COMPLETE 3-DIMENSIONAL $\lambda$-TRANSLATORS IN THE MINKOWSKI SPACE $\mathbb{R}^4_1$

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ABSTRACT. In this paper, we obtain the classification theorem for three-dimensional complete space-like $\lambda$-translators $x: M^3 \to \mathbb{R}^4_1$ with constant norm of the second fundamental form and constant $f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li}$ with $h_{ij}$ being the components of the second fundamental form in the Minkowski space $\mathbb{R}^4_1$.

1. INTRODUCTION

Let $x: M^n \to \mathbb{R}^{n+1}_1$ be an immersed space-like hypersurface in the Minkowski space $\mathbb{R}^{n+1}_1$. Fix a constant vector $T \neq 0$ in $\mathbb{R}^{n+1}_1$ and $\lambda$ a real number. In this paper we study orientable hypersurface $M^n$ in $\mathbb{R}^{n+1}_1$ whose mean curvature vector $\vec{H}$ satisfies

(1.1) \[ \vec{H} + T^\perp = \lambda \vec{n}, \]

where $\vec{H} = H \vec{n}$ and $\vec{n}$ is the unit normal vector. Then $x$ is called a $\lambda$-translating soliton or simply a $\lambda$-translator of the mean curvature flow (MCF). The constant vector $T$ will be called the corresponding translating vector or density vector. In particular, if we denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}^{n+1}_1$, then the equation (1.1) is equivalent to

(1.2) \[ H - \langle T, \vec{n} \rangle = \lambda, \quad \langle \vec{n}, \vec{n} \rangle = -1, \]

where $x$ is space-like.

The interest of these equations is due to its relation with manifolds with density. So it naturally makes sense to study the $\lambda$-translator in $\mathbb{R}^{n+1}_1$. Indeed, as described in [18], considering $\mathbb{R}^{n+1}_1$ with the conformal metric $\tilde{g} = e^{-\frac{2\phi}{n}} \langle \cdot, \cdot \rangle$ for a positive density function $e^{-\frac{2\phi}{n}}, \phi \in C^\infty(\mathbb{R}^{n+1}_1)$, which defines a weighted volume element $d\tilde{V}_\phi$ on $\mathbb{R}^{n+1}_1$ and thus induces a weighted volume $V_\phi(x)$ on $M^n$ for each given immersion $x$. The first variation of the weighted volume $V_\phi$ under compactly supported variations $x_t$ of $x$ is

(1.3) \[ \frac{d}{dt} \Big|_{t=0} V_\phi(t) = -\int_{M^n} H_\phi \langle \vec{n}, \xi \rangle dV_\phi, \]

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where $\xi$ is the variation vector and $H_\phi = H + \langle \nabla_{\xi} \phi, n \rangle = H + \frac{d\phi}{dn}$. So that $X$ is a critical point of the functional $V_\phi(t)$ for a given weighted volume if and only if $H_\phi$ is a constant function $H_\phi \equiv \lambda$: see ([7], [19]). In particular, if we take $\phi : \mathbb{R}^{n+1}_1 \to \mathbb{R}$ to be the height function $\phi(x) := \langle x, T \rangle$, then the expression $H_\phi \equiv \lambda$ is exactly the equation (1.2). Secondly, a special case of (1.1) is when $\lambda = 0$. In such a case the immersion $x$ is called a translating soliton of the mean curvature flow, or simply a translator ([26]). Translators play an important role in the study of mean curvature flow. On the one hand, a translating soliton is a solution of the mean curvature flow that evolves purely by translations along the direction $T$. On the other hand, they arise as blow-up solutions of MCF at type II singularities([11], [13]). For instance, Huisken and Sinestrari ([12]) proved that at a type II singularity of a mean convex flow, there exists a blow-up solution which is a convex translating solution. Besides, in the nonparametric form, the equation $H_\phi = 0$ appeared in the classical article of Serrin ([24]) and it was studied in the context of the maximum principle of elliptic equations. As we know, translating soliton have been widely studied and various interesting results have been obtained in recent years. For more information about translating soliton, please refer to the literatures ([1], [6], [9], [10], [14], [20], [21], [22], [23], [25], [27], [28]).

In [17], López classified all $\lambda$-translators in $\mathbb{R}^3$ that are invariant by a one-parameter group of translations and a one-parameter group of rotations. He also studied in [18] the shape of a compact $\lambda$-translator of $\mathbb{R}^3$ in terms of the geometry of its boundary, obtaining some necessary conditions for the existence of two-dimensional compact $\lambda$-translators with a given closed boundary curve. In particular, he proved that there do not exist any closed $\lambda$-translators of dimension two. In fact, just as that $\lambda$-translating solitons are generalizations of the translators of mean curvature flow, $\lambda$-hypersurfaces defined by Cheng and Wei in [4] are generalization of the self-shrinkers of mean curvature flow. Self-shrinking solutions are important in the study of type-I singularities of MCF. For instance, by proving the monotonicity formula, at a given type-I singularity of the MCF, Huisken [8] proved that the flow is asymptotically self-similar, which implies that in this situation the flow can be modeled by self-shrinking solutions. As is known, there have been many rigidity theorems and classification theorems for self-shrinkers in the Euclidean s-space and the pseudo-Euclidean space. Furthermore, there have been, up to now, several interesting and important results in the study of $\lambda$-hypersurfaces. In particular, Cheng and Wei recently obtained a classification theorem using their own generalized maximum principle ([5]) specially for $\lambda$-hypersurfaces, which generalizes an interesting classification theorem in [2] for self-shrinkers.

The classification theorem also exists for $\lambda$-translators. For example, canonical examples of $\lambda$-translators in $\mathbb{R}^{n+1}_1$ are the space-like affine hyperplanes, and the right hyperbolic cylinders $\mathbb{H}^k(r) \times \mathbb{R}^{n-k}$ with $1 \leq k \leq n-1$, where $\mathbb{H}^k(r)$ is the hyperbolic $k$-space defined by

$$\mathbb{H}^k(r) = \{ x \in \mathbb{R}^{k+1}_1; \langle x, x \rangle = -r^2 \}.$$  

Recently, Li, Qiao and Liu [15] have classified complete $\lambda$-translators in the Euclidean space $\mathbb{R}^3$ and the Minkowski space $\mathbb{R}^3_1$ with second fundamental form of constant length $S$. For the higher dimension $n$, it is not easy to classify $\lambda$-translator
in \( \mathbb{R}^n \) and \( \mathbb{R}^n_1 \) with constant squared norm \( S \) of the second fundamental form. In this paper, under the assumption that \( f_4 \) is constant, we give a complete classification for 3-dimensional complete \( \lambda \)-translators in \( \mathbb{R}^4_1 \) with constant squared norm \( S \) of the second fundamental form. In fact, we prove the following result.

**Theorem 1.1.** Let \( x : M^3 \to \mathbb{R}^4_1 \) be a 3-dimensional complete space-like \( \lambda \)-translator in \( \mathbb{R}^4_1 \). If the squared norm \( S \) of the second fundamental form and \( f_4 \) are constant, then \( x : M^3 \to \mathbb{R}^4_1 \) is isometric to one of

1. \( \mathbb{R}^3_1 \),
2. \( \mathbb{H}^1(\frac{1}{\lambda}) \times \mathbb{R}^2 \),
3. \( \mathbb{H}^2(\frac{2}{\lambda}) \times \mathbb{R}^1 \).

In particular, \( S \) must be 0, \( \frac{1}{2} \lambda^2 \); \( f_4 \) must be 0, \( \lambda^4 \) and \( \frac{1}{8} \lambda^4 \), where \( \lambda \neq 0 \), \( S = \sum_{i,j} h_{ij}^2 \) and \( f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li} \) with \( h_{ij} \) being the components of the second fundamental form.

**Remark 1.1.** We also obtain a similar classification for 3-dimensional complete \( \lambda \)-translators in \( \mathbb{R}^4 \) (see [16]). That is, for a 3-dimensional complete \( \lambda \)-translator in the Euclidean space \( \mathbb{R}^4 \), if the squared norm \( S \) of the second fundamental form and \( f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li} \) are constant, then hypersurface is isometric to one of \( \mathbb{R}^3; \mathbb{S}^1(\frac{1}{\lambda}) \times \mathbb{R}^2; \mathbb{S}^2(\frac{2}{\lambda}) \times \mathbb{R}^1 \).

## 2. Preliminaries

Let \( x : M^n \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional space-like hypersurface of the \((n + 1)\)-dimensional Minkowski space \( \mathbb{R}^{n+1}_1 \). Around each point of \( M^n \), we choose a local orthonormal frame field \( \{e_A\}_{A=1}^{n+1} \) in \( \mathbb{R}^{n+1}_1 \) with dual coframe field \( \{\omega_A\}_{A=1}^{n+1} \), such that, restricted to \( M^n \), \( \{e_1, \cdots, e_n\} \) are tangent to \( M^n \).

From now on, we use the following conventions on the ranges of indices:

\[
1 \leq i, j, k, l \leq n
\]

and \( \sum_i \) means taking summation from 1 to \( n \) for \( i \). Then we have

\[
\begin{align*}
\text{d}x &= \sum_i \omega_i e_i, \\
\text{d}e_i &= \sum_j \omega_{ij} e_j + \omega_{in+1} e_{n+1}, \\
\text{d}e_{n+1} &= \omega_{n+1 i} e_i, \quad \omega_{n+1 i} = \omega_{in+1},
\end{align*}
\]

where \( \omega_{ij} = -\omega_{ji} \) is the Levi-Civita connection of the hypersurface.

By restricting these forms to \( M^n \), we get

\[
\omega_{n+1} = 0.
\]

Taking exterior derivatives of (2.1), we obtain

\[
0 = \text{d}\omega_{n+1} = \sum_i \omega_{n+1 i} \wedge \omega_i.
\]
By Cartan’s lemma, we know that there exist local smooth functions $h_{ij}$, $1 \leq i, j \leq n$, such that

\begin{equation}
\omega_{in+1} = \sum_{j} h_{ij} \omega_j, \quad h_{ij} = h_{ji},
\end{equation}

\begin{equation}
h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j, \quad H = \sum_i h_{ii}
\end{equation}

are called the second fundamental form and the mean curvature of $x: M \to \mathbb{R}^{n+1}_1$, respectively. Let $S = \sum_{i,j} (h_{ij})^2$ be the squared norm of the second fundamental form of $x: M \to \mathbb{R}^{n+1}_1$. The induced structure equations of $M^n$ are given by

\begin{equation}
d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji},
\end{equation}

\begin{equation}
d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,
\end{equation}

where $R_{ijkl}$ denote components of the curvature tensor of the hypersurface. Hence, the Gauss equations of the space-like hypersurface $x$ in $\mathbb{R}^{n+1}_1$ are as follows:

\begin{equation}
R_{ijkl} = -(h_{ik} h_{jl} - h_{il} h_{jk}).
\end{equation}

Defining the covariant derivative of $h_{ij}$ by

\begin{equation}
\sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{kj} \omega_{ki} + \sum_k h_{ik} \omega_{kj},
\end{equation}

we obtain the Codazzi equations

\begin{equation}
h_{ijk} = h_{ikj}.
\end{equation}

By taking exterior differentiation of (2.4), and defining

\begin{equation}
\sum_l h_{ijkl} \omega_l = dh_{ijk} + \sum_l h_{lj} \omega_{li} + \sum_l h_{il} \omega_{lj} + \sum_l h_{ij} \omega_{lk},
\end{equation}

we have the following Ricci identities:

\begin{equation}
h_{ijkl} - h_{ijlk} = \sum_m h_{mij} R_{mikl} + \sum_m h_{im} R_{mjkl}.
\end{equation}

Defining

\begin{equation}
\sum_m h_{ijklm} \omega_m = dh_{ijkl} + \sum_m h_{mjk} \omega_{mil} + \sum_m h_{imk} \omega_{mlj} + \sum_m h_{ijm} \omega_{mkl}
\end{equation}

\begin{equation}
+ \sum_m h_{ijkm} \omega_{ml}
\end{equation}

and taking exterior differentiation of (2.6), we get

\begin{equation}
h_{ijklm} - h_{ijkln} = \sum_m h_{mjk} R_{mln} + \sum_m h_{imk} R_{mjln} + \sum_m h_{ijm} R_{mkln}.
\end{equation}
For a smooth function $f$, we define
\begin{equation}
\sum_i f_{,i} \omega_i = df,
\end{equation}
\begin{equation}
\sum_j f_{,ij} \omega_j = df_{,i} + \sum_j f_{,j} \omega_{ji},
\end{equation}
\begin{equation}
|\nabla f|^2 = \sum_i (f_{,i})^2, \quad \Delta f = \sum_i f_{,ii}.
\end{equation}

