ON THE MULTICANONICAL SYSTEMS OF QUASI-ELLIPTIC SURFACES

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ABSTRACT. We consider the multicanonical systems $|mK_S|$ of quasi-elliptic surfaces with Kodaira dimension 1 in characteristic 2. We show that for any $m \geq 6$ $|mK_S|$ gives the structure of quasi-elliptic fiber space, and 6 is the best possible number to give the structure for any such surfaces.

1. INTRODUCTION

Let $k$ be an algebraically closed field of characteristic $p \geq 0$, and let $S$ be a nonsingular complete algebraic surface with Kodaira dimension 1 defined over $k$. Then, $S$ has a structure of genus 1 fibration $\varphi : S \to B$. We denote by $K_S$ a canonical divisor of $S$ and we consider the multicanonical system $|mK_S|$. As is well known, the multicanonical system $|mK_S|$ gives the genus 1 fibration if $m$ is large enough. In Katsura and Ueno [5] and Katsura [3] (see also Iitaka [2]), we considered the following question:

Question 1.1. (1) Does there exist a positive integer $M$ such that if $m \geq M$, the multicanonical system $|mK_S|$ gives a structure of genus 1 fibration for any elliptic surface $S$ over $k$ with Kodaira dimension 1?

(2) What is the smallest $M$ which satisfies this property?

For this question, we have the following theorem.

Theorem 1.2. (1) For the complex analytic elliptic surfaces, $M = 86$ and 86 is best possible (cf. Iitaka [2]).

(2) For the algebraic elliptic surfaces, if the characteristic $p = 0$ or $p \geq 3$, then $M = 14$ and 14 is best possible (Katsura and Ueno [5] and Katsura [3]).

(3) For the algebraic elliptic surfaces, if the characteristic $p = 2$, then $M = 12$ and 12 is best possible (Katsura [3]).
If $p = 2$ or 3, there are two kinds of genus 1 fibrations, namely, the elliptic fibration and the quasi-elliptic fibration (cf. Bombieri and Mumford [1]). In these cases, we can also consider the same question for quasi-elliptic surfaces with Kodaira dimension 1. In characteristic 3, we already showed the following results (Katsura [4]).

**Theorem 1.3.** For the quasi-elliptic surfaces in characteristic 3, we have $M = 5$, and 5 is best possible.

Therefore, the remaining case of the question for the surfaces with Kodaira dimension 1 is the one in characteristic 2, and in this paper we show the following theorem. It finishes the answer to the question above for surfaces with Kodaira dimension 1 which S. Iitaka considered in the case of complex analytic elliptic surfaces in 1970 (cf. [2]).

**Theorem 1.4.** For the quasi-elliptic surfaces in characteristic 2, we have $M = 6$ and 6 is best possible.

In Section 2, we summarize basic facts on the theory of vector fields in positive characteristic and some results on quasi-elliptic surfaces. In Section 3, we give a criterion for a vector field that makes a singularity on the quotient of curve. In Section 4, we construct a quasi-elliptic surface over an elliptic curve with only one tame multiple fiber and examine the structure of its multicanonical system. In Section 5, we examine the multicanonical system of quasi-elliptic surfaces in characteristic 2 and show our main theorem.

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2. **Preliminaries**

Let $k$ be an algebraically closed field of characteristic $p > 0$, and let $S$ be a nonsingular complete algebraic surface defined over $k$. A non-zero rational vector field $D$ on $S$ is called $p$-closed if there exists a rational function $f$ on $S$ such that $D^p = fD$.

We use a vector field to construct a quotient surface of $S$. Let $\{U_i = \text{Spec}A_i\}$ be an affine open covering of $S$ and we set $A_i^D = \{D(\alpha) = 0 | \alpha \in A_i\}$. Then, affine surfaces $\{U_i^D = \text{Spec}A_i^D\}$ glue together to define a normal quotient surface $S^D$.

