An upper bound for higher order eigenvalues of symmetric graphs

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Abstract. In this paper, we derive an upper bound for higher eigenvalues of the normalized Laplace operator associated with a symmetric finite graph in terms of lower eigenvalues.

1. Introduction

Let $G$ be a connected, finite, simple and undirected graph on $N$ vertices. Let $\Delta$ be the normalized Laplace operator associated with $G$. The operator $-\Delta$ is identified with a positive semidefinite real symmetric matrix of size $N$. Denote by $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$ all eigenvalues of $\Delta$ counted with multiplicities. For any connected graph, we have $\lambda_0 = 0$ and its multiplicity is 1. All the eigenvalues lie in the interval $[0, 2]$. We consider the following question: Are there other constraints on the spectrum $\{\lambda_i\}_{i=0}^{N-1}$? In particular, is $\lambda_{k+1}$ controlled by preceding eigenvalues, $\lambda_1, \ldots, \lambda_k$? This question is a discrete analogue of the so-called Payne-Pólya-Weinberger’s inequality. For the Dirichlet eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \uparrow \infty$ of the Laplacian on a bounded domain in the Euclidean plane, Payne-Pólya-Weinberger [5,6] proved that

$$\lambda_{k+1} - \lambda_k \leq \frac{2}{k} \sum_{i=1}^{k} \lambda_i.$$ 

This result is extended to arbitrary dimension by Thompson [7]. Later, Hile and Protter [3] and Yang [8] proved sharper inequalities. In particular, Yang [8] proved that

$$(1.1) \quad \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i.$$ 

Chung and Oden proposed to study of the discrete analogue of these results. For the Dirichlet eigenvalues $\{\lambda_i\}_{i\geq 1}$ of the normalized Laplacian on a connected finite subgraph in the integer lattice of rank $n$, Hua, Lin and Su [4] proved that

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 (1 - \lambda_i) \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i.$$ 

What can be said about the Laplacian without boundary conditions? Unlike the
case of the Dirichlet boundary condition, 0 is always an eigenvalue. For the eigenvalues \( \{\lambda_i\}_{i \geq 0} \) with \( \lambda_0 := 0 \) of the Laplacian on a compact Riemannian homogeneous manifold, Cheng and Yang [1] proved that

\[
\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(4\lambda_i + \lambda_1).
\]

In this paper, we consider a discrete analogue of (1.2). More precisely, for a finite symmetric graph, i.e., a graph whose isomorphism group acts transitively on the set of pairs of adjacency vertices, we prove a discrete analogue of (1.2).

**Theorem 1.1.** Let \( G \) be an symmetric finite graph with \( N \) vertices. Denote by \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1} \) the all eigenvalues of the normalized Laplace operator. Then, for any non-zero eigenvalue \( \lambda \) of \( \Delta \), we have

\[
\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2(1 - \lambda_i) \leq \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)(2(2 - \lambda)\lambda_i + \lambda).
\]

By using Chebyshev’s sum inequality, we obtain an upper bound of \( \lambda_{k+1} \) in terms of \( \lambda_1, \ldots, \lambda_k \).

**Theorem 1.2.** In the same setting as Theorem 1.1, we have

\[
\lambda_{k+1} \leq \frac{(k + 1)\lambda_1 + \sum_{i=1}^{k}((5 - 2\lambda_1)\lambda_i - \lambda_i^2)}{\sum_{i=0}^{k}(1 - \lambda_i)}.
\]

Let \( \mu_1 := \lambda_1 \) and \( m \) be the multiplicity of \( \mu_1 \). If \( G \) is not a complete graph, then we can consider \( \mu_2 := \lambda_{m+1} \), i.e., the second smallest positive eigenvalue. We have a upper bound for the ratio \( \mu_2/\mu_1 \) in terms of the multiplicity of \( \mu_1 \).

