pairs of integers (b_j, c_j) ($1 \leq j \leq (n - 6)/2$) satisfying the following condition:*

There exist infinitely many prime numbers l and infinitely many pairs of integers (b_0, c_0) such that the equation

$$(8) \quad (X^3 + PY^3)(b_0X^3 + lc_0Y^3) \prod_{j=1}^{(n-6)/2} (b_j^2X^2 + b_jc_jXY + c_j^2Y^2) = l^2Z^n$$

define non-singular plane curves which violate the local-global principle.

Moreover, for each n , there exists a set of such $(n - 4)/2$ -tuples $((b_j, c_j))_{0 \leq j \leq (n-6)/2}$ which gives infinitely many geometrically non-isomorphic classes of such curves.

6. NUMERICAL EXAMPLES OF DEGREE 8

In this section, we demonstrate that Theorems 1.7 and 5.2 actually give explicit equations of non-singular plane curve which violate the local-global principle. We emphasize that our construction has a character in contrast to examples obtained by Nguyen in [19, 20] (cf. Theorems 1.5 and 1.6). Recall that Nguyen constructed infinitely many plane curves of even degree which violate the local-global principle but explained by the Brauer-Manin obstruction on certain hyperelliptic curves covered by the plane curves. Here, we give two numerical examples for Theorems 1.7 and 5.2 respectively, both of which are $\mathbb{Z}/4\mathbb{Z}$ -coverings of hyperelliptic curves with \mathbb{Q} -rational points. This means that the violation of the local-global principle for the former plane curves cannot be explained by the Brauer-Manin obstruction on the latter hyperelliptic curves.

6.1. Example for Theorem 1.7 with $u = 7$. First, we construct an example for Theorem 1.7 in the case of $n = 8$ and $u = 7$. As a special value of $f(X, Y) = 14^2(3X + 1)^3 + (3Y + 1)^3$, we obtain a prime number $197 = 14^2 \cdot 1^3 + 1^3$ with $(b_1, c_1) = (1, 1)$. Moreover, as a special value of $h(A, B) = (6A + 1)^3 + 14(6B - 1)^3$, we obtain another prime number $l = 419 = (-11)^3 + 14 \cdot 5^3$, which is prime to $\{P, b_1, c_1\}$. Finally, in order to generate the coefficients $b_0 = -1 + 6b$ and c_0 , we solve the equation

$$14^2(-1 + 6b) - 419c_0 = \pm 3^k.$$

It has solutions $(b_0, c_0, \pm 3^k) = (365, 13^2, 3^6)$ with square c_0 . Therefore, for every $m = 3, 5, 7$, the equation

$$(X^3 + 14^2 Y^3)(365X^3 + 419 \cdot 13^2 Y^3)(X^2 + XY + Y^2) = 419^m Z^8$$

defines a non-singular plane curve which violates the local-global principle. However, its quotient by the automorphism $Z \rightarrow \zeta_4 Z$ gives a hyperelliptic curve defined by

$$(X^3 + 14^2 Y^3)(365X^3 + 419 \cdot 13^2 Y^3)(X^2 + XY + Y^2) = 419^m Z^2$$

which has a \mathbb{Q} -rational point $[X : Y : Z] = [0 : 1 : 13 \cdot 14/419^{(m-1)/2}]$.

6.2. Example for Theorem 5.2 with $u = 79$. Next, we construct an example for Theorem 5.2 again for $n = 8$ and $u = 79$.⁷ As a special value of $f(X, Y) = 158(3X - 1)^3 + (3Y + 1)^3$, we obtain a prime number $19751 = 158 \cdot 5^3 + 1^3$ with $(b_1, c_1) = (5, 1)$. Moreover, as a special value of $h(A, B) = (6A+1)^3 + 158(6B-1)^3$, we obtain another prime number $l = 4919 = (-59)^3 + 158 \cdot 11^3$, which is prime to $\{P, b_1, c_1\}$. Then, by solving the equation

$$158(1 + 6b) - 4919c_0 = \pm 3^k$$

with $b_0 = 1 + 6b$, we obtain a solution $(b_0, c_0, \pm 3^k) = (271^2, 2359, -3^5)$ with square b_0 . Therefore, the equation