We now recall some results on vector fields by Rudakov and Shafarevich [8, Section 1]. Now, we assume that $D$ is $p$-closed. Then, we know that the natural morphism $\pi : S \rightarrow S^D$ is a purely inseparable morphism of degree $p$. If the affine open covering $\{U_i\}$ of $S$ is fine enough, then taking local coordinates $x_i, y_i$ on $U_i$, we see that there exist $f_i, g_i \in A_i$ and a rational
function $h_i$ such that the divisors defined by $f_i = 0$ and by $g_i = 0$ have no common divisor and that the vector field $D$ is expressed as

$$D = h_i \left( f_i \frac{\partial}{\partial x_i} + g_i \frac{\partial}{\partial y_i} \right)$$
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on $U_i$.

Divisors $(h_i)$ on $U_i$ give a global divisor $(D)$ on $S$, and zero-cycles defined by the ideal $(f_i, g_i)$ on $U_i$ give a global zero cycle $(D)$ on $S$. A point contained in the support of $(D)$ is called an isolated singular point of $D$. ([8, Theorem 1, Corollary]). Rudakov and Shafarevich showed that $S^D$ is nonsingular if and only if $(D) = 0$. When $S^D$ is nonsingular, they also showed a canonical divisor formula

\begin{equation}
K_S \sim \pi^* K_{S^D} + (p - 1)(D),
\end{equation}

where $\sim$ means linear equivalence.

Now, we consider an irreducible curve $C$ on $S$ and we set $C' = \pi(C)$. Take an affine open set $U_i$ above such that $C \cap U_i$ is non-empty. The curve $C$ is said to be integral with respect to the vector field $D$ if $D$ is tangent to $C$ at a general point of $C \cap U_i$. Rudakov-Shafarevich showed the following proposition (cf. [8, Proposition 1]):

**Proposition 2.1.** (i) If $C$ is integral, then $C = \pi^{-1}(C')$ and $C^2 = pC'^2$.

(ii) If $C$ is not integral, then $pC = \pi^{-1}(C')$ and $pC^2 = C'^2$.

Now, let $\varphi : S \longrightarrow B$ be a quasi-elliptic surface. We denote by $g$ the genus of the curve $B$. As was shown in Katsura [4], we have $\text{Alb}(S) \cong J(B)$, and $\chi(\mathcal{O}_S) \geq (1 - g)/3$ (See also Lang [6] and Raynaud [7]). Here, $\text{Alb}(S)$ is the Albanese variety of $S$ and $J(B)$ is the Jacobian variety of $B$. As a corollary, we know that if $g = 1$, then $\chi(\mathcal{O}_S) \geq 0$, and that if $g = 0$, then $\chi(\mathcal{O}_S) \geq 1$. We will freely use these inequalities in Section 5.

3. CUSPIDAL POINTS

From here on, let $k$ be an algebraically closed field of characteristic 2, if otherwise mentioned. Let $S$ be a nonsingular complete algebraic surface over $k$, and let $D$ be a non-zero 2-closed rational vector field on $S$. Let $U$ be an affine open set of $S$, and $x, y$ be local coordinates of $U$. Then, as in Section 2, $D$ is given by

$$D = h(f \partial/\partial x + g \partial/\partial y),$$

where $f, g$ are regular functions on $U$ such that $f = 0$ has no common curves with $g = 0$, and where $h$ is a rational function on $S$.

**Lemma 3.1.** Under the assumption above, $D(fg) = 0$ holds.
Proposition 3.4. Under the notation in Definition 3.3, we consider the projection \( P \) of the cuspidal point \( P \) on \( C \) and integral at a point \( P \) on \( C \), we call \( P \) a cuspidal point of the vector field \( D \).