**Corollary 1.3.** In the same setting as Theorem 1.1, let \( m \) be the multiplicity of \( \mu_1 \) and put \( \mu_2 := \lambda_{m+1} \). Then, we have

\[
\frac{\mu_2}{\mu_1} \leq 3m + 1.
\]

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## 2. Preliminaries

In this section, unless otherwise stated, we assume that all graphs are connected, finite, simple and undirected. We recall some basic facts on the theory of eigenvalues of a regular graph. For details, see e.g. [2]. Let \( G = (V, E) \) be a \( d \)-regular graph, \( d \geq 1 \), and put \( N := \#V \). If two vertices \( x, y \in V \) are adjacent, then we denote this situation by \( x \sim y \). Note that since \( G \) is undirected, \( x \sim y \) if and only if \( y \sim x \). The normalized
Laplace operator $\Delta$ acting on the space $C(V)$ of functions on $V$ is defined by

$$\Delta u(x) := \frac{1}{d} \sum_{y \sim x} (u(y) - u(x)), \quad u \in C(V), \quad x \in V.$$  

The normalized Laplace operator is identified with the real-symmetric matrix $D^{-1} A - I$, where $D$ is the scalar matrix with diagonal entries $d$, $A$ is the adjacency matrix of $G$ and $I$ is the identity matrix. A complex number $\lambda$ is called an eigenvalue of $\Delta$ if there exists $u \in C(V) \setminus \{0\}$ such that $\Delta u + \lambda u = 0$ holds. In this case, the function $u$ is called an eigenfunction with eigenvalue $\lambda$. For an eigenvalue $\lambda$ of $\Delta$, we denote by $W_\lambda$ the space of all functions $u \in C(V)$ satisfying $\Delta u + \lambda u = 0$ and we call the dimension of $W_\lambda$ multiplicity of $\lambda$. Let us denote the eigenvalues of $\Delta$ by $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$, counted with multiplicity. We define an inner product $\langle \cdot, \cdot \rangle$ on $C(V)$ by

$$\langle u, v \rangle := \sum_{x \in V} u(x)v(x)d.$$  

We denote by $\|\cdot\|$ the norm induced by the inner product $\langle \cdot, \cdot \rangle$. We list up some elementary facts on eigenvalues and eigenfunctions without proofs.

- 0 is an eigenvalue of multiplicity 1 and constant functions are eigenfunctions with eigenvalue 0.
- All eigenvalues lie in the interval $[0, 2] \subset \mathbb{R}$.
- There exists an orthonormal basis $\{u_i\}_{i=0}^{N-1}$ of $C(V)$ such that each function $u_i$ is an eigenfunction with eigenvalue $\lambda_i$.

By the min-max formula, each eigenvalue $\lambda_k$ has a variational characterization:

$$\lambda_k = \inf \left\{ \frac{\sum_{x \sim y} (u(y) - u(x))^2}{2d \sum_v u^2} \middle| u \neq 0, \langle u, u_i \rangle = 0, \quad i = 0, \ldots, k-1 \right\},$$  

where the symbol $\sum_{x \sim y}$ means the summation over all unordered pairs $(x, y)$ such that $x \sim y$. In particular, we have

$$\lambda_1 = \inf \left\{ \frac{\sum_{x \sim y} (u(y) - u(x))^2}{2d \sum_v u^2} \middle| u \neq 0, \quad \sum_v u = 0 \right\}.$$  

The following lemma is well-known. For the proof, see e.g. [2].

**Lemma 2.1.** For any regular graph $G$ but a complete graph, we have

$$\lambda_1 \leq 1.$$  

**Remark 2.2.** If $G$ is the complete graph of degree $d$, then $\lambda_1 = 1 + 1/d$.

Let $\Gamma : C(V) \times C(V) \to C(V)$ be the carré du champ operator associated to $\Delta$, i.e., for $u, v \in C(V)$,

$$\Gamma(u, v) := \frac{1}{2} (\Delta(uv) - (\Delta u)v - u\Delta v).$$  

For two vertices \(x, y \in V \) with \(x \sim y\), we define the difference operator \(\nabla_{xy} : C(V) \to C(V)\) by

\[
\nabla_{xy} u := u(y) - u(x), \quad u \in C(V).
\]

By a simple calculation, we have

\[
\Gamma(u, v)(x) = \frac{1}{2d} \sum_{y \sim x} (\nabla_{xy} u)(\nabla_{xy} v), \quad x \in V.
\]

The carré du champ \(\Gamma(u, v)\) is an analogy of \(\langle \nabla u, \nabla v \rangle\) in the context of Riemannian geometry, where \(\nabla\) is the gradient operator. We list up some identities for \(\Gamma\).