Let $V$ be a tangent $C^1$-vector field on $M^n$, and denote by $\text{Ric}_V := \text{Ric} - \frac{1}{2} L_V g$ the Bakry-Emery Ricci tensor with $L_V$ to be the Lie derivative along the vector field $V$. Define a differential operator
\[ \Delta_V f = \Delta f + \langle V, \nabla f \rangle, \]
where $\Delta$ and $\nabla$ denote the Laplacian and the gradient operator, respectively. Then we have the following maximum principle of Omori-Yau type which was proved by Chen-Qiu [3] and Li-Qiao-Liu[15]:

**Lemma 2.1.** Let $(M^n, g)$ be a complete Riemannian manifold, and $V$ is a $C^1$ vector field on $M^n$. If the Bakry-Emery Ricci tensor $\text{Ric}_V$ is bounded from below, then for any $f \in C^2(M^n)$ bounded from above, there exists a sequence $\{p_m\} \subset M^n$, such that
\[ \lim_{m \to \infty} f(p_m) = \sup f, \quad \lim_{m \to \infty} |\nabla f|(p_m) = 0, \quad \lim_{m \to \infty} \Delta_V f(p_m) \leq 0. \]

Suppose that the given hypersurface $x : M \to \mathbb{R}_1^{n+1}$ is a $\lambda$-translator with a translating vector $T$, and let $\{e_i\}$ be an orthonormal tangent frame on $M^n$. Then from the definition (1.2) of $\lambda$-translator in $\mathbb{R}_1^{n+1}$, we have the following basic formulas for covariant derivatives:
\begin{equation}
\nabla_i H = \sum_k h_{ik} \langle T, e_k \rangle,
\end{equation}
\begin{equation}
\nabla_j \nabla_i H = \sum_k h_{ijk} \langle T, e_k \rangle + (H - \lambda) \sum_k h_{ik} h_{kj}.
\end{equation}

Moreover, we define three functions $f_3$, $f_4$ and $f_5$ as follows:
\[ f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}, \quad f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li}, \quad f_5 = \sum_{i,j,k,l,m} h_{ij} h_{jk} h_{kl} h_{lm} h_{mi}. \]
If we denote $V = T^T$, the tangent component of the translating vector $T$ when restricted to $M^n$, then direct computations using above formulas and the Ricci identities easily give the following Lemma (cf. [4] and [15]):

**Lemma 2.2.** Let $x : M^n \to \mathbb{R}_1^{n+1}$ be an $n$-dimensional complete $\lambda$-translator in $\mathbb{R}_1^{n+1}$, we have
\begin{equation}
\Delta_{-V} H = S(H - \lambda).
\end{equation}
\[\frac{1}{2} \Delta_V H^2 = |\nabla H|^2 + S(H - \lambda) H.\] (2.15)

\[\frac{1}{2} \Delta_V S = \sum_{i,j,k} h_{ijk}^2 + S^2 - \lambda f_3.\] (2.16)

\[\frac{1}{4} \Delta_V f_4 = 2 \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{klh} + \sum_{i,j,k,l,m} h_{ijm} h_{jkh} h_{kl} h_{hi} + S f_4 - \lambda f_5.\] (2.17)

**Lemma 2.3.** Let \( x : M^n \rightarrow \mathbb{R}^{n+1}_+ \) be an \( n \)-dimensional complete \( \lambda \)-translator in \( \mathbb{R}^{n+1} \). If \( S \) is constant, we have

\[\frac{1}{2} \Delta_V \sum_{i,j,k} (h_{ijk})^2 = \sum_{i,j,k,l} (h_{ijkl})^2 + S \sum_{i,j,k} (h_{ijk})^2 - 6 \sum_{i,j,k,l,m} h_{ijk} h_{ih} h_{jph} h_{kp} + 3 \sum_{i,j,k,l} h_{ijk} h_{ijh} h_{kl} - 3 \sum_{i,j,k,l} h_{ijk} h_{ijh} h_{kl}.\] (2.18)

For \( n = 3 \), if diagonalized \( (h_{ij}) \) at some point, it is easy to get the following formula.

\[\frac{1}{2} \Delta_V \sum_{i,j,k} (h_{ijk})^2 = -3 \lambda H |\nabla H|^2 + \frac{3}{4} \lambda S (S - H^2) (H - \lambda) - 3 \lambda \sum_k (h_{11} h_{23k}^2 + h_{22} h_{13k}^2 + h_{33} h_{12k}^2) + \frac{9}{2} \lambda S h_{11} h_{22} h_{33} - \frac{3}{2} \lambda^2 \sum_k (h_{22} h_{33} h_{1k}^2 + h_{11} h_{33} h_{2k}^2 + h_{11} h_{22} h_{3k}^2) + 3 \lambda \sum_k (h_{11} h_{22} h_{33k} + h_{22} h_{11k} h_{33} + h_{33} h_{11k} h_{22k}).\] (2.19)

**Proof.** By making use of the Ricci identities (2.7), (2.9) and a direct calculation, we can obtain (2.18).

For \( n = 3 \), we have

\[f_3 = \frac{H}{2} (3S - H^2) - 3h_{11} h_{23}^2 - 3h_{22} h_{13}^2 - 3h_{33} h_{12}^2 + 3h_{11} h_{22} h_{33} + 6h_{12} h_{13} h_{23}.\]

From (2.16) in Lemma 2.2, we have

\[\sum_{i,j,k} h_{ijk}^2 = -(S^2 - \lambda f_3).\]
Then, by making use of the Ricci identities (2.7), we obtain

\[
- \frac{1}{2} \Delta_{-V}(S^2 - \lambda f_3) = \frac{1}{2} \lambda \Delta_{-V} f_3 \\
= -\frac{3}{2} \lambda H|\nabla H|^2 + \frac{3}{4} \lambda S(S - H^2)(H - \lambda) - \frac{9}{2} \lambda S(h_{11}h_{23}^2 + h_{22}h_{13}^2 + h_{33}h_{12}^2) \\
+ \frac{3}{2} \lambda^2 \sum_k (h_{23}^2h_{1k}^2 + h_{13}h_{2k}^2 + h_{12}h_{3k}^2 + 2h_{11}h_{23}h_{2k}h_{3k} + 2h_{22}h_{13}h_{1k}h_{3k} \\
+ 2h_{33}h_{12}h_{1k}h_{2k}) = \lambda \sum_k (h_{11}h_{23}^2 + h_{22}h_{13}^2 + h_{33}h_{12}^2 + 2h_{23}h_{23}h_{11k} + 2h_{13}h_{13k}h_{22k} + 2h_{12}h_{12k}h_{33k} + \frac{9}{2} \lambda Sh_{11}h_{22}h_{33} - \frac{3}{2} \lambda^2 \sum_k (h_{22}h_{33}^2h_{1k}^2) \\
+ h_{11}h_{33}^2h_{2k}^2 + h_{11}h_{22}h_{33}^2k) + 3\lambda \sum_k (h_{11}h_{22k}h_{33k} + h_{22}h_{11k}h_{33k} + h_{33}h_{11k}h_{22k}) + 9\lambda Sh_{12}h_{13}h_{23} - \frac{3}{2} \lambda^2 \sum_k (h_{12}h_{13}h_{23}h_{2k}h_{3k} + h_{12}h_{23}h_{1k}h_{3k} + h_{13}h_{23}h_{1k}h_{2k}) \\
+ 6\lambda \sum_k (h_{12}h_{13}h_{23} + h_{13}h_{12k}h_{23k} + h_{23}h_{12k}h_{13k}).
\]

Finally, it is easy to get the conclusion by diagonalizing \((h_{ij})\). \(\square\)

**Lemma 2.4.** Let \(x : M^3 \to \mathbb{R}^4\) be a 3-dimensional complete \(\lambda\)-translator in \(\mathbb{R}^4\). Then we can choose a local field of orthonormal frames on \(M^3\) such that, at the point, \(h_{ij} = \lambda_i \delta_{ij}\),

\[
f_3 = \frac{H}{2}(3S - H^2) + 3\lambda_1 \lambda_2 \lambda_3, \\
f_4 = \frac{4}{3} Hf_3 - H^2S + \frac{1}{6}H^4 + \frac{1}{2}S^2, \\
f_5 = \frac{5}{6}H^2f_3 + \frac{5}{6}Sf_3 - \frac{5}{6}H^3S + \frac{1}{6}H^5,
\]

and

\[
\nabla_m f_4 = 4 \sum_{i,j,k,l} h_{ijm}h_{jk}h_{kl}h_{li}, \quad \text{for } m = 1, 2, 3,
\]

\[
\nabla_p \nabla_m f_4 = 4 \sum_{i,j,k,l} h_{ijmp}h_{jk}h_{kl}h_{li} \\
+ 4 \sum_{i,j,k,l} h_{ijm}(2h_{jkp}h_{klh_{li}} + h_{jk}h_{klp}h_{li}), \quad \text{for } m, p = 1, 2, 3.
\]

\begin{align*}
\nabla_k f_4 &= \frac{4}{3} f_3 H_{,k} + \frac{4}{3} H \nabla_k f_3 - 2SHH_{,k} + \frac{2}{3} H^3H_{,k}, \\
\nabla_l \nabla_k f_4 &= \frac{4}{3} f_3 H_{,kl} - 2SHH_{,kl} + \frac{2}{3} H^3H_{,kl} + \frac{4}{3} H \nabla_l \nabla_k f_3 + \frac{4}{3} \nabla_l f_3 H_{,k} \\
&+ \frac{4}{3} H_{,l} \nabla_k f_3 - 2SH_{,k} H_{,l} + 2H^2H_{,k} H_{,l}, \quad \text{for } k, l = 1, 2, 3.
\end{align*}
Proof. For 3-dimensional hypersurfaces, according to the definition of $f_3$, $f_4$ and $f_5$, the details are as follows:

$$f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}, \quad f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li}, \quad f_5 = \sum_{i,j,k,l,m} h_{ij} h_{jk} h_{kl} h_{lm} h_{mi},$$

we have that

$$f_3 = \frac{H}{2} (3S - H^2) - 3h_{11} h_{22}^2 - 3h_{22} h_{13}^2 - 3h_{33} h_{12}^2 + 3h_{11} h_{22} h_{33} + 6h_{12} h_{13} h_{23},$$

$$f_4 = \frac{4}{3} H f_3 - H^2 S + \frac{1}{6} H^4 + \frac{1}{2} S^2, \quad f_5 = \frac{5}{6} H^2 f_3 + \frac{5}{6} S f_3 - \frac{5}{6} H^3 S + \frac{1}{6} H^5.$$

If diagonalized $(h_{ij})$ at some point, it is easy to get that

$$h_{ij} = \lambda_i \delta_{ij}, \quad f_3 = \frac{H}{2} (3S - H^2) + 3\lambda_1 \lambda_2 \lambda_3.$$

Taking the first and second order covariant derivatives for $f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li}$, and

$$f_4 = \frac{4}{3} H f_3 - H^2 S + \frac{1}{6} H^4 + \frac{1}{2} S^2,$$

it’s easy to draw the conclusion. $\square$

To make use of the maximum principle of Omori-Yau type, we prove the following lemma.

**Lemma 2.5.** For a space-like complete $\lambda$-translator $x : M^n \to \mathbb{R}^{n+1}$ with the translating vector $T$ and non-zero constant squared norm $S$ of the second fundamental form, the Bakry-Emery Ricci tensor $\text{Ricc}_- V$ is bounded from below, where $V = T^T$.

**Proof.** Let $e$ be an arbitrary unit eigenvector of the symmetric two-tensor $\text{Ricc}_- V$. Choose an orthonormal tangent frame field $\{e_i\}_{i=1}^n$ such that $e_1 = e$. Then, by the definition of $\lambda$-translator, we have

$$-\frac{1}{2} L_- g(e, e) = \frac{1}{2} V(g(e_1, e_1)) - g([V, e_1], e_1)$$

$$= \frac{1}{2} \{g(\nabla_V e_1, e_1) + g(e_1, \nabla_V e_1)\} - g(\nabla_V e_1 - \nabla_{e_1} V, e_1)$$

$$= g(\nabla_{e_1} (T - T^\perp), e_1)$$

$$= - g(\nabla_{e_1} T^\perp, e_1)$$

$$= (H - \lambda) g(\nabla_{e_1} n, e_1),$$

and

$$g(\nabla_{e_1} n, e_1) = g(dn(e_1), e_1)$$

$$= g(\omega_{n+1}^i (e_1) e_i, e_1)$$

$$= g(\omega_{n+1}^i (e_1) e_i, e_1)$$

$$= h_{11}.$$
Therefore,
\[-\frac{1}{2}L_{-\nu}g(e,e) = (H - \lambda)h_{11},\]
\[\text{Ricc}_{-\nu}(e,e) = \text{Ricc}(e,e) - \frac{1}{2}L_{-\nu}g(e,e)\]
\[= - (Hh_{11} - \sum h_{1k}^2) + (H - \lambda)h_{11} = \sum h_{1k}^2 - \lambda h_{11} \geq \sum h_{1k}^2 - \frac{1}{2}h_{11}^2 - \frac{1}{2}\lambda^2 \geq -\frac{1}{2}\lambda^2.\]

The proof of Lemma 2.5 is finished. \(\square\)

3. Proof of the main result

If \(S = 0\), we know that \(x : M^3 \to \mathbb{R}_4^4\) is \(\mathbb{R}^3\). We next consider \(S > 0\). Equations (2.13), (2.16) and Lemma 2.4 are sufficient to prove that \(\inf H^2 > 0\). If \(\inf H^2 > 0\), we can apply the generalized maximum principle for the operator \(\Delta_{-\nu}\) to the function \(H^2\) (or \(-H^2\)), discuss the relationship of the values of the three principal curvature and get the conclusion of Theorem 3.2 (or Theorem 3.3). Then, the conclusion of Theorem 1.1 can be obtained. We now prove the following theorems.

