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Regularity of ends of zero mean curvature surfaces in \( \mathbb{R}^{2,1} \)

By Naoya Ando, Kohei Hamada, Kaname Hashimoto and Shin Kato

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Abstract. In this paper, we analyze ends of zero mean curvature surfaces of mixed (or non-mixed) type in the Lorentzian 3-space \( \mathbb{R}^{2,1} \). Among these, we show that spacelike or timelike planar ends are \( C^\infty \) in the compactification \( \hat{\mathbb{L}} \) of \( \mathbb{R}^{2,1} \) as in the case of minimal surfaces in the Euclidean 3-space \( \mathbb{R}^3 \). On the other hand, lightlike planar ends are not \( C^\infty \). Each lightlike planar end of a mixed type surface has the following additional parts: the converging part (a lightlike line in \( \mathbb{R}^{2,1} \)), the diverging part (the set of the points in \( \hat{\mathbb{L}} \setminus \mathbb{R}^{2,1} \) corresponding to zero-divisors), and the border of these two parts. We show that such an end is \( C^\infty \) on the first two parts almost everywhere, while there appears an isolated singularity in the form of \((x^3, x\tau + "higher order terms", \tau)\) on the border. We also show that conelike singularities of mixed type appear on the lightlike lines in special cases.

1. Introduction

It is known that a complete minimal surface in the Euclidean 3-space \( \mathbb{R}^3 \) with finite total curvature is conformal to a compact Riemann surface punctured at a finite number of points. The image of a sufficiently small punctured neighborhood of each point or the point itself is called an end. Here we also call the punctured point the origin of the end. If the end has no self-intersections, then it is said to be embedded. An embedded end is asymptotic to either a plane or a half of a catenoid. In the former case, the embedded end has no logarithmic term and such an end is said to be planar. In the latter case, the embedded end has a nonzero logarithmic term and such an end is said to be catenoidal. We consider an embedded end to be a surface in \( S^3 \) through the stereographic projection. Bryant [7] proved that the surface in \( S^3 \) given by an embedded end extends to the point corresponding to its origin as a \( C^1 \) surface if the end is catenoidal, and a \( C^\infty \) surface if the end is planar. See [9], [5] for results on the regularity of ends of surfaces and submanifolds in Euclidean spaces.

Now, let us consider a spacelike maximal surface in the Lorentzian 3-space \( \mathbb{R}^{2,1} \). In this case, the ends corresponding to the embedded ends in the previous paragraph are said to be simple and are classified into six types by Imaizumi [12]. Among these, simple ends of two types have no logarithmic terms. We call them spacelike and lightlike planar ends respectively (see §3). Any spacelike planar end is asymptotic to a spacelike plane. Since its asymptotic behavior is quite similar to those of planar ends in \( \mathbb{R}^3 \), we can show that the surface in the compactification \( \hat{\mathbb{L}} \) of \( \mathbb{R}^{2,1} \) (see §2, §6.3) given by the end extends to the point corresponding to its origin as a \( C^\infty \) surface as in the Euclidean case. Simultaneously, we also show the corresponding result for a branching case (Theorem 3.1).

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In general, the metric of a spacelike maximal surface has a singularity everywhere the normal direction is lightlike. However, it is known that if the singularity is a fold singularity, then the surface extends real-analytically to a timelike minimal surface across the singularity (cf. [15, 14]). Such an extended surface is said to be of mixed type. Hence it is natural to consider the regularity of ends of timelike minimal surfaces together. In the case of timelike minimal surfaces, the surfaces are expressed by using parameromorphic functions. If such a function has a pole at a point on a coordinate neighborhood of a Lorentz surface, then it also has a “pole” at each of the corresponding zero-divisors of the point, since its principal part is decomposed into two parts as follows:

\[
\sum_{k=-K}^{-1} c_k z^k = \sum_{k=-K}^{-1} (a_k + b_k)(x + y)^k \frac{1+j}{2} + \sum_{k=-K}^{-1} (a_k - b_k)(x - y)^k \frac{1-j}{2},
\]

where \( c_k = a_k + j b_k \), \( z = x + jy \) is a coordinate function and we assume that the pole is \( z = 0 \) (see §A). In this situation, we consider an end to be the image of a sufficiently narrow cross-shaped slit neighborhood of the union of the pole and its zero-divisors in some coordinate neighborhood \((U, z)\), for instance

\[
\{ \hat{z} \in U \mid |x + y| < \epsilon \text{ or } |x - y| < \epsilon \} \setminus \{ \hat{z} \in U \mid x + y = 0 \text{ or } x - y = 0 \},
\]

or the union itself. Here we also call the original point of the Lorentz surface corresponding to a pole as \( \hat{z} = 0 \) above the origin of the end. Also in this case, ends corresponding to the simple ends of two types of a spacelike maximal surface in the previous paragraph have no logarithmic terms. We call them timelike and lightlike planar ends respectively (see §3). We note here that a timelike planar end has four connected components if we choose \((U, \hat{z})\) small enough, but they can be extended analytically to each other by using bicomplex extensions (cf. [10, Theorem 9.1]). Based on this extension, we can show that the surface in \( \hat{L} \) given by the end extends to the point corresponding to its origin as a \( C^{\infty} \) surface (Theorem 3.2).

On the other hand, the origin of any lightlike planar end is located on a singular set in the sense that its limit normal is lightlike, and the end is asymptotic to a lightlike line. In particular, any spacelike maximal surface with a lightlike planar end is not complete. However, it is known that if the singularity is a fold singularity, then the surface extends real-analytically to a timelike minimal surface across not only the fold singularity but also the lightlike line of the lightlike planar end so that it is of mixed type (cf. [14, 10]). In general, lightlike planar ends of timelike minimal surfaces have complicated structures. Each lightlike planar end of a timelike minimal surface has the following additional parts:

(i) the converging part: a lightlike line in \( \mathbb{R}^{2,1} \),
(ii) the diverging part: four curves in \( \hat{L} \setminus \mathbb{R}^{2,1} \) which consist of points corresponding to zero-divisors,
(iii) the border of the converging part and the diverging part, which also appears in \( \hat{L} \setminus \mathbb{R}^{2,1} \).

We note here that if the singularity is a fold singularity, then the number of curves in (ii) becomes two, and the fold singularity consists of two curves in \( \mathbb{R}^{2,1} \) such that one endpoint of each of them is the border in (iii). In this paper, we prove that any lightlike planar end of a mixed type zero mean curvature surface in \( \mathbb{R}^{2,1} \) extends to the first two parts as a \( C^{\infty} \) surface in \( \hat{L} \) almost everywhere, while it extends to the last part with an
isolated singularity \((x^3, x\tau + \text{"higher order terms"}, \tau)\) (Theorems 4.1, 4.3 and 4.4). We also give a similar result for the regularity on lightlike lines in the non-mixed type case (Theorem 4.2).

Note here that the derivative of the Gauss map vanishes at any planar end in general. Hence a singular set on which normal directions are lightlike goes across the lightlike line of converging part in many cases. In generic cases, such singular sets are cuspidal edges. However, in special cases, the singular set is conelike. In this paper, we also give a criterion for a conelike singularity of mixed (or non-mixed) type to appear on the lightlike line (Theorems 5.2 and 5.3).

The result in the first paragraph also describes the regularity of compact Willmore surfaces in \(\mathbb{R}^3\) or \(\mathbb{S}^3\), since for any minimal surface in \(\mathbb{R}^3\), its image by the inversion (resp. inverse image by the stereographic projection) is a Willmore surface in \(\mathbb{R}^3\) (resp. \(\mathbb{S}^3\)). By using this fact, Kusner [16] constructed Willmore projective planes in \(\mathbb{R}^3\) or \(\mathbb{S}^3\), and determined the minimum of energies of Willmore projective planes. A Willmore surface can be characterized by a condition that the conformal Gauss map has a zero mean curvature vector. Also in the Lorentzian case, a zero mean curvature surface, or its image by the inversion, can be regarded as a Willmore type surface in \(\mathbb{R}^{2,1}\), \(\mathbb{S}^{2,1}\) or \(\mathbb{L}\), and we can characterize it by using its conformal Gauss map (see §6). By applying the compactification we discussed in the previous paragraphs, we can construct typical examples of Willmore type surfaces in \(\mathbb{L}\) with singularities (see §7).

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2. Representation formulas and compactification

In this paper, we use a representation formula of the following type for spacelike maximal surfaces in \(\mathbb{R}^{2,1}\):

\[
X(z) = \text{Re} \int_z \left(1 - g^2, 2g, 1 + g^2\right) f dz \quad (z \in M).
\]

Here \(f\) is a holomorphic function on a Riemann surface \(M\), and \(g\) is a meromorphic function on \(M\) such that \(g^2f\) is holomorphic on \(M\). The reason to employ this type of formula is the fact that this formula is valid also for timelike minimal surfaces given as an image of a map defined on a Lorentz surface \(\tilde{M}\) (see the appendix of [4] for Lorentz surfaces) if we replace the complex parameter \(z \in \mathbb{C}\) by paracomplex parameter \(\tilde{z} \in \mathbb{C}\) as follows:

\[
\tilde{X}(\tilde{z}) = \text{Re} \int_{\tilde{z}} \left(1 - \tilde{g}^2, 2\tilde{g}, 1 + \tilde{g}^2\right) \tilde{f} d\tilde{z} \quad (\tilde{z} \in \tilde{M}).
\]

Note here that the normal direction is null at the point \(X(z)\) (resp. \(\tilde{X}(\tilde{z})\)) if and only if \(g(z) \in \mathbb{R}\) (resp. \(\tilde{g}(\tilde{z}) \in \mathbb{R}\)). Hence we can join \(X(z)\) and \(\tilde{X}(\tilde{z})\) naturally on the real axis \(\mathbb{R}\) by applying the formulas above. We consider the case of the surfaces of mixed type...
an end only by an inversion, which is a coordinate transformation of of points, and called the light cone at infinity. Hence we cannot describes the behavior of Indeed, is more complicated than that of the Euclidean case. We regard of this map. It is the correspondent of . We calculate the rank of instead of . Let be the nondegenerate metric of . Set \[ L := \{ x \in \mathbb{R}^3 \setminus \{0\} \mid \langle x, x \rangle_{3,2} = 0 \} \] and \( \hat{L} := \{ [x] = [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{R}P^4 \mid \langle x, x \rangle_{3,2} = 0 \} \). The identification of with \( L \setminus \{ x_4 = x_0 + 1 \} \) implies that of with an open subset of \( \hat{L} \). It is realized by the following map: 

\[ \mathbb{R}^2,1 \to \mathbb{R}P^4 \]

\[ (x_1, x_2, x_3) \mapsto \left[ \frac{\langle x, x \rangle_{2,1} - 1}{2} : x_1 : x_2 : x_3 : \frac{\langle x, x \rangle_{2,1} + 1}{2} \right]. \]

We regard \( \hat{L} \) as a compactification of \( \mathbb{R}^2,1 \) in the sense that \( \hat{L} \) is the closure of the image of this map. It is the correspondent of \( S^3 \) in the case of \( \mathbb{R}^3 \), but the set of points of infinity \( \{ [x] \in \hat{L} \mid x_0 = 0 \} \) added under the compactification consists of infinite number of points, and called the light cone at infinity. Hence we cannot describes the behavior of an end only by an inversion, which is a coordinate transformation of \( \hat{L} \), and the problem is more complicated than that of the Euclidean case.

Set 

\[ F_\ell(z) := \int^z g^\ell f dz, \quad R_\ell(z) := \text{Re} F_\ell(z) \quad (\ell = 0, 1, 2). \]

Then 

\[ X = (R_0 - R_2, 2R_1, R_0 + R_2), \]

\[ \langle X, X \rangle_{2,1} = 4(R_1^2 - R_0 R_2). \]

We calculate the rank of \( X \) by 

\[ [R_1^2 - R_0 R_2 : R_0 : R_1 : R_2 : 1] \]

instead of 

\[ \left[ \frac{4(R_1^2 - R_0 R_2) - 1}{2} : R_0 - R_2 : 2R_1 : R_0 + R_2 : \frac{4(R_1^2 - R_0 R_2) + 1}{2} \right]. \]

Indeed, 

\[ \psi_0 : [x_0 : x_1 : x_2 : x_3 : x_4] \mapsto \left[ \frac{x_0 + x_4}{4} : \frac{x_1 + x_3}{2} : \frac{x_2}{2} : \frac{-x_1 + x_3}{2} : -x_0 + x_4 \right] \]

is a diffeomorphism, and maps (2.4) to (2.3). On the other hand, both of 

\[ \varphi_{\pm} : \{(x_0, x_2, x_4) \mid x_0^2 + x_2^2 < 1, -1 < x_4 < 1\} \]

\[ \to \{ [x_0 : x_1 : x_2 : x_3 : x_4] \in \hat{L} \mid x_1 \neq 0, x_3 \neq 0 \} \]

\[ (x_0, x_2, x_4) \mapsto \left[ x_0 : \sqrt{1 - x_0^2 - x_2^2} : x_2 : \pm \sqrt{1 - x_4^2} : x_4 \right] \]
and

\[ \psi : \{(y_0 : y_1 : y_2 : y_3 : y_4) \mid y_1 \neq 0\} \to \mathbb{R}^4 \]

\[ [y_0 : y_1 : y_2 : y_3 : y_4] \mapsto \left( \frac{y_0}{y_1}, \frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{y_4}{y_1} \right) \]

are also diffeomorphisms. Set \( \pi(y_0, y_2, y_3, y_4) := (y_0, y_2, y_4) \). Now, since \(|J(\psi \circ \psi_0 \circ \varphi_0)| = 2/(x_1 x_3(x_1 + x_3)^3)\), \( \psi \circ \psi_0 \circ \varphi_0 \) is a diffeomorphism from \([x_0 : x_1 : x_2 : x_3 : x_4] \in L \mid x_1 \neq 0, x_3 \neq 0, x_1 + x_3 \neq 0\) to its image. Hence we can clarify the regularity or the type of singularity of \( X \) as a map into \( \mathbb{RP}^4 \) by clarifying that of \( \psi \circ \psi_0 \circ \varphi \circ X \) everywhere \( X_1 \neq 0, X_3 \neq 0, X_1 + X_3 \neq 0 \).

We denote all the correspondents to the above \( F_\ell, R_\ell \) in the case of timelike minimal surfaces by \( \hat{F}_\ell, \hat{R}_\ell \) \((\ell = 0, 1, 2)\). The same assertion as above holds also for \( \hat{X} \) or the surface of mixed type defined by joining \( X \) and \( \hat{X} \).

Here we give some remarks on the correspondence between symmetries and fold singularities or degenerations of null curves to a point.

First, assume that a coordinate neighborhood \( D \subset \mathbb{C} \) satisfies \( \mathbb{R} \cap D \neq \emptyset \). If \( F_\ell(z) \) \((\ell = 0, 1, 2)\) are equivariant with respect to \( I(z) = \overline{z} \), that is, \( F_\ell(\overline{z}) = \overline{F_\ell(z)} \) \((\ell = 0, 1, 2)\), then \( X|_D \) has a fold singularity on the real axis. The condition above is described by means of the Weierstrass data as follows:

\[ g(\overline{z}) = \overline{g(z)}, \quad f(\overline{z}) = \overline{f(z)}. \]

As for the Taylor or Laurent expansions around any point on the real axis, this condition is equivalent with the condition that all the coefficient are real numbers. It is known that, if \( \hat{X} \) also satisfies the same condition and \( X(x) = \hat{X}(x) \) holds for \( x \in \mathbb{R} \cap D \cap D \), then \( X \) and \( \hat{X} \) are real-analytic extensions with each other across the image of the real axis.

Secondly, assume that a coordinate neighborhood \( D \subset \mathbb{C} \) satisfies \( i\mathbb{R} \cap D \neq \emptyset \). If \( F_\ell(z) \) \((\ell = 0, 1, 2)\) are equivariant with respect to \( I(z) = -\overline{z} \), that is, \( F_\ell(-\overline{z}) = -\overline{F_\ell(z)} \) \((\ell = 0, 1, 2)\), then the image of the imaginary axis by \( X|_D \) degenerates to a point. The condition above is described by means of the Weierstrass data as follows:

\[ g(-\overline{z}) = \overline{g(z)}, \quad f(-\overline{z}) = \overline{f(z)}. \]

As for the Taylor or Laurent expansions around any point on the imaginary axis, this condition is equivalent with the condition that all the coefficient of the terms of even (resp. odd) order are real (resp. pure imaginary) numbers. If \( \hat{X} \) also satisfies the same condition, then we get the same conclusion also for \( \hat{X} \).

3. Spacelike and timelike planar ends

3.1. Spacelike planar ends

Let \( X \) be a spacelike maximal surfaces in \( \mathbb{R}^{2,1} \). For any end of \( X \) of finite total curvature, we may choose the local coordinate \( z \in \mathbb{C} \) such that the end is \( z = 0 \) without loss of generality. Let \((g, f)\) be the Weierstrass data of \( X \) in the meaning of the previous
section, and set
\[ \Phi(z) := \int^z (1 - g(z)^2, 2g(z), 1 + g(z)^2)f(z)dz. \]
The surface is given by \( X(z) = \text{Re} \Phi(z) \). We may also assume here that \( g(0) \neq \infty \). Denote the expansion of \( \Phi \) around \( z = 0 \) by the following:
\[ \Phi(z) = \sum_{k=-K}^{-1} z^k V_k + \log z V_{\log} + \Phi_{\text{hol}}(z), \]
where \( K \) is a nonnegative integer, \( V_k \in C^3 \) \( (k = -K, \ldots, -1) \), \( V_{\log} \in C^3 \) and \( \Phi_{\text{hol}}(z) \) is a \( C^3 \) valued holomorphic function defined on a neighborhood of \( z = 0 \). In particular, \( V_{-K} \neq 0 \) if \( K \geq 1 \), \( V_{\log} \neq 0 \) if \( K = 0 \). \( X(z) = \text{Re} \Phi(z) \) is well-defined around \( z = 0 \) if and only if \( V_{\log} \in R^3 \). We call the end \( z = 0 \) a simple end if \( \Phi \) has a pole at \( z = 0 \) whose order is at most 1, that is, \( K \leq 1 \). Simple ends with \( K = 1 \) correspond to embedded ends of minimal surfaces in \( R^3 \). Since we study the regularity of the end at infinity, we assume \( V_{\log} = 0 \) to exclude the clearly nonregular case. By using this expansion, simple ends without logarithmic term are classified to two types as follows (cf. Imaizumi [12, Theorem 2.7]):

1. If \( V_{-1} \neq 0 \), \( V_{\log} = 0 \) and \( g(0) \notin R \) (i.e. the limit normal is timelike and the flux vanishes), then the end is asymptotic to a spacelike plane.
2. If \( V_{-1} \neq 0 \), \( V_{\log} = 0 \) and \( g(0) \in R \) (i.e. the limit normal is null and the flux vanishes), then the end is asymptotic to a lightlike line.