**Lemma 2.3.** Let \(u, v, v_1, v_2 \in C(V)\).

1. \(\langle u, \Delta v \rangle = -\sum_{x \in V} \Gamma(u, v)d.\)
2. For any \(x \in V\), we have

\[
\Gamma(u, v_1v_2)(x) = \Gamma(u, v_1)v_2(x) + \Gamma(u, v_2)v_1(x)
+ \frac{1}{2d} \sum_{y \sim x} (\nabla_{xy} u)(\nabla_{xy} v_1)(\nabla_{xy} v_2).
\]

In particular,

\[
\sum_{x \in V} \Gamma(u, v_1v_2) = \sum_{x \in V} (\Gamma(u, v_1)v_2 + \Gamma(u, v_2)v_1).
\]

Making use of the min-max formula and appropriate trial functions, we have the following lemma.

**Lemma 2.4.** Let \(k \geq 1\) be an integer. For any function \(h \in C(V)\), we have

\[
\frac{1}{2} \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \Phi_i(h) \leq \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) \|2\Gamma(h, u_i) + u_i \Delta h\|^2,
\]

where \(\Phi_i(h) = \sum_{x \sim y} u_i(x)u_i(y)(\nabla_{xy} h)^2\).

**Proof.** Let \(h \in C(V)\). For \(i = 0, \ldots, k\), define \(\varphi_i \in C(V)\) as the orthogonal projection of \(hu_i\) to the subspace spanned by \(\{u_{k+1}, \ldots, u_{N-1}\}\), i.e.,

\[
\varphi_i := hu_i - \sum_{j=0}^{k} a_{ij}u_j, \quad a_{ij} := \langle hu_i, u_j \rangle.
\]

Clearly the function \(\varphi_i\) is perpendicular to \(u_0, \ldots, u_k\). The min-max formula yields

\[
\lambda_{k+1} \|\varphi_i\|^2 \leq \frac{1}{2} \sum_{x \sim y} (\nabla_{xy} \varphi_i)^2 = \sum_{x \in V} \Gamma(\varphi_i, \varphi_i)d.
\]
From (1) in Lemma 2.3 and the fact that \( \langle \phi_i, u_j \rangle = 0 \) for \( j = 0, \ldots, k \), we have
\[
\sum_{V} \Gamma(\phi_i, \phi_i) d = -\langle \phi_i, \Delta \phi_i \rangle \\
= -\langle \phi_i, 2\Gamma(h, u_i) + u_i \Delta h - \lambda_i u_i h + \sum_{j=0}^{k} a_{ij} \lambda_j u_j \rangle \\
= -\langle \phi_i, 2\Gamma(h, u_i) + u_i \Delta h - \lambda_i u_i h \rangle \\
= -\langle \phi_i, 2\Gamma(h, u_i) + u_i \Delta h \rangle + \lambda_i ||\phi_i||^2.
\]

From (2.2), we obtain
\[
(2.3) \quad (\lambda_{k+1} - \lambda_i) ||\phi_i||^2 \leq -\langle \phi_i, 2\Gamma(h, u_i) + u_i \Delta h \rangle.
\]

Let \( A_i \) be the right hand side of (2.3). We estimate \( A_i \) in two ways. First, we claim that
\[
(2.4) \quad A_i = \frac{1}{2} \sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2 + \sum_{j=0}^{k} (\lambda_i - \lambda_j) a_{ij}^2.
\]

To see (2.4), we use Lemma 2.3. By the definition of \( \phi_i \),
\[
A_i = \sum_{j=0}^{k} a_{ij} \langle u_j, u_i \Delta h + 2\Gamma(h, u_i) \rangle - d \sum_{V} (h u_i^2 \Delta h + 2 h u_i \Gamma(h, u_i)).
\]