**Theorem 3.1.** For a 3-dimensional complete \(\lambda\)-translator \(x : M^3 \to \mathbb{R}_4^4\) with non-zero constant squared norm \(S\) of the second fundamental form and constant \(f_4\), then \(\inf H^2 > 0\), where \(S = \sum_{i,j} h_{ij}^2\) and \(f_4 = \sum_{i,j,k,l} h_{ij}h_{jk}h_{kl}h_{li}\).

**Proof.** If \(\inf H^2 = 0\), there exists a sequence \(\{p_t\} \subset M^3\) such that \(\lim_{t \to \infty} H^2(p_t) = \inf H^2 = \bar{H}^2 = 0\).

From (2.16) and \(S\) being constant, we know that \(\{h_{ij}(p_t)\}\) and \(\{h_{ijk}(p_t)\}\) are bounded sequences for \(i, j, k = 1, 2, 3\).

By \(S = \text{constant}\) and (2.16) in Lemma 2.2, we have
\[0 = 2\sum_{i,j,k} h_{ijk}h_{ijkl} - \lambda \nabla_{l} f_3 = 2\sum_{i,j,k} h_{ijk}h_{ijkl} - 3\lambda \sum_{i,j,k} h_{ijl}h_{jkl}h_{ki}, \quad l = 1, 2, 3.\]

Again, according to \(\{h_{ij}(p_t)\}\) and \(\{h_{ijk}(p_t)\}\) are bounded sequences, we know that \(\{h_{ijkl}(p_t)\}\) is also a bounded sequence for \(i, j, k, l = 1, 2, 3\). One can assume
\[\lim_{t \to \infty} h_{ij}(p_t) = \bar{h}_{ij} = \lambda_i \delta_{ij}, \quad \lim_{t \to \infty} h_{ijk}(p_t) = \bar{h}_{ijk}, \quad \lim_{t \to \infty} h_{ijkl}(p_t) = \bar{h}_{ijkl}, \quad i, j, k, l = 1, 2, 3.\]

Then,
\[\bar{H} = \sum_i \bar{h}_{ii} = \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 0, \quad S = \sum_{i,j} \bar{h}_{ij}^2 = \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_3^2 = 2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_1 \bar{\lambda}_2).\]

From
\[H_{,i} = \sum_k h_{ik}(T, e_k), \quad i = 1, 2, 3,\]
we have
\[ h_{11i} + h_{22i} + h_{33i} = \lambda_i \lim_{t \to \infty} \langle T, e_i(p_t) \rangle, \quad i = 1, 2, 3. \]

Since
\[ \nabla_j \nabla_i H = \sum_k h_{ijk} \langle T, e_k(p_t) \rangle + (H - \lambda) \sum_k h_{ik} h_{kj}, \]
we conclude
\[ \bar{H}_{ij} = \sum_k \bar{h}_{ijk} \lim_{t \to \infty} \langle T, e_k(p_t) \rangle + (\bar{H} - \lambda) \bar{\lambda}_i \bar{\lambda}_j \delta_{ij}, \]
that is,
\[
\begin{cases}
\sum_k \bar{h}_{kkii} = \sum_k h_{iik} \lim_{t \to \infty} \langle T, e_k(p_t) \rangle - \lambda \bar{\lambda}_i^2, & i = 1, 2, 3, \\
\sum_k \bar{h}_{kkij} = \sum_k h_{ijk} \lim_{t \to \infty} \langle T, e_k(p_t) \rangle, & i \neq j, \quad i, j = 1, 2, 3.
\end{cases}
\]

Since \( S \) is constant, we know
\[ \sum_{i,j} h_{ij} h_{ijk} = 0, \quad k = 1, 2, 3. \]

Thus,
\[ \sum_{i,j} \bar{h}_{ij} \bar{h}_{ijk} = 0, \quad k = 1, 2, 3. \]

Specifically,
\[ \bar{\lambda}_1 \bar{h}_{11k} + \bar{\lambda}_2 \bar{h}_{22k} + \bar{\lambda}_3 \bar{h}_{33k} = 0, \quad k = 1, 2, 3. \]

For principal curvature, there are only three cases we consider.

1. The principal curvature \( \bar{\lambda}_1, \bar{\lambda}_2 \) and \( \bar{\lambda}_3 \) are all equal.
   From \( \bar{H} = \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 0, \bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3 = 0 \), we get \( S = 0 \). It is impossible since \( S > 0 \).

2. Two of the values of the principal curvature \( \bar{\lambda}_1, \bar{\lambda}_2 \) and \( \bar{\lambda}_3 \) are equal.
   Without loss of generality, we assume that \( \bar{\lambda}_1 = \bar{\lambda}_2 \neq \bar{\lambda}_3 \).
   From \( \bar{H} = \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 0 \), we infer that \( \bar{\lambda}_1 = \bar{\lambda}_2 \neq 0 \) and \( \bar{\lambda}_3 \neq 0 \).
   From the first equation of Lemma 2.4 and (2.20), we obtain
   \[ \lim_{t \to \infty} f_3(p_t) \neq 0, \quad \bar{H}_k = 0, \quad k = 1, 2, 3. \]
   By (3.1) and \( \bar{H}_k = 0 \) for \( k = 1, 2, 3 \), we have
   \[ \lim_{t \to \infty} \langle T, e_k(p_t) \rangle = 0, \quad k = 1, 2, 3. \]
   From (3.3), we have that
   \[ \bar{H}_{kk} = \bar{h}_{11kk} + \bar{h}_{22kk} + \bar{h}_{33kk} = -\lambda \bar{\lambda}_k^2, \quad k = 1, 2, 3. \]
   From (2.21) in Lemma 2.4 and \( \bar{H}_k = 0 \) for \( k = 1, 2, 3 \), we have
   \[ \bar{H}_{kl} = 0, \quad k, l = 1, 2, 3. \]
Then, it follows from (3.5) that \( \bar{H}_{kk} = -\lambda\bar{\lambda}_k^2 = 0 \) for \( k = 1, 2, 3 \). It is a contradiction.

3. The values of the principal curvature \( \bar{\lambda}_1, \bar{\lambda}_2 \) and \( \bar{\lambda}_3 \) are not equal to each other.

Case 1: \( \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 = 0 \).
Without loss of generality, we assume that \( \bar{\lambda}_3 = 0 \). That is, \( \bar{\lambda}_1 \neq 0, \bar{\lambda}_2 \neq 0 \) and \( \bar{\lambda}_1 \neq \bar{\lambda}_2 \).
From \( \bar{H} = 0, \bar{\lambda}_3 = 0 \) and \( f_3 = \frac{H}{2}(3S - H^2) + 3\lambda_1\lambda_2\lambda_3 \), we have
\[
\lim_{t \to \infty} f_3(p_t) = 0.
\]
From (2.16) in Lemma 2.2, we have
\[
\sum_{i,j,k} \bar{h}_{ijk}^2 + S^2 - \lambda f_3 = 0.
\]
Since \( \lim_{t \to \infty} f_3(p_t) = 0 \), we have
\[
\sum_{i,j,k} \bar{h}_{ijk}^2 + S^2 = 0,
\]
and then,
\[
S = 0.
\]
It is a contradiction.

Case 2: \( \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 \neq 0 \).
From \( S > 0, \bar{H} = 0 \) and \( f_3 = \frac{H}{2}(3S - H^2) + 3\lambda_1\lambda_2\lambda_3 \), we have
\[
\lim_{t \to \infty} f_3(p_t) \neq 0.
\]
Since \( f_4 = \frac{4}{3}H f_3 - H^2 S + \frac{1}{6}H^4 + \frac{1}{2}S^2 \) and \( \lim_{t \to \infty} f_3(p_t) \neq 0 \), we get
\[
0 = \nabla_k f_4 = \frac{4}{3}f_3 H_k + \frac{4}{3}H \nabla_k f_3 - 2SH H_k + \frac{2}{3}H^3 H_k,
\]
\[
0 = \nabla_l \nabla_k f_4 = \frac{4}{3}f_3 H_{kl} - 2SH H_{kl} + \frac{2}{3}H^3 H_{kl} + \frac{4}{3}H \nabla_l \nabla_k f_3 + \frac{4}{3} \nabla_l f_3 H_k
\]
\[
+ \frac{4}{3}H_l \nabla_k f_3 - 2SH_{,kl} H_{,l} + 2H^2 H_{,k} H_{,l}, \quad k, l = 1, 2, 3.
\]
Then, \( \bar{H}_{\cdot k} = 0 \) and \( \bar{H}_{\cdot kl} = 0 \) for \( k, l = 1, 2, 3 \).
Especially,
\[
(3.7) \quad \bar{H}_{, k} = \bar{\lambda}_k \lim_{t \to \infty} \langle T, e_k \rangle(p_t) = 0, \quad k = 1, 2, 3.
\]
and
\[
(3.8) \quad \bar{H}_{, ii} = \sum_k \bar{h}_{iik} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) - \lambda \bar{\lambda}_i^2 = 0, \quad i = 1, 2, 3.
\]
From (3.7) and \( \bar{\lambda}_k \neq 0 \) for \( k = 1, 2, 3 \), one has
\[
\lim_{t \to \infty} \langle T, e_k \rangle(p_t) = 0, \quad k = 1, 2, 3.
\]
By (3.8), we know that \( \bar{\lambda}_k = 0 \) for \( k = 1, 2, 3 \). It is a contradiction.
To sum up, under the assumption of $\inf H^2 = H^2 = 0$, the above three situations are impossible to exist. Therefore, we can see that the hypothesis doesn’t exist. That is to say, $\inf H^2 > 0$. \hfill \Box

**Theorem 3.2.** For a 3-dimensional complete $\lambda$-translator $x : M^3 \rightarrow \mathbb{R}^4_1$ with non-zero constant squared norm $S$ of the second fundamental form and constant $f_4$, where $S = \sum h^2_{ij}$ and $f_4 = \sum h_{ijkl} h_{ijkl}$, we have either

1. $\lambda^2 = S$ and $\sup H^2 = S$, or
2. $\lambda^2 = 2S$ and $\sup H^2 = 2S$, or
3. $\lambda^2 = 3S$ and $\sup H^2 = 3S$.

**Proof.** At each point $p \in M^3$, we choose $e_1$, $e_2$ and $e_3$ such that

$$h_{ij} = \lambda_i \delta_{ij}.$$  

From $2ab \leq \alpha a^2 + \frac{1}{\alpha} b^2$, we obtain

$$S = \lambda^2_1 + \lambda^2_2 + \lambda^2_3, \quad H^2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 \leq 3(\lambda^2_1 + \lambda^2_2 + \lambda^2_3) = 3S.$$  

Hence, we have on $M^3$

$$H^2 \leq 3S$$

and the equality holds if and only if $\lambda_1 = \lambda_2 = \lambda_3$.