In these case, we can expect some regularity for their images of inversions. In this paper, we call an end of type (1) (resp. (2)) a spacelike (resp. lightlike) planar end. On the other hand, we define a non-embedded spacelike flat end by the condition \( K \geq 2 \), \( \text{rank}(V_{-K}, \ldots, V_{-1}) = 1 \), \( V_{\log} = 0 \) and \( g(0) \notin R \) (see [8] for the case of \( R^3 \)).

As for spacelike planar ends, we can show their regularity by the quite similar way as in the case of planar ends in \( R^3 \) because their asymptotic behaviors are quite similar to each other, and the end is a one point set at infinity in both cases. We also show a branching result for non-embedded spacelike flat ends.

**Theorem 3.1.** Any spacelike planar end of spacelike maximal surfaces in \( R^2,1 \) extends as a \( C^\infty \) surface in \( \hat{L} \). On the other hand, any non-embedded spacelike flat end extends in \( \hat{L} \) with a branch point.

(Proof.) In this case, we may assume that the end is \( z = 0 \) and \( g(0) = i \), \( g'(0) = \cdots = g^{(K)}(0) = 0 \) without loss of generality, and \( (g, f) \) is of the following form:
\[
\begin{cases}
g = i + z^{K+1}g_{\text{hol}}, \\
f = \sum_{k=-K}^{-1} \alpha_k z^{k-1} + f_{\text{hol}},
\end{cases}
\]
where \( K \geq 1 \), \( \alpha_{-K} \in C \setminus \{0\} \), and \( g_{\text{hol}} \) and \( f_{\text{hol}} \) are holomorphic functions around \( z = 0 \).
Since
\[ g^2 = -1 + z^{K+1}(2i\varphi_{01} + z^K\varphi_{01}^2), \]
\[ gf = i\sum_{k=-K}^{-1} \alpha_k z^{k-1} + \left( \varphi_{01} \sum_{k=-K}^{-1} \alpha_k z^{k+K} + g\varphi_{01} \right), \]
\[ g^2f = -\sum_{k=-K}^{-1} \alpha_k z^{k-1} + \left\{ (2i\varphi_{01} + z^K\varphi_{01}^2) \sum_{k=-K}^{-1} \alpha_k z^{k+K} + g^2\varphi_{01} \right\}, \]
we have
\[ F_0 = F_{\text{pri}} + F_{\text{hol}0}, \quad F_1 = iF_{\text{pri}} + F_{\text{hol}1}, \quad F_2 = -F_{\text{pri}} + F_{\text{hol}2} \]
for some holomorphic functions \( F_{\text{hol}0}, F_{\text{hol}1} \) and \( F_{\text{hol}2} \) around \( z = 0 \), where
\[ F_{\text{pri}} = \sum_{k=-K}^{-1} \frac{\alpha_k}{K} z^k = \frac{1}{z^K} \left( -\frac{\alpha-K}{K} + O(z) \right). \]
Hence we get
\[ R_0 = \text{Re} F_{\text{pri}} + \text{Re} F_{\text{hol}0} = \frac{1}{|z|^{2K}} \left\{ -\frac{1}{K} \text{Re}(\alpha-Kz^K) + O_{K+1} \right\}, \]
\[ R_1 = -\text{Im} F_{\text{pri}} + \text{Re} F_{\text{hol}1} = \frac{1}{|z|^{2K}} \left\{ \frac{1}{K} \text{Im}(\alpha-Kz^K) + O_{K+1} \right\}, \]
\[ R_2 = -\text{Re} F_{\text{pri}} + \text{Re} F_{\text{hol}2} = \frac{1}{|z|^{2K}} \left\{ \frac{1}{K} \text{Re}(\alpha-Kz^K) + O_{K+1} \right\}, \]
where we set \( O_{K+1} := \sum_{l=0}^{K+1} O(x^{K+1-l}y^l) \). On the other hand, since
\[ R_1^2 - R_0 R_2 = (\text{Re} F_{\text{pri}})^2 + (\text{Im} F_{\text{pri}})^2 + \text{Re} F_{\text{pri}}(\text{Re} F_{\text{hol}0} - \text{Re} F_{\text{hol}2}) \]
\[ -2\text{Im} F_{\text{pri}} \cdot \text{Re} F_{\text{hol}1} + (\text{Re} F_{\text{hol}})^2 - \text{Re} F_{\text{hol}} \cdot \text{Re} F_{\text{hol}2} \]
\[ = |F_{\text{pri}}|^2 + \text{Re} F_{\text{pri}} \cdot O(1) + \text{Im} F_{\text{pri}} \cdot O(1) + O(1), \]
and
\[ |F_{\text{pri}}|^2 = \frac{1}{|z|^{2K}} \left| -\frac{\alpha-K}{K} + O(z) \right|^2 = \frac{1}{|z|^{2K}} \left( \frac{|\alpha-K|^2}{K^2} + O_1 \right), \]
we also get
\[ R_1^2 - R_0 R_2 = \frac{1}{|z|^{2K}} \left( \frac{|\alpha-K|^2}{K^2} + O_1 \right), \]
where \( O_1 = O(x) + O(y) \). Hence it follows that
\[ [R_1^2 - R_0 R_2 : R_0 : R_1 : R_2 : 1] \]
\[ = \left[ \frac{1}{|z|^{2K}} \left( \frac{|\alpha-K|^2}{K^2} + O_1 \right) : \frac{1}{|z|^{2K}} \left\{ -\frac{1}{K} \text{Re}(\alpha-Kz^K) + O_{K+1} \right\} \right] \]
\[ = \left[ \frac{1}{|z|^{2K}} \left( \frac{|\alpha-K|^2}{K^2} + O_1 \right) : 0 : 0 : 0 : 1 \right] \]
\[ \begin{align*}
&: \left\{ \frac{1}{|z|^{2K}} \left\{ \frac{1}{K} \Im(\alpha_{-K}z^K) + O_{K+1} \right\} : \frac{1}{|z|^{2K}} \left\{ \frac{1}{K} \Re(\alpha_{-K}z^K) + O_{K+1} \right\} : 1 \right\} \\
&= \left[ \alpha_{-K}^2 + O_1 : -K\Re(\alpha_{-K}z^K) + O_{K+1} \right] \\
&: K\Im(\alpha_{-K}z^K) + O_{K+1} : K\Re(\alpha_{-K}z^K) + O_{K+1} : K^2|z|^{2K} \right].
\end{align*} \]

Since all the terms \( O_1 \) and \( O_{K+1} \) are real analytic functions, \([R_1^2 - R_0 R_2 : R_0 : R_1 : R_2 : 1]\) is also analytic as a map defined around \( z = 0 \). Now we get our assertion by the local coordinate expression
\[
\frac{1}{R_1^2 - R_0 R_2} (R_0 - R_2, 2R_1) = \frac{2K}{|\alpha_{-K}|^2 + O_1} \left( -\Re(\alpha_{-K}z^K) + O_{K+1}, \Im(\alpha_{-K}z^K) + O_{K+1} \right).
\]

\[ \square \]

3.2. Timelike planar ends

Let \( \tilde{X} \) be a real analytic timelike minimal surfaces in \( \mathbb{R}^{2,1} \). Let \((\tilde{g}, \tilde{f})\) be the Weierstrass data of \( \tilde{X} \) in the meaning of the previous section, i.e. \( \tilde{X}(\tilde{z}) = \Re \Phi(\tilde{z}) \) where
\[
\Phi(\tilde{z}) := \int_{\tilde{z}} (1 - \tilde{g}(\tilde{z})^2, 2\tilde{g}(\tilde{z}), 1 + \tilde{g}(\tilde{z})^2) \tilde{f}(\tilde{z}) d\tilde{z}.
\]
If \( \tilde{g} \) is paraholomorphic and if \( \tilde{f} \) is parameromorphic on \( \tilde{z} = 0 \), then we can denote the expansion of \( \Phi \) around \( \tilde{z} = 0 \) as follows:
\[
\Phi(\tilde{z}) = \sum_{k=-K}^{-1} \tilde{z}^k V_k + \log \tilde{z} V_{\log} + \Phi_{\text{hol}}(\tilde{z}),
\]
where \( K \) is a nonnegative integer, \( V_k \in \mathcal{C}^3 \) \((k = -K, \ldots, -1)\), \( V_{\log} \in \mathcal{C}^3 \) and \( \Phi_{\text{hol}}(\tilde{z}) \) is a \( \mathcal{C}^3 \) valued paraholomorphic function defined on a neighborhood of \( \tilde{z} = 0 \). In particular, \( V_{-K} \neq 0 \) if \( K \geq 1 \), \( V_{\log} \neq 0 \) if \( K = 0 \). As we have already mentioned in §1, the domain of each end is not connected in general, but its connected components can be extended analytically to each other by using bicomplex extensions (cf. [10, Theorem 9.1]). We call the end \( \tilde{z} = 0 \) a simple end if \( \Phi \) has a pole at \( \tilde{z} = 0 \) whose order is at most 1, that is, \( K \leq 1 \). Among these, we can find planar ends of the following two types:

(1) If \( V_{-1} \notin e_1 \mathbb{R}^3 \cup e_2 \mathbb{R}^3 \), \( V_{\log} = 0 \) and \( \tilde{g}(0) \notin \mathbb{R} \), then the end is asymptotic to a timelike plane.

(2) If \( V_{-1} \notin e_1 \mathbb{R}^3 \cup e_2 \mathbb{R}^3 \), \( V_{\log} = 0 \) and \( \tilde{g}(0) \in \mathbb{R} \), then the end is asymptotic to a lightlike line.

Here we set \( e_1 := (1 + j)/2 \) and \( e_2 := (1 - j)/2 \). In this paper, we call an end of type (1) (resp. (2)) a timelike (resp. lightlike) planar end. On the other hand, we define a high-order timelike flat end by the condition \( K \geq 2 \), \( V_{-K} \notin e_1 \mathbb{R}^3 \cup e_2 \mathbb{R}^3 \), \( \text{rank} (V_{-K}, \ldots, V_{-1}) = 1 \), \( V_{\log} = 0 \) and \( \tilde{g}(0) \notin \mathbb{R} \). We can show the same conclusion as Theorem 3.1 also for timelike planar ends and high-order timelike flat ends.
Theorem 3.2. Any timelike planar end of real analytic timelike minimal surfaces in $\mathbb{R}^{2,1}$ extends as a $C^\infty$ surface in $\mathring{L}$. On the other hand, any high-order timelike flat end extends in $\mathring{L}$ with a branch point in the paracomplex sense.

(Proof.) In this case, we may assume that the end is $\hat{z} = 0$ and $\hat{g}(0) = j$, $\hat{g}'(0) = \cdots = \hat{g}^{(K)}(0) = 0$ without loss of generality, and $(\hat{g}, \hat{f})$ is of the following form:

\[
\begin{aligned}
\hat{g} &= j + \hat{z}^{K+1}\hat{g}_{\text{hol}}, \\
\hat{f} &= \sum_{k=-K}^{-1} \hat{\alpha}_k \hat{z}^{k-1} + \hat{f}_{\text{hol}},
\end{aligned}
\]

where $K \geq 1$, $\hat{\alpha}_{-K} \in \mathbb{C}$, $|\hat{\alpha}_{-K}|^2 \neq 0$, and $\hat{g}_{\text{hol}}$ and $\hat{f}_{\text{hol}}$ are paraholomorphic functions around $\hat{z} = 0$. Since

\[
\hat{g}^2 = 1 + \hat{z}^{K+1}(2j\hat{g}_{\text{hol}} + \hat{z}^{K+1}\hat{g}_{\text{hol}}),
\]

\[
\hat{g}\hat{f} = j \sum_{k=-K}^{-1} \hat{\alpha}_k \hat{z}^{k-1} + \left(\hat{g}_{\text{hol}} \sum_{k=-K}^{-1} \hat{\alpha}_k \hat{z}^{k+K} + \hat{g}_{\text{hol}} \hat{f}_{\text{hol}}\right),
\]

\[
\hat{g}^2 \hat{f} = \sum_{k=-K}^{-1} \hat{\alpha}_k \hat{z}^{k-1} + \left(2j\hat{g}_{\text{hol}} + \hat{z}^{K+1}\hat{g}_{\text{hol}}^2 \sum_{k=-K}^{-1} \hat{\alpha}_k \hat{z}^{k+K} + \hat{g}^2 \hat{f}_{\text{hol}}\right),
\]

we have

\[
\hat{F}_0 = \hat{F}_{\text{pri}} + \hat{F}_{\text{hol0}}, \quad \hat{F}_1 = j\hat{F}_{\text{pri}} + \hat{F}_{\text{hol1}}, \quad \hat{F}_2 = \hat{F}_{\text{pri}} + \hat{F}_{\text{hol2}}
\]

for some paraholomorphic functions $\hat{F}_{\text{hol0}}$, $\hat{F}_{\text{hol1}}$ and $\hat{F}_{\text{hol2}}$ around $\hat{z} = 0$, where

\[
\hat{F}_{\text{pri}} = \sum_{k=-K}^{-1} \frac{\hat{\alpha}_k}{k} \hat{z}^k = \frac{1}{\hat{z}^K} \left(-\frac{\hat{\alpha}_{-K}}{K} + O(\hat{z})\right).
\]

Hence we get

\[
\check{R}_0 = \Re \hat{F}_{\text{pri}} + \Re \hat{F}_{\text{hol0}} = \frac{1}{|\hat{z}|^{2K}} \left\{ -\frac{1}{K} \Re(\hat{\alpha}_{-K} \hat{z}^K) + \mathcal{O}_{K+1} \right\},
\]

\[
\check{R}_1 = \Im \hat{F}_{\text{pri}} + \Re \hat{F}_{\text{hol1}} = \frac{1}{|\hat{z}|^{2K}} \left\{ -\frac{1}{K} \Im(\hat{\alpha}_{-K} \hat{z}^K) + \mathcal{O}_{K+1} \right\},
\]

\[
\check{R}_2 = \Re \hat{F}_{\text{pri}} + \Re \hat{F}_{\text{hol2}} = \frac{1}{|\hat{z}|^{2K}} \left\{ -\frac{1}{K} \Re(\hat{\alpha}_{-K} \hat{z}^K) + \mathcal{O}_{K+1} \right\},
\]

where $\mathcal{O}_{K+1}$ is as before. On the other hand, since

\[
\check{R}_1^2 - \check{R}_0 \check{R}_2 = -(\Re \hat{F}_{\text{pri}})^2 + (\Im \hat{F}_{\text{pri}})^2 - \Re \hat{F}_{\text{pri}}(\Re \hat{F}_{\text{hol0}} + \Re \hat{F}_{\text{hol2}})
\]

\[
+ 2 \Im \hat{F}_{\text{pri}} \cdot \Re \hat{F}_{\text{hol1}} + (\Re \hat{F}_{\text{hol1}})^2 - \Re \hat{F}_{\text{hol0}} \cdot \Re \hat{F}_{\text{hol2}}
\]

\[
= -|\hat{F}_{\text{pri}}|^2 + \Re \hat{F}_{\text{pri}} \cdot O(1) + \Im \hat{F}_{\text{pri}} \cdot O(1) + O(1),
\]

Regularity of ends of zero mean curvature surfaces in $\mathbb{R}^{2,1}$
\[ |F_{\text{pt}}|^2 = \frac{1}{|z|^{2K}} \left| \frac{\bar{\alpha} - K}{K} + O(\bar{z}) \right|^2 = \frac{1}{|z|^{2K}} \left( \frac{|\bar{\alpha} - K|^2}{K^2} + O_1 \right), \]

we also get
\[ \bar{R}_1^2 - \bar{R}_0 \bar{R}_2 = \frac{1}{|z|^{2K}} \left( -\frac{|\bar{\alpha} - K|^2}{K^2} + O_1 \right), \]

where \( O_1 = O(x) + O(y) \). Hence it follows that
\[ [\bar{R}_1^2 - \bar{R}_0 \bar{R}_2 : \bar{R}_0 : \bar{R}_1 : \bar{R}_2 : 1] \]
\[ = \left[ \frac{1}{|z|^{2K}} \left\{ -\frac{|\bar{\alpha} - K|^2}{K^2} + O_1 \right\} : \frac{1}{|z|^{2K}} \left\{ -\frac{1}{K} \text{Re}(\bar{\alpha} - K\bar{z}^K) + O_{K+1} \right\} \right. \]
\[ : \frac{1}{|z|^{2K}} \left\{ -\frac{1}{K} \text{Im}(\bar{\alpha} - K\bar{z}^K) + O_{K+1} \right\} : \frac{1}{|z|^{2K}} \left\{ -\frac{1}{K} \text{Re}(\bar{\alpha} - K\bar{z}^K) + O_{K+1} \right\} : 1 \]
\[ = \left[ |\bar{\alpha} - K|^2 + O_1 : K \text{Re}(\bar{\alpha} - K\bar{z}^K) + O_{K+1} \right. \]
\[ : K \text{Im}(\bar{\alpha} - K\bar{z}^K) + O_{K+1} : K \text{Re}(\bar{\alpha} - K\bar{z}^K) + O_{K+1} : -K^2 |\bar{z}|^{2K} \right]. \]

Since all the terms \( O_1 \) and \( O_{K+1} \) are real analytic functions, \([\bar{R}_1^2 - \bar{R}_0 \bar{R}_2 : \bar{R}_0 : \bar{R}_1 : \bar{R}_2 : 1]\) is also analytic as a map defined around \( \bar{z} = 0 \). Now we get our assertion by the local coordinate expression
\[ \frac{1}{\bar{R}_1^2 - \bar{R}_0 \bar{R}_2}(2\bar{R}_1, \bar{R}_0 + \bar{R}_2) = \frac{2K}{|\bar{\alpha} - K|^2 + O_1} \left( \text{Im}(\bar{\alpha} - K\bar{z}^K) + O_{K+1}, \text{Re}(\bar{\alpha} - K\bar{z}^K) + O_{K+1} \right). \]

\[ \square \]

4. Lightlike planar ends

4.1. Fold singularities and lightlike lines

The structure of lightlike planar ends is more complicated. At first, since the limit normal at the end is null, the surface has a singular set goes across the end. Moreover, this type of end is not complete and asymptotic to some lightlike line, and its closure is not \( C^\infty \). Hence we employ the fact that if a spacelike maximal surface in \( \mathbb{R}^{2,1} \) has a fold singularity, then the surface extends analytically to timelike minimal surface across the singularity, and the following fact in addition to the compactification of \( \mathbb{R}^{2,1} \).