The first term is equal to \( \sum_{j=0}^{k} (\lambda_i - \lambda_j) a_{ij}^2 \). Indeed, by the definition of \( \Gamma(h, u_i) \) and Lemma 2.3, we have
\[
\langle u_j, u_i \Delta h + 2\Gamma(h, u_i) \rangle = \langle u_j, \Delta(h u_i) + \lambda_i h u_i \rangle = \lambda_i a_{ij} - \langle \lambda_i u_j, h u_i \rangle = (\lambda_i - \lambda_j) a_{ij}.
\]

The second term is equal to \( \sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2 / 2 \). Indeed,
\[
-\langle h u_i^2, \Delta h \rangle = \sum_{V} \Gamma(h u_i^2, h) d \\
= \sum_{V} (h \Gamma(u_i^2, h) + u_i^2 \Gamma(h, h)) d \\
= \frac{1}{2} \sum_{x \sim y} ((\nabla_{xy} u_i)^2 (h)(\nabla_{xy} h) + u_i(x)^2 (\nabla_{xy} h)^2) \\
+ \sum_{V} 2 h u_i \Gamma(h, u_i) d \\
= \frac{1}{2} \sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2 + \sum_{V} 2 h u_i \Gamma(h, u_i) d.
\]
Second, we claim that

\[(\lambda_{k+1} - \lambda_i)A_i \leq \|u_i \Delta h + 2\Gamma(h, u_i)\|^2 - \sum_{j=0}^k (\lambda_i - \lambda_j)^2 a_{ij}^2,\]

From the definition of $A_i$, we have

\[A_i = -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h - \sum_{j=0}^k (\lambda_i - \lambda_j) a_{ij}u_j \rangle.\]

Applying the Cauchy-Schwartz inequality to the definition of $A_i$ and taking (2.3) and (2.5) into account, we have

\[(\lambda_{k+1} - \lambda_i)A_i^2 \leq A_i(\|2\Gamma(h, u_i) + u_i \Delta h\|^2 - \sum_{j=0}^k (\lambda_i - \lambda_j)^2 a_{ij}^2).\]

From (2.4) and (2.6), we obtain

\[
\begin{align*}
\frac{1}{2} & \sum_{i=0}^k (\lambda_{k+1} - \lambda_i)^2 \sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2 + \sum_{i,j=0}^k (\lambda_{k+1} - \lambda_i)^2 (\lambda_i - \lambda_j) a_{ij}^2 \\
& \leq \sum_{i=0}^k (\lambda_{k+1} - \lambda_i)\|2\Gamma(h, u_i) + u_i \Delta h\|^2 - \sum_{i,j=0}^k (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2 a_{ij}^2.
\end{align*}
\]

Since $\sum_{i,j=0}^k (\lambda_{k+1} - \lambda_i)^2 (\lambda_i - \lambda_j) a_{ij}^2 = -\sum_{i,j=0}^k (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2 a_{ij}^2$, we complete the proof. \(\square\)

3. Proof of main results

In this section, we prove Theorem 1.1, Theorem 1.2 and Corollary 1.3. In the subsection 3.1, we show some symmetries of eigenfunctions on a symmetric graph, which are used to prove Theorem 1.1.

3.1. Symmetries of eigenfunctions on a symmetric graph

We derive some properties of eigenfunctions on a symmetric graph. In particular, it seems that symmetry is essential in Lemma 3.2. A graph $G = (V, E)$ is said to be symmetric if for any two edges $(x, y), (x', y') \in E$, there exists an automorphism $\gamma$ of $G$ such that $x' = \gamma x$ and $y' = \gamma y$ hold. We denote by $\text{Aut}(G)$ the group of automorphisms of $G$. Note that symmetric graphs are vertex-transitive, i.e., $\text{Aut}(G)$ acts transitively on $V$, and thus regular. We say that a vector subspace $W$ of $C(V)$ is invariant if for any $u \in W$ and $\gamma \in \text{Aut}(G)$, $\gamma u \in W$, where $\gamma u$ is defined by $\gamma u(x) := u(\gamma x), x \in V$.

**Lemma 3.1.** Let $G = (V, E)$ be a vertex-transitive graph. Let $W$ be an invariant vector subspace of $C(V)$ of dimension $m$ and let $\{u_{\alpha}\}_{\alpha=1}^m$ be an orthonormal basis of $W$. Then, the function $|u_1|^2 + \cdots + |u_m|^2$ is constant and its value is $m/d\#V$.