From Lemma 2.5, we know that the Bakry-Emery Ricci tensor $Ric_V$ of $x : M^3 \rightarrow \mathbb{R}^4_1$ is bounded from below. We can apply the generalized maximum principle and Lemma 2.2 for the operator $\Delta V$ to the function $H^2$. Thus, there exists a sequence $\{p_t\}$ in $M^3$ such that

$$\lim_{t \rightarrow \infty} H^2(p_t) = \sup H^2, \quad \lim_{t \rightarrow \infty} |\nabla H^2(p_t)| = 0.$$  

For $S \neq 0$, from Theorem 3.1, we know that $\sup H^2 \geq \inf H^2 > 0$. Using the similar proof of Theorem 3.1, by (2.16) and $S = constant$, we know that $\{h_{ij}(p_t)\}$, $\{h_{ijk}(p_t)\}$ and $\{h_{ijkl}(p_t)\}$ are bounded sequences for $i, j, k, l = 1, 2, 3$. Without loss of the generality, we can assume

$$\lim_{t \rightarrow \infty} f_3(p_t) = \bar{f}_3, \quad \lim_{t \rightarrow \infty} f_5(p_t) = \bar{f}_5, \quad \lim_{t \rightarrow \infty} h_{ij}(p_t) = \bar{h}_{ij} = \lambda_i \delta_{ij}, \quad \lim_{t \rightarrow \infty} h_{ijk}(p_t) = \bar{h}_{ijk}, \quad \lim_{t \rightarrow \infty} h_{ijkl}(p_t) = \bar{h}_{ijkl}, \quad i, j, k = 1, 2, 3.$$  

From $\lim_{t \rightarrow \infty} |\nabla H^2(p_t)| = 0$ and $|\nabla H^2|^2 = 4 \sum k (HH_k)^2$, we have

$$\bar{H}_k = 0, \quad k = 1, 2, 3,$$

that is,

$$\bar{h}_{11k} + \bar{h}_{22k} + \bar{h}_{33k} = 0, \quad k = 1, 2, 3.$$  

Since $x$ is a $\lambda$-translator, from (2.13), we have

$$H_i = \sum_k h_{ik} \langle T, e_k \rangle, \quad i = 1, 2, 3,$$

$$\nabla_j \nabla_i H = \sum_k h_{ijk} \langle T, e_k \rangle + (H - \lambda) \sum_k h_{ikj}, \quad i, j = 1, 2, 3.$$
Thus,
$$
\bar{H}_i = \bar{h}_{1ii} + \bar{h}_{2ii} + \bar{h}_{33ii} = \lambda_i \lim_{t \to \infty} \langle T, e_i \rangle(p_t), \quad i = 1, 2, 3,
$$
and
$$
\bar{H}_{ij} = \sum_k \bar{h}_{ijk} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) + (\bar{H} - \lambda)\bar{\lambda}_i \delta_{ij}, \quad i, j = 1, 2, 3.
$$
Especially,
(3.11) \quad \bar{H}_i = \bar{\lambda}_i \lim_{t \to \infty} \langle T, e_i \rangle(p_t), \quad i = 1, 2, 3,
and
(3.12) \quad \left\{
\begin{align*}
\bar{h}_{11ii} + \bar{h}_{22ii} + \bar{h}_{33ii} &= \sum_k \bar{h}_{iik} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) + (\bar{H} - \lambda)\bar{\lambda}_i^2, \quad i = 1, 2, 3, \\
\bar{h}_{11ij} + \bar{h}_{22ij} + \bar{h}_{33ij} &= \sum_k \bar{h}_{ijk} \lim_{t \to \infty} \langle T, e_k \rangle(p_t), \quad i \neq j, \quad i, j = 1, 2, 3.
\end{align*}
\right.
$$

Since $S$ is constant, we know
$$
\sum_{i,j} h_{ij} h_{ijk} = 0, \quad k = 1, 2, 3,
$$
and
$$
\sum_{i,j} h_{ij} h_{ijkl} + \sum_{i,j} h_{ijk} h_{ijl} = 0, \quad k, l = 1, 2, 3.
$$
Thus,
$$
\sum_{i,j} \bar{h}_{ij} \bar{h}_{ijk} = 0, \quad k = 1, 2, 3,
$$
and
$$
\sum_{i,j} \bar{h}_{ij} \bar{h}_{ijkl} + \sum_{i,j} \bar{h}_{ijk} \bar{h}_{ijl} = 0, \quad k, l = 1, 2, 3.
$$
Specifically,
(3.13) \quad \bar{\lambda}_1 \bar{h}_{11k} + \bar{\lambda}_2 \bar{h}_{22k} + \bar{\lambda}_3 \bar{h}_{33k} = 0, \quad k = 1, 2, 3,
(3.14) \quad \sum_i \bar{\lambda}_i \bar{h}_{iikl} = -\sum_i \bar{h}_{ijk} \bar{h}_{ijl}, \quad k, l = 1, 2, 3.
$$
From Ricci identities (2.7), we obtain
$$
\bar{h}_{ijkl} - \bar{h}_{ijlk} = -\bar{\lambda}_i \bar{\lambda}_j \bar{\lambda}_k \delta_{il} \delta_{jk} - \bar{\lambda}_i \bar{\lambda}_j \bar{\lambda}_l \delta_{ik} \delta_{jl} + \bar{\lambda}_i \bar{\lambda}_j \bar{\lambda}_k \delta_{il} \delta_{jk} - \bar{\lambda}_i \bar{\lambda}_j \bar{\lambda}_l \delta_{ik} \delta_{jl},
$$
that is,
(3.15) \quad \left\{
\begin{align*}
\bar{h}_{1212} - \bar{h}_{1221} &= -\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 - \bar{\lambda}_2), \quad \bar{h}_{1313} - \bar{h}_{1331} = -\bar{\lambda}_1 \bar{\lambda}_3 (\bar{\lambda}_1 - \bar{\lambda}_3), \\
\bar{h}_{2323} - \bar{h}_{2332} &= -\bar{\lambda}_2 \bar{\lambda}_3 (\bar{\lambda}_2 - \bar{\lambda}_3), \quad \bar{h}_{iikt} - \bar{h}_{iilk} = 0, \quad i, k, l = 1, 2, 3.
\end{align*}
\right.
$$
Since $f_4$ is constant, we know from the Lemma 2.4,
$$
0 = \nabla_m f_4 = 4 \sum_{i,j,k,l} h_{ijm} h_{jk} h_{kl} h_{li},
$$
and
$$
0 = \nabla_p \nabla_m f_4 = 4 \sum_{i,j,k,l} h_{ijmp} h_{jk} h_{kl} h_{li} + 4 \sum_{i,j,k,l} h_{ijm} (2 h_{jkp} h_{kl} h_{li} + h_{jk} h_{klp} h_{li}),
$$
for $m, p = 1, 2, 3$. Thus,
\[(3.16) \bar{\lambda}^3_1 \bar{h}_{11k} + \bar{\lambda}^3_2 \bar{h}_{22k} + \bar{\lambda}^3_3 \bar{h}_{33k} = 0, \quad k = 1, 2, 3,\]
\[(3.17) \sum_i \bar{\lambda}^3_i \bar{h}_{iikl} = -\sum_{i,j} (2\bar{\lambda}^2_i + \bar{\lambda}^2_i \bar{\lambda}^2_j) \bar{h}_{ijk} \bar{h}_{ijl}, \quad k, l = 1, 2, 3.\]

For principal curvature, there are only three cases we consider.

1. **The principal curvature \(\bar{\lambda}_1, \bar{\lambda}_2\) and \(\bar{\lambda}_3\) are all equal.**

   From \(\bar{H} = \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 \neq 0\), \(\bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3 \neq 0\), we get

   \(\bar{H}^2 = 3S.\)

   From (3.9), (3.11) and \(\bar{\lambda}_k \neq 0\) for \(k = 1, 2, 3\), we have

   \[(3.18) \lim_{t \to \infty} \langle T, e_k \rangle(p_t) = 0, \quad k = 1, 2, 3.\]

   From (3.14), (3.17) and \(\bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3\), we have

   \[(3.19) \bar{\lambda}_1 \sum_i \bar{h}_{iikk} = -\sum_{i,j} \bar{h}^2_{ijk}, \quad k = 1, 2, 3,\]

   and

   \[(3.20) \bar{\lambda}_3 \sum_i \bar{h}_{iikk} = -3\bar{\lambda}_1 \sum_{i,j} \bar{h}^2_{ijk}, \quad k = 1, 2, 3.\]

   Then it's quite clear that (3.19) and (3.20) imply

   \[\sum_{i,j} \bar{h}^2_{ijk} = 0, \quad k = 1, 2, 3,\]

   and thus,

   \[(3.21) \bar{h}_{ijk} = 0, \quad i, j, k = 1, 2, 3.\]

   From (3.12) and (3.18), we know that

   \[\sum_i \bar{h}_{iikk} = \bar{h}_{11kk} + \bar{h}_{22kk} + \bar{h}_{33kk} = (\bar{H} - \lambda)\bar{\lambda}^2_k, \quad k = 1, 2, 3.\]

   From (3.19) and (3.21),

   \[\sum_i \bar{h}_{iikk} = 0, \quad k = 1, 2, 3.\]

   And then, by \(\bar{H}^2 = 3S\), we have

   \[\lambda = \bar{H}, \quad \lambda^2 = \bar{H}^2 = \sup H^2, \quad \lambda^2 = 3S.\]

2. **Two of the values of the principal curvature \(\bar{\lambda}_1, \bar{\lambda}_2\) and \(\bar{\lambda}_3\) are equal.**

   Without loss of generality, we assume that \(\bar{\lambda}_1 \neq \bar{\lambda}_2 = \bar{\lambda}_3\), and then,

   \(\bar{H} = \bar{\lambda}_1 + 2\bar{\lambda}_2 \neq 0.\)

   From (3.10) and (3.13), we get

   \[(3.22) \bar{h}_{11k} = 0, \quad \bar{h}_{22k} + \bar{h}_{33k} = 0, \quad k = 1, 2, 3.\]

   **Case 1:** \(\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 = 0.\)
Subcase 1.1: $\bar{\lambda}_1 \neq 0$, $\bar{\lambda}_2 = \bar{\lambda}_3 = 0$.
Since $\bar{\lambda}_1 \neq 0$ and $\bar{\lambda}_2 = \bar{\lambda}_3 = 0$, we have that $H^2 = S$ and $\bar{f}_3 = \bar{\lambda}_1 S$.
From (3.17) with $k = l = 1$ and (3.22), we have
$$\bar{h}_{1111} = 0,$$
and then, by (3.14) and $\bar{\lambda}_2 = \bar{\lambda}_3 = 0$, we know
$$0 = \bar{\lambda}_1 \bar{h}_{1111} + \bar{\lambda}_2 \bar{h}_{2211} + \bar{\lambda}_3 \bar{h}_{3311}$$
$$= -\bar{h}_{221}^2 - \bar{h}_{331}^2 - 2\bar{h}_{121}^2 - 2\bar{h}_{131}^2 - 2\bar{h}_{231}^2,$$
$$= -\bar{h}_{221}^2 - \bar{h}_{331}^2 - 2\bar{h}_{231}^2.$$
Thus,
$$\bar{h}_{221} = \bar{h}_{331} = \bar{h}_{231} = 0.$$  
From $\bar{h}_{221} = \bar{h}_{331} = \bar{h}_{231} = 0$, (3.17) with $k = l = 2$ and (3.22), we have
$$\bar{h}_{1122} = 0,$$
and then, by (3.14) and $\bar{\lambda}_2 = \bar{\lambda}_3 = 0$, we know
$$0 = \bar{\lambda}_1 \bar{h}_{1122} + \bar{\lambda}_2 \bar{h}_{2222} + \bar{\lambda}_3 \bar{h}_{3322}$$
$$= -\bar{h}_{112}^2 - \bar{h}_{222}^2 - \bar{h}_{332}^2 - 2\bar{h}_{122}^2 - 2\bar{h}_{132}^2 - 2\bar{h}_{232}^2,$$
$$= -\bar{h}_{222}^2 - \bar{h}_{332}^2 - 2\bar{h}_{232}^2.$$
Thus,
$$\bar{h}_{222} = \bar{h}_{332} = \bar{h}_{232} = \bar{h}_{333} = 0.$$  
That is,
$$\bar{h}_{ijk} = 0, \ i, j, k = 1, 2, 3.$$  
From (2.16) in Lemma 2.2, we have
$$0 = S^2 - \lambda \bar{f}_3,$$
then, we obtain
$$\lambda^2 = \bar{H}^2 = \sup H^2, \quad \lambda^2 = S.$$  
Subcase 1.2: $\bar{\lambda}_1 = 0$, $\bar{\lambda}_2 = \bar{\lambda}_3 \neq 0$.
Since $\bar{\lambda}_1 = 0$ and $\bar{\lambda}_2 = \bar{\lambda}_3 \neq 0$, we have that $H^2 = 2S$ and $\bar{f}_3 = \bar{\lambda}_2 S$.
From (3.9), (3.11) and $\bar{\lambda}_2 = \bar{\lambda}_3 \neq 0$, we have
$$\lim_{t \to \infty} \langle T, e_2 \rangle(p_t) = 0, \quad \lim_{t \to \infty} \langle T, e_3 \rangle(p_t) = 0.$$  
From (3.22), we have
$$\bar{h}_{111} = \bar{h}_{112} = \bar{h}_{113} = 0, \quad \bar{h}_{221} = -\bar{h}_{331}, \quad \bar{h}_{222} = -\bar{h}_{332}, \quad \bar{h}_{223} = -\bar{h}_{333}.$$  
By (3.14) and (3.17), we have that
$$\begin{cases}
\bar{\lambda}_2(\bar{h}_{2211} + \bar{h}_{3311}) = -2\bar{h}_{221}^2 - 2\bar{h}_{123}, \\
\bar{\lambda}_2(\bar{h}_{2211} + \bar{h}_{3311}) = -6\bar{\lambda}_2^2\bar{h}_{221}^2 - 6\bar{\lambda}_2^2\bar{h}_{123}, \\
\bar{\lambda}_2(\bar{h}_{2222} + \bar{h}_{3322}) = -2\bar{h}_{221}^2 - 2\bar{h}_{222}^2 - 2\bar{h}_{232}^2 - 2\bar{h}_{132}^2, \\
\bar{\lambda}_2^2(\bar{h}_{2222} + \bar{h}_{3322}) = -2\bar{\lambda}_2^2(\bar{h}_{221}^2 + 3\bar{h}_{222}^2 + 3\bar{h}_{232}^2 + \bar{h}_{132}^2),
\end{cases}$$
then,
\[ \tilde{h}_{221} = 0, \quad \tilde{h}_{123} = 0, \quad \tilde{h}_{222} = 0, \quad \tilde{h}_{223} = 0. \]
Therefore,
\[ \tilde{h}_{ijk} = 0, \quad i, j, k = 1, 2, 3. \]
From (2.16) in Lemma 2.2, we have
\[ 0 = S^2 - \lambda \tilde{f}_3, \]
then, we obtain
\[ \lambda^2 = \bar{H}^2 = \sup H^2, \quad \lambda^2 = 2S. \]