$$(X^3 + 158Y^3)(271^2 X^3 + 4919 \cdot 2359Y^3)(5^2 X^2 + 5XY + Y^2) = 4919^2 Z^8$$

defines a non-singular plane curve which violates the local-global principle. However, its quotient by the automorphism $Z \rightarrow \zeta_4 Z$ gives a hyperelliptic curve defined by

$$(X^3 + 158Y^3)(271^2 X^3 + 4919 \cdot 2359Y^3)(5^2 X^2 + 5XY + Y^2) = 4919^2 Z^2,$$

which again has a \mathbb{Q} -rational point $[X : Y : Z] = [1 : 0 : 5 \cdot 271/4919]$.

7. NON-SINGULARITY OF SOME NGUYEN'S SEXTIC CURVES

The following Proposition 7.1, which is implicit in Nguyen's original article [19], gives an infinite family of non-singular plane curves of degree 6 which violate the local-global principle. In fact, we can prove similar results in more general settings. However, we restrict us in the following very special case because proofs in more general settings require some elaborated arguments including algebraic geometry on the thinness of singular loci and analytic number theory on prime distributions.

Proposition 7.1. *In the notation of Theorem 1.5, suppose that $(n, p, d, m) = (6, 17, 7, 6)$ with $q = 661$. Then, the plane curve defined by*

$$F(X, Y, Z) := sX^6 + Y^2(tX^2 + uY^2)(vX^2 + wY^2) - Z^2(AX^2 + BY^2 + CXY + \alpha Z^2)^2 = 0.$$

is geometrically non-singular for every $\alpha \in \mathbb{Q}^\times$ such that $AC \neq 0$.

⁷There are many positive integers satisfying the whole of the conditions in Theorem 5.2, say $u = 21, 35, 39, 79, 89, \dots$. Among them, $u = 79$ is the minimal one having no prime divisors $q = 3$ or $q \equiv 2 \pmod{3}$. Recall that, due to Proposition 2.1, if u has a prime divisor $q = 3$, then we need a congruence $Pb_0 - lc_0 = \pm 1$ stronger than $Pb_0 - lc_0 = \pm 3^k$. On the other hand, due to the proof of Theorem 5.2 (cf. the proof of Theorem 1.7 in §3), if u has a prime divisor $q \equiv 2 \pmod{3}$, then we need a congruence $b_0 \equiv 1 \pmod{6q}$ stronger than $b_0 \equiv 1 \pmod{6}$.

Proof. We prove the assertion by contradiction. Suppose that the curve has a singular point $[X : Y : Z] = [X_0 : Y_0 : Z_0]$. Then, we have

$$F(X_0, Y_0, Z_0) = \frac{\partial F}{\partial X}(X_0, Y_0, Z_0) = \frac{\partial F}{\partial Y}(X_0, Y_0, Z_0) = \frac{\partial F}{\partial Z}(X_0, Y_0, Z_0) = 0.$$

Here, we may assume that $X_0, Y_0, Z_0 \in \overline{\mathbb{Q}}$ and at least one of X_0, Y_0, Z_0 is non-zero. Since

$$\begin{aligned} \frac{\partial F}{\partial Z} &= -2Z(AX^2 + BY^2 + CXY + \alpha Z^2)^2 - 4\alpha Z^3(AX^2 + BY^2 + CXY + \alpha Z^2) \\ &= -2Z(AX^2 + BY^2 + CXY + \alpha Z^2)(AX^2 + BY^2 + CXY + 3\alpha Z^2), \end{aligned}$$

the condition $(\partial F / \partial Z)(X_0, Y_0, Z_0) = 0$ implies that

$$g := Z_0(AX_0^2 + BY_0^2 + CX_0Y_0 + \alpha Z_0^2) = 0$$

or

$$h := AX_0^2 + BY_0^2 + CX_0Y_0 + 3\alpha Z_0^2 = 0.$$

If $g = 0$, then the other conditions

$$F(X_0, Y_0, Z_0) = \frac{\partial F}{\partial X}(X_0, Y_0, Z_0) = \frac{\partial F}{\partial Y}(X_0, Y_0, Z_0) = 0$$