**Proof.** Put $f(x) := |u_1(x)|^2 + \cdots + |u_m(x)|^2$. By the invariance of $W$, the family $\{\gamma u_{\alpha}\}_{\alpha=1}^m$ is also an orthonormal basis of $W$ for any $\gamma \in \text{Aut}(G)$. For fixed $x \in V$, it
is easy to see that the sum $|u_1(x)|^2 + \cdots + |u_m(x)|^2$ is independent of the choice of an orthonormal basis $\{u_\alpha\}$. Thus,

$$f(\gamma x) = \sum_{\alpha=1}^m |\gamma u_\alpha(x)|^2 = \sum_{\alpha=1}^m |u_\alpha(x)|^2 = f(x).$$

The transitivity of the action of $\text{Aut}(G)$ yields that $f$ is constant. Let $C$ be the value of $|u_1(x)|^2 + \cdots + |u_m(x)|^2$. By multiplying $d$ and summing over $x \in V$, we have

$$Cd\#V = \sum_{\alpha=1}^m \sum_{x \in V} |u_\alpha(x)|^2 d = m.$$

□

**Lemma 3.2.** Let $G$ be a symmetric graph. Let $\lambda$ be an eigenvalue of $\Delta$ and let $\{u_\alpha\}_{\alpha=1}^m$ be an orthonormal basis of $W_\lambda$. Then, the function $g(x, y) := \sum_{\alpha=1}^m |\nabla_{xy} u_\alpha|^2$, $x \sim y$, is constant and its value is $m\lambda/\#E$.

**Proof.** Since $W_\lambda$ is an invariant vector subspace of $C(V)$, the family $\{\gamma u_\alpha\}_{\alpha=1}^m$ is also an orthonormal basis of $W_\lambda$ for any $\gamma \in \text{Aut}(G)$. Since the sum $\sum_{\alpha=1}^m |\nabla_{xy} u_\alpha|^2$ is independent of the choice of an orthonormal basis $\{u_\alpha\}$, we have

$$g(\gamma x, \gamma y) = \sum_{\alpha=1}^m |\nabla_{xy} (\gamma u_\alpha)|^2 = \sum_{\alpha=1}^m |\nabla_{xy} u_\alpha|^2 = g(x, y).$$

The symmetry of $G$ yields that $g$ is constant. Let $C'$ be the value of $g$. By summing over $x \sim y$, we have

$$2C' \#E = \sum_{\alpha=1}^m \sum_{x \sim y} |\nabla_{xy} u_\alpha|^2 = 2\lambda m.$$

□

**Corollary 3.3.** Let $G$ be a symmetric graph. Let $\lambda$ and $\{u_\alpha\}$ be as in Lemma 3.2. Then, the function $f_3(x, y) = \sum_{\alpha=1}^m u_\alpha(x) \nabla_{xy} u_\alpha$ is constant and its value is $-\lambda m/2\#E$.

**Proof.** The constancy of $f_3$ immediately follows from Lemma 3.1 and Lemma 3.2. Let $C_3$ be the value of $f_3$. By summing over $x \sim y$, we have

$$2C_3 \#E = \sum_{\alpha=1}^m \sum_{x \sim y} u_\alpha(x) \nabla_{xy} u_\alpha.$$

By interchanging $x$ and $y$,

$$\sum_{x \sim y} u_\alpha(x) \nabla_{xy} u_\alpha = - \sum_{x \sim y} u_\alpha(y) \nabla_{xy} u_\alpha.$$
Thus, we obtain

\[ 2C_3 \# E = \frac{1}{2} \sum_{\alpha=1}^{m} \sum_{x \sim y} (u_{\alpha}(x) - u_{\alpha}(y)) \nabla_{xy} u_{\alpha} = -\lambda m. \]

\[ \square \]