**Case 2:** \( \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 \neq 0. \)
According to Theorem 3.1, we have that
\[ \bar{H} = \bar{\lambda}_1 + 2\bar{\lambda}_2 \neq 0, \quad \bar{\lambda}_1 \neq \bar{\lambda}_2 = \bar{\lambda}_3, \quad \bar{\lambda}_k \neq 0, \quad k = 1, 2, 3. \]
From (3.9), (3.11) and \( \lambda_k \neq 0 \) for \( k = 1, 2, 3 \), we get
\[ \lim_{t \to \infty} (T, e_k)(p_t) = 0, \quad k = 1, 2, 3. \]
From (3.12), (3.14), (3.17), (3.22) and (3.23), we know that
\[
\begin{cases}
\tilde{h}_{1111} + \tilde{h}_{2211} + \tilde{h}_{3311} = (\bar{H} - \lambda)\bar{\lambda}_1^2, \\
\tilde{h}_{1122} + \tilde{h}_{2222} + \tilde{h}_{3322} = (\bar{H} - \lambda)\bar{\lambda}_2^2, \\
\tilde{h}_{1133} + \tilde{h}_{2233} + \tilde{h}_{3333} = (\bar{H} - \lambda)\bar{\lambda}_3^2, \\
\tilde{h}_{1112} + \tilde{h}_{2212} + \tilde{h}_{3312} = 0, \\
\tilde{h}_{1113} + \tilde{h}_{2213} + \tilde{h}_{3313} = 0,
\end{cases}
\]
(3.24)
\[
\begin{cases}
\bar{\lambda}_1 \tilde{h}_{1111} + \bar{\lambda}_2 (\tilde{h}_{2211} + \tilde{h}_{3311}) = -2(\tilde{h}_{221} + \tilde{h}_{123})^2, \\
\bar{\lambda}_1 \tilde{h}_{1122} + \bar{\lambda}_2 (\tilde{h}_{2222} + \tilde{h}_{3322}) = -2(\tilde{h}_{222} + \tilde{h}_{223})^2, \\
\bar{\lambda}_1 \tilde{h}_{1133} + \bar{\lambda}_2 (\tilde{h}_{2233} + \tilde{h}_{3333}) = -2(\tilde{h}_{223} + \tilde{h}_{123})^2, \\
\bar{\lambda}_1 \tilde{h}_{1112} + \bar{\lambda}_2 (\tilde{h}_{2212} + \tilde{h}_{3312}) = -2(\tilde{h}_{221} + \tilde{h}_{223} + \tilde{h}_{123}), \\
\bar{\lambda}_1 \tilde{h}_{1113} + \bar{\lambda}_2 (\tilde{h}_{2213} + \tilde{h}_{3313}) = -2(\tilde{h}_{221} + \tilde{h}_{223} - \tilde{h}_{123}),
\end{cases}
\]
(3.25)
and
\[
\begin{cases}
\bar{\lambda}_1^3 \tilde{h}_{1111} + \bar{\lambda}_2^3 (\tilde{h}_{2211} + \tilde{h}_{3311}) = -6\bar{\lambda}_2^2 (\tilde{h}_{221} + \tilde{h}_{123})^2, \\
\bar{\lambda}_1^3 \tilde{h}_{1122} + \bar{\lambda}_2^3 (\tilde{h}_{2222} + \tilde{h}_{3322}) = -6\bar{\lambda}_2^2 (\tilde{h}_{222} + \tilde{h}_{223})^2 - 2(\bar{\lambda}_1 + \bar{\lambda}_2)^2 \\
+ \bar{\lambda}_1 \bar{\lambda}_2 (\tilde{h}_{221} + \tilde{h}_{123})^2, \\
\bar{\lambda}_1^3 \tilde{h}_{1133} + \bar{\lambda}_2^3 (\tilde{h}_{2233} + \tilde{h}_{3333}) = -6\bar{\lambda}_2^2 (\tilde{h}_{223} + \tilde{h}_{123})^2 - 2(\bar{\lambda}_1 + \bar{\lambda}_2)^2 \\
+ \bar{\lambda}_1 \bar{\lambda}_2 (\tilde{h}_{221} + \tilde{h}_{123})^2, \\
\bar{\lambda}_1^3 \tilde{h}_{1112} + \bar{\lambda}_2^3 (\tilde{h}_{2212} + \tilde{h}_{3312}) = -6\bar{\lambda}_2^2 (\tilde{h}_{221} + \tilde{h}_{223} + \tilde{h}_{123}), \\
\bar{\lambda}_1^3 \tilde{h}_{1113} + \bar{\lambda}_2^3 (\tilde{h}_{2213} + \tilde{h}_{3313}) = -6\bar{\lambda}_2^2 (\tilde{h}_{221} - \tilde{h}_{223} + \tilde{h}_{123}).
\end{cases}
\]
(3.26)
From (3.24), we get
\[ \tilde{h}_{2212} + \tilde{h}_{3312} = -\tilde{h}_{1112}, \quad \tilde{h}_{2213} + \tilde{h}_{3313} = -\tilde{h}_{1113}, \]
and then from $\lambda_1 \neq \lambda_2$, (3.25) and (3.26), we get
\begin{align*}
(3.27) \quad & h_{221}h_{222} + h_{223}H_{123} = 0, \quad h_{221}h_{222} - h_{223}H_{123} = 0, \quad h_{1112} = 0, \quad h_{1113} = 0.
\end{align*}
Besides, by (3.25) and (3.26), we get
\begin{align*}
(3.28) \quad & \left\{ \begin{array}{l}
\lambda_1(\lambda_2^2 - \lambda_3^2)h_{1112} = 4\lambda_2^2(h_{221}^2 + h_{123}^2), \\
\lambda_1(\lambda_2^2 - \lambda_3^2)h_{1112} = 4\lambda_2^2(h_{222}^2 + h_{223}^2) + 2(\lambda_1^2 + \lambda_1\lambda_2)(h_{221}^2 + h_{123}^2), \\
\lambda_1(\lambda_2^2 - \lambda_3^2)h_{1113} = 4\lambda_2^2(h_{222}^2 + h_{223}^2) + 2(\lambda_1^2 + \lambda_1\lambda_2)(h_{221}^2 + h_{123}^2).
\end{array} \right.
\end{align*}
Now we consider four subcases.

**Subcase 2.1:** $h_{221}^2 + h_{123}^2 \neq 0, \quad h_{222}^2 + h_{223}^2 \neq 0$. From (3.27), it is a contradiction.

**Subcase 2.2:** $h_{221}^2 + h_{123}^2 = 0, \quad h_{222}^2 + h_{223}^2 = 0$. From (3.22), we know
\begin{align*}
(3.29) \quad & h_{ijk} = 0, \quad i, j, k = 1, 2, 3,
\end{align*}
and then, by (2.16) in Lemma 2.2, we have
\begin{align*}
(3.30) \quad & 0 = S^2 - \lambda \bar{f}_3.
\end{align*}
If $\lambda_1 + \lambda_2 = 0$, we have
\begin{align*}
(3.31) \quad & \bar{H} = -\lambda_1, \quad S = 3\lambda_1^2, \quad \bar{f}_3 = -\lambda_1^3.
\end{align*}
From (3.30) and (3.31), we know
\begin{align*}
(3.32) \quad & \lambda = -9\lambda_1 = 9\bar{H}.
\end{align*}
From (2.18), (2.19), (3.29), (3.31) and (3.32), we know
\begin{align*}
\frac{1}{2} \lim_{t \to \infty} \Delta_V \sum_{i,j,k}(h_{ijk})^2(p_t) &= \sum_{i,j,k,l}(\bar{h}_{ijkl})^2, \\
\frac{1}{2} \lim_{t \to \infty} \Delta_V \sum_{i,j,k}(h_{ijk})^2(p_t) &= \frac{3}{4} \lambda S(S - H^2)(H - \lambda) + \frac{9}{2} \lambda S h_{111}h_{222}h_{333} - \frac{3}{2} \lambda^2 \sum_k (h_{222}h_{333}h_{11k}^2 + h_{111}h_{333}h_{22k}^2 + h_{111}h_{222}h_{33k}^2), \\
&= \frac{3}{4} \lambda S(S - H^2)(H - \lambda) + \frac{9}{2} \lambda \lambda_1 \lambda_2 \lambda_3 S - \frac{3}{2} \lambda^2 \lambda_1 \lambda_2 \lambda_3 H \\
&= -324\lambda_1^6, \\
\end{align*}
and then,
\begin{align*}
\sum_{i,j,k,l}(\bar{h}_{ijkl})^2 &= -324\lambda_1^6 < 0.
\end{align*}
It is a contradiction.
If $\lambda_1 + \lambda_2 \neq 0$, from (3.25) and (3.26), we know that
\begin{align*}
\left\{ \begin{array}{l}
\bar{h}_{1111} = 0, \quad h_{2211} + h_{3311} = 0, \\
\bar{h}_{1122} = 0, \quad h_{2222} + h_{3322} = 0, \\
\bar{h}_{1133} = 0, \quad h_{2233} + h_{3333} = 0,
\end{array} \right.
\end{align*}
and then, by (3.24), we have

(3.33) \[ \lambda = \bar{H}. \]

From (3.30) and (3.33), we know

\[ \bar{\lambda}_1 = \bar{\lambda}_2, \]

where \( \bar{H} = \bar{\lambda}_1 + 2\bar{\lambda}_2, S = \bar{\lambda}_1^2 + 2\bar{\lambda}_2^2 \) and \( \bar{f}_3 = \bar{\lambda}_1^3 + 2\bar{\lambda}_2^3 \). It is a contradiction.

**Subcase 2.3:** \( \bar{h}_{221}^2 + \bar{h}_{123}^2 = 0, \quad \bar{h}_{222}^2 + \bar{h}_{223}^2 \neq 0 \).

From (3.28), we know

\[
\begin{cases}
\bar{\lambda}_1 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2) \bar{h}_{1111} = 0, \\
\bar{\lambda}_1 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2) \bar{h}_{1122} = 4\bar{\lambda}_2^2 (\bar{h}_{222}^2 + \bar{h}_{223}^2), \\
\bar{\lambda}_1 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2) \bar{h}_{1133} = 4\bar{\lambda}_2^2 (\bar{h}_{222}^2 + \bar{h}_{223}^2),
\end{cases}
\]

and then,

(3.34) \[ \bar{\lambda}_1 + \bar{\lambda}_2 \neq 0, \quad \bar{h}_{1111} = 0, \quad \bar{h}_{1122} = \frac{4\bar{\lambda}_2^2}{\bar{\lambda}_1 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2)} (\bar{h}_{222}^2 + \bar{h}_{223}^2). \]

From \( \bar{h}_{1111} = 0 \) and the first equation in (3.25), we know

\[ \bar{h}_{2211} + \bar{h}_{3311} = 0, \]

and then, by (3.24), we have

(3.35) \[ \bar{H}_{11} = 0, \quad \lambda = \bar{H}, \quad \bar{H}_{22} = 0. \]

From (3.25) and (3.35), we know

(3.36) \[ \bar{h}_{1122} = \frac{2}{\bar{\lambda}_2 - \bar{\lambda}_1} (\bar{h}_{222}^2 + \bar{h}_{223}^2), \]

From (3.34) and (3.36), we have

\[ \bar{\lambda}_1 = \bar{\lambda}_2. \]

It is a contradiction.