**Corollary 3.2.** \( D(f/g) = 0. \)

**Proof.** We have \( D(f/g) = D(fg/g^2) = (1/g^2)D(fg) = 0. \)

**Definition 3.3.** Let \( D \) be a non-zero rational vector field on a nonsingular surface \( S \), and \( C \) be a nonsingular irreducible curve on \( S \). Let \( P \) be a point on \( C \) which is not an isolated singular point of \( D \). If \( D \) is non-integral on \( C \) and integral at a point \( P \) on \( C \), we call \( P \) a cuspidal point of the vector field \( D \).

**Proposition 3.4.** Under the notation in Definition 3.3, we consider the projection \( \pi : S \rightarrow S^D \). Then, the image \( \pi(P) \) of the cuspidal point \( P \) is a singular point of the curve \( \pi(C) \).

**Proof.** Let \( O_P \) be the local ring of the cuspidal point \( P \) and let \( x, y \) be a system of parameters of \( O_P \). Let \( x = 0 \) be a local equation of \( C \) at the point \( P \). By the definition of cuspidal points, there exist elements \( \alpha, \beta, \gamma \) and \( \delta \) of \( O_P \) and a constant \( c \in k \) such that \( \beta \neq 0 \) and \( c \neq 0 \), and such that \( f = \alpha x + \beta y \) and \( g = \gamma x + \delta y + c \). Since the situation is local, we may omit \( h \) from \( D \). By Corollary 3.2, we see that \( D(x + (f/g)y) = 0. \) Since \( g(P) \neq 0 \), \( x + (f/g)y \) is contained in \( O_P \). Considering the completion \( \hat{O}_P \) of \( O_P \), we have \( \hat{O}_P \cong k[[x, y]] \). Since \( k[[x, y]]^D \supset k[[y^2, x + (f/g)y]] \) and \( \dim_k k[[x, y]]^D/k[[x^2, y^2]] = \dim_k k[[y^2, x + (f/g)y]]/k[[x^2, y^2]] = 2, \) we have \( k[[x, y]]^D = k[[y^2, x + (f/g)y]]. \) Although by the general theory of the vector field the point \( \pi(P) \) is a nonsingular point of \( S^D \), this result also shows that \( S^D \) is nonsingular at \( \pi(P) \). We set \( X = x^2, Y = y^2 \) and \( Z = x + (f/g)y \), and let \( \tilde{f}, \tilde{g} \) be elements of \( O_P \) whose coefficients are the squares of the ones of \( f, g \), respectively. Let \( S' \) be a surface defined by the equation

\[ Z^2 = X + (\tilde{f}/\tilde{g})Y. \]

Since the degrees of the algebraic extensions \( k(S)/k(S^D) \) and \( k(S)/k(S') \) of fields are 2 and \( k(S^D) \supset k(S') \) holds, we have \( k(S^D) = k(S') \), that is, \( S' \) is birationally equivalent to \( S^D \). Since \( \tilde{g}(P) = c^2 \neq 0 \), \( S' \) is nonsingular at the point \((X, Y, Z) = (0, 0, 0)\). Therefore, by the Zariski main theorem the surface \( S^D \) is isomorphic to \( S' \) around \( \pi(P) \). The curve \( \pi(C) \) is defined
by $X = 0$ at the point $\pi(P)$. Therefore, the equation of the curve $\pi(C)$ at $\pi(P)$ on the plane $X = 0$ is given by

$$Z^2(\delta|_{X=0}Y + c^2) = \beta|_{X=0}Y^2.$$ 

Here, the notation of $\beta$ and $\delta$ are similar to $\tilde{f}$ and $\tilde{g}$. This equation for the curve $\pi(C)$ shows that $\pi(P)$ is a singular point of $\pi(C)$. □