3.2. Proof of main results

We prove Theorem 1.1, Theorem 1.2 and Corollary 1.3. First, we prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( \{u_{\alpha}\}_\alpha \) be an orthonormal basis of \( E_{\mu} \). Then, we have

\[ \sum_{\alpha=1}^{m} \sum_{x \sim y} u_{\alpha}(x) u_{\alpha}(y) |\nabla_{xy} u_{\alpha}|^2 = \frac{\lambda m}{\# E} \sum_{x \sim y} u_{\alpha}(x) u_{\alpha}(y) \]

\[ = \frac{\lambda m}{\# E} \sum_{x \in V} u_{\alpha}(x) d \sum_{y \sim x} u_{\alpha}(y) \]

\[ = \frac{\lambda m}{\# E} (1 - \lambda_1) \sum_{x \in V} u_{\alpha}(x)^2 d \]

\[ = \frac{\lambda m}{\# E} (1 - \lambda_1). \quad (3.1) \]

Next, we evaluate \( \sum_{\alpha} \|2\Gamma(u_{\alpha}, u_{\alpha}) + u_{\alpha} \Delta u_{\alpha}\|^2 \). By Jensen’s inequality, we have

\[ 4\Gamma(u_{\alpha}, u_{\alpha})(x)^2 = \left( \frac{1}{d} \sum_{y \sim x} (\nabla_{xy} u_{\alpha})(\nabla_{xy} u_{\alpha}) \right)^2 \leq \frac{1}{d} \sum_{y \sim x} (\nabla_{xy} u_{\alpha})^2 (\nabla_{xy} u_{\alpha})^2, \]

which yields

\[ 4 \sum_{\alpha=1}^{m} \sum_{x \in V} \Gamma(u_{\alpha}, u_{\alpha})(x)^2 d \leq \frac{2\lambda_1 m}{\# E}. \quad (3.2) \]

By Lemma 3.1, we have

\[ \sum_{\alpha=1}^{m} \sum_{x \in V} (u_{\alpha}(x) \Delta u_{\alpha}(x))^2 d = \frac{\lambda^2 m}{2 \# E}. \quad (3.3) \]

By Lemma 3.3,

\[ -4\lambda \sum_{\alpha=1}^{m} \sum_{x \in V} u_{\alpha}(x) u_{\alpha}(x) \Gamma(u_{\alpha}, u_{\alpha}) d = \frac{\lambda^2 m}{\# E} \sum_{x \in V} u_{\alpha}(x) \sum_{y \sim x} \nabla_{xy} u_{\alpha} \]

\[ = -\frac{\lambda^2 \lambda_1 m}{\# E}. \quad (3.4) \]

By letting \( h = u_{\alpha} \) in Lemma 2.4, summing over \( \alpha = 1, \ldots, m \) and taking (3.1), (3.2),
(3.3) and (3.4) into account, we obtain
\[ \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 (1 - \lambda_i) \leq \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)(2(2 - \lambda)\lambda_i + \lambda). \]

In order to prove Theorem 1.2, we need some lemmas.

**Lemma 3.4 (Chebyshev’s sum inequality).** Let \( N \geq 1 \) be an integer and \( \{a_i\}_{i=1}^{N}, \{b_i\}_{i=1}^{N} \) two sequences of real numbers. If both of \( \{a_i\}_{i=1}^{N}, \{b_i\}_{i=1}^{N} \) are non-increasing, then
\[ \frac{1}{N} \sum_{i=1}^{N} a_i b_i \geq \left( \frac{1}{N} \sum_{i=1}^{N} a_i \right) \left( \frac{1}{N} \sum_{i=1}^{N} b_i \right). \]

**Lemma 3.5.** For any \( 0 \leq k \leq N - 1 \),
\[ \sum_{i=0}^{k} (1 - \lambda_i) \geq 0 \]
and the equality holds if and only if \( k = N - 1 \).

**Proof.** Let \( A \) be the adjacency matrix of \( G \) and \( \mu_0 \geq \mu_1 \geq \cdots \geq \mu_{N-1} \) be all eigenvalues of \( A \). Since any diagonal entry of \( A \) is 0, \( \sum_{i=0}^{N-1} \mu_i \) is also 0 and \( \sum_{i=0}^{k} \mu_i \geq 0 \) for any \( k \), with the equality holds if and only if \( k = N - 1 \). By the relation between \( \Delta \) and \( A \), we have
\[ \sum_{i=0}^{k} (1 - \lambda_i) = \frac{1}{d} \sum_{i=0}^{k} \mu_i \geq 0 \]
and the equality holds if and only if \( k = N - 1 \).