**Theorem 4.1.** If a spacelike maximal surface in \( \mathbb{R}^{2,1} \) with a fold singularity has an end on the singular set whose principal part consists of terms of odd order only, then the end also extends to a timelike minimal surface across a lightlike line as a map.

This theorem is a corollary to [10, Theorem 3.3]. The case of lightlike planar ends is
the case that the order of the ends is $-1$. Here we may assume that the end is $z = 0$ and $g(0) = 0$ without loss of generality. Now, by applying the criterion in [10, Theorem 3.3], we see that, if the Gauss map $g$ satisfies $g''(0) \neq 0$, then the image of the map is $C^\infty$ on the lightlike line except for a point, and if $g''(0) = \cdots = g^{(K+1)}(0) = 0$ and $g^{(K+2)}(0) \neq 0$, then the image of the map is $C^\infty$ everywhere on the lightlike line, where $-K$ is the order of the end, i.e. $f$ has a pole of order $K + 1$ at 0. In the other cases, the surface has a singular set of various type on the lightlike line.

The similar results to Theorem 4.1 hold also in the case without fold singularities under suitable conditions.

**Theorem 4.2.** If a spacelike maximal (resp. real analytic timelike minimal) surface in $\mathbb{R}^{2,1}$ with the Weierstrass data $(g, f)$ (resp. $(\tilde{g}, \tilde{f})$) has an end on the singular set \( \{ z \mid g(z) \in \mathbb{R} \} \) (resp. \( \{ \tilde{z} \mid \tilde{g}(\tilde{z}) \in \mathbb{R} \} \) ) such that the principal parts of \( \int z^\ell g^\ell f dz \) (resp. \( \int \tilde{z}^\ell \tilde{g}^\ell \tilde{f} d\tilde{z} \)) \( (\ell = 0, 1, 2) \) consist of terms of odd order with real coefficients and terms of even order with purely imaginary coefficients only, then the end also extends to itself across a lightlike line as a map.

This theorem is a corollary to Theorems B.1 and B.3. The case of lightlike planar ends is the case that the order of the ends is $-1$. Here we may assume that the end is $z = 0$ and $g(0) = 0$ without loss of generality. Now, by applying the criterion in Theorem 4.1, we see that, if the Gauss map $g$ satisfies $g''(0) \neq 0$, then the image of the map is $C^\infty$ on the lightlike line except for a point, and if $g''(0) = \cdots = g^{(K+1)}(0) = 0$ and $g^{(K+2)}(0) \neq 0$, then the image of the map is $C^\infty$ everywhere on the lightlike line, where $-K$ is the order of the end, i.e. $f$ has a pole of order $K + 1$ at 0. In the other cases, the surface has a singular set of various type on the lightlike line. We note here that the condition $g''(0) = 0$ and $g''(0) \notin \mathbb{R}$ is generic in the case $K = 1$, since the logarithmic term of any lightlike planar end vanishes. This observation is valid also for timelike minimal surfaces.

**4.2. Around zero-divisors**

One of the most typical difference of timelike minimal surfaces in $\mathbb{R}^{2,1}$ from spacelike maximal surfaces or minimal surfaces in $\mathbb{R}^3$ is the fact that any end cannot consist of only one point in its domain. This arises from the fact that if any parameromorphic function has a pole at $z_0 \in \mathbb{C}$, then it also has a “pole” also at $z_0 + S$ where $S$ is the set of zero-divisors, i.e. $S = (1 + j)\mathbb{R} \cup (1 - j)\mathbb{R} \subset \mathbb{C}$ as we have already mentioned in Introduction. We call an end asymptotic to a lightlike line an incomplete end. Any incomplete end has the following additional parts:

(i) A lightlike line.
(ii) Points at infinity corresponding to zero-divisors.
(iii) The border of the above two parts.

Figure 1 shows the structure of a lightlike planar end of a mixed-type surface. Here the upper (resp. lower) half-plane is the domain of the spacelike (resp. timelike) part, and the real axis is that of the corresponding fold singularity. The end is located on the
We will discuss about special points later. As for (ii), we have the following:

(i). Broken half-lines are zero-divisors which are mapped to points at infinity (ii).

As for (i), we have already discussed about generic points in the previous subsection. We will discuss about special points later. As for (ii), we have the following:

**Theorem 4.3.** If a real analytic timelike minimal surface $X(z)$ in $\mathbb{R}^{2,1}$ with the Weierstrass data $(\hat{g}, \hat{f})$ has an end of order $-K$ at $z = z_0$ in the sense that $V_{-K} \notin e_1 \mathbb{R}^3 \cup e_2 \mathbb{R}^3$, and if $\hat{g}$ has a $\hat{g}(z_0)$-point of order $M \geq 1$ at $z = z_0$ in the sense that $\hat{g}'(z_0) = \cdots = \hat{g}^{(M-1)}(z_0) = 0$ and $|\hat{g}^{(M)}(z_0)|^2 \neq 0$, then the following holds:

1. If $K = 1$ and $M \geq 2$, namely the end is a planar end, then the surface extends as a $C^{\infty}$ surface almost everywhere on the zero-divisors $z_0 + S \setminus \{0\}$.
2. If $K \geq 2$ and $M \geq 1$, then the surface does not extend as a $C^{\infty}$ surface on the zero-divisors. In particular, if $K = 2$ and $M = 3$, or if $K = 3$ and $M = 2$, then the image of the zero-divisors is a cuspidal edge, and if $2M \leq K - 1$, then the image of the zero-divisors is a one point set.

(Proof.) Also in this case, we may assume that the end is $z_0 = 0$ and $\hat{g}(0) = 0$ without loss of generality. For a neighborhood of any $t_0 e_2 \in e_2 \mathbb{R} \setminus \{0\}$, we use the following null coordinate centered at $t_0 e_2$.

$$(x - \frac{t_0}{2}) + (y + \frac{t_0}{2}) = x + y =: \xi_1, \quad (x - \frac{t_0}{2}) - (y + \frac{t_0}{2}) = x - y - t_0 =: \xi_2.$$

Then, for a generic $t_0$, $(\hat{g}, \hat{f})$ has the following expansion around $z = t_0 e_2$:

$$\begin{align*}
\hat{g} &= (\beta_1 \xi_1^M + O(\xi_1^{M+1}))e_1 + (\beta_2 + O(\xi_2))e_2, \\
\hat{f} &= (\alpha_1 \xi_1^{-K-1} + O(\xi_1^{-K}))e_1 + (\alpha_2 + O(\xi_2))e_2,
\end{align*}$$

where $K \geq 1$, $M \geq 1$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R} \setminus \{0\}$, and $K \neq M, 2M$, i.e., the terms of $\xi_1^{-1}$ of $f, g f, g^2 \hat{f}$ vanish. Since

$$\begin{align*}
\hat{g}^2 &= (\beta_1^2 \xi_1^{2M} + O(\xi_1^{2M+1}))e_1 + (\beta_2^2 + O(\xi_2))e_2, \\
\hat{g} \hat{f} &= (\beta_1 \alpha_1 \xi_1^{-K-1+M} + O(\xi_1^{-K+M}))e_1 + (\beta_2 \alpha_2 + O(\xi_2))e_2,
\end{align*}$$

Figure 1

origin in (a), and blows up to the thick line in (b), which is mapped to a lightlike line (i). Broken half-lines are zero-divisors which are mapped to points at infinity (ii).

(a) Before blowing up (b) After blowing up
\[ \tilde{g}^2 \tilde{f} = (\beta_1^2 \alpha_1 \xi_1^{-K-1+2M} + O(\xi_1^{-K+2M}))e_1 + (\beta_2^2 \alpha_2 + O(\xi_2))e_2, \]

we have

\[
F_0 = \left( -\frac{\alpha_1}{K} \xi_1^{-K} + O(\xi_1^{-K+1}) \right) e_1 + (\alpha_2 \xi_2 + O(\xi_2^2))e_2, \\
F_1 = \left( -\frac{\beta_1 \alpha_1}{K-M} \xi_1^{-K+M} + O(\xi_1^{-K+1+M}) \right) e_1 + (\beta_2 \alpha_2 \xi_2 + O(\xi_2^2))e_2, \\
F_2 = \left( -\frac{\beta_2 \alpha_1}{K-2M} \xi_1^{-K+2M} + O(\xi_1^{-K+1+2M}) \right) e_1 + (\beta_2^2 \alpha_2 + O(\xi_2^2))e_2, \\
R_0 = \frac{1}{2} \left( \frac{\alpha_1}{K} \xi_1^{-K} + O(\xi_1^{-K+1}) + \alpha_2 \xi_2 + O(\xi_2^2) \right), \\
R_1 = \frac{1}{2} \left( -\frac{\beta_1 \alpha_1}{K-M} \xi_1^{-K+M} + O(\xi_1^{-K+1+M}) + \beta_2 \alpha_2 \xi_2 + O(\xi_2^2) \right), \\
R_2 = \frac{1}{2} \left( -\frac{\beta_2 \alpha_1}{K-2M} \xi_1^{-K+2M} + O(\xi_1^{-K+1+2M}) + \beta_2^2 \alpha_2 + O(\xi_2^2) \right),
\]

and

\[
4(R_1^2 - \tilde{R}_0 \tilde{R}_2) = \left\{ \frac{\beta_1^2 \alpha_1^2}{(K-M)^2} \xi_1^{-2K+2M} + O(\xi_1^{-2K+1+2M}) + \beta_2^2 \alpha_2 \xi_2^2 + O(\xi_2^3) \right\} \\
- \left\{ \frac{2\beta_1 \beta_2 \alpha_1 \alpha_2}{K-M} \xi_1^{-K+M} \xi_2 + O(\xi_1^{-K+M})O(\xi_2^2) + O(\xi_1^{-K+1+M})O(\xi_2) \right\} \\
- \left\{ \frac{\beta_2^2 \alpha_1 \alpha_2}{K(K-2M)} \xi_1^{-K+2M} + O(\xi_1^{-K+1+2M}) + \beta_2^2 \alpha_2 \xi_2^2 + O(\xi_2^3) \right\} \\
- \left\{ \frac{\beta_2^2 \alpha_1 \alpha_2}{K-2M} \xi_1^{-K+2M} + \beta_1^2 \alpha_1 \alpha_2 \xi_1^{-K+2M} \xi_2 + O(\xi_1^{-K})O(\xi_2^2) + O(\xi_1^{-K+1})O(\xi_2) \right\} \\
- \left\{ \frac{\beta_2^2 \alpha_1 \alpha_2}{K-2M} \xi_1^{-K+2M} + \beta_1^2 \alpha_1 \alpha_2 \xi_1^{-K+2M} \xi_2 + O(\xi_1^{-K+1+2M}) + O(\xi_2^2) + O(\xi_1)O(\xi_2) \right\}.
\]

Hence it follows that, if \( K < 2M \), then

\[
[R_1^2 - \tilde{R}_0 \tilde{R}_2 : \tilde{R}_0 : \tilde{R}_1 : \tilde{R}_2 : 1] = \frac{\beta_2^2 \alpha_1 \alpha_2}{4K} \xi_2 + O(\xi_1^{-K+2M}) + O(\xi_2^2) + O(\xi_1)O(\xi_2).
\]
\[
\begin{align*}
: - \frac{\alpha_1}{2K} + O(\xi_1) : - \frac{\beta_1 \alpha_1}{2(K-M)} \xi_1^M + O(\xi_1^{M+1}) + O(\xi_1^K) O(\xi_2) \\
: - \frac{\beta_1 \alpha_1}{2(K-2M)} \xi_1^{2M} + O(\xi_1^{2M+1}) + O(\xi_1^K) O(\xi_2) : \xi_1^K \end{align*}
\]

Since the second element does not vanish on the zero-divisors, we get our assertions in (1) and the former part of (2). On the other hand, if \( K > 2M \), then

\[
[\tilde{R}_1^2 - \tilde{R}_0 \tilde{R}_2 : \tilde{R}_0 : \tilde{R}_1 : \tilde{R}_2 : 1] = \left[ - \frac{M^2 \beta_1^2 \alpha_1^2}{4(K-M)^2 K(K-2M)} + O(\xi_1) \right.
\]

\[
- \frac{\alpha_1}{2K} \xi_1^{K-2M} + O(\xi_1^{K-1-2M}) \quad \left| \frac{\beta_1 \alpha_1}{2(K-M)} \xi_1^{K-M} + O(\xi_1^{K-1-M}) \right|
\]

\[
- \frac{\beta_1 \alpha_1}{2(K-2M)} \xi_1^K + O(\xi_1^{K+1}) : \xi_1^{2K-2M} \right],
\]

and hence we get

\[
[\tilde{R}_1^2 - \tilde{R}_0 \tilde{R}_2 : \tilde{R}_0 : \tilde{R}_1 : \tilde{R}_2 : 1]|_{\xi_1=0} = \left[ - \frac{M^2 \beta_1^2 \alpha_1^2}{4(K-M)^2 K(K-2M)} : 0 : 0 : 0 : 0 \right].
\]

\[ \square \]

### 4.3. Borders

In this subsection, we consider (iii) in the previous subsection. As we have already observed, the image of the zero-divisors of an end is \( C^\infty \) only if the end is planar, namely the order of the end is \(-1\). Moreover, the corresponding lightlike line is \( C^\infty \) only if its Gauss map \( \tilde{g} \) and the Gauss map \( g \) of the corresponding spacelike maximal surface satisfy \( g''(x_0) = \tilde{g}''(x_0) \neq 0 \) or \( g'''(x_0) = \tilde{g}'''(x_0) \neq 0 \), where \( x_0 \in \mathbb{R} = \mathbb{C} \cap \mathbb{C} \) is the common coordinate of the end. However, if \( g'''(x_0) = 0 \), then the singular set \( \{ z \mid g(z) \in \mathbb{R} \} \) goes across the border, and hence we cannot expect some regularity. Hence the case that the order of the end is \(-1\) and \( g''(x_0) = \tilde{g}''(x_0) \neq 0 \) is nontrivial. As for this case, we get the following:

**Theorem 4.4.** If a zero mean curvature surface in \( \mathbb{R}^{2,1} \) of mixed type joined by a fold singularity with the Weierstrass data \((g,f)\) and \((\tilde{g},\tilde{f})\) has an end of order \(-1\) at \( z = \bar{z} = x_0 \), and if \( g \) and \( \tilde{g} \) have a \( g(x_0) \)-point of order 2 at \( z = \bar{z} = x_0 \), then the image of the border between the lightlike line and the zero-divisors is an isolated singularity whose top term is given by \((x^3, x\tau, \tau)\).

(Proof.) Also in this case, we may assume that the end is \( x_0 = 0 \) and \( g(0) = \tilde{g}(0) = 0 \) without loss of generality, and \((g,f)\) and \((\tilde{g},\tilde{f})\) have the following expansions around \( z = 0 \) and \( \bar{z} = 0 \) respectively:

\[
\begin{align*}
g &= z^2 g_{\text{hol}}, \\
f &= \alpha z^{-2} + f_{\text{hol}}, \\
\tilde{g} &= z^2 \tilde{g}_{\text{hol}}, \\
\tilde{f} &= \alpha z^{-2} + \tilde{f}_{\text{hol}},
\end{align*}
\]
where $\alpha \in \mathbb{R} \setminus \{0\}$, and $g_{\text{hol}}$ and $f_{\text{hol}}$ (resp. $\check{g}_{\text{hol}}$ and $\check{f}_{\text{hol}}$) are holomorphic (resp. paraholomorphic) functions around $z = 0$ (resp. $\check{z} = 0$). In particular, $g_{\text{hol}}(0) = g_{\text{hol}}(0) \in \mathbb{R} \setminus \{0\}$ holds under our assumption. Hence we can find local coordinates such that $g_{\text{hol}} \equiv 1$ and $\check{g}_{\text{hol}} \equiv 1$. Set

$$O_k := \begin{cases} 
\sum_{\ell=0, \ell \text{ even}}^{k-1} O(x^{k-\ell}y^\ell) + O(y^{k+1}) & (k : \text{odd}), \\
\sum_{\ell=0, \ell \text{ even}}^k O(x^{k-\ell}y^\ell) & (k : \text{even}), 
\end{cases}$$

for any $k \geq 1$. We note here that $\text{Re } z^k = O_k$ and $\text{Re } \check{z}^k = \check{O}_k$ hold. Under the assumption (2.5), since

$$R_0 = -\alpha \frac{x}{x^2 + y^2} + O_1, \\
R_1 = \alpha x + O_3, \\
R_2 = \frac{1}{3} \alpha x(x^2 - 3y^2) + O_5,$$

we have

$$R_1^2 - R_0 R_2 = \frac{4}{3} \alpha^2 \frac{x^4}{x^2 + y^2} + \frac{x}{x^2 + y^2} O_5 + O_4.$$ 

On the other hand, since

$$\check{R}_0 = -\alpha \frac{x}{x^2 - y^2} + O_1, \\
\check{R}_1 = \alpha x + O_3, \\
\check{R}_2 = \frac{1}{3} \alpha x(x^2 + 3y^2) + O_5,$$

we have

$$\check{R}_1^2 - \check{R}_0 \check{R}_2 = \frac{4}{3} \alpha^2 \frac{x^4}{x^2 - y^2} + \frac{x}{x^2 - y^2} O_5 + O_4.$$ 

Now, if we employ the coordinate transformation for the blowing-up around $(x, y) = (0, 0)$

$$(x, \tau) := \begin{cases} 
(x, (x^2 + y^2)/x) & (x + iy \in \mathbb{C}), \\
(x, (x^2 - y^2)/x) & (x + jy \in \mathbb{C}), 
\end{cases}$$

then

$$O_k = \begin{cases} 
\sum_{\ell=0}^{(k-1)/2} O(x^{k-\ell}\tau^\ell) & (k : \text{odd}), \\
\sum_{\ell=0}^{k/2} O(x^{k-\ell}\tau^\ell) & (k : \text{even}), 
\end{cases}$$
and hence we have
\[
R_0, \hat{R}_0 = -\frac{\alpha}{\tau} + O_1 = \frac{1}{\tau}(-\alpha + O(x\tau)),
\]
\[
R_1, \hat{R}_1 = \alpha x + O_3 = \frac{1}{\tau}(\alpha x\tau + O(x^3\tau) + O(x^2\tau^2)),
\]
\[
R_2, \hat{R}_2 = \frac{1}{3}\alpha x^2(4x - 3\tau) + O_5 = \frac{1}{\tau}(O(x^3\tau) + O(x^2\tau^2)),
\]
\[
R_1^2 - R_0R_2, \quad \hat{R}_1^2 - \hat{R}_0\hat{R}_2 = \frac{4}{3}\alpha^2 x^3 + \frac{1}{\tau}O_5 + O_4
\]
\[
= \frac{1}{\tau}\left(\frac{4}{3}\alpha^2 x^3 + O(x^5) + O(x^4\tau) + O(x^3\tau^2) + O(x^2\tau^3)\right).
\]

Therefore we see that the image around the border \((x, \tau) = (0, 0)\) is given by the following:
\[
[R_1^2 - R_0R_2 : R_0 : R_1 : R_2 : 1], \quad [\hat{R}_1^2 - \hat{R}_0\hat{R}_2 : \hat{R}_0 : \hat{R}_1 : \hat{R}_2 : 1]
\]
\[
= \left[\frac{4}{3}\alpha^2 x^3 + O(x^5) + O(x^4\tau) + O(x^3\tau^2) + O(x^2\tau^3)
\right.
\]
\[
- \alpha + O(x\tau) : \alpha x\tau + O(x^3\tau) + O(x^2\tau^2) : O(x^3\tau) + O(x^2\tau^2) : \tau
\].