4. A CONSTRUCTION OF A QUASI-ELLIPTIC SURFACE

Let $E$ be an elliptic curve and $\{U_0, U_\infty\}$ be an affine open covering and let $U_0$ (resp. $U_\infty$) be given by the equation

$$y^2 + y = x^3 \text{ (resp. } z^2 + z = w^3).$$

The change of coordinates is given by

$$y = 1/z, \; x = w/z.$$ 

Let $\{V_0, V_\infty\}$ ($V_0 \cong V_\infty \cong \mathbb{A}^1$: an affine line) be affine open covering of the projective line $\mathbb{P}^1$ and $t$ (resp.$s$) be a coordinate of $V_0$ (resp. $V_\infty$). The change of coordinates is given by

$$t = 1/s.$$ 

We consider the algebraic surface $S = E \times \mathbb{P}^1$. Then, $\{U_i \times V_j \mid i = 0, \infty; j = 0, \infty\}$ gives an affine open covering of $S$. We have a projection

$$\psi : S \rightarrow E.$$ 

Let $C_\infty$ be the curve on $S$ defined by $s = 0$. We consider the following rational vector field $D$ on $U_0 \times V_0$.

$$(I) \quad D = y \frac{\partial}{\partial x} + (x^2 + x^2t + t^4) \frac{\partial}{\partial t}.$$ 

Then, $D$ gives a rational vector field on $S$ and on each affine chart it is concretely given as follows:

$$(II) \quad D = \frac{1}{x} \{z \frac{\partial}{\partial w} + (w^2 + w^2t + z^2t^4) \frac{\partial}{\partial t}\}$$
$$= \frac{1}{w} \{(z + 1)w \frac{\partial}{\partial w} + ((z + 1)^2 + (z + 1)^2t + w^4t^4) \frac{\partial}{\partial t}\} \quad \text{on } U_\infty \times V_0$$

$$(III) \quad D = \frac{1}{s} \{ys^2 \frac{\partial}{\partial x} + (x^2s^4 + x^2s^3 + 1) \frac{\partial}{\partial s}\} \quad \text{on } U_0 \times V_\infty$$

$$(IV) \quad D = \frac{1}{x} \{z \frac{\partial}{\partial w} + (w^2s^4 + w^2s^3 + z^2) \frac{\partial}{\partial s}\}$$
$$= \frac{1}{w} \{(z + 1)ws^2 \frac{\partial}{\partial w} + ((z + 1)^2s^4 + (z + 1)^2s^3 + w^4) \frac{\partial}{\partial s}\} \quad \text{on } U_\infty \times V_\infty.$$
Since \( \frac{\partial y}{\partial x} = x^2 \), we have \( D^2 = x^2 D \). Therefore, the rational vector field \( D \) is 2-closed. The isolated singularities of \( D \) on each affine chart are as follows.

- On \( U_0 \times V_0 \) \( P : (x, y, t) = (0, 0, 0) \)
- On \( U_\infty \times V_0 \) \( Q_1 : (w, z, t) = (0, 0, 1) \)
- On \( U_0 \times V_\infty \) No isolated singular point
- On \( U_\infty \times V_\infty \) \( R : (w, z, s) = (0, 0, 0), Q_2 : (w, z, s) = (0, 0, 1) \).

On the surface \( S \), \( Q_1 \) and \( Q_2 \) give the same point, and we denote it by \( Q \). We set

\[
\psi(P) = P', \psi(Q) = \psi(R) = Q', \psi^{-1}(P') = F_0, \psi^{-1}(Q') = F_\infty.
\]

From here on, we use the same notation for the curve and the proper transform of the curve, if no confusion can occur. We blow-up at \( P \), and denote the exceptional curve by \( G_1 \). Then, on the exceptional curve \( G_1 \) there exists one isolated singular point of the rational vector field \( D \). We blow-up at the singular point, and denote the exceptional curve by \( G_2 \). Then, the vector field has no isolated singular point on \( G_2 \). Now, we blow-up at \( Q \), and denote the exceptional curve by \( E_1 \). Then, the vector field has no isolated singular point on \( E_1 \). We again blow-up at \( R \), and denote the exceptional curve by \( E_2 \). On the surface \( \tilde{S} \) which we got by these blowing-ups the rational vector field \( D \) has no isolated singularities. We have the morphism

\[
\psi : \tilde{S} \longrightarrow E
\]

which is induced by \( \psi \). Then, on \( \tilde{S} \), by our construction we have the following lemma.