**Subcase 2.4:** \( \bar{h}_{221}^2 + \bar{h}_{123}^2 \neq 0, \quad \bar{h}_{222}^2 + \bar{h}_{223}^2 = 0 \).

From (3.28), we know

\[
\begin{cases}
\bar{\lambda}_1 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2) \bar{h}_{1111} = 4\bar{\lambda}_2^2 (\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2) \bar{h}_{1122} = 2(\bar{\lambda}_1^2 + \bar{\lambda}_1 \bar{\lambda}_2) (\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2) \bar{h}_{1133} = 2(\bar{\lambda}_1^2 + \bar{\lambda}_1 \bar{\lambda}_2) (\bar{h}_{221}^2 + \bar{h}_{123}^2),
\end{cases}
\]

and then,

(3.37) \[
\begin{cases}
\bar{\lambda}_1 + \bar{\lambda}_2 \neq 0, \quad \bar{h}_{1111} = \frac{4\bar{\lambda}_2^2}{\bar{\lambda}_1 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2)} (\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{h}_{1122} = \bar{h}_{1133} = \frac{2}{\bar{\lambda}_2 - \bar{\lambda}_1} (\bar{h}_{221}^2 + \bar{h}_{123}^2). \]
\]
From (3.24), (3.25) and (3.37), we know
\[-2(\tilde{h}_{221}^2 + \tilde{h}_{123}^2) = \tilde{\lambda}_1 \tilde{h}_{1122} + \tilde{\lambda}_2((\tilde{H} - \lambda)\tilde{\lambda}_2^2 - \tilde{h}_{1122})
\]
\[= (\tilde{H} - \lambda)\tilde{\lambda}_2^3 + (\tilde{\lambda}_1 - \tilde{\lambda}_2)\tilde{h}_{1122}
\]
\[= (\tilde{H} - \lambda)\tilde{\lambda}_2^3 + (\tilde{\lambda}_1 - \tilde{\lambda}_2) \cdot \frac{2}{\tilde{\lambda}_2 - \tilde{\lambda}_1}(\tilde{h}_{221}^2 + \tilde{h}_{123}^2)
\]
\[= (\tilde{H} - \lambda)\tilde{\lambda}_2^3 - 2(\tilde{h}_{221}^2 + \tilde{h}_{123}^2),
\]
and then, by (3.24), we have
\[(3.38) \quad \lambda = \tilde{H}, \quad \tilde{H}_{11} = 0.
\]
From (3.25) and (3.38), we have
\[(3.39) \quad \tilde{h}_{1111} = \frac{2}{\tilde{\lambda}_2 - \tilde{\lambda}_1}(\tilde{h}_{221}^2 + \tilde{h}_{123}^2).
\]
From (3.37) and (3.39), we know
\[\tilde{\lambda}_1 = \tilde{\lambda}_2.
\]
It is a contradiction.

3. The values of the principal curvature $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ are not equal to each other.

Case 1: $\tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_3 = 0$.

Without loss of generality, we assume that $\tilde{\lambda}_3 = 0$, we know that $\tilde{\lambda}_1 \neq 0$, $\tilde{\lambda}_2 \neq 0$, $\tilde{\lambda}_1 - \tilde{\lambda}_2 \neq 0$ and $\tilde{H} = \tilde{\lambda}_1 + \tilde{\lambda}_2 \neq 0$.

From (3.13) and (3.16), we have that
\[(3.40) \quad \tilde{h}_{11k} = \tilde{h}_{22k} = 0, \quad k = 1, 2, 3.
\]
From (3.10) and (3.40), we have
\[(3.41) \quad \tilde{h}_{33k} = 0, \quad k = 1, 2, 3.
\]
By (3.14), (3.17), (3.40) and (3.41), we have
\[\begin{cases} 
\tilde{\lambda}_1 \tilde{h}_{1111} + \tilde{\lambda}_2 \tilde{h}_{2211} = -2\tilde{h}_{123}^2, \\
\tilde{\lambda}_1 \tilde{h}_{1122} + \tilde{\lambda}_2 \tilde{h}_{2222} = -2\tilde{h}_{123}^2,
\end{cases}
\]
and then,
\[(3.42) \quad \tilde{h}_{1111} = 0, \quad \tilde{h}_{2211} = -\frac{2\tilde{h}_{123}^2}{\tilde{\lambda}_2}, \quad \tilde{h}_{1122} = -\frac{2\tilde{h}_{123}^2}{\tilde{\lambda}_1}, \quad \tilde{h}_{2222} = 0.
\]
From (3.15) and (3.42), we know
\[\tilde{h}_{1122} - \tilde{h}_{2211} = -\tilde{\lambda}_1 \tilde{\lambda}_2 (\tilde{\lambda}_1 - \tilde{\lambda}_2) = \frac{2(\tilde{\lambda}_1 - \tilde{\lambda}_2)\tilde{h}_{123}^2}{\tilde{\lambda}_1 \tilde{\lambda}_2},
\]
and then,
\[\tilde{h}_{123}^2 = -\frac{\tilde{\lambda}_1 \tilde{\lambda}_2}{2}, \quad \tilde{h}_{123}^2 = 0, \quad \tilde{\lambda}_1 \tilde{\lambda}_2 = 0.
\]
It is a contradiction.

Case 2: $\tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_3 \neq 0$. 

From (3.11), we have that

\[(3.43) \quad \lim_{m \to \infty} \langle T, e_k \rangle(p_t) = 0, \quad k = 1, 2, 3.\]

From (3.10), (3.13) and (3.16), we have that

\[(3.44) \quad \bar{h}_{11k} = \bar{h}_{22k} = \bar{h}_{33k} = 0, \quad k = 1, 2, 3.\]

From (3.12), (3.14), (3.17), (3.43) and (3.44), we have that

\[(3.45) \begin{cases} \sum_k \bar{h}_{kkii} = (\bar{H} - \lambda)\bar{\lambda}_i^2, & i = 1, 2, 3, \\ \sum_k \bar{h}_{kkij} = 0, & i \neq j, \quad i, j = 1, 2, 3. \end{cases}\]

\[(3.46) \begin{cases} \sum_i \bar{\lambda}_i \bar{h}_{iikk} = -2\bar{h}_{123}^2, & k = 1, 2, 3, \\ \sum_i \bar{\lambda}_i \bar{h}_{iikl} = 0, & k \neq l, \quad k, l = 1, 2, 3. \end{cases}\]

and

\[(3.47) \begin{cases} \sum_i \bar{\lambda}_i^3 \bar{h}_{iikk} = -\sum_{i,j} (2\bar{\lambda}_i^2 + \bar{\lambda}_i \bar{\lambda}_j)\bar{h}_{ijk}, & i \neq j \neq k \neq i, \quad k = 1, 2, 3, \\ \sum_i \bar{\lambda}_i^3 \bar{h}_{iikl} = 0, & k \neq l, \quad k, l = 1, 2, 3. \end{cases}\]
Therefore,

\[
\begin{align*}
\bar{h}_{1111} &= \frac{\bar{\lambda}_2 \bar{\lambda}_3 (\bar{\lambda}_2 + \bar{\lambda}_3)}{\bar{H}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot (\bar{H} - \lambda) \bar{\lambda}_1^2, \\
\bar{h}_{2211} &= \frac{-2\bar{h}_{123}^2}{\lambda_2 - \lambda_3} + \frac{\bar{\lambda}_1 \bar{\lambda}_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{\bar{H}(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot (\bar{H} - \lambda) \bar{\lambda}_1^2, \\
\bar{h}_{3311} &= \frac{-2\bar{h}_{123}^2}{\lambda_3 - \lambda_2} + \frac{\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 + \bar{\lambda}_2)}{\bar{H}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot (\bar{H} - \lambda) \bar{\lambda}_1^2, \\
\bar{h}_{2222} &= \frac{\bar{\lambda}_1 \bar{\lambda}_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{\bar{H}(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot (\bar{H} - \lambda) \bar{\lambda}_2^2, \\
\bar{h}_{1122} &= \frac{-2\bar{h}_{123}^2}{\lambda_1 - \lambda_3} + \frac{\bar{\lambda}_2 \bar{\lambda}_3 (\bar{\lambda}_2 + \bar{\lambda}_3)}{\bar{H}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot (\bar{H} - \lambda) \bar{\lambda}_2^2, \\
\bar{h}_{3322} &= \frac{-2\bar{h}_{123}^2}{\lambda_3 - \lambda_1} + \frac{\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 + \bar{\lambda}_2)}{\bar{H}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot (\bar{H} - \lambda) \bar{\lambda}_2^2, \\
\bar{h}_{3333} &= \frac{\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 + \bar{\lambda}_2)}{\bar{H}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot (\bar{H} - \lambda) \bar{\lambda}_3^2, \\
\bar{h}_{1133} &= \frac{-2\bar{h}_{123}^2}{\lambda_1 - \lambda_2} + \frac{\bar{\lambda}_2 \bar{\lambda}_3 (\bar{\lambda}_2 + \bar{\lambda}_3)}{\bar{H}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot (\bar{H} - \lambda) \bar{\lambda}_3^2, \\
\bar{h}_{2333} &= \frac{-2\bar{h}_{123}^2}{\lambda_2 - \lambda_1} + \frac{\bar{\lambda}_1 \bar{\lambda}_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{\bar{H}(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot (\bar{H} - \lambda) \bar{\lambda}_3^2, \\
\bar{h}_{1112} = \bar{h}_{2212} = \bar{h}_{3312} &= 0, \quad \bar{h}_{1113} = \bar{h}_{2213} = \bar{h}_{3313} = 0, \\
\bar{h}_{1123} = \bar{h}_{2223} = \bar{h}_{3333} &= 0.
\end{align*}
\]

From (3.15) and (3.48), we have that

\[
\begin{align*}
\frac{2\bar{h}_{123}^2 (\bar{\lambda}_1 - \bar{\lambda}_2)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} + \frac{(\bar{H} - \lambda) \bar{\lambda}_3 \left( \bar{\lambda}_3^3 (\bar{\lambda}_1^2 - \bar{\lambda}_3^2) + \bar{\lambda}_1^2 (\bar{\lambda}_2^2 - \bar{\lambda}_3^2) \right)}{\bar{H}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \\
\frac{-2\bar{h}_{123}^2 (\bar{\lambda}_1 - \bar{\lambda}_3)}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} - \frac{(\bar{H} - \lambda) \bar{\lambda}_1 \left( \bar{\lambda}_1^3 (\bar{\lambda}_2^2 - \bar{\lambda}_3^2) - \bar{\lambda}_3^2 (\bar{\lambda}_1^2 - \bar{\lambda}_3^2) \right)}{\bar{H}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \\
\frac{2\bar{h}_{123}^2 (\bar{\lambda}_2 - \bar{\lambda}_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} - \frac{(\bar{H} - \lambda) \bar{\lambda}_2 \left( \bar{\lambda}_2^3 (\bar{\lambda}_1^2 - \bar{\lambda}_3^2) + \bar{\lambda}_3^2 (\bar{\lambda}_1^2 - \bar{\lambda}_3^2) \right)}{\bar{H}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}
\end{align*}
\]
and then,
\[
\begin{align*}
2\vec{h}_{123}^2 & \cdot \left( \lambda_3(\lambda_1 - \lambda_2) + \lambda_2(\lambda_1 - \lambda_3) \right) + \frac{(\bar{H} - \lambda)}{H} \cdot \left( \frac{\lambda_1^2 \lambda_2^2 (\lambda_1^2 - \lambda_2^2) + \lambda_2^2 \lambda_3^2 (\lambda_2^2 - \lambda_3^2)}{\lambda_1 - \lambda_2} \right) \\
+ & \frac{\lambda_2^2 \lambda_3^2 (\lambda_3^2 - \lambda_2^2) + \lambda_1^2 \lambda_2^2 (\lambda_1^2 - \lambda_2^2)}{\lambda_1 - \lambda_3} = 0, \\
2\vec{h}_{123}^2 & \cdot \left( \lambda_1(\lambda_3 - \lambda_2) + \lambda_2(\lambda_3 - \lambda_1) \right) + \frac{(\bar{H} - \lambda)}{H} \cdot \left( \frac{\lambda_1^2 \lambda_3^2 (\lambda_1^2 - \lambda_3^2) + \lambda_3^2 \lambda_2^2 (\lambda_3^2 - \lambda_2^2)}{\lambda_3 - \lambda_2} \right) \\
+ & \frac{\lambda_2^2 \lambda_3^2 (\lambda_3^2 - \lambda_2^2) + \lambda_1^2 \lambda_3^2 (\lambda_1^2 - \lambda_3^2)}{\lambda_3 - \lambda_1} = 0, \\
2\vec{h}_{123}^2 & \cdot \left( \lambda_3(\lambda_2 - \lambda_1) + \lambda_1(\lambda_2 - \lambda_3) \right) + \frac{(\bar{H} - \lambda)}{H} \cdot \left( \frac{\lambda_1^2 \lambda_3^2 (\lambda_1^2 - \lambda_3^2) + \lambda_3^2 \lambda_2^2 (\lambda_3^2 - \lambda_2^2)}{\lambda_2 - \lambda_1} \right) \\
+ & \frac{\lambda_1^2 \lambda_3^2 (\lambda_3^2 - \lambda_1^2) + \lambda_2^2 \lambda_3^2 (\lambda_2^2 - \lambda_1^2)}{\lambda_2 - \lambda_3} = 0.
\end{align*}
\]
That is,
\[
AX = 0,
\]
where
\[
A = \begin{pmatrix}
\lambda_3(\lambda_1 - \lambda_2) + \lambda_2(\lambda_1 - \lambda_3) & \frac{\lambda_1^2 \lambda_2^2 (\lambda_1^2 - \lambda_2^2) + \lambda_2^2 \lambda_3^2 (\lambda_2^2 - \lambda_3^2)}{\lambda_1 - \lambda_2} \\
\lambda_1(\lambda_3 - \lambda_2) + \lambda_2(\lambda_3 - \lambda_1) & \frac{\lambda_1^2 \lambda_3^2 (\lambda_1^2 - \lambda_3^2) + \lambda_3^2 \lambda_2^2 (\lambda_3^2 - \lambda_2^2)}{\lambda_3 - \lambda_2}
\end{pmatrix},
\]
and
\[
X = \begin{pmatrix}
\frac{2\vec{h}_{123}^2}{H - \lambda}
\end{pmatrix}.
\]
By a direct calculation, we have
\[
\det(A) = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \cdot \left( (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\lambda_1^5 + \lambda_2^5 + \lambda_3^5) - (\lambda_1^3 + \lambda_2^3 + \lambda_3^3)(\lambda_1^4 + \lambda_2^4 + \lambda_3^4) \right).
\]
When
\[
(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\lambda_1^5 + \lambda_2^5 + \lambda_3^5) - (\lambda_1^3 + \lambda_2^3 + \lambda_3^3)(\lambda_1^4 + \lambda_2^4 + \lambda_3^4) \neq 0,
\]
that is
\[
S\bar{f}_5 - \bar{f}_3\bar{f}_4 \neq 0,
\]
we have that the matrix \( A \) is nondegenerate, and then
\[
\vec{h}_{123}^2 = 0, \quad \lambda = \bar{H}.
\]
That is,
\[
\vec{h}_{ijk} = 0, \quad i, j, k = 1, 2, 3.
\]
From (2.16) and (2.17) in Lemma 2.2, we obtain

\[ \sum_{i,j,k} h_{ijk}^2 + S^2 - \lambda f_3 = 0, \]

\[ 2 \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{kl} h_{li} + \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{kl} h_{li} + S f_4 - \lambda f_5 = 0. \]

Specifically,

\[ S^2 - \lambda \bar{f}_3 = 0, \quad S f_4 - \lambda \bar{f}_5 = 0, \]

and then, \( S \bar{f}_5 - \bar{f}_3 f_4 = 0 \). This contradicts the hypothesis.

When

\[ (\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_3^2)(\bar{\lambda}_1^5 + \bar{\lambda}_2^5 + \bar{\lambda}_3^5) - (\bar{\lambda}_1^3 + \bar{\lambda}_2^3 + \bar{\lambda}_3^3)(\bar{\lambda}_1^4 + \bar{\lambda}_2^4 + \bar{\lambda}_3^4) = 0, \]

that is

(3.50)

\[ S \bar{f}_5 - \bar{f}_3 f_4 = 0. \]

From (2.16) and (2.17) in Lemma 2.2, we have

\[ \sum_{i,j,k} h_{ijk}^2 + S^2 - \lambda f_3 = 0, \]

\[ 2 \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{kl} h_{li} + \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{kl} h_{li} + S f_4 - \lambda f_5 = 0. \]

Thus,

\[ \sum_{i,j,k} \bar{h}_{ijk}^2 + S^2 - \lambda \bar{f}_3 = 0, \]

\[ 2 \sum_{i,j,k,l,m} \bar{h}_{ijm} \bar{h}_{jkm} \bar{h}_{kl} \bar{h}_{li} + \sum_{i,j,k,l,m} \bar{h}_{ijm} \bar{h}_{jkm} \bar{h}_{kl} \bar{h}_{li} + S f_4 - \lambda \bar{f}_5 = 0. \]

Especially,

(3.51)

\[ \bar{h}_{123}^2 = -\frac{1}{6}(S^2 - \lambda \bar{f}_3), \]

\[ \bar{h}_{123}^2 = \frac{-(S f_4 - \lambda \bar{f}_5)}{H^2 + 3S}. \]

From (3.51), we obtain

\[ \lambda \left( 6 \bar{f}_5 - \bar{f}_3 (H^2 + 3S) \right) = 6 S f_4 - S^2 (H^2 + 3S). \]

Supposing

\[ 6 \bar{f}_5 - \bar{f}_3 (H^2 + 3S) = 0, \]

we obtain

(3.52)

\[ 6 f_4 = S (H^2 + 3S). \]
From Lemma 2.4, we have

\[
\begin{align*}
    f_4 &= \frac{4}{3} \bar{H} \bar{f}_3 - \bar{H}^2 S + \frac{1}{6} \bar{H}^4 + \frac{1}{2} S^2, \\
    \bar{f}_5 &= \frac{5}{6} \bar{H}^2 \bar{f}_3 + \frac{5}{6} S \bar{f}_3 - \frac{5}{6} \bar{H}^3 S + \frac{1}{6} \bar{H}^5.
\end{align*}
\]  

(3.53)

From (3.50) and (3.53), we obtain

\[
8 \bar{H} \bar{f}_3^2 + (\bar{H}^4 - 11 \bar{H}^2 S - 2S^2) \bar{f}_3 + 5 \bar{H}^3 S^2 - \bar{H}^5 S = 0.
\]  

(3.54)

From (3.52) and (3.53), we obtain

\[
8 \bar{H} \bar{f}_3 - 7 \bar{H}^2 S + \bar{H}^4 = 0,
\]

that is,

\[
\bar{f}_3 = \frac{7}{8} \bar{H} S - \frac{1}{8} \bar{H}^3.
\]  

(3.55)

From (3.54) and (3.55), we obtain

\[
\bar{H} S(2\bar{H}^4 - 7\bar{H}^2 S + 7S^2) = \bar{H} S \left(2(\bar{H}^2 - \frac{7}{4} S)^2 + \frac{7}{8} S^2 \right) = 0,
\]

which is impossible. Then we have

\[
6 \bar{f}_5 - \bar{f}_3 (\bar{H}^2 + 3S) \neq 0, \quad \lambda = \frac{6S f_4 - S^2 (\bar{H}^2 + 3S)}{6 \bar{f}_5 - \bar{f}_3 (\bar{H}^2 + 3S)}.
\]  

(3.56)

From (3.51) and (3.56), we obtain

\[
\bar{h}_{123}^2 = -\frac{1}{6} (S^2 - \lambda \bar{f}_3)
= -\frac{1}{6} \left( S^2 - \bar{f}_3 \cdot \frac{6S f_4 - S^2 (\bar{H}^2 + 3S)}{6 \bar{f}_5 - \bar{f}_3 (\bar{H}^2 + 3S)} \right)
= -S \left( \frac{S \bar{f}_5 - f_4 \bar{f}_3}{6 \bar{f}_5 - \bar{f}_3 (\bar{H}^2 + 3S)} \right)
= 0,
\]

where \( S \bar{f}_5 - \bar{f}_3 f_4 = 0 \).

That is,

\[
\bar{h}_{123} = 0, \quad \bar{h}_{ijk} = 0, \quad i, j, k = 1, 2, 3.
\]

Supposing \( \bar{H} - \lambda = 0 \), from \( \bar{h}_{123} = 0 \) and (3.48), we obtain

\[
\bar{h}_{ijkl} = 0, \quad i, j, k = 1, 2, 3.
\]
From (2.18) and (2.19) in lemma 2.3, we have
\[
0 = \lim_{t \to \infty} \frac{1}{2} \Delta - V \sum_{i,j,k} (h_{i,j,k})^2(p_t) \\
= \frac{9}{2} \lambda S \tilde{h}_{11} \tilde{h}_{22} \tilde{h}_{33} - \frac{3}{2} \lambda^2 \sum_k (\tilde{h}_{22} \tilde{h}_{33} \tilde{h}_{1k} + \tilde{h}_{11} \tilde{h}_{22} \tilde{h}_{3k}) \\
= \frac{3}{2} \lambda \tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_3 (3S - \tilde{H}) \\
= \frac{3}{2} \lambda \tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_3 (3S - \tilde{H}^2),
\]
where \( \tilde{H} - \lambda = 0 \) and \( \tilde{h}_{ijk} = 0, \tilde{h}_{ijkl} = 0 \), \( i,j,k,l = 1,2,3 \).

Therefore,
\[
3S - \tilde{H}^2 = 0, \quad \tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3.
\]
This contradicts the hypothesis. We have
\[
\tilde{H} - \lambda \neq 0.
\]

From \( \tilde{h}_{123} = 0 \) and (3.51), we have
\[
(3.57) \quad \lambda = \frac{S^2}{\bar{f}_3}, \quad \frac{\tilde{H} - \lambda}{\tilde{H}} = \frac{\bar{f}_3 - S^2}{\bar{f}_3}.
\]

From \( S = \text{constant} \) and (2.16) in Lemma 2.2, we have
\[
2 \sum_{i,j,k} h_{i,j,k} h_{i,jkl} - \lambda \nabla_l \bar{f}_3 = 0,
\]
\[
2 \sum_{i,j,k} h_{i,j,k} h_{i,jklm} + 2 \sum_{i,j,k} h_{i,jkm} h_{i,jkl} - \lambda \nabla_m \nabla_l \bar{f}_3 = 0, \quad l, m = 1, 2, 3.
\]

Thus,
\[
\sum_{i,j,k} \tilde{h}_{i,j,k} \tilde{h}_{i,jkl} + \sum_{i,j,k} \tilde{h}_{i,jkm} \tilde{h}_{i,jkl} - \frac{1}{2} \lambda \lim_{t \to \infty} \nabla_m \nabla_l \bar{f}_3(p_t) = 0, \quad l, m = 1, 2, 3.
\]

Especially,
\[
(3.58) \quad \sum_{i,j,k} \tilde{h}_{i,j,k}^2 - \frac{1}{2} \lambda \lim_{t \to \infty} \nabla_l \nabla_l \bar{f}_3(p_t) = 0, \quad l = 1, 2, 3.
\]

From \( \bar{f}_4 = \text{constant} \) and (2.21) in Lemma 2.4, we have
\[
(4 \bar{f}_3 - 2S \bar{H} + \frac{2}{3} \bar{H}^3) \bar{H}_{,k} + 4 \bar{H} \lim_{t \to \infty} \nabla_l \nabla_k \bar{f}_3(p_t) = 0, \quad k, l = 1, 2, 3,
\]
and then,
\[
\lim_{t \to \infty} \nabla_l \nabla_k \bar{f}_3(p_t) = -\frac{\bar{f}_3 - \frac{2}{3} S \bar{H} + \frac{1}{2} \bar{H}^3}{\bar{H}} \cdot \bar{H}_{,kl} = -\frac{3 \tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_3}{\bar{H}} \cdot \bar{H}_{,kl}, \quad k, l = 1, 2, 3.
\]
Therefore,
\[
(3.59) \quad -\frac{1}{2} \lambda \lim_{t \to \infty} \nabla_k \nabla_k \bar{f}_3(p_t) = \lambda \cdot \frac{3 \tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_3}{2 \bar{H}} \cdot \bar{H}_{,kk} = \frac{3 \lambda (\bar{H} - \lambda)}{2 \bar{H}} \cdot \tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_3 \lambda^2, \quad k = 1, 2, 3.
\]
From $h_{123} = 0$ and (3.48), we have

$$
\begin{align*}
\bar{h}_{1111} &= \frac{\lambda_2 \lambda_3 (\bar{\lambda}_2 + \bar{\lambda}_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda) \bar{\lambda}_2^2}{H}, \\
\bar{h}_{2211} &= \frac{\lambda_1 \lambda_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda) \bar{\lambda}_1^2}{H}, \\
\bar{h}_{3311} &= \frac{\lambda_1 \lambda_2 (\bar{\lambda}_1 + \bar{\lambda}_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot \frac{(\bar{H} - \lambda) \bar{\lambda}_1^2}{H}, \\
\bar{h}_{2222} &= \frac{\lambda_1 \lambda_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda) \bar{\lambda}_2^2}{H}, \\
\bar{h}_{1122} &= \frac{\lambda_2 \lambda_3 (\bar{\lambda}_2 + \bar{\lambda}_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda) \bar{\lambda}_3^2}{H}, \\
\bar{h}_{3322} &= \frac{\lambda_1 \lambda_2 (\bar{\lambda}_1 + \bar{\lambda}_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot \frac{(\bar{H} - \lambda) \bar{\lambda}_3^2}{H}, \\
\bar{h}_{3333} &= \frac{\lambda_2 \lambda_3 (\bar{\lambda}_2 + \bar{\lambda}_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda) \bar{\lambda}_3^2}{H}, \\
\bar{h}_{1133} &= \frac{\lambda_1 \lambda_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda) \bar{\lambda}_3^2}{H}, \\
\bar{h}_{2233} &= \frac{\lambda_1 \lambda_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda) \bar{\lambda}_3^2}{H}.
\end{align*}
$$

From (3.57), (3.58), (3.59) and (3.60), we have

$$
\begin{align*}
\lambda_1 (\bar{H} f_3 - S^2) &= \frac{\lambda_1 \lambda_2^2 (\bar{\lambda}_2 + \bar{\lambda}_3)^2}{2 f_3} \cdot \left( \frac{\lambda_2 \lambda_3 (\bar{\lambda}_2 + \bar{\lambda}_3)^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{3 \lambda_1 \lambda_3^2 (\bar{\lambda}_1 + \bar{\lambda}_3)^2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)^2} \right) \\
&\quad + \frac{3 \lambda_1^2 \lambda_2^2 (\bar{\lambda}_1 + \bar{\lambda}_2)^2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)^2} + \frac{3 \lambda_2 \lambda_3 S^2}{2 f_3} = 0, \\
\lambda_2 (\bar{H} f_3 - S^2) &= \frac{\lambda_1 \lambda_3^2 (\bar{\lambda}_1 + \bar{\lambda}_3)^2}{2 f_3} \cdot \left( \frac{\lambda_2 \lambda_3 (\bar{\lambda}_2 + \bar{\lambda}_3)^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{3 \lambda_2 \lambda_3^2 (\bar{\lambda}_2 + \bar{\lambda}_3)^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)^2} \right) \\
&\quad + \frac{3 \lambda_1 \lambda_3^2 (\bar{\lambda}_1 + \bar{\lambda}_3)^2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)^2} + \frac{3 \lambda_1 \lambda_3 S^2}{2 f_3} = 0, \\
\lambda_3 (\bar{H} f_3 - S^2) &= \frac{\lambda_2 \lambda_3^2 (\bar{\lambda}_2 + \bar{\lambda}_3)^2}{2 f_3} \cdot \left( \frac{\lambda_1 \lambda_3^2 (\bar{\lambda}_1 + \bar{\lambda}_3)^2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)^2} + \frac{3 \lambda_1 \lambda_2^2 (\bar{\lambda}_1 + \bar{\lambda}_2)^2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)^2} \right) \\
&\quad + \frac{3 \lambda_1 \lambda_3^2 (\bar{\lambda}_1 + \bar{\lambda}_3)^2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)^2} + \frac{3 \lambda_1 \lambda_2 S^2}{2 f_3} = 0.
\end{align*}
$$
And then,

$$
\begin{align*}
&2\lambda_1(\bar{H}\bar{f}_3 - S^2) \cdot \left( \bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_3^2(\bar{\lambda}_1^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2 \right) \\
&+ 3\bar{\lambda}_2\bar{\lambda}_3\bar{H}S^2(\bar{\lambda}_1 - \bar{\lambda}_2)^2(\bar{\lambda}_1 - \bar{\lambda}_3)^2(\bar{\lambda}_2 - \bar{\lambda}_3)^2 = 0, \quad (1) \\
&2\lambda_2(\bar{H}\bar{f}_3 - S^2) \cdot \left( \bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2 \right) \\
&+ 3\bar{\lambda}_1\bar{\lambda}_3\bar{H}S^2(\bar{\lambda}_1 - \bar{\lambda}_2)^2(\bar{\lambda}_1 - \bar{\lambda}_3)^2(\bar{\lambda}_2 - \bar{\lambda}_3)^2 = 0, \quad (2) \\
&2\lambda_3(\bar{H}\bar{f}_3 - S^2) \cdot \left( \bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_3^2(\bar{\lambda}_1^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 \right) \\
&+ 3\bar{\lambda}_1\bar{\lambda}_2\bar{H}S^2(\bar{\lambda}_1 - \bar{\lambda}_2)^2(\bar{\lambda}_1 - \bar{\lambda}_3)^2(\bar{\lambda}_2 - \bar{\lambda}_3)^2 = 0. \quad (3)
\end{align*}
$$

By computing $\bar{\lambda}_1 \times (1) - \bar{\lambda}_2 \times (2)$, we have

$$
2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)(\bar{H}\bar{f}_3 - S^2) \left( 3\bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_3^2(\bar{\lambda}_1^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 + 2\bar{\lambda}_1^2\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 - 2\bar{\lambda}_3^2) \right) = 0.
$$

Supposing $\bar{H}\bar{f}_3 - S^2 = 0$, from (3.54), we obtain

$$
0 = 8\bar{H}^2f_3^2 + (\bar{H}^4 - 11\bar{H}^2S - 2S^2)f_3 + 5\bar{H}^3S^2 - \bar{H}^5S
$$

(3.62)

$$
= \frac{S}{\bar{H}}(6S^3 - 11\bar{H}^2S^2 + 6\bar{H}^4S - \bar{H}^6)
$$

$$
= \frac{S}{\bar{H}}(S - \bar{H}^2)(2S - \bar{H}^2)(3S - \bar{H}^2).
$$

From $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, we obtain

$$
H^2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 < 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = 3S.
$$

Hence,

$$
H^2 < 3S.
$$

From (3.62), we obtain that $S - \bar{H}^2 = 0$ or $2S - \bar{H}^2 = 0$. Besides, for $n = 3$, we have $f_3 = \frac{H}{2}(3S - \bar{H}^2) + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3$.

When $S - \bar{H}^2 = 0$, we have that

$$
\bar{f}_3 = \frac{S^2}{\bar{H}} = \bar{H}^3,
$$

$$
\bar{f}_3 = \frac{H}{2}(3S - \bar{H}^2) + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 = \bar{H}^3 + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3.
$$

And then, $\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 = 0$. This contradicts the hypothesis.

When $2S - \bar{H}^2 = 0$, we have that

$$
\bar{f}_3 = \frac{S^2}{\bar{H}} = \frac{H^3}{4},
$$

$$
\bar{f}_3 = \frac{H}{2}(3S - \bar{H}^2) + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 = \frac{H^3}{4} + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3.
And then, $\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 = 0$. This contradicts the hypothesis. Hence,

$$\bar{H} \bar{f}_3 - S^2 \neq 0.$$ 

Supposing $\bar{\lambda}_1^2 - \bar{\lambda}_2^2 = 0$, that is $\bar{\lambda}_1 = -\bar{\lambda}_2$. From (3.50), we obtain

$$\lambda^2 = (2\bar{\lambda}_1^2 + \bar{\lambda}_2^2) \bar{\lambda}_3^2 - (2\bar{\lambda}_1^4 + \bar{\lambda}_2^4) \bar{\lambda}_3^2 = 2\lambda_1^2 \lambda_3^2 (\lambda_2^2 - \lambda_1^2),$$

which implies $\bar{\lambda}_1^2 = \bar{\lambda}_3^2$. Then $\bar{\lambda}_1 = \bar{\lambda}_3$ or $\bar{\lambda}_1 = -\bar{\lambda}_3 = -\bar{\lambda}_2$, which is a contradiction. Hence,

$$3\lambda_1^2 \lambda_2^2 (\lambda_1^2 - \lambda_2^2) + 3\lambda_1^2 \lambda_3^2 (\lambda_1^2 - \lambda_3^2) + 3\lambda_2^2 \lambda_3^2 (\lambda_2^2 - \lambda_3^2) + 2\lambda_1^2 \lambda_2^2 \lambda_3 (\lambda_1^2 + \lambda_2^2 - 2\lambda_3^2) = 0.$$ 

Similarity, by computing $\bar{\lambda}_2 \times (2) - \bar{\lambda}_3 \times (3)$, we have

$$2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)(\bar{H} \bar{f}_3 - S^2) \left(3\lambda_1^2 \lambda_2^2 (\lambda_1^2 - \lambda_2^2) + 3\lambda_1^2 \lambda_3^2 (\lambda_1^2 - \lambda_3^2) + 3\lambda_2^2 \lambda_3^2 (\lambda_2^2 - \lambda_3^2) + 2\lambda_1^2 \lambda_2^2 \lambda_3 (\lambda_1^2 + \lambda_2^2 - 2\lambda_3^2) \right) = 0,$$

which implies

$$3\lambda_1^2 \lambda_2^2 (\lambda_1^2 - \lambda_2^2) + 3\lambda_1^2 \lambda_3^2 (\lambda_1^2 - \lambda_3^2) + 3\lambda_2^2 \lambda_3^2 (\lambda_2^2 - \lambda_3^2) + 2\lambda_1^2 \lambda_2^2 \lambda_3 (\lambda_1^2 + \lambda_2^2 - 2\lambda_3^2) = 0.$$ 

From (3.63) and (3.64), we have

$$\bar{\lambda}_1^2 + \bar{\lambda}_2^2 - 2\bar{\lambda}_3^2 = \lambda_2^2 + \lambda_3^2 - 2\lambda_1^2.$$

That is, $\bar{\lambda}_1 = -\bar{\lambda}_3$.

From (3.50) and $\bar{\lambda}_1 = -\bar{\lambda}_3$, we obtain

$$0 = S\bar{f}_5 - \bar{f}_3 f_4$$

$$= (2\bar{\lambda}_1^2 + \bar{\lambda}_2^2) \bar{\lambda}_3^2 - (2\bar{\lambda}_1^4 + \bar{\lambda}_2^4) \bar{\lambda}_3^2$$

$$= 2\lambda_1^2 \lambda_3^2 (\lambda_2^2 - \lambda_1^2),$$

Then $\bar{\lambda}_1 = -\bar{\lambda}_3 = -\bar{\lambda}_2$ and $\bar{\lambda}_2 = \bar{\lambda}_3$, which is a contradiction. 

\textbf{Theorem 3.3.} For a 3-dimensional complete $\lambda$-translator $x : M^3 \to \mathbb{R}^4$ with non-zero constant squared norm $S$ of the second fundamental form and constant $f_4$, where $S = \sum_i h_{ij}^2$ and $f_4 = \sum_i h_{ij} h_{ij} h_{kl} h_{kl}$, we have either

1. $\lambda^2 = S$ and $\inf H^2 = S$, or
2. $\lambda^2 = 2S$ and $\inf H^2 = 2S$, or
3. $\lambda^2 = 3S$ and $\inf H^2 = 3S$. 
Proof. We apply the generalized maximum principle for the operator $\Delta_W$ to the function $-H^2$. Thus, there exists a sequence $\{p_t\}$ in $M^3$ such that

$$\lim_{t \to \infty} H^2(p_t) = \inf H^2 = \bar{H}^2, \quad \lim_{t \to \infty} |\nabla H^2(p_t)| = 0.$$ 

By Theorem 3.1, we have $\inf H^2 > 0$. Using the similar proof of Theorem 3.1, by (2.16) and $S = \text{constant}$, we know that $\{h_{ij}(p_t)\}$, $\{h_{ijk}(p_t)\}$ and $\{h_{ijkl}(p_t)\}$ are bounded sequences for $i, j, k, l = 1, 2, 3$. Without loss of the generality, we can assume

$$\lim_{t \to \infty} h_{ij}(p_t) = \bar{h}_{ij}, \quad \lim_{t \to \infty} h_{ij}(p_t) = \bar{h}_{ij} = \bar{\lambda}_{ij}, \quad \lim_{t \to \infty} h_{ijkl}(p_t) = \bar{h}_{ijkl}, \quad i, j, k, l = 1, 2, 3.$$ 

By making use of the same assertion as in the proof of the Theorem 3.2, we know that at least two principal curvatures are equal.

1. **Two of the values of the principal curvature $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ are equal.**

According to the proof of Theorem 3.2, there are two cases. First, one is not zero and the other two are equal to zero. We have

$$\lambda^2 = \bar{H}^2 = \inf H^2, \quad \lambda^2 = S;$$

Second, one is zero and the other two are equal and not zero. We have

$$\lambda^2 = \bar{H}^2 = \inf H^2, \quad \lambda^2 = 2S;$$

2. **The values of the principal curvature $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ are all equal.**

From $\bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3$, we have

$$\inf H^2 = (\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3)^2 = 3S.$$ 

Since

$$0 \leq 3S - H^2 \leq \sup(3S - H^2) = 3S - \inf H^2 = 0,$$

we obtain

$$3S = \inf H^2.$$ 

According to case one of Theorem 3.2, we know that $3S = \sup H^2$. Namely, $H$ is constant. Hence, we conclude from (2.15)

$$\lambda = H, \quad \lambda^2 = 3S.$$ 

The proof of Theorem 3.3 is finished. \qed

Proof of Theorem 1.1. If $S = 0$, we know that $x : M^3 \to \mathbb{R}^4_1$ is a space-like affine plane $\mathbb{R}^3_1$, not necessarily passing through the origin. If $S \neq 0$, from Theorem 3.2 and Theorem 3.3, we have

1. $\lambda^2 = S$ and $\sup H^2 = \inf H^2 = S$, or
2. $\lambda^2 = 2S$ and $\sup H^2 = \inf H^2 = 2S$, or
3. $\lambda^2 = 3S$ and $\sup H^2 = \inf H^2 = 3S$.

It follows that the mean curvature $H