Set
\[
Y(x, \tau) = (Y_0(x, \tau), Y_2(x, \tau), Y_4(x, \tau)) := \begin{cases}
\pi \circ \psi \circ \psi_0 \circ X|_{x^2 + y^2 = x\tau} \quad (x\tau - x^2 \geq 0), \\
\pi \circ \psi \circ \psi_0 \circ X|_{x^2 - y^2 = x\tau} \quad (x\tau - x^2 \leq 0)
\end{cases}
\]

for any \((x, \tau)\) close to \((0, 0)\), where \(\pi, \psi\) and \(\psi_0\) are as in §2. By the estimate above and
\[
1/(-\alpha + O(x\tau)) = -(1/\alpha) + O(x\tau),
\]
we have
\[
Y(x, \tau) = \left(\begin{array}{c}
-\frac{4}{3}\alpha x^3 + O(x^5) + O(x^4\tau) + O(x^3\tau^2) + O(x^2\tau^3), \\
-x\tau + O(x^3\tau) + O(x^2\tau^2), -\frac{1}{\alpha} + O(x\tau)
\end{array}\right).
\]

Set
\[
\psi_I(y_0, y_2, y_4) := \left(\begin{array}{c}
-\frac{3}{4\alpha}y_0, -y_2, -\alpha y_4
\end{array}\right).
\]

Then \(\psi_I\) is a diffeomorphism, and we get
\[
Y_I(x, \tau) = (Y_{I0}(x, \tau), Y_{I2}(x, \tau), Y_{I4}(x, \tau)) := \psi_I \circ Y(x, \tau)
\]
\[
= (x^3 + O(x^5) + O(x^4\tau) + O(x^3\tau^2) + O(x^2\tau^3), \\
x\tau + O(x^3\tau) + O(x^2\tau^2), \tau + O(x\tau^2)).
\]

Set
\[
(x, \tau_A) := \varphi_A(x, \tau) := (x, Y_{I4}(x, \tau)) = (x, \tau + O(x\tau^2)).
\]
There exists a real analytic function $h_{A0}(\tau_A)$ defined near $\tau_A = 0$ satisfying

$$Y_{I\!A0}(x,\tau_A) = x^3 + O(x^5) + O(x^4\tau_A) + O(x^3\tau_A^2) + O(x^2\tau_A^3),$$

$$x\tau_A + O(x^3\tau_A) + O(x^2\tau_A^2), \tau_A).$$

Set

$$\psi_{II}(y_0, y_2, y_4) := (y_0 - y_2^2 h_{A0}(y_4), y_2, y_4).$$

Since $|J\psi_{II}(y_0, y_2, y_4)| = 1$, $\psi_{II}$ is a local diffeomorphism around $(y_0, y_2, y_4) = (0, 0, 0)$, and we get

$$Y_{I\!I\!A}(x,\tau_A) = (Y_{I\!A0}(x,\tau_A), Y_{I\!A2}(x,\tau_A), Y_{I\!A4}(x,\tau_A)) := \psi_{II} \circ Y_{I\!A}(x,\tau_A)$$

$$= (x^3 + O(x^5) + O(x^4\tau_A) + O(x^3\tau_A^2), \tau_A)$$

Set

$$(x_B, \tau_A) = \varphi_B(x,\tau_A) := (Y_{I\!I\!A0}(x,\tau_A)^{1/3}, \tau_A) = (x + O(x^3) + O(x^2\tau_A) + O(x\tau_A^2), \tau_A).$$

Since $|J\varphi_B(0,0)| = 1$, $\varphi_B$ is a local diffeomorphism around $(x,\tau_A) = (0, 0)$. Moreover we have the estimate $x = x_B + O(x_B^3) + O(x_B^2\tau_A) + O(x_B\tau_A^2)$, from which we get

$$Y_{I\!I\!B}(x_B,\tau_A) = (Y_{I\!I\!B0}(x_B,\tau_A), Y_{I\!I\!B2}(x_B,\tau_A), Y_{I\!I\!B4}(x_B,\tau_A))$$

$$:= Y_{I\!I\!A} \circ \varphi_B^{-1}(x_B,\tau_A)$$

$$= (x_B^3, x_B^2\tau_A + O(x_B^3\tau_A) + O(x_B^2\tau_A^2) + O(x_B\tau_A^3), \tau_A).$$

There exist real analytic functions $h_{B\ell}(\chi,\tau_A)$ $(\ell = 1, 2, 3)$ defined near $(\chi,\tau_A) = (0, 0)$ satisfying

$$Y_{I\!I\!B2}(x_B,\tau_A) = x_B\tau_A h_{B1}(x_B^3,\tau_A) + x_B^2\tau_A h_{B2}(x_B^3,\tau_A) + x_B^3\tau_A h_{B3}(x_B^3,\tau_A),$$

$h_{B1}(0,0) = 1$ and $h_{B2}(0,0) = 0$. Set

$$\psi_{III}(y_0, y_2, y_4) := \left( y_0 - \frac{y_2 y_4 h_{B3}(y_0, y_4)}{h_{B1}(y_0, y_4)}, y_4 \right).$$

Since $|J\psi_{III}(0,0,0)| = 1$, $\psi_{III}$ is a local diffeomorphism around $(y_0, y_2, y_4) = (0, 0, 0)$. Set $h_{B2}(\chi,\tau_A) := h_{B2}(\chi,\tau_A)/h_{B1}(\chi,\tau_A)$. Then $h_{B2}(\chi,\tau_A)$ is a real analytic function
defined near \((\chi, \tau_A) = (0, 0)\), and we get
\[
Y_{III}B(x_B, \tau_A) = (Y_{III}B_0(x_B, \tau_A), Y_{III}B_2(x_B, \tau_A), Y_{III}B_4(x_B, \tau_A))
\]
\[
:= \psi_{III} \circ Y_{III}B(x_B, \tau_A) = (x_B^3, x_B \tau_A + x_B^2 \tau_A h_{B^2}(x_B^3, \tau_A), \tau_A).
\]
Now, there exist real analytic functions \(h_{B^2}(\chi, \tau_A)\) and \(h_{B^2\prime\prime}(\chi, \tau_A)\) defined near \(\chi = 0\) and \((\chi, \tau_A) = (0, 0)\) respectively and satisfying \(h_{B^2}(\chi, \tau_A) = \chi h_{B^2\prime\prime}(\chi) + \tau_A h_{B^2\prime\prime\prime}(\chi, \tau_A)\) and hence
\[
Y_{III}B_2(x_B, \tau_A) = x_B \tau_A + x_B^5 \tau_A h_{B^2\prime\prime}(x_B^3, \tau_A) + x_B^2 \tau_A^2 h_{B^2\prime\prime\prime}(x_B^3, \tau_A)
\]
also. Set
\[
h_{IV}(y_0, y_2, y_4) := \frac{y_2^2 - 2y_0^2 y_4^2 h_{B^2\prime\prime}(y_0) - y_0^3 y_2 y_4 h_{B^2\prime\prime\prime}(y_0)^2}{1 - y_0^4 h_{B^2\prime\prime\prime}(y_0)^3}.
\]
Then \(h_{IV}\) is a real analytic function around \((y_0, y_2, y_4) = (0, 0, 0)\), and we get
\[
x_B^2 \tau_A^2 = h_{IV}(x_B^3, x_B \tau_A + x_B^5 \tau_A h_{B^2\prime\prime}(x_B^3, \tau_A), \tau_A).
\]
Set
\[
\psi_{IV}(y_0, y_2, y_4) := (y_0, y_2 + h_{IV}(y_0, y_2, y_4) h_{B^2\prime\prime\prime}(y_0, y_4)) y_4).
\]
Since \(|J\psi_{IV}(0, 0, 0)| = 1\), \(\psi_{IV}\) is a local diffeomorphism around \((y_0, y_2, y_4) = (0, 0, 0)\), and we get
\[
Y_{IV}B(x_B, \tau_A) = (Y_{IV}B_0(x_B, \tau_A), Y_{IV}B_2(x_B, \tau_A), Y_{IV}B_4(x_B, \tau_A))
\]
\[
:= \psi_{IV}^{-1} \circ Y_{III}B(x_B, \tau_A) = (x_B^3, x_B \tau_A + x_B^5 \tau_A h_{B^2\prime\prime}(x_B^3, \tau_A), \tau_A).
\]

\[\square\]

**Remark.** Suppose that \((g, f)\) and \((\tilde{g}, \tilde{f})\) have the expansions
\[
\begin{cases}
g = z^2 g_{hol3}(z^3), \\
f = \alpha z^{-2} + z f_{hol3}(z^3),
\end{cases}
\]
\[
\begin{cases}
\tilde{g} = \tilde{z}^2 \tilde{g}_{hol3}(\tilde{z}^3), \\
\tilde{f} = \alpha \tilde{z}^{-2} + \tilde{z} \tilde{f}_{hol3}(\tilde{z}^3),
\end{cases}
\]
for some holomorphic functions \(g_{hol3}\) and \(f_{hol3}\), and some paraholomorphic functions \(\tilde{g}_{hol3}\) and \(\tilde{f}_{hol3}\). Also in this case, we can find local coordinates such that \(g_{hol3} \equiv 1\) and \(\tilde{g}_{hol3} \equiv 1\). Now we have
\[
\begin{align*}
R_0, & \quad \hat{R}_0 = -\frac{\alpha}{\tau} + O(x^2) + O(x \tau) \quad = \frac{1}{\tau}(-\alpha + O(x^2 \tau) + O(x \tau^2)), \\
R_1, & \quad \hat{R}_1 = \alpha x + x^4 h_1(x^3) + O(x^3 \tau) + O(x^2 \tau^2) \\
& \quad = \frac{1}{\tau}(\alpha x + x^4 h_1(x^3) \tau + O(x^3 \tau^2) + O(x^2 \tau^3)), \\
R_2, & \quad \hat{R}_2 = \frac{4}{3} \alpha x^3 - \alpha x^2 \tau + x^6 h_2(x^3) + O(x^5 \tau) + O(x^4 \tau^2) + O(x^3 \tau^3) \\
& \quad = \frac{1}{\tau}(O(x^3 \tau) + O(x^2 \tau^2)),
\end{align*}
\]
\[ R_1^2 - R_0 R_2, \quad \tilde{R}_1^2 - \tilde{R}_0 \tilde{R}_2 \]
\[ = \frac{4}{3} \alpha^2 \frac{x^3}{\tau} + \alpha \frac{x^6}{\tau} h_2(x^3) + O(x^5) + O(x^4 \tau) + O(x^3 \tau^2) \]
\[ = \frac{1}{\tau} \left( \frac{4}{3} \alpha^2 x^3 + \alpha x^6 h_2(x^3) + O(x^5 \tau) + O(x^4 \tau^2) + O(x^3 \tau^3) \right), \]

where \( h_1 \) and \( h_2 \) are also analytic functions. Hence

\[ Y_I(x, \tau) = \left( x^3 + \frac{3}{4 \alpha} x^6 h_2(x^3) + O(x^3 \tau), x\tau + \frac{1}{\alpha} x^4 h_1(x^3) \tau + O(x^2 \tau^2), \tau + O(x \tau^2) \right), \]
\[ Y_{IA}(x, \tau_A) = Y_{IA}(x, \tau_A) \]
\[ = \left( x^3 + \frac{3}{4 \alpha} x^6 h_2(x^3) + O(x^3 \tau_A), x\tau_A + \frac{1}{\alpha} x^4 h_1(x^3) \tau_A + O(x^2 \tau_A^2), \tau_A \right), \]
\[ x_B = Y_{IA}(x, \tau_A) \]
\[ = x + x^4 h_3(x^3) + O(x \tau A), \]

where \( h_3 \) is also an analytic function. In this case, since \( x = x_B + x_B^4 h_4(x_B^3) + O(x_B^3 \tau_A) \) holds for some analytic function \( h_4 \), we have \( h_{B2}(x_B^3, 0) \equiv 0 \) which implies \( h_{B2'}(x_B^3, 0) \equiv 0 \) and hence \( h_{B2'}(x_B^3) \equiv 0 \). Finally we get

\[ Y_{IVB}(x_B, \tau_A) = (x_B^3, x_B \tau_A, \tau_A). \]

5. Cone-like singularities of mixed type

As we have already observed, the lightlike line of converging part of spacelike maximal (resp. timelike minimal) surfaces in \( \mathbb{R}^{2,1} \) is \( C^\infty \) on a generic point. However this does not hold on all points in general. Indeed if \( g(0) = 0 \) at an end \( z = 0 \), then the singular set \( \{ z \mid g(z) \in \mathbb{R} \} \) (resp. \( \{ \hat{z} \mid \hat{g}(\hat{z}) \in \mathbb{R} \} \) ) passes through the range of the coordinate changing \( x = s(t^{k+1})^{1/2}, t = y^2 \) (resp. \( t = -y^2 \) ) \( (s, t \in \mathbb{R}) \) at \( t = 0 \), and a cuspidal edge intersects with the lightlike line in the case that \( g \) (resp. \( \hat{g} \) ) has a zero of even order. On the other hand, if \( g \) (resp. \( \hat{g} \) ) has a zero of odd order, then the cuspidal edge intersects with the lightlike line at the point of infinity \( (s, t) = (\pm \infty, 0) \) that is the border we discussed in the previous section.

In the former case, if the surface itself is given by odd functions, namely if its Weierstrass data \( (g, f) \) is given by even functions, then the image of the imaginary axis is a one point set, and this point is a cone-like singularity under some admissible condition. To show this, we first prepare the following criterion:

**Lemma 5.1.** Let \( X = (X_1, X_2, X_3) : U \rightarrow \mathbb{R}^3 \) be a \( C^\infty \) map, where \( U = (-\epsilon, \epsilon) \times S^1 \) for some \( \epsilon > 0 \) and \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \). Set \( X_{12} := (X_1, X_2) \) and \( \Theta := \arg(X_1 + iX_2) \). Suppose that \( X \) satisfies the following conditions:

1. \( X(0, \theta) = (0, 0, 0) \) \( (\forall \theta) \).
2. \( X_{12} \) is a local diffeomorphism on \( U \ \setminus \ \{ \{0\} \times S^1 \} \).
3. \( |X_{12}|(\rho, \theta) \) is \( C^1 \) on \( \rho \in [0, \epsilon) \) (resp. \( (-\epsilon, 0] \)), and \( |X_{12}|(\rho, [0, \theta]) > 0 \) (resp. \( < 0 \))
\((\forall \theta)\).

(4) \(\Theta : ((-\epsilon, 0) \cup (0, \epsilon)) \times \mathbb{R} \to \mathbb{R}\) can be extended to \(\{0\} \times \mathbb{R}\). \(\Theta(0, \theta)\) is \(C^1\) with respect to \(\theta\), \(\Theta(0, 0) = 0\) \((\forall \theta)\), and \(\Theta(0, 2\pi) - \Theta(0, 0) = 2\pi\) \((\forall \theta)\).

(5) \(X_3(\rho, \theta)\) is \(C^1\) with respect to \(\rho\), and \((X_3)_\rho(0, [\theta]) > 0\) \((\forall \theta)\).

(6) \(X_1^2 + X_2^2 - X_3^2 = o(X_1^2 + X_2^2)\).

Then \(X\) has a conelike singularity on \(\{0\} \times S^1\).

(Proof.) At first, by replacing \(\epsilon\) by a suitable positive number small enough, we may assume that the inequalities in (3) and (5) hold on \(U\), namely \(|X_{12}(\rho, \theta)| > 0\) \((\forall (\rho, \theta) \in [0, \epsilon] \times S^1)\), \(|X_{12}(\rho, \theta)| < 0\) \((\forall (\rho, \theta) \in (-\epsilon, 0] \times S^1)\), \((X_3)_\rho(\rho, \theta) > 0\) \((\forall \rho, \theta) \in (-\epsilon, 0) \times S^1\). Now, by (1) and (5) with \((X_3)_\rho > 0\), we see that \((X_3(\rho, \theta)) > 0\) \((\forall \rho > 0)\). By (1) and (3), we also see that \(|X_{12}(\rho, \theta)| > 0\) \((\forall \rho > 0)\). Set \(U_+ := [0, \epsilon] \times S^1\) and denote \(U_+ \sim := U_+/(\{0\} \times S^1)\). Then \(X_+ : U_+ \sim \to \mathbb{R}^2\), \([(\rho, \theta)] \mapsto X_{12}(\rho, \theta)\) is a well-defined continuous map. Now, by (4) with \(\Theta_0 > 0\) \((\text{resp. } < 0)\), there exists a \(\rho_0 > 0\) such that \(X_+([\rho, \theta])] (0 \leq \theta \leq 2\pi)\) is a loop in \(\mathbb{R}^2\) surrounding the origin \((0, 0) = X_{12}(0, [\theta]) \in \mathbb{R}^2\) anticlockwisely \((\text{resp. } \text{clockwisely})\) for any \(\rho \in (0, \rho_0)\). Set \(U_{\rho_0} := ([0, \rho_0] \times S^1) / \sim \subset U_+ \sim\). Then \(X_+(U_{\rho_0})\) includes some closed neighborhood \(V_+\) of the origin \((0, 0)\).

Choose \(r_0 > 0\) such that \(B_{r_0} := \{(x_1, x_2) \in \mathbb{R}^2 \mid \|x\| \leq r_0^2\} \subset V_+\). By (3), again, for any \(r \in (0, r_0]\), there exists a unique \(\rho = \rho(r, \theta) \in (0, \rho_0]\) such that \(|X_{12}(\rho, \theta)| = \rho\) \((\forall \theta)\). By (2), \(\rho(r, \theta)\) in continuous with respect to \(\theta\). Now, if \(X_+([\rho(r, \theta), \theta)])\) \((\theta \in [0, 2\pi])\) is not injective, then \(\Theta(\rho(r, \theta), \theta)\) has a local maximum or minimum. This contradicts (2). Hence \(X_+([\rho(r, \theta), \theta)])\) \((\theta \in [0, 2\pi])\) is injective, and there exists \(X_+^{-1}|_{B_{r_0}}\) and \(X_3 \circ X_+^{-1}|_{B_{r_0}}\) is a \(C^\infty\) function on \(B_{r_0} \setminus \{(0, 0)\}\). Finally, by (6), we see that this function is asymptotic to \(|X_{12}| \circ X_+^{-1}|_{B_{r_0}}\).