Next, we prove Theorem 1.2 and Corollary 1.3.

**Proof of Theorem 1.2.** By letting \( \lambda = \lambda_1 \) in Theorem 1.1, we have
\[ \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i^2 - (\lambda_{k+1} - 2\lambda_1 + 5)\lambda_i + \lambda_{k+1} - \lambda) \leq 0. \]

Clearly, \( \lambda_{k+1} - \lambda_i \) is non-increasing in \( i \). Put \( f(x) := x^2 - (\lambda_{k+1} - 2\lambda_1 + 5)x \). Then, the function \( f \) is non-increasing in the interval \((-\infty, (\lambda_{k+1} - 2\lambda_1 + 5)/2]\). From Lemma 2.1, \( (\lambda_{k+1} - 2\lambda_1 + 5)/2 \geq 2 \). Since \( 0 \leq \lambda_i \leq 2 \), \( \lambda_i^2 - (\lambda_{k+1} - 2\lambda_1 + 5)\lambda_i + \lambda_{k+1} - \lambda \) is non-increasing in \( i \). We may use Lemma 3.4 and thus
\[ \left( \lambda_{k+1} - \sum_{i=0}^{k} \frac{\lambda_i}{k+1} \right) \left( \frac{1}{k+1} \sum_{i=0}^{k} (1 - \lambda_i)\lambda_{k+1} + \lambda_i^2 - (5 - 2\lambda_1)\lambda_i - \lambda \right) \leq 0. \]
If $k \geq m(\lambda_1)$, then \( \lambda_{k+1} - \sum_{i=0}^{k} \lambda_i/(k+1) \) is strictly positive. In this case, we have

\[
\frac{1}{k+1} \sum_{i=0}^{k} ((1 - \lambda_i)\lambda_{k+1} + \lambda_i^2 - (5 - 2\lambda_1)\lambda_i - \lambda_1) \leq 0.
\]

By Lemma 3.5, we obtain

\[
\lambda_{k+1} \leq \frac{(k+1)\lambda_1 + \sum_{i=1}^{k} ((5 - 2\lambda_1)\lambda_i - \lambda_i^2)}{\sum_{i=0}^{k} (1 - \lambda_i)}.
\]

This inequality also holds for $k < m(\lambda_1)$. □

**Proof of Corollary 1.3.** If $k = m(\lambda_1)$, then $\lambda_{k+1} = \mu_2$ and $\lambda_1 = \cdots = \lambda_{k-1} = \mu_1$. By Theorem 1.2, we have

\[
\frac{\mu_2}{\mu_1} \leq \frac{6m + 1 - 3m\mu_1}{m + 1 - m\mu_1}.
\]

Let $g(x) := (6m + 1 - 3mx)/(m + 1 - mx)$. The function $g$ is increasing. By Lemma 2.1,

\[
\frac{\mu_2}{\mu_1} \leq g(1) = 3m + 1.
\]

☐

### 4. On the non-triviality of Corollary 1.3

In this section, we consider symmetric graphs, other than complete graphs. Let $\mu_1$ and $\mu_2$ be the first and the second smallest positive eigenvalue, respectively. If $(3m+1)\mu_1$ is not less than 2, then the inequality in Corollary 1.3 is trivial since $\mu_2 \leq 2$ always holds. In this section, we see that there exist infinitely many graphs such that $(3m + 1)\mu_1$ is strictly less than 2.

Let $C_N, N \geq 3$, be the cycle graph with $N$ vertices. Cycle graphs are symmetric. The spectra of cycle graphs are well-known.

**Lemma 4.1.** The smallest positive eigenvalue of the normalized Laplace operator associated with $C_N$ is $1 - \cos(2\pi/N)$ and its multiplicity is 2.

Since $1 - \cos(2\pi/N)$ is decreasing in $N$ and tends to 0 as $N \to \infty$, there exists a number $N_0$ such that $(3m + 1)\mu_1 = 7(1 - \cos(2\pi/N))$ is strictly less than 2 for any $N \geq N_0$. In fact, we can take $N_0 = 9$.

**References**


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