**Lemma 4.1.** On \( \tilde{S} \), we have the following results.

1. \( \psi^{-1}(P') = F_0 + G_1 + 2G_2, \psi^{-1}(Q') = F_\infty + E_1 + E_2 \).
2. The curves \( F_0, G_1 \) and \( F_\infty \) are integral with respect to the vector field \( D \). The curves \( G_2, E_1, E_2 \) and \( C_\infty \) are non-integral with respect to the vector field \( D \).
3. \( F_0^2 = -2, G_1^2 = -2, G_2^2 = -1, F_\infty^2 = -2, E_1^2 = -1, E_2^2 = -1 \).
4. \( (F_0, G_2) = (G_2, G_1) = 1, (F_0, G_1) = 0 \).
5. \( (F_\infty, E_1) = (F_1, E_2) = (C_\infty, E_2) = 1, (F_\infty, C_\infty) = (E_1, E_2) = (C_\infty, E_1) = 0 \).
6. There is a cuspidal point of the vector field \( D \) on \( G_2 \). There is also a cuspidal point of the vector field \( D \) on \( E_2 \) where it intersects with \( C_\infty \).

We consider the quotient surface \( \tilde{S}^D \) of \( \tilde{S} \) by \( D \). We have the projection

\[
\pi : \tilde{S} \longrightarrow \tilde{S}^D
\]
and a commutative diagram

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\pi} & \tilde{S}^D \\
\downarrow \psi & & \downarrow \psi' \\
E & \xrightarrow{F} & E^{(2)}.
\end{array}
\]

Here, \( F \) is the Frobenius morphism and \( E^{(2)} \) is the Frobenius image. We set \( B = E^{(2)}, P'' = F(P') \) and \( Q'' = F(Q') \). For a curve \( C \) on \( \tilde{S} \), we denote the curve \( \pi(C) \) on \( \tilde{S}^D \) again by \( C \), if no confusion can occur. By Lemma 4.1 and Proposition 3.4, we have the following lemma.

**Lemma 4.2.** On \( \tilde{S}^D \), we have the following results.

1. \( \psi'^{-1}(P'') = 2F_0 + 2G_1 + 2G_2, \psi'^{-1}(Q'') = 2F_\infty + E_1 + E_2. \)
2. \( F_0^2 = -1, G_1^2 = -1, G_2^2 = -2, F_\infty^2 = -1, E_1^2 = -2, E_2^2 = -2, C_\infty^2 = -2. \)
3. \( (F_0, G_2) = (G_2, G_1) = 1, (F_0, G_1) = 0. \)
4. \( (F_\infty, E_1) = (F_1, E_2) = 1, (C_\infty, E_2) = 2, (F_\infty, C_\infty) = (E_1, E_2) = (C_\infty, E_1) = 0. \)
5. \( G_2 \) and \( E_2 \) are rational cuspidal curves.

First, we blow-down \( F_0, G_1 \) and \( F_\infty \), and then \( E_1 \) becomes an exceptional curve of the first kind and so we blow-down it:

\( \eta : \tilde{S}^D \rightarrow X. \)

Then, we have a quasi-elliptic surface

\( \varphi : X \rightarrow B. \)

The fiber \( \varphi^{-1}(P'') \) is the only one multiple fiber, and we have no other singular fiber.