imply that the binary form $sX^6 + Y^2(tX^2 + uY^2)(vX^2 + wY^2)$ has a multiple factor $Y_0X - X_0Y$ in $\overline{\mathbb{Q}}[X, Y]$. Since $suw \neq 0$, the binary cubic form $sX^3 + Y(tX + uY)(vX + wY)$ must have a multiple factor in $\mathbb{Q}[X, Y]$. However, it is impossible because for every odd prime divisor l of d

$$sX^3 + Y(tX + uY)(vX + wY) \equiv pqX(qX^2 - pm^2Y^2) \pmod{l}$$

has no multiple factors in $\mathbb{F}_l[X, Y]$. Here, note that d actually has an odd prime divisor l because $p \equiv 1 \pmod{8}$ and d is a quadratic non-residue in \mathbb{F}_p^\times .

Thus, we may assume that

$$g \neq 0 \quad \text{and} \quad h = 0.$$

In this case, since $AX_0^2 + BY_0^2 + CX_0Y_0 + \alpha Z_0^2 = -2\alpha Z_0^2$, we have

$$F(X_0, Y_0, Z_0) = sX_0^6 + Y_0^2(tvX_0^4 + (tw + uv)X_0^2Y_0^2 + uwY_0^4) - 4\alpha^2Z_0^6 = 0.$$

Moreover, by using $h = 0$, we also have

$$\begin{aligned} 0 &= \frac{\partial F}{\partial X}(X_0, Y_0, Z_0) \\ &= 6sX_0^5 + X_0Y_0^2(4tvX_0^2 + 2(tw + uv)Y_0^2) + 4\alpha Z_0^4(2AX_0 + CY_0) \\ &= \frac{2}{9\alpha}(27s\alpha X_0^5 + 18tv\alpha X_0^3Y_0^2 + 9(tw + uv)\alpha X_0Y_0^4 \\ &\quad + 2(AX_0^2 + BY_0^2 + CX_0Y_0)^2(2AX_0 + CY_0)) \end{aligned}$$

Similarly, we also have

$$\begin{aligned}
0 &= \frac{\partial F}{\partial Y}(X_0, Y_0, Z_0) \\
&= Y_0(2tvX_0^4 + 4(tw + uv)X_0^2Y_0^2 + 6uwY_0^4) + 4\alpha Z_0^4(2BY_0 + CX_0) \\
&= \frac{2}{9\alpha}(9tv\alpha X_0^4Y_0 + 18(tw + uv)\alpha X_0^2Y_0^3 + 27uw\alpha Y_0^5 \\
&\quad + 2(AX_0^2 + BY_0^2 + CX_0Y_0)^2(2BY_0 + CX_0))
\end{aligned}$$

Now, we can check that $Y_0 \neq 0$ as follows. Indeed, if $Y_0 = 0$, then the condition $(\partial F/\partial Y)(X_0, Y_0, Z_0) = 0$ implies that $2A^2CX_0^5 = 0$, hence $X_0 = 0$. However, $h = 0$ implies that $Z_0 = 0$, which contradicts that at least one of X_0, Y_0, Z_0 is non-zero. Thus, we see that the two polynomials

$$27s\alpha x^5 + 18tv\alpha x^3 + 9(tw + uv)\alpha x + 2(Ax^2 + B + Cx)^2(2Ax + C)$$

and

$$9tv\alpha x^4 + 18(tw + uv)\alpha x^2 + 27uw\alpha + 2(Ax^2 + B + Cx)^2(2B + Cx)$$

has a common root $x_0 := X_0/Y_0$. However, the resultant of these two polynomials is

$$\begin{aligned}
&- 3^{12}\alpha^4 \cdot (37497075264030176093761516863755475310533474414045775364269884859527041449 \\
&\quad 2885208200593655822193598505700583601848088542180882946410298687488\alpha^{18} \\
&\quad + 47438632527983487701937506887768090856575961426590017104794262649537596 \\
&\quad 03534688739357716544340865167285385077432212371049215428777297018880\alpha^{17} \\
&\quad + \dots),
\end{aligned}$$

where we can check by Magma [3] that the second factor is an irreducible polynomial of α in $\mathbb{Q}[\alpha]$. In particular, the above resultant cannot be zero for any $\alpha \in \mathbb{Q}^\times$, a contradiction. This completes the proof. \square

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