We can show the similar assertion also for \(\rho < 0\) by the same way. Now we conclude that \(X\) has a conelike singularity on \(\{0\} \times S^1\). \(\square\)

Now, we give estimates which enable us to apply Lemma 5.1 to various cases.

**Case 1.** (The imaginary axis \(i\mathbb{R}\)).

For any function \(\varphi(z)\) holomorphic on a neighborhood of some point on the imaginary axis \(i\mathbb{R} \subset \mathbb{C}\), we see that

\[
\varphi(x + iy) = \sum_{k=0}^{+\infty} \frac{1}{k!} \varphi^{(k)}(iy)x^k.
\]

If \(\varphi\) satisfies

\[
(5.1) \quad \varphi(-z) = -\overline{\varphi(z)},
\]
then \(\varphi(i\mathbb{R}) \subset i\mathbb{R}\), and hence

\[
\varphi^{(k)}(iy) \in \begin{cases} \mathbb{R} & (k : \text{odd}), \\ i\mathbb{R} & (k : \text{even}). \end{cases}
\]
Therefore we have

\[ \text{Re } \varphi(x + iy) = \sum_{k=1, k \text{ odd}}^{+\infty} \frac{1}{k!} \varphi^{(k)}(iy)x^k. \]

Applying this equality to \( F_\ell \) (\( \ell = 0, 1, 2 \)) defined by \((g, f)\) satisfying (2.6), we get

\[ R_\ell = \text{Re } F_\ell = \sum_{k=1, k \text{ odd}}^{+\infty} \frac{1}{k!} (g^\ell f)^{(k-1)}(iy)x^k \]

\[ = g(iy)^\ell f(iy)x + O(x^3) \quad (\ell = 0, 1, 2). \]

By using these expansions, we have the following estimates:

\[ |X_{12}|^2 = (R_0 - R_2)^2 + 4R_1^2 = 2x^2 \{ (1 + g(iy)^2)^2 f(iy)^2 + O(x^2) \}, \]

\[ |X_{12}|x(\pm 0 + iy) = \pm (1 + g(iy)^2)|f(iy)|, \]

\[ X_3 = R_0 + R_2 = x\{ (1 + g(iy)^2)f(iy) + O(x^2) \}, \]

\[ (X_3)_x(iy) = (1 + g(iy)^2)f(iy), \]

\[ \tan \Theta = \frac{2R_1}{R_0 - R_2} = \frac{2g(iy)}{1 - g(iy)^2} + O(x^2) \quad (\text{if } (1 - g(iy)^2)f(iy) \neq 0), \]

\[ (\tan \Theta)_y = \frac{i \cdot 2g'(iy)(1 + g(iy)^2)}{(1 - g(iy)^2)^2} + O(x^2), \]

\[ 1 + \tan^2 \Theta = \frac{(1 + g(iy)^2)^2}{(1 - g(iy)^2)^2} + O(x^2), \]

\[ \Theta_y = \frac{(\tan \Theta)_y}{1 + \tan^2 \Theta} = \frac{i \cdot 2g'(iy)}{1 + g(iy)^2} + O(x^2), \]

\[ |X_{12}|^2 - X_3^2 = 4(R_1^2 - R_0R_2) = O(x^4), \]

\[ |X_{12}| \pm X_3 = \frac{|X_{12}|^2 - X_3^2}{|X_{12}| \pm X_3} = O(x^3) = O(|X_{12}|^3) \quad \text{(where } \pm f(iy) > 0). \]

**Case 1'. (The imaginary axis \( j\mathbb{R} \)).**

For any function \( \tilde{\varphi}(z) \) paraholomorphic on a neighborhood of some point on the imaginary axis \( j\mathbb{R} \subset \mathbb{C} \), we have the same estimates as in Case 1 for \( |X_{12}|x(\pm 0 + jy) \), \((X_3)_x(jy)\), \( \Theta_y \) and \( |X_{12}| \pm X_3 \) by the quite similar way.

**Case 2. (Fold singularities).**

If \( \varphi \) satisfies both (5.1) and

\[ \varphi(\overline{z}) = \overline{\varphi(z)}, \]

then \( \varphi \) is an odd function, and hence \( \varphi^{(k)} \) is an even function for any positive odd number \( k \). Suppose that \( \tilde{\varphi} \) coincides with \( \varphi \) on \( \mathbb{R} \). Then \( \tilde{\varphi} \) also satisfies the same condition as
above. Set
\begin{equation}
\varphi_{nd}(x, t) := \begin{cases} 
\varphi(x + i\sqrt{t}) & (t \geq 0), \\
\varphi(x + j\sqrt{-t}) & (t \leq 0).
\end{cases}
\end{equation}

Suppose that \((g, f)\) satisfies both (2.5) and (2.6), and that \((\hat{g}, \hat{f})\) coincides with \((g, f)\) on \(\mathbb{R}\). Define \((X_{12})_{nd}(x, t), (X_3)_{nd}(x, t), \) and \(\Theta_{nd}(x, t)\) in the same manner. Then we have the same estimates as in Case 1 for \(j(X_{12})_{nd, 0}(x, t), j(X_3)_{nd, 0}(x, t), \) and \(j(X_{12})_{nd} \pm (X_3)_{nd}\). Moreover we also have the following estimates:

\[
\tan \Theta_{nd} = \frac{2g_{nd}(0, t)}{1 - g_{nd}(0, t)^2} + O(x^2) \quad \text{(if } (1 - g_{nd}(0, t)^2)f_{nd}(0, t) \neq 0),
\]

\[
(tan \Theta_{nd})_t = \frac{2(g_{nd})_t(0, t)(1 + g_{nd}(0, t)^2)}{(1 - g_{nd}(0, t)^2)^2} + O(x^2),
\]

\[
(\Theta_{nd})_t = \frac{2(g_{nd})_t(0, t)}{1 + g_{nd}(0, t)^2} + O(x^2),
\]

where

\[
2(g_{nd})_t(0, t) = \begin{cases} 
\frac{i\hat{g}'(i\sqrt{t})}{\sqrt{t}} & (t > 0), \\
\frac{-j\hat{g}'(j\sqrt{-t})}{\sqrt{-t}} & (t < 0),
\end{cases}
\]

\[
2(g_{nd})_t(0, 0) = \lim_{t \to +0} \frac{i\hat{g}'(i\sqrt{t})}{\sqrt{t}} = -g''(0) = -\hat{g}''(0) = \lim_{t \to -0} \frac{-j\hat{g}'(j\sqrt{-t})}{\sqrt{-t}}.
\]

**Case 3.** (Lightlike lines on \(0 \in \mathbb{C}\))

Suppose that \(\varphi\) satisfies (5.1). If \(\varphi\) has a pole of order \(K\) at the origin \(z = 0\), then, for any positive odd number \(k\), \(\varphi^{(k)}\) has a pole of order \(K + k\) at the origin \(z = 0\), and

\[
\varphi^{(k)}(iy)g^{(K+1)k} = O(g^{K(k-1)})
\]

holds near \(z = 0\). Set

\[
\varphi_{end}(s, y) := \varphi(sy^{K+1} + iy).
\]

Then we have

\[
\text{Re } \varphi_{end}(s, y) = \sum_{k=1, k \text{ odd}}^{+\infty} \frac{1}{k!} \varphi^{(k)}(iy)g^{(K+1)k}s^k.
\]

Suppose that \((g, f)\) satisfies (2.6), and that \(f\) has a pole of order \(K + 1\) at the origin \(z = 0\). Applying the equality above to \(F_\ell (\ell = 0, 1, 2)\) defined by such \((g, f)\), we get

\[
(R_\ell)_{\text{end}} = \text{Re } (F_\ell)_{\text{end}} = \sum_{k=1, k \text{ odd}}^{+\infty} \frac{1}{k!} (g^{(k)}f^{(k-1)}(iy)g^{(K+1)k}s^k)
\]
\[ g(iy)^\ell f(iy)y^{K+1}s + O(s^3y^{2K}) \]

for \( \ell = 0, 1, 2 \). By using these expansions, we have the following estimates:

\[
|X_{12}\rangle_{\text{end}}^2 = s^2\{(1 + g(iy)^2)^2 f(iy)^2 y^{2(K+1)} + O(s^2y^{2K})\},
\]

\[
|X_{12}\rangle_{\text{end}}|s(\pm 0, y) = \pm(1 + g(iy)^2)|f(iy)y^{K+1}|,
\]

\[
(X_{3})_{\text{end}}(s, y) = s\{(1 + g(iy)^2) f(iy)y^{K+1} + O(s^2y^{2K})\},
\]

\[
((X_{3})_{\text{end}})_s(0, y) = (1 + g(iy)^2)f(iy)y^{K+1},
\]

\[
\tan \Theta_{\text{end}} = \frac{2g(iy)}{1 - g(iy)^2} + O(s^2y^{2K}) \quad \text{(if } 1 - g(iy)^2 \neq 0\text{)},
\]

\[
\tan \Theta_{\text{end}} = \frac{i \cdot 2g(iy)(1 + g(iy)^2)}{(1 - g(iy)^2)^2} + O(s^2y^{2K-1}),
\]

\[
(\Theta_{\text{end}})_s = \frac{i \cdot 2g(iy)}{1 + g(iy)^2} + O(s^2y^{2K-1}),
\]

\[
|(X_{12})_{\text{end}}| - (X_3)_{\text{end}} = O(s^4y^{2K}),
\]

\[
|(X_{12})_{\text{end}}| + (X_3)_{\text{end}} = O(|(X_{12})_{\text{end}}|^3y^{2K}) \quad \text{(where } \pm sf(iy)y^{K+1} > 0\text{)}.
\]

**Case 3'**. (Lightlike lines on fold singularities)

For any \( \tilde{\varphi} \) which has a pole of order \( K \) at the origin \( \tilde{z} = 0 \), we have the same estimates as in Case 3 for \( |X_{12}\rangle_{\text{end}}|s(\pm 0, y), (X_3)_{\text{end}}s(0, y), (\Theta_{\text{end}})_s \) and \( |X_{12}\rangle_{\text{end}}| + (X_3)_{\text{end}} \) by the quite similar way.

**Case 4**. (Lightlike lines on fold singularities)

Suppose that \( \varphi \) satisfies both (5.1) and (5.2), and that \( \varphi \) has a pole of odd order \( K \) at the origin \( z = 0 \). Suppose that \( \tilde{\varphi} \) coincides with \( \varphi \) on \( R \). Set

\[
\varphi_{\text{fed}}(s, t) := \begin{cases} \varphi(st^{(K+1)/2} + i\sqrt{t}) & (t > 0), \\ \varphi(st^{(K+1)/2} + j\sqrt{t}) & (t < 0). \end{cases}
\]

Suppose that \( (g, f) \) satisfies both (2.5) and (2.6), and that \( f \) has a pole of order \( K + 1 \) at the origin \( z = 0 \). Suppose that \( (\tilde{g}, \tilde{f}) \) coincides with \( (g, f) \) on \( R \). Define \((X_{12})_{\text{fed}}(s, t), (X_3)_{\text{fed}}(s, t), \) and \( \Theta_{\text{fed}}(s, t) \) in the same manner. Then we have the same estimates as in Case 3 for \( |X_{12}\rangle_{\text{fed}}|s(\pm 0, t), (X_3)_{\text{fed}}s(0, t) \) and \( |X_{12}\rangle_{\text{fed}}| + (X_3)_{\text{fed}} \). Moreover we also have the following estimates:

\[
\tan \Theta_{\text{fed}} = \frac{2g_{\text{fed}}(0, t)}{1 - g_{\text{fed}}(0, t)^2} + O(s^2t^K) \quad \text{(if } 1 - g_{\text{fed}}(0, t)^2 \neq 0\text{)},
\]

\[
(\tan \Theta_{\text{fed}})_t = \frac{2(g_{\text{fed}})_t(0, t)(1 + g_{\text{fed}}(0, t)^2)}{(1 - g_{\text{fed}}(0, t)^2)^2} + O(s^2t^{K-1}),
\]

\[
(\Theta_{\text{fed}})_t = \frac{2(g_{\text{fed}})_t(0, t)}{1 + g_{\text{fed}}(0, t)^2} + O(s^2t^{K-1}),
\]

where \( 2(g_{\text{fed}})_t(0, t) \) is the same as \( 2(g_{\text{fed}})_t(0, t) \) in Case 2.
Let Ω (resp. ˇΩ ) be a neighborhood of the imaginary axis iR (resp. jR) in the compactification C = S^2 of C (resp. C = T^2 of C), K_0 and K_∞ be odd numbers greater than or equal to −1, and (g, f) (resp. (ˇg, ˇf)) be a pair of meromorphic (resp. paraholomorphic) functions such that

(1) g is holomorphic on Ω \ C, g(0) = 0 and g has a pole at ∞ (resp. ˇg is paraholomorphic on Ω \ C, ˇg(0) = 0 and ˇg has a pole at ∞).

(2) f is holomorphic on Ω \ C \ {0}, f has a pole of order K_0 + 1 at 0, and g^2 f dz has a pole of order K_∞ + 1 at ∞ (resp. ˇf is paraholomorphic on Ω ∩ C \ {0}, ˇf has a pole of order K_0 + 1 at 0, and ˇg^2 ˇf dz has a pole of order K_∞ + 1 at ∞. (Here a pole of order −1 + 1 means a regular point.)

Applying the estimates above to Lemma 5.1, we get the following:

**Theorem 5.2.** Let (g, f) (resp. (ˇg, ˇf)) be as above. Suppose that it satisfy both (2.5) and (2.6), and that (g, f) and (ˇg, ˇf) coincide with each other on Ω ∩ R ∩ Ω. Then X(iR) ∪ X(jR) is a conelike singularity, if the following condition hold:

\[ g'(z) \neq 0, \quad f(z) \neq 0 \quad (z \in iR \setminus \{0\}) \]
\[ \hat{g}'(\hat{z}) \neq 0, \quad \hat{f}(\hat{z}) \neq 0 \quad (\hat{z} \in jR \setminus \{0\}) \]
\[ g''(0) \neq 0, \quad \frac{d^2}{dz^2} \frac{1}{g(z-1)} \neq 0 \]
\[ (K_0 + 1) + (K_\infty + 1) \equiv 2 \quad (\text{mod} \ 4) \]

(Proof.) We may restrict the domain of X (resp. ˇX) to Ω ∩ {Im z ≥ 0} (resp. ˇΩ ∩ {Im ˇz ≤ 0}) since X(z) = X(¯z) (resp. ˇX(ˇz) = ˇX(¯z)) holds.

Let η be a continuous function on R such that 0 ≤ η(t) ≤ 1 (t ∈ R), η(t) = 0 (t ≤ 0), η(t) = 1 (t ≥ 1), η is C^∞ on R, and η'(t) ≥ 0 (0 ≤ t ≤ 1). Choose a positive numbers T_1 < T_2 < T_3 < T_4 such that T_2 is small enough and T_3 is large enough, and set

\[ \eta_{+0}(t) := 1 - \eta \left( \frac{t - T_1}{T_2 - T_1} \right), \quad \eta_{-0}(t) := \eta_{+0}(-t), \]
\[ \eta_{+\infty}(t) := \eta \left( \frac{t - T_3}{T_4 - T_3} \right), \quad \eta_{-\infty}(t) := \eta_{+\infty}(-t). \]

Moreover set

\[
x(s, t) := \begin{cases} \frac{s t^{(K_\infty + 1)/2}}{s^2 + t K_\infty} \eta_{-\infty}(t) - (-1)^{(K_\infty + 1)/2}s(1 - \eta_{-\infty}(t)) & (-∞ \leq t \leq -T_2), \\
\frac{s t^{(K_\infty + 1)/2}}{s^2 + t K_\infty} \eta_{-0}(t) + (-1)^{(K_\infty + 1)/2}s(1 - \eta_{-0}(t)) & (-T_3 \leq t \leq 0), \\
\frac{s t^{(K_\infty + 1)/2}}{s^2 + t K_\infty} \eta_{+0}(t) + s(1 - \eta_{+0}(t)) & (0 \leq t \leq T_3), \\
\frac{s t^{(K_\infty + 1)/2}}{s^2 + t K_\infty} \eta_{\infty}(t) + s(1 - \eta_{\infty}(t)) & (T_2 \leq t \leq +∞), \end{cases}
\]
point of \( \Phi \) is an even function, there exists an even number \( g \) order is at least 2. Since \( g \)
yielding the same as in Case 4, since

\[
\begin{align*}
\frac{x}{x^2 - y^2}, \frac{y}{x^2 - y^2} &= (st - (K_\infty + 1/2)(-t)^{-1/2}) \quad (t \in [-\infty,-T_4]), \\
\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} &= (st - (K_\infty + 1/2),t^{-1/2}) \quad (t \in [T_4, +\infty]),
\end{align*}
\]

and we can apply the estimates in Case 4. Moreover, \((-1)^{(K_\infty + 1)/2} x_4(0,t) > 0 \quad (t < 0), \)
\(x_4(0,t) > 0 \quad (t > 0), \) \(y_4(0,t) > 0 \quad (t \neq 0), \) and \((x(s,t),y(s,t))\) is a diffeomorphism from \((-\epsilon, +\epsilon) \times ((-T_4, -T_4) \cup (T_4, +\infty))\) to its image in \(\mathbb{R}^2\) for an \(\epsilon > 0\) small enough. Now consider \(X\) as a map defined on \(U\) by

\[
X(\rho, [\theta]) := \begin{cases} 
X(x \left(\rho, \tan \frac{\theta}{2}\right) + iy \left(\rho, \tan \frac{\theta}{2}\right)), & (\theta \in [0, \pi]) \\
\bar{X}(x \left(\rho, \tan \frac{\theta}{2}\right) + jy \left(\rho, \tan \frac{\theta}{2}\right)), & (\theta \in [\pi, 2\pi]).
\end{cases}
\]

Then \(X\) satisfies all the assumption in Lemma 5.1, and it has a conelike singularity on \(\rho = 0\). \(\square\)

**Remark.** Each assumption on \(g, f, K_0\) and \(K_\infty\) in Theorem 5.2 is also a necessary condition for \(X(i\mathbb{R}) \cup \bar{X}(i\mathbb{R})\) to be a conelike singularity.

Indeed, if \(g'(z_1) = 0\) holds for some \(z_1 \in i\mathbb{R} \setminus \{0\}\), then \(z_1\) is a \(g(z_1)\) point of \(g\) whose order is at least 2. Since \(g(z_1) \in \mathbb{R}\), the set \(\{z \in \Omega \mid g(z) \in \mathbb{R}\} \setminus (i\mathbb{R} \setminus \{z_1\})\) includes a curve which goes across \(i\mathbb{R}\) at \(z = z_1\), and its image by \(X\) is singular.

On the other hand, if \(f(z_2) = 0\) holds for some \(z_2 \in i\mathbb{R} \setminus \{0\}\), then \(z_2\) is a \(\Phi_3(z_2)\) point of \(\Phi_3 := F_0 + F_2\) whose order is at least 2. Hence the set \(\{z \in \Omega \mid \Phi_3(z) - \Phi_3(z_2) \in i\mathbb{R} \setminus (i\mathbb{R} \setminus \{z_2\})\}\) includes a curve which goes across \(i\mathbb{R}\) at \(z = z_2\), and its image by \(X\) satisfies \(X_3(z) = X_3(z_2)\).

Now, consider the case that \(g''(0) = 0\) holds. As we have already mentioned after Theorem 4.1, the relating lightlike line becomes a singular set in this case. Indeed, since \(g\) is an even function, there exists an even number \(m \geq 4\) satisfying \(g(0) = g'(0) = \cdots = g^{(m-1)}(0) = 0\) and \(g^{(m)}(0) \neq 0\). By substituting \(x = s^{1/(K_0+1/2)}\), the expansions of
Let $R_0 \pm R_2$ and $R_1$ around $z = 0$ are expressed as

\[ s(\alpha + O(t) + O(s^2t^K)) \quad \text{and} \quad st^{m/2}(\beta + O(t) + O(s^2t^K)) \]

for some $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ respectively. Except for the special case that $(g,f)$ are the data of $(m/2)$-tuple covering of some other $X$, these expansions imply that the lightlike line of the blowing up at $z = 0$ is singular (cf. [10, Theorem 3.3]).

By (5.4), if $K_0 + 1 \equiv 0 \pmod{2}$, then each side of $i\mathbb{R}$ extends to the same (resp. opposite) side of $j\mathbb{R}$ across (the blowing up at) $z = 0$. On the other hand, by (5.5), if $K_\infty + 1 \equiv 0 \pmod{2}$, then each side of $i\mathbb{R}$ extends to the opposite (resp. same) side of $j\mathbb{R}$ across (the blowing up at) $z = \infty$. Hence the assumption on $K_0$ and $K_\infty$ is also necessary.

By the quite similar way as Theorem 5.2, we can also get the following (see the comment after Theorem 4.2 and §B):

**Theorem 5.3.** Let $(g,f)$ (resp. $(\tilde{g},\tilde{f})$) be as above without the assumptions on the parities of $K_0$ and $K_\infty$. Suppose that it satisfy (2.6) and that the expansions of $g^f$ (resp. $\tilde{g}^\tilde{f}$) ($\ell = 0, 1, 2$) around $z$ (resp. $\tilde{z}$) $= 0$ and $\infty$ do not have terms of order $-1$. Then $X(i\mathbb{R})$ (resp. $X(j\mathbb{R})$) is a conelike singularity, if the following condition hold:

\[ g^f(z) \neq 0, \quad f(z) \neq 0 \quad (z \in i\mathbb{R} \setminus \{0\}), \]

\[ g^f(0) \neq 0, \quad \frac{d}{dz}
 \left. \frac{1}{g(z^{-1})} \right|_{z=0} \neq 0, \]

(resp. $\tilde{g}^{\tilde{f}}(z) \neq 0, \quad \tilde{f}(z) \neq 0 \quad (z \in j\mathbb{R} \setminus \{0\})$)

\[ g^{\tilde{f}}(0) \neq 0, \quad \frac{d}{dz}
 \left. \frac{1}{g(z^{-1})} \right|_{z=0} \neq 0, \]

$(K_0 + 1) + (K_\infty + 1) : \text{even}$.

6. The conformal Gauss maps

6.1. The conformal Gauss maps of spacelike surfaces in 3-dimensional Lorentz space forms

Let $\langle \cdot, \cdot \rangle_{p,q}$ be the nondegenerate metric of $\mathbb{R}^{p,q}$ $(p \geq 0, q \geq 0)$:

\[ \langle x, y \rangle_{p,q} := x_0 y_0 + x_1 y_1 + \cdots + x_{p-1} y_{p-1} - x_p y_p - \cdots - x_{p+q-1} y_{p+q-1} \]

\[ (x = (x_0, x_1, \ldots, x_{p+q-1}), \quad y = (y_0, y_1, \ldots, y_{p+q-1})). \]

Let $M$ be an oriented 2-dimensional manifold and $\iota$ a spacelike immersion of $M$ into $N^{2,1} = S^{2,1} \times \mathbb{R}^{2,1}$ or $H^{2,1}$. We set

\[ L := \{ x \in \mathbb{R}^{3,2} \setminus \{0\} \mid \langle x, x \rangle_{3,2} = 0 \}. \]

We identify $S^{2,1}$, $\mathbb{R}^{2,1}$, $H^{2,1}$ with $L \cap \{ x_4 = 1 \}$, $L \cap \{ x_4 = x_0 + 1 \}$, $L \cap \{ x_0 = 1 \}$, respectively and therefore we consider $\iota : M \rightarrow N^{2,1}$ to be an $L$-valued function. Let $e_3$ be a normal vector field of $\iota$ in $N^{2,1}$ satisfying $\langle e_3, e_3 \rangle_{3,2} = -1$. We can naturally choose $e_3$ for the orientations of $M$ and $N^{2,1}$. Let $H$ be the mean curvature of $\iota$ with respect to $e_3$. Let $\gamma : M \rightarrow H^{3,1}$ be a map from $M$ into $H^{3,1}$ defined by $\gamma := -e_3 + H\iota$. Then $\gamma|_{\text{Reg}(\iota)}$ is a spacelike
immersion satisfying \( \tilde{g} = \varepsilon^2 g \), where \( \varepsilon := \sqrt{H^2 + K} - \delta \) with \( \delta = \pm 1 \) according to \( N^{2,1} = S^{2,1}, \mathbb{R}^{2,1} \) or \( H^{2,1} \), and \( K \) is the curvature of \( g \). The map \( \gamma_\iota : M \to H^{3,1} \) is called the conformal Gauss map of a spacelike immersion \( \iota : M \to N^{2,1} \). We see that \( \iota \) is an \( L \)-valued normal vector field of \( \gamma_\iota|_{\text{Reg}(\iota)} \). Principal curvatures of \( \gamma_\iota|_{\text{Reg}(\iota)} \) with respect to \( \iota \) are given by \( \pm 1/\varepsilon \) and the trace of the shape operator of \( \gamma_\iota|_{\text{Reg}(\iota)} \) with respect to \( \iota \) changes the signature of the 0-th component of each point of \( \mathbb{R}^4 \).

Let \( \nu \) be a \( L \)-valued normal vector field of \( \gamma_\iota|_{\text{Reg}(\iota)} \) satisfying \( \langle \nu, \iota \rangle = -1 \). Then the trace of the shape operator of \( \gamma_\iota|_{\text{Reg}(\iota)} \) with respect to \( \nu \) is given by \( -(1/\varepsilon^2)(\Delta H - 2\varepsilon^2 H) \) ([3]), where \( \Delta \) is the Laplacian on \( M \) with respect to \( g \). Since the Euler-Lagrange equation for Willmore surfaces in \( N^{2,1} \) is given by \( \Delta H - 2\varepsilon^2 H = 0 \), we obtain

**Proposition 6.1** ([3]). The mean curvature vector of \( \gamma_\iota|_{\text{Reg}(\iota)} \) vanishes if and only if \( \iota : M \to N^{2,1} \) is Willmore.

Let \( \gamma : M \to H^{3,1} \) be a spacelike immersion with zero mean curvature vector and nowhere zero shape operator with respect to a lightlike normal vector field \( \iota \). Then on a neighborhood of each point of \( M \), we can choose \( \iota \) so that one of the following holds:

1. \( \iota_1 = 1 \) for \( k = 3 \) or 4;
2. \( \iota_1 = \iota_2 + 1 \) for \( k = 0, 1 \) or 2, \( l = 3 \) or 4;
3. \( \iota_1 = 1 \) for \( k = 0, 1 \) or 2.

If \( \iota \) satisfies (i), (ii) or (iii), then \( \iota \) is a spacelike and Willmore immersion into \( S^{2,1}, \mathbb{R}^{2,1} \) or \( H^{2,1} \), and \( \gamma \) or \( -\gamma \) is equal to the conformal Gauss map \( \gamma_\iota \) of \( \iota \). Hence we obtain

**Proposition 6.2** ([3]). Let \( \gamma : M \to H^{3,1} \) be a spacelike immersion with zero mean curvature vector and nowhere zero shape operator with respect to a lightlike normal vector field \( \iota \). Then \( \gamma \) or \( -\gamma \) is the conformal Gauss map of a Willmore immersion of a neighborhood of each point of \( M \) into each of \( N^{2,1} = S^{2,1}, \mathbb{R}^{2,1} \) and \( H^{2,1} \) given by \( \iota \).

There exists a natural one-to-one correspondence between \( S^{2,1} \setminus \{x_0 = 1\} = L \cap \{x_4 = 1, x_0 \neq 1\} \) and \( \mathbb{R}^{2,1} \setminus \{x_4 = 0\} = L \cap \{x_4 = x_0 + 1, x_4 \neq 0\} \) given by the map \( \text{pr} : S^{2,1} \setminus \{x_0 = 1\} \to \mathbb{R}^{2,1} \setminus \{x_4 = 0\}, \text{pr}(x) = x/(1 - x_0) \). We call \( \text{pr} \) the stereographic projection. We see that \( \text{pr} \) is conformal. Let \( \iota \) be a spacelike immersion of \( M \) into \( S^{2,1} \setminus \{x_0 = 1\} \). Then \( \text{pr} \circ \iota \) is a spacelike immersion. The conformal Gauss map \( \gamma_{\text{pr} \circ \iota} \) of \( \text{pr} \circ \iota \) coincides with the conformal Gauss map \( \gamma_\iota \) of \( \iota \). In particular, from Proposition 6.1 and Proposition 6.2, we obtain

**Proposition 6.3.** A spacelike immersion \( \iota : M \to S^{2,1} \setminus \{x_0 = 1\} \) is Willmore if and only if \( \text{pr} \circ \iota : M \to \mathbb{R}^{2,1} \setminus \{x_4 = 0\} \) is Willmore.

Let \( I : \mathbb{R}^{2,1} \setminus \{(x, x)_2, 1 = 0\} \to \mathbb{R}^{2,1} \setminus \{(x, x)_2, 1 = 0\} \) be the standard inversion: \( I(x) := x/(x, x)_{2,1} \). Then \( I \) is given by \( \text{pr} \circ R \circ \text{pr}^{-1} \), where \( R \) is a reflection of \( \mathbb{R}^{2,1} \) which changes the signature of the 0-th component of each point of \( \mathbb{R}^{3,2} \). Therefore from Proposition 6.3, we obtain
Proposition 6.4. Let \( \iota : M \to \mathbb{R}^{2,1} \setminus \{ (x, x)_{2,1} = 0 \} \) be a spacelike and Willmore immersion. Then \( I \circ \iota \) is Willmore.

Let \( M \) be a Riemann surface and \( \iota \) a spacelike and conformal immersion of \( M \) into \( N^{2,1} = S^{2,1}, \mathbb{R}^{2,1} \) or \( H^{2,1} \). Let \( e_3 \) be a normal vector field of \( \iota \) in \( N^{2,1} \) satisfying \( (e_3, e_3) = -1 \) and \( H \) the mean curvature of \( \iota \) with respect to \( e_3 \). Let \( h \) be the second fundamental form of \( \iota \) with respect to \( e_3 \). Let \( \Xi \) be a 4-tensor field on \( M \) given by

\[
\Xi := 2h \otimes \text{Hess}_H - (H^2 - \delta) h \otimes h - 2dH \otimes \nabla h,
\]

where \( \text{Hess}_H \) is the Hessian of \( H \) with respect to the Levi-Civita connection \( \nabla \) of \( g \) and \( \delta = +1, 0 \) or \(-1\) according to \( N^{2,1} = S^{2,1}, \mathbb{R}^{2,1} \) or \( H^{2,1} \). Let \( z \) be a local complex coordinate on a neighborhood of each point of \( M \). If \( \iota : M \to N^{2,1} \) is Willmore, then a complex quartic differential

\[
\hat{Q} := \Xi(\partial_z, \partial_z, \partial_{\bar{z}}, \partial_{\bar{z}}) dz \otimes dz \otimes dz \otimes dz
\]

is holomorphic ([3]). Let \( Q \) be the holomorphic quartic differential on \( \text{Reg}(\iota) \) for a spacelike immersion \( F := \gamma_{\iota|\text{Reg}(\iota)} \) with zero mean curvature vector (see [3]). The following holds:

Proposition 6.5 ([3]). Let \( \iota : M \to N^{2,1} \) be a conformal and Willmore immersion. Then on \( \text{Reg}(\iota) \), the following hold:
(a) \( Q \) coincides with \( \hat{Q} \) up to a nonzero constant;
(b) \( Q \equiv 0 \) if and only if a lightlike normal vector field \( \nu \) of \( F \) is contained in a constant direction.

Let \( \iota : M \to S^{2,1} \) be a conformal and Willmore immersion satisfying \( \hat{Q} \equiv 0 \). Then a lightlike normal vector field \( \nu \) of \( F = \gamma_{\iota|\text{Reg}(\iota)} \) is contained in a constant direction. Then \( \nu \) gives a unique point \( p_0 \) of \( S^{2,1} \). For each point \( p \) of \( M \), let \( F^\perp(p) \) be a hyperplane in \( \mathbb{R}^{3,2} \) through the origin of \( \mathbb{R}^{3,2} \) which is perpendicular to a vector \( F(p) \). Then \( F^\perp(p) \) contains \( p_0, \iota(p) \) and the tangent plane of \( \iota(M) \) at \( \iota(p) \). The intersection of \( S^{2,1} \) and \( F^\perp(p) \) is a totally geodesic sphere \( S(p) \) in \( S^{2,1} \) such that the mean curvature vector of \( S(p) \) coincides with the mean curvature vector of \( \iota \) at \( a \). Therefore noticing Proposition 6.3, Proposition 6.4 and Proposition 6.5, we obtain

Proposition 6.6. Let \( \iota : M \to \mathbb{R}^{2,1} \) be a conformal and Willmore immersion. Then \( \hat{Q} \equiv 0 \) if and only if \( \iota \) is given by a maximal surface and an inversion of \( \mathbb{R}^{2,1} \).

6.2. The conformal Gauss maps of timelike surfaces in 3-dimensional Lorentz space forms

Let \( \tilde{M} \) be an oriented 2-dimensional manifold and \( \iota \) a timelike immersion of \( \tilde{M} \) into \( N^{2,1} = S^{2,1}, \mathbb{R}^{2,1} \) or \( H^{2,1} \). Let \( e_3 \) be a unit normal vector field of \( \iota \) in \( N^{2,1} \). Let \( H \) be the mean curvature of \( \iota \) with respect to \( e_3 \). Let \( \gamma_{\iota} : \tilde{M} \to S^{2,2} \) be a map from \( \tilde{M} \) into \( S^{2,2} \) defined by \( \gamma_{\iota} := e_3 + H \iota \). At an umbilical point of \( \iota \), the tangential component of \( d\gamma_{\iota} \) is zero. A lightlike umbilical point of \( \iota \) is a point of \( \tilde{M} \) where the tangential
component of $d\gamma_i$ is lightlike. The complement of the set of umbilical or lightlikely uniblial points of $\iota$ is denoted by $\text{Reg}(\iota)$. If we set $\Lambda := H^2 - K + \delta$, then $\text{Reg}(\iota)$ is the set of nonzero points of $\Lambda$ and $\gamma(\iota)$, a timelike immersion which induces a Lorentz metric $\tilde{g}$ given by $\tilde{g} = \Lambda g$. The map $\gamma(\iota) : M \rightarrow S^{2,2}$ is called the conformal Gauss map of a timelike immersion $\iota : \tilde{M} \rightarrow \tilde{N}^{2,1}$. We see that $\iota$ is an $L$-valued normal vector field of $F := \gamma(\iota)$ such that the shape operator is neither zero nor lightlike, and traceless. Let $\nu$ be as in § 6.1. Then the trace of the shape operator of $F$ with respect to $\nu$ is given by $-(1/\Lambda)(\Delta H + 2\Lambda H)$ ([4]). Since $\Lambda \equiv 0$ implies $\Delta H = 0$, we obtain

**Proposition 6.7** ([4]). The mean curvature vector of $F = \gamma(\iota)$ vanishes if and only if $\iota : \tilde{M} \rightarrow \tilde{N}^{2,1}$ satisfies $\Delta H + 2\Lambda H = 0$.

A timelike immersion $\iota : \tilde{M} \rightarrow \tilde{N}^{2,1}$ is said to be of Willmore type if $\iota$ satisfies $\Delta H + 2\Lambda H = 0$.

The following is an analogue of Proposition 6.2:

**Proposition 6.8** ([4]). Let $\gamma : \tilde{M} \rightarrow S^{2,2}$ be a timelike immersion with zero mean curvature vector and neither zero nor lightlike shape operator with respect to a lightlike normal vector field $\iota$. Then $\gamma$ or $-\gamma$ is the conformal Gauss map of a timelike immersion of Willmore type of a neighborhood of each point of $M$ into each of $\tilde{N}^{2,1} = \mathbb{R}^{2,1}$ and $H^{2,1}$ given by $\iota$.

Let $\iota$ be a timelike immersion of $\tilde{M}$ into $S^{2,1} \setminus \{x_0 = 1\}$. Then $\iota_0 := pr \circ \iota$ is a timelike immersion. The conformal Gauss map $\gamma(\iota)$ of $\iota_0$ coincides with the conformal Gauss map $\gamma(\iota)$ of $\iota$. In particular, from Proposition 6.7 and Proposition 6.8, we obtain

**Proposition 6.9**. A timelike immersion $\iota : \tilde{M} \rightarrow S^{2,1} \setminus \{x_0 = 1\}$ is of Willmore type if and only if $\iota_0 := pr \circ \iota : \tilde{M} \rightarrow \mathbb{R}^{2,1} \setminus \{x_0 = 0\}$ is of Willmore type.

Let $I : \mathbb{R}^{2,1} \setminus \{(x, x)_{2,1} = 0\} \rightarrow \mathbb{R}^{2,1} \setminus \{(x, x)_{2,1} = 0\}$ be the inversion as in the previous subsection. Then by Proposition 6.9, we obtain

**Proposition 6.10**. Let $\iota : \tilde{M} \rightarrow \mathbb{R}^{2,1} \setminus \{(x, x)_{2,1} = 0\}$ be a timelike immersion of Willmore type. Then $I \circ \iota$ is of Willmore type.

Let $\tilde{M}$ be a Lorentz surface and $\iota$ a conformal immersion of $\tilde{M}$ into $\tilde{N}^{2,1} = S^{2,1}$, $\mathbb{R}^{2,1}$ or $H^{2,1}$. Let $e_3, H, h$ be as above. Let $\Xi$ be a 4-tensor field on $\tilde{M}$ given by

\begin{equation}
\Xi := 2h \otimes \text{Hess}_H + (H^2 + \delta)h \otimes h - 2dH \otimes \nabla h.
\end{equation}

We consider $\Xi$ to be a paracomplex 4-linear function on the paracomplexification of the tangent plane of $\tilde{M}$ at each point. Let $\tilde{z}$ be a local paracomplex coordinate on a neighborhood of each point of $\tilde{M}$. If $\iota : \tilde{M} \rightarrow \tilde{N}^{2,1}$ is of Willmore type, then a paracomplex quartic differential as in (6.1) with (6.2) is holomorphic ([4]). Let $Q$ be the holomorphic quartic differential on $\text{Reg}(\iota)$ for a timelike immersion $F := \gamma(\iota)$ with
zero mean curvature vector (see [4]). The following holds:

**Proposition 6.11 ([4]).** Let \( \iota : \tilde{M} \to N^{2,1} \) be a conformal immersion of Willmore type. Then on \( \text{Reg}(\iota) \), the following hold:

(a) the null points of \( Q \) coincide with the null points of \( \tilde{Q} \), and a null point of \( Q \) is just given by a condition that the shape operator of \( F \) with respect to \( \nu \) is lightlike;

(b) except the null points, \( Q \) coincides with \( \tilde{Q} \) up to a nonzero constant;

(c) \( Q \equiv 0 \) if and only if a lightlike normal vector field \( \nu \) of \( F \) is contained in a constant direction.

Referring to the discussions on Proposition 6.6, we obtain

**Proposition 6.12.** Let \( \iota : M \to \mathbb{R}^{2,1} \) be a conformal immersion of Willmore type. Then \( \tilde{Q} \equiv 0 \) if and only if \( \iota \) is given by a minimal surface and an inversion of \( \mathbb{R}^{2,1} \).

### 6.3. The conformal Gauss maps of spacelike surfaces and timelike surfaces in the compactification of \( \mathbb{R}^{2,1} \)

Let \( \pi : \mathbb{R}^5 \setminus \{0\} \to \mathbb{RP}^4 \) be the projection from \( \mathbb{R}^5 \setminus \{0\} \) onto the 4-dimensional real projective space \( \mathbb{RP}^4 \). Then we can consider \( \mathbb{L} := \pi(L) \) for \( L \subset \mathbb{R}^{3,2} \) given in §6.1 to be a 3-dimensional manifold diffeomorphic to \((S^2 \times S^1) / \{ \pm \text{Id} \}\). Let \( M \) (resp. \( \tilde{M} \)) be a Riemann (resp. Lorentz) surface and \( \iota : M \) (resp. \( \tilde{M} \)) into a spacelike (resp. timelike) immersion. Then \( i := \pi \circ \iota \) is an immersion of \( M \) (resp. \( \tilde{M} \)) into \( \mathbb{L} \). We call such an immersion as \( i \) a spacelike (resp. timelike) immersion of \( M \) (resp. \( \tilde{M} \)) into \( \mathbb{L} \) if \( i \) is spacelike (resp. timelike). We call \( \iota \) a lift of \( i \). Whether \( i \) is spacelike or timelike does not depend on the choice of a lift of \( \iota \).

Let \( \mathcal{P} \) be the set of hyperplanes in \( \mathbb{R}^5 \) and \( \mathcal{P}_0 \) a subset of \( \mathcal{P} \) such that for each \( P \in \mathcal{P}_0 \), \( N := P \cap L \) is isomorphic to \( \mathbb{R}^{2,1} \). Let \( \tilde{\iota} : M \to \tilde{\mathbb{L}} \) be a spacelike immersion. Let \( \mathcal{P}_0(\iota) \) be a subset of \( \mathcal{P}_0 \) such that for each \( P \in \mathcal{P}_0(\iota) \), the direction given by \( \iota \) at each point of \( M \) does not become parallel to \( P \). For \( P \in \mathcal{P}_0(\iota) \), let \( \iota : M \to N = P \cap L \) be a lift of \( \iota \).

Then the conformal Gauss map \( \gamma_{\iota} : M \to H^{3,1} \) of \( \iota \) does not depend on the choice of \( P \in \mathcal{P}_0(\iota) \) and it is determined by \( \iota \). Therefore we can define the conformal Gauss map \( \gamma_{\iota} \) of a spacelike immersion \( \iota : M \to \tilde{\mathbb{L}} \) by \( \gamma_{\tilde{\iota}} := \gamma_{\iota} \), where \( \iota : M \to N = P \cap L \) is a lift of \( \iota \) for \( P \in \mathcal{P}_0(\iota) \). Noticing Proposition 6.1 and Proposition 6.2, we see that whether a lift \( \iota : M \to N = P \cap L \) is Willmore does not depend on the choice of \( P \in \mathcal{P}_0(\iota) \) and it is determined by \( \iota \). Hence we say that a spacelike immersion \( \iota : M \to \tilde{\mathbb{L}} \) is Willmore if a lift \( \iota : M \to N = P \cap L \) is Willmore for \( P \in \mathcal{P}_0(\iota) \).

Similarly, for a timelike immersion \( \tilde{\iota} : \tilde{M} \to \tilde{\mathbb{L}} \), we can define the conformal Gauss map \( \gamma_{\tilde{\iota}} : \tilde{M} \to S^{2,2} \), and we say that \( \tilde{\iota} : \tilde{M} \to \tilde{\mathbb{L}} \) is of Willmore type if a lift \( \iota : \tilde{M} \to N = P \cap L \) is of Willmore type for \( P \in \mathcal{P}_0(\iota) \).

**Remark.** For the conformal Gauss map of a spacelike surface in \( \tilde{\mathbb{L}} \) and Willmore surfaces in \( \tilde{\mathbb{L}} \), we can refer to [2].
We call field of \( \hat{\imath} \) of \( \hat{\gamma} \) map is defined as a map to \( \hat{X} \) a function \( f \) not preserve their orientations. In the case that \( \hat{M} \) the closures of \( \hat{M}_+ \) and \( \hat{M} \setminus (M_+ \cup M_-) \) are not empty and that the interior of \( \hat{M} \setminus (M_+ \cup M_-) \) is empty. Let \( q_0 \) be a point of the intersection of the closures of \( M_+ \) and \( M_- \). Then the image of a neighborhood \( U \) of \( q_0 \) by \( \hat{\iota} \) is represented as the graph of a function \( f \) of two variables \( x_1, x_2 \). We set \( e := (f_{x_1}, f_{x_2}, 1) \). Then \( e \) is a normal vector field of \( \hat{\iota} \) on \( U_+ := U \cap M_+ \) and \( U_- := U \cap M_- \). On \( U_+ \cup U_- \), we set

\[
e_3 := \frac{1}{\sqrt{\langle e, e \rangle_{2,1}}} e.
\]

Then \( e_3 \) satisfies \( \langle e_3, e_3 \rangle_{2,1} = -1 \) on \( U_+ \) and \( \langle e_3, e_3 \rangle_{2,1} = 1 \) on \( U_- \). Let \( H \) be the mean curvature of \( \hat{\iota} \) with respect to \( e_3 \). Then the conformal Gauss map \( \gamma_3 \) is given by \(-e_3 + H e_3 \) or \( e_3 - H e_3 \), where \(-e_3 \) and \( -e_3 \) mean \(-e_3, e_3 \), respectively. Therefore, if the mean curvature of \( \hat{\iota} \) is bounded, then \( \gamma'_3 := \pi' \circ \gamma_3 \) is continuously extended to \( q_0 \) and therefore we obtain a continuous map \( \gamma'_3 : \hat{M} \to \hat{\mathcal{L}} \). We call \( \gamma'_3 \) the conformal Gauss map of \( \hat{\iota} : \hat{M} \to \mathbb{R}^{2,1} \).

**Remark.** The mean curvature vector field of a real-analytic, mixed type surface in \( \mathbb{R}^{2,1} \) with bounded mean curvature can be analytically extended across the set of type change, under a suitable condition (see [11]).

Let \( \hat{\iota} : \hat{M} \to \hat{\mathcal{L}} \) be an immersion. Let \( M_+, M_- \) be as above for \( \hat{\iota} \) and suppose that \( M_+, M_- \) and \( \hat{M} \setminus (M_+ \cup M_-) \) are not empty and that the interior of \( \hat{M} \setminus (M_+ \cup M_-) \) is empty. If the mean curvature of a lift \( \iota : \hat{M} \to N = \mathbb{P} \cap L \) of \( \hat{\iota} \) for a hyperplane \( P \in \mathbb{P}_0(\hat{\iota}) \) is bounded, then \( \gamma'_3 := \pi' \circ \gamma_3 \) is continuously extended to the intersection of the closures of \( M_+ \) and \( M_- \), and therefore we obtain a continuous map \( \gamma'_3 : \hat{M} \to \hat{\mathcal{L}} \). We call \( \gamma'_3 \) the conformal Gauss map of \( \hat{\iota} : \hat{M} \to \mathbb{R}^{2,1} \).

Now, let \( X \) and \( \hat{X} \) be as in \$2$ with (2.5), and let \( \hat{M} \) be the mixed type surface as above. Then

\[
\iota(M_+) = X(M \setminus \{g \in \hat{R}\}), \quad \iota(M_-) = \hat{X}(M \setminus \{g \in \hat{R}\}).
\]

Note here that the maps \( X|_{M \setminus \{g \in \hat{R}\}} \) and \( \hat{X}|_{\hat{M} \setminus \{g \in \hat{R}\}} \) are double covering maps which do not preserve their orientations. In the case that \( \hat{M} \) is nonorientable, its conformal Gauss map is defined as a map to \( \mathbb{L}'/\{ \pm \text{Id} \} \).
7. Examples

Let us observe typical examples of mixed type each of whose compactifications gives a global example of Willmore type surface in \( \hat{L} \) with singularities.

**Example 7.1.** Let \( X \) and \( \hat{X} \) be a pair of zero mean curvature surfaces whose Weierstrass data are given by the following:

\[
(g, f) = \left(-z^2, -\frac{1}{z^2}\right), \quad (\hat{g}, \hat{f}) = \left(-\hat{z}^2, -\frac{1}{\hat{z}^2}\right).
\]

Substituting the data above into (2.1) and (2.2), we have

\[
X_{\text{fld}}(x, t) = \left(\frac{x}{x^2 + t} + \frac{1}{3}(x^3 - 3xt), 2x, \frac{x}{x^2 + t} - \frac{1}{3}(x^3 - 3xt)\right),
\]

\[
X_{\text{fed}}(s, t) = \left(\frac{s}{s^2t + 1} + \frac{1}{3}(s^3t^3 - 3st^2), 2st, \frac{s}{s^2t + 1} - \frac{1}{3}(s^3t^3 - 3st^2)\right),
\]

where \( X_{\text{fld}} \) and \( X_{\text{fed}} \) are defined by (5.3) and (5.4) with \( \varphi = X, \hat{\varphi} = \hat{X} \) and \( K = 1 \) (cf. [10, Example 9.3]), and the joined surface is of mixed type. It has two ends at 0 and \( \infty \). The end 0 is a lightlike planar end, and the end \( \infty \) is an incomplete end of order \(-3\). Its compactification includes two lightlike line on which it is \( C^\infty \) as in Theorem 4.1 except for a point. By Theorem 4.3, the image of the zero-divisors of the end 0 (resp. \( \infty \)) is smooth (resp. a cuspidal edge) in \( \hat{L} \). The border of the lightlike line and the zero-divisors of the end 0 is an isolated singularity as in Theorem 4.4. On the other hand, the singularity located on the intersection of two lightlike lines is conelike. Indeed, by direct computation, we can see that this surface is a pair of entire graphs of the following functions which are joined by a conelike singularity:

\[
X_3 = \pm \sqrt{X_1^2 + X_2^2 - \frac{1}{3}X_1X_2^3 + \frac{1}{36}X_2^6 - \frac{1}{6}X_2^3}.
\]

There is an example with a conelike singularity of mixed type and only one end.

**Example 7.2.** Let \( X \) and \( \hat{X} \) be a pair of zero mean curvature surfaces whose Weierstrass data are given by the following:

\[
(g, f) = (z^2, 1), \quad (\hat{g}, \hat{f}) = (\hat{z}^2, 1).
\]

Substituting the data above into (2.1) and (2.2), we have

\[
X_{\text{fld}}(x, t) = \left(x - \frac{1}{5}(x^5 - 10x^3t + 5xt^2), \frac{1}{3}(x^3 - 3xt), x + \frac{1}{5}(x^5 - 10x^3t + 5xt^2)\right).
\]

It has only one end at \( \infty \). The end \( \infty \) is an incomplete end of order \(-5\). Its compactification includes a lightlike line on which it is \( C^\infty \) as in Theorem 4.1 except for a point. By Theorem 4.3, the image of the zero-divisors of the end \( \infty \) is not smooth in \( \hat{L} \). It has a conelike singularity on the intersection of the unique lightlike line and the null curve.
Figure 2 shows Examples 7.1 and 7.2. (a) (resp. (b)) shows a neighborhood of the conelike singularity of Example 7.1 (resp. 7.2). (c) shows a neighborhood of the singularity \((x^1, x^2, \tau)\) which is located at the border of the lightlike line and the zero-divisors of the end 0 of Example 7.1. (d) shows a neighborhood of the intersection of the zero-divisors of the ends 0 (regular points) and \(\infty\) (cuspidal edges) of Example 7.1. In each figure, spacelike parts are colored in light gray, and timelike parts are colored in gray or dark gray.

In general, we can regard that any intersection of the zero-divisors of two ends as above and the corresponding zero-divisors compose another end, since partial fraction...
decompositions of paracomplex rational functions are not unique.

Remark. In each of the examples above, the domain of $X$ and $X$ as a joined map into $L$ is the union of the compactifications of $C$ and $C$, where we identify $R \subset C$ and $R \subset C$. The quotient space of the domain by complex and paracomplex conjugations is a projective plane. Note here that, if $(K + 1)/2$ is odd (e.g. $K = 1, 5$, and so on), then the reparametrization in (5.4) causes a blowing up, where $-K$ is the order of the end. Hence the compactification of the corresponding mixed type surface is homeomorphic to a Klein bottle with a node.

A. Paracomplex functions

A.1. Paracomplex and holomorphic functions

For $\hat{z}_1 = (x_1, y_1), \hat{z}_2 = (x_2, y_2)$, we set $\hat{z}_1 \cdot \hat{z}_2 := (x_1x_2 + y_1y_2, x_1y_2 + y_1x_2)$. Then we have an operation and we call this operation the paracomplex product. We call $\hat{z}_1 \cdot \hat{z}_2$ the paracomplex product of $\hat{z}_1$ and $\hat{z}_2$. We consider $R^2$ equipped with the paracomplex product to be a commutative algebra. In this paper, this algebra is denoted by $C$. Each element of $C$ is called a paracomplex number. We denote $(1, 0) \in C$ by $j$ and therefore we consider $R$ to be a subset of $C$. We denote $(0, 1) \in C$ by $j$. Then each $\hat{z} = (x, y) \in C$ is also denoted by $x + jy$ and we have $j^2 = j \cdot j = 1$. For $\hat{z} = x + jy$, $x, y$ are denoted by $\text{Re} \hat{z}, \text{Im} \hat{z}$ and called the real and the imaginary parts of $\hat{z}$, respectively. The conjugate number $\overline{\hat{z}}$ of $\hat{z}$ is given by $\overline{\hat{z}} = x - jy$. We set $|\hat{z}|^2 := \hat{z} \cdot \overline{\hat{z}} = x^2 - y^2$. We say that $\hat{z}$ is null or a zero-divisor if $\hat{z}$ is not zero and if $|\hat{z}|^2 = 0$.

Let $f$ be a smooth function of two variables $x, y$ defined on an open set $D$ of $C$ and valued in $C$. Then there exist real-valued functions $a, b$ on $D$ satisfying $f = u + jv$. Such a function as $f$ is called a paracomplex-valued function. We say that $f$ is holomorphic with respect to a paracomplex variable $\hat{z} = x + jy$ if $u, v$ satisfy $u_x = v_y$ and $u_y = v_x$ on $D$. For a paracomplex-valued function $\hat{f} = u + jv$, we set

$$
\hat{f}_{\hat{z}} := \frac{1}{2}(u_x + v_y) + \frac{j}{2}(v_x + u_y),
$$

$$
\hat{f}_{\overline{\hat{z}}} := \frac{1}{2}(u_x - v_y) + \frac{j}{2}(v_x - u_y).
$$

Then $\hat{f}$ is holomorphic with respect to $\hat{z}$ if and only if $\partial \hat{f}/\partial \overline{\hat{z}} = 0$. If $\hat{f}$ is holomorphic with respect to $\hat{z}$, then $f(x, y)$ is also denoted by $f(\hat{z})$.

Suppose that $\hat{f}$ is holomorphic with respect to $\hat{z}$. Then

$$
\hat{f}_{\hat{z}} = u_x + jv_x = v_y + ju_y.
$$

In addition, we obtain $u_{xx} = u_{yy}$ and $v_{xx} = v_{yy}$. Therefore there exist real-valued functions $u_1, u_2, v_1, v_2$ of one real variable satisfying

$$
u(x, y) = \frac{1}{2}(v_1(x + y) + v_2(x - y)), \quad v(x, y) = \frac{1}{2}(v_1(x + y) + v_2(x - y)).$$

In addition, using $u_x = v_y$ and $u_y = v_x$, we see that $u_1 - v_1$ and $u_2 + v_2$ are constant.
Therefore we obtain
\[ \hat{f}(\hat{z}) = u_1(x+y)e_1 + u_2(x-y)e_2 + \hat{c} \]
for a paracomplex number \( \hat{c} \), where we set \( e_1 := (1+j)/2 \) and \( e_2 := (1-j)/2 \). Conversely, such a paracomplex-valued function is holomorphic with respect to \( \hat{z} \): if \( u_1, u_2 \) are real-valued functions of one real variable and if \( \hat{f} \) is a paracomplex-valued function defined by (A.1), then \( \hat{f} \) satisfies \( \partial\hat{f}/\partial\bar{z} = 0 \) and therefore \( \hat{f} \) is holomorphic with respect to \( \hat{z} \).

Let \( \hat{f} = u + jv \) be a holomorphic function with respect to \( \hat{z} = x+jy \). We say that \( \hat{f} \) is nondegenerate if \( |\partial\hat{f}/\partial\bar{z}|^2 \) does not vanish; we say that \( \hat{f} \) is orientation-preserving if \( |\partial\hat{f}/\partial\bar{z}|^2 > 0 \). We represent \( u \) as \( u(x,y) = (u_1(x+y) + u_2(x-y))/2 \). Then \( \hat{f} \) is nondegenerate (respectively, orientation-preserving) if and only if \( u_1(x+y)u_2'(x-y) \neq 0 \) (respectively, > 0), where \( u_k' \) is the derivative of \( u_k \) for \( k \in \{1, 2\} \). If \( \hat{f} = u + jv \) is a null holomorphic function, then \( u_2 \equiv 0 \) and \( |\partial\hat{f}/\partial\bar{z}|^2 \equiv 0 \).

### A.2. Power series expansions

Since our main interest is in the case of type changing by an analytic extension, we assume that each of paracomplex functions \( \hat{f}(\hat{z}), \hat{g}(\hat{z}) \) etc. is holomorphic or meromorphic with respect to \( \hat{z} \), also called paraholomorphic or parameromorphic, and that it can be expressed as a power series, for instance,

\[ \hat{f}(\hat{z}) = \sum_{k=-\infty}^{+\infty} \hat{c}_k(\hat{z} - \hat{z}_0)^k \]

near any regular point or any pole \( \hat{z} = \hat{z}_0 \), where \( K = 0 \) if \( \hat{f} = u + jv \) is paraholomorphic.

As we have already mentioned in §A.1, \( u + v \) (resp. \( u - v \) ) depends only on \( \xi_1 = x + y \) (resp. \( \xi_2 = x - y \)), and we can set
\[
\begin{align*}
  u_1(\xi_1) & := u(\xi_1 e_1 + \xi_2 e_2) + v(\xi_1 e_1 + \xi_2 e_2), \\
  u_2(\xi_2) & := u(\xi_1 e_1 + \xi_2 e_2) - v(\xi_1 e_1 + \xi_2 e_2).
\end{align*}
\]

Set \( a_k := \text{Re} \hat{c}_k, b_k := \text{Im} \hat{c}_k \) for any \( k \), and set \( \xi_{10} := x_0 + y_0, \xi_{20} := x_0 - y_0 \) for any \( \hat{z}_0 = x_0 + jy_0 \). Then we have
\[
\begin{align*}
  u_1(\xi_1)e_1 + u_2(\xi_2)e_2 & = u(\xi_1 e_1 + \xi_2 e_2) + jv(\xi_1 e_1 + \xi_2 e_2) \\
  & = \hat{f}(\hat{z}) = \sum_{k=-\infty}^{+\infty} \hat{c}_k(\hat{z} - \hat{z}_0)^k \\
  & = \sum_{k=-\infty}^{+\infty} \text{Re}\{\hat{c}_k(\hat{z} - \hat{z}_0)^k\} + j \sum_{k=-\infty}^{+\infty} \text{Im}\{\hat{c}_k(\hat{z} - \hat{z}_0)^k\} \\
  & = \sum_{k=-\infty}^{+\infty} [\text{Re}\{\hat{c}_k(\hat{z} - \hat{z}_0)^k\} + \text{Im}\{\hat{c}_k(\hat{z} - \hat{z}_0)^k\}]e_1 \\
  & \quad + \sum_{k=-\infty}^{+\infty} [\text{Re}\{\hat{c}_k(\hat{z} - \hat{z}_0)^k\} - \text{Im}\{\hat{c}_k(\hat{z} - \hat{z}_0)^k\}]e_2
\end{align*}
\]
By our assumption, both the radii \( r_1 \) and \( r_2 \) of convergence of \( \sum_{k=-K}^{+\infty} (a_k + b_k)(\xi_1 - \xi_{10})^k e_1 + \sum_{k=-K}^{+\infty} (a_k - b_k)(\xi_2 - \xi_{20})^k e_2 \) are positive (or infinity), and hence the domain of convergence of the expansion of \( \hat{f}(z) \) around \( \hat{z} = \hat{z}_0 \) is

\[
\begin{align*}
\left| \xi_1 - \xi_{10} \right| &< r_1, \quad \left| \xi_2 - \xi_{20} \right| < r_2 \quad (\hat{z}_0 \text{ : a regular point}), \\
0 &< \left| \xi_1 - \xi_{10} \right| < r_1, 0 < \left| \xi_2 - \xi_{20} \right| < r_2 \quad (\hat{z}_0 \text{ : a pole}), \\
0 &< \left| \xi_1 - \xi_{10} \right| < r_1, \quad 0 < \left| \xi_2 - \xi_{20} \right| < r_2 \quad (\hat{z}_0 \text{ : “a pole” } \in \mathbb{R}e_1), \\
0 &< \left| \xi_1 - \xi_{10} \right| < r_1, \left| \xi_2 - \xi_{20} \right| < r_2 \quad (\hat{z}_0 \text{ : “a pole” } \in \mathbb{R}e_2).
\end{align*}
\]

In particular in the case that \( \hat{z}_0 = 0 \) and that \( \hat{f}(\hat{z}) = \overline{\hat{f}(\hat{z})} \) holds, \( \hat{c}_k \in \mathbb{R} \) (i.e. \( b_k = 0 \)) holds for any \( k \), and hence \( r_1 = r_2 \).

**B. Lightlike lines without type-changing**

Let \( \varphi^n(z) \) be a holomorphic function defined on an open subset \( \Omega \) of \( \mathbb{C} \) for \( n = 1, 2, \ldots, N \). Set

\[
\Phi : = (\varphi^1, \varphi^2, \ldots, \varphi^N), \tag{B.1}
\]

and denote the projection of \( \Phi \) to \( \mathbb{R}^N \) by \( F \), namely

\[
F : \mathbb{C} \ni \Omega \to \mathbb{R}^N, \quad z \mapsto \text{Re } \Phi(z). \tag{B.2}
\]

**Theorem B.1.** Let \( F \) (resp. \( \Phi \)) be a map defined by (B.2) (resp. (B.1)). Assume that \( \Omega = \Omega_{\infty} \setminus \{ q \} \) for some domain \( \Omega_{\infty} \subset \mathbb{C} \) and \( q \in \Omega_{\infty} \), and that \( \Phi \) has a pole of order \( K \) at \( q \) and \( \Phi \) can be written as

\[
\Phi(z) = \sum_{k=1, \text{K,odd}}^{K_0} \frac{1}{(z-q)^k} C_k + i \sum_{k=2, \text{K,even}}^{K_e} \frac{1}{(z-q)^k} C_k + \Phi_{\text{hol}}(z),
\]

where

\[
(K_0, K_e) := \begin{cases} (K, K-1) & (K : \text{odd}), \\
(K-1, K) & (K : \text{even}), \end{cases}
\]

\( C_k \in \mathbb{R}^N \) (1 \( \leq k \leq K \)), \( C_K \neq (0, \ldots, 0) \) and \( \Phi_{\text{hol}} \) is a holomorphic map from \( \Omega_{\infty} \) to \( \mathbb{C}^N \). Then \( F \) is an analytic extension of itself across a subset of a line parallel to \( C_K \). The image of the extension is nondegenerate on an open subset of the line if \( C_K \) and either \( \text{Im}(\Phi_{\text{hol}})_x(q) \) or \( C_{K-1} \) (\( K \geq 2 \)) are linear independent.

(Proof.) We may assume \( q = 0 \) without loss of generality. In this case, \( F \) is of the
following form:

\[
F(x + iy) = \sum_{k=1; k, \text{odd}}^{K_2} \left\{ \sum_{\tau=0; \tau, \text{even}}^{k-1} \frac{(-1)^{\tau/2} k^{k-\tau} y^{(K-1)k-K\tau}}{(x^2 + y^2)^k} \right\} C_k \\
+ \sum_{k=2; k, \text{even}}^{K_2} \left\{ \sum_{\tau=1; \tau, \text{odd}}^{k-1} \frac{(-1)^{(\tau-1)/2} k^{k-\tau} y^{(K-1)k-K\tau}}{(x^2 + y^2)^k} \right\} C_k + F_{\text{hol}}(x + iy),
\]

where \( F_{\text{hol}} := \text{Re} \, \Phi_{\text{hol}} \). Now, for any \( s \in \mathbb{R} \) such that \( sy^{K+1} + iy \in \Omega_\infty \) holds for \( y \in \mathbb{R} \) near to 0, \( F \) satisfies

\[
F(sy^{K+1} + iy) \\
= \sum_{k=1; k, \text{odd}}^{K_2} \left\{ \sum_{\tau=0; \tau, \text{even}}^{k-1} \frac{(-1)^{\tau/2} k^{k-\tau} y^{(K-1)k-K\tau}}{(s^2y^{2K} + 1)^k} \right\} C_k \\
+ \sum_{k=2; k, \text{even}}^{K_2} \left\{ \sum_{\tau=1; \tau, \text{odd}}^{k-1} \frac{(-1)^{(\tau-1)/2} k^{k-\tau} y^{(K-1)k-K\tau}}{(s^2y^{2K} + 1)^k} \right\} C_k + F_{\text{hol}}(sy^{K+1} + iy) \\
= F_{\text{hol}}(0) - \text{Im} \, (\Phi_{\text{hol}})_z(0)y + O(y^2) \\
+ \begin{cases} 
(1)^{(K-1)/2} KsC_K + (1)^{(K-3)/2}(K-1)sC_{K-1} & (K \geq 1; K : \text{odd}), \\
(1)^{(K-2)/2} KsC_K + (1)^{(K-2)/2}(K-1)sC_{K-1} & (K \geq 2; K : \text{even}),
\end{cases}
\]

and hence \( F(sy^{K+1} + iy) \) is an analytic extension of itself in the sense that

\[ F_{\text{end}}(s, y) := \begin{cases} 
F(sy^{K+1} + iy) & (t \neq 0, sy^{K+1} + iy \in \Omega_\infty), \\
(1)^{(K-1)/2} KsC_K & (t = 0, K \geq 1; K : \text{odd}), \\
(1)^{(K-2)/2} KsC_K & (t = 0, K \geq 2; K : \text{even})
\end{cases} \]

is real analytic with respect to \((s, y)\). The image of \( F_{\text{end}} \) is nondegenerate at least on the following subset of the borderline:

\[
\{(1)^{(K-1)/2} KsC_K + F_{\text{hol}}(0) \mid s \in \mathbb{R}, \\
-\text{Im} \, (\Phi_{\text{hol}})_z(0) + (1)^{(K-3)/2}(K-1)sC_{K-1} \text{and} C_K \text{ are linear independent.}\}
\]

\((K \geq 1; K : \text{odd}),
\]

\[
\{(1)^{(K-2)/2} KsC_K + F_{\text{hol}}(0) \mid s \in \mathbb{R}, \\
-\text{Im} \, (\Phi_{\text{hol}})_z(0) + (1)^{(K-2)/2}(K-1)sC_{K-1} \text{and} C_K \text{ are linear independent.}\}
\]

\((K \geq 2; K : \text{even})\).

Example B.2. Let us observe the self-extension of a spacelike maximal surface in \( \mathbb{R}^{2,1} \) which has a simple end of zero flux on a singular set (cf. [12], [13]). This is a typical case of Theorem B.1. We may assume that both the end and the stereographic image of
the limit normal at the end are 0 without loss of generality. Then the Weierstrass data of the surface around the end is of the following form:

\[ g = z^{m+2}g_{\text{hol}}, \quad f = \frac{\alpha}{z^2} + f_{\text{hol}}, \]

where \( g_{\text{hol}} \) and \( f_{\text{hol}} \) are holomorphic functions on a domain \( \Omega_\infty \) including 0 satisfying \( m \in \mathbb{N} \cup \{0\}, \alpha \in \mathbb{C} \setminus \{0\} \) and \( g_{\text{hol}}(0) \in \mathbb{C} \setminus \{0\} \). We may assume that \( \alpha \in \mathbb{R} \setminus \{0\} \) also by applying a suitable coordinate changing. Substituting the data above into (2.1), we have

\[
\Phi(z) = \int z \left\{ \left( \frac{\alpha}{z}, 0, \frac{\alpha}{z^2} \right) + (f_{\text{hol}}, 2\alpha z^m g_{\text{hol}}, f_{\text{hol}}) \right. \\
+ z^{m+2} (-z^m g_{\text{hol}}(\alpha + z^2 f_{\text{hol}}), 2g_{\text{hol}} f_{\text{hol}}, z^m g_{\text{hol}}^2(\alpha + z^2 f_{\text{hol}})) \right\} dz
\]

\[
= \left( -\frac{\alpha}{z}, 0, -\frac{\alpha}{z} \right) + \Phi_{\text{hol}}(z),
\]

where \( \Phi_{\text{hol}} \) is a holomorphic map. Hence

\[
X(x + iy) = \left( -\frac{\alpha x}{x^2 + y^2}, 0, -\frac{\alpha x}{x^2 + y^2} \right) + X_{\text{hol}}(x + iy),
\]

where \( X_{\text{hol}} := \text{Re} \, \Phi_{\text{hol}}. \)

For any \( s \in \mathbb{R} \), \( X \) satisfies

\[
X(sy^2 + iy) = \left( -\frac{\alpha s}{s^2y^2 + 1}, 0, -\frac{\alpha s}{s^2y^2 + 1} \right) + X_{\text{hol}}(sy^2 + iy)
\]

\[
(x \neq 0, sy^2 + iy \in \Omega_\infty).
\]

Note here that

\[
X(sy^2 + iy) = (-\alpha s, 0, -\alpha s) + X_{\text{hol}}(0) + yC(s) + O(y^2)
\]

holds for \( C(s) = (C^1(s), C^2(s), C^3(s)) \in \mathbb{R}^3 \), where

\[
C^1(s) = C^3(s) = -\text{Im} \, f_{\text{hol}}(0),
\]

\[
C^2(s) = \begin{cases} \\
-2\alpha \text{Im} \, g_{\text{hol}}(0) & (m = 0), \\
0 & (m \geq 1).
\end{cases}
\]

Now, if \( m = 0 \) and \( g_{\text{hol}}(0) \notin \mathbb{R} \), then the image of \( X(x + iy)|_{y<0} \) can be regarded as an analytic extension of that of \( X(x + iy)|_{y>0} \) across the subset of a lightlike line \{\{-\alpha s, 0, -\alpha s\} + X_{\text{hol}}(0) | s \in \mathbb{R} \} \) in the sense that \( X_{\text{end}}(s, y) \) defined by (B.3) with \( F = X \) is analytic with respect to \( (s, y) \) and nondegenerate on the subset above.

The simplest example is given by \( m = 0 \), \( g_{\text{hol}} = -\beta = -i(\beta_1 + i\beta_2) \notin \mathbb{R} \), \( f_{\text{hol}} = 0 \) and \( \alpha = -1 \). In this case, we have

\[
X_{\text{end}}(s, y) = \left( \frac{s}{s^2y^2 + 1} + \frac{1}{3}\{(\beta_1^2 - \beta_2^2)(s^3y^6 - 3sy^4) - 2\beta_1\beta_2(3s^2y^5 - y^3)\}, \\
2(\beta_1sy^2 - \beta_2y),
\right)
\]
where we set $X_{\text{hol}}(0) := (0, 0, 0)$.

We note here that the existence of examples of spacelike maximal surfaces including a nondegenerate lightlike line was first pointed out in [1].

We can show the quite similar result as Theorem B.1 also for real analytic timelike minimal surfaces by the similar way. Indeed, let $\check{\phi}^n(\check{z})$ be a paraholomorphic function defined on an open subset $\check{\Omega}$ of $\mathbb{C}$ for $n = 1, 2, \ldots, N$. Set

\begin{equation}
\check{\Phi} := (\check{\phi}^1, \check{\phi}^2, \ldots, \check{\phi}^N),
\end{equation}

and denote the projection of $\check{\Phi}$ to $\mathbb{R}^N$ by $\check{F}$, namely

\begin{equation}
\check{F} : \mathbb{C} \supset \check{\Omega} \to \mathbb{R}^N \quad \check{z} \mapsto \text{Re} \check{\Phi}(\check{z}).
\end{equation}

**Theorem B.3.** Let $\check{F}$ (resp. $\check{\Phi}$) be a map defined by (B.5) (resp. (B.4)). Assume that $\check{\Omega} = \check{\Omega}_\infty \setminus \{\check{q}\}$ for some domain $\check{\Omega}_\infty \subset \mathbb{C}$ and $\check{q} \in \check{\Omega}_\infty$, and that $\check{\Phi}$ has a pole of order $K$ at $\check{q}$ and $\check{\Phi}$ can be written as

\begin{equation}
\check{\Phi}(\check{z}) = \sum_{k=1, k: \text{odd}}^{K_o} \frac{1}{(\check{z} - \check{q})^k} C_k + \sum_{k=2, k: \text{even}}^{K_e} \frac{1}{(\check{z} - \check{q})^k} C_k + \check{\Phi}_{\text{hol}}(\check{z}),
\end{equation}

where

\begin{equation}
(K_o, K_e) := \begin{cases} (K, K - 1) & (K: \text{odd}), \\ (K - 1, K) & (K: \text{even}), \end{cases}
\end{equation}

$C_k \in \mathbb{R}^N (1 \leq k \leq K)$, $C_K \neq t(0, \ldots, 0)$ and $\check{\Phi}_{\text{hol}}$ is a paraholomorphic map from $\check{\Omega}_\infty$ to $\mathbb{C}^N$. Then $\check{F}$ is an analytic extension of itself across a subset of a line parallel to $C_K$. The image of the extension is nondegenerate on an open subset of the line if $C_K$ and either $\text{Im} (\check{\Phi}_{\text{hol}}) \check{z}(\check{q})$ or $C_{K-1}$ ($K \geq 2$) are linear independent.

**References**


Naoya ANDO
Faculty of Advanced Science and Technology,
Kumamoto University,
2-39-1 Kurokami, Kumamoto 860-8555, Japan
E-mail: andonaoya@kumamoto-u.ac.jp

Kohei HAMADA
Osaka Ibaraki High School,
12-1 Shinjocho, Ibaraki, Osaka 567-8523, Japan
E-mail: k.hamada3221@gmail.com

Kaname HASHIMOTO
Advanced Mathematical Institute,
Osaka City University,
3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
E-mail: h-kaname@sci.osaka-cu.ac.jp

Shin KATO
Department of Mathematics,
Osaka City University,
3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
E-mail: shinkato@sci.osaka-cu.ac.jp