Now, let’s calculate the canonical divisor \( K_X \). First, we have

\[ K_{\tilde{S}^D} \sim \eta^* K_X + F_0 + G_1 + E_1 + 2F_\infty. \]

Therefore, we have

\[ \pi^* K_{\tilde{S}^D} \sim \pi^* \eta^* K_X + F_0 + G_1 + 2E_1 + 2F_\infty. \]

On \( \tilde{S} \), by a direct calculation of \( D \) and \( K_{\tilde{S}} \), we have

\[
(D) = -2C_\infty - 4F_\infty + G_1 + 4G_2 - 3E_1 - 3E_2,
\]

\[ K_{\tilde{S}} \sim -2C_\infty + G_1 + 2G_2 + E_1 - E_2. \]

Putting these data in the canonical bundle formula by Rudakov-Shafarevich:

\[ K_{\tilde{S}} \sim (D) + \pi^* K_{\tilde{S}^D}, \]

we have

\[ \pi^* \eta^* K_X \sim 2(F_\infty + E_1 + E_2) - (F_0 + G_1 + 2G_2). \]
Therefore, we have

\[ \eta^* K_X \sim (2F_\infty + E_1 + E_2) - (F_0 + G_1 + G_2). \]

Hence, we have

\[ K_X \sim E_2 - G_2 \approx G_2, \]

where \( \approx \) means numerical equivalence. This means that there exists a divisor \( L \) on \( B \) such that

\[ (4.1) \quad K_X \sim \varphi^*(L) + G_2. \]

Therefore, the fiber \( \varphi^{-1}(P'') \) is a tame multiple fiber.

**Proposition 4.3.** The surface \( \varphi : X \to B \) which we constructed above is a quasi-elliptic surface with only one tame multiple fiber. It has no more singular fibers and \( \chi(\mathcal{O}_X) = 0 \) holds. The linear system \( |6K_X| \) gives the structure of the quasi-elliptic surface, and the linear system \( |5K_X| \) does not give the structure of the quasi-elliptic surface.

**Proof.** Take a general fiber \( G \). Then, we have \( G^2 = 0 \) and \( (K_X, G) = 0 \). Therefore, by the genus formula the virtual genus of \( G \) is 1. On the other hand, \( \tilde{\psi} : \tilde{S} \to E \) is a ruled surface. Therefore, \( G \) is not an elliptic curve. This means that \( \varphi : X \to B \) is a quasi-elliptic surface. By our construction, we have Betti numbers \( b_1(X) = 2 \) and \( b_2(X) = 2 \). Therefore, the Euler number \( c_2(X) = 1 - 2 + 2 - 2 + 1 = 0 \). Since \( K_X^2 = 0 \), we have \( \chi(\mathcal{O}_X) = 0 \) by Noether’s formula. Since we have \( H^0(X, \mathcal{O}_X(6K_X)) \cong H^0(B, \mathcal{O}_B(3P'')) \) and the divisor \( 3P'' \) is very ample on \( B \), the linear system \( |6K_X| \) gives the structure of the quasi-elliptic surface. Since \( H^0(X, \mathcal{O}_X(5K_X)) \cong H^0(B, \mathcal{O}_B(2P'')) \) and the divisor \( 2P'' \) is not very ample on \( B \), the linear system \( |5K_X| \) does not give the structure of the quasi-elliptic surface. \( \square \)

**Remark 4.4.** In the above, we calculate the canonical divisor \( K_X \) by the construction of our quasi-elliptic surface. We give here one more proof for (4.1). On the quasi-elliptic surface \( \varphi : X \to B \), the cusp locus \( C_\infty \) is an elliptic curve and we have \( C_\infty^2 = -1 \) by considering the structure of blow-down. Therefore, by the genus formula, we have \( (K_X, C_\infty) = 1 \). On the other hand, by the canonical bundle formula for the quasi-elliptic surface \( X \), we have

\[ K_X \sim \varphi^*(\mathcal{L}) + aG_2 \]

with a line bundle \( \mathcal{L} \) on \( B \) and \( a = 0 \) or 1. Since \( 1 = (K_X, C_\infty) = 2\deg \mathcal{L} + a \), we conclude \( a = 1 \) and \( \deg \mathcal{L} = 0 \), which shows (4.1).
5. Multicanonical systems

Let \( \varphi : S \to B \) be a quasi-elliptic surface over an algebraically closed field \( k \) of characteristic \( p > 0 \). Such a surface exists only in characteristic \( p = 2 \) or \( 3 \). In this case, the multiplicity of a multiple fiber is equal to \( p \) (cf. Bombieri-Mumford [1]). We denote by \( pF_i \) \((i = 1, \ldots, \lambda)\) the multiple fibers. Then, the canonical divisor formula is given by

\[
K_S \sim \varphi^*(K_B - f) + \sum_{i=1}^{\lambda} a_i F_i,
\]

where \( f \) is a divisor on \( B \) and \( -\deg f = \chi(\mathcal{O}_S) + t \) with \( t = \text{length of the torsion part of } R^1\varphi_*\mathcal{O}_S \), and \( 0 \leq a_i \leq p - 1 \). For details, see Bombieri-Mumford [1].

We denote by \( g \) the genus of the base curve \( B \). Then, we have the following theorem.

**Theorem 5.1.** Assume \( p = 2 \). Then, for any quasi-elliptic surface \( \varphi : S \to B \) with Kodaira dimension \( \kappa(S) = 1 \) over \( k \) and for any \( m \geq 6 \) \( |mK_S| \) gives the unique structure of quasi-elliptic surface, and \( 6 \) is the best possible number.

**Proof.** The method of the proof is similar to the one in Iitaka [2] and Katsura-Ueno [5] (see also Katsura [3] [4]). The Kodaira dimension of \( S \) is equal to \( 1 \) if and only if

\[
(*) \quad 2g - 2 + \chi(\mathcal{O}_S) + t + \sum_{i=1}^{\lambda} (a_i/m_i) > 0.
\]

Therefore, we need to find the least integer \( m \) such that

\[
(**) \quad m(2g - 2 + \chi(\mathcal{O}_S) + t) + \sum_{i=1}^{\lambda} [ma_i/m_i] \geq 2g + 1
\]

holds under the condition \((*)\). Here, \([r]\) means the integral part of a real number \( r \). We have the following 6 cases:

Case (I) \( g \geq 2 \)
Case (II-1) \( g = 1, \chi(\mathcal{O}_S) + t \geq 1 \)
Case (II-2) \( g = 1, \chi(\mathcal{O}_S) = 0, t = 0 \)
Case (III-1) \( g = 0, \chi(\mathcal{O}_S) + t \geq 3 \)
Case (III-2) \( g = 0, \chi(\mathcal{O}_S) + t = 2 \)
Case (III-3) \( g = 0, \chi(\mathcal{O}_S) = 1, t = 0 \)

Case (I) We have \( 2g - 2 + \chi(\mathcal{O}_S) \geq 5(g - 1)/3 \). Hence, if \( m \geq 3 \), \((**\) holds.
Case (II-1) If \( m \geq 3 \), \((**\) holds.
Case (II-2) All multiple fibers are tame in this case. If \( m \geq 6 \), \((**)\) holds by \( p = 2 \). As we constructed in Section 4, there exists a quasi-elliptic surface with only one tame multiple fiber of type II and \( \chi(O_S) = 0 \) over an elliptic curve. Therefore, we need \( m \geq 6 \).  
Case (III-1) \((**)\) holds for \( m \geq 1 \).  
Case (III-2) Since \( \chi(O_S) \geq 1 \), we have \( t \leq 1 \). Therefore, the number of wild fibers is less than or equal to 1. If there exists at least one tame multiple fiber then \((**)\) holds for \( m \geq 2 \). If there exist no tame fibers and only one wild fiber, then by Katsura-Ueno [5] Lemma 2.4, this case is excluded in the case of \( p = 2 \).  
Case (III-3) By \( p = 2 \), \((**)\) holds for \( m \geq 4 \). The result on the best possible number follows from the example in Section 4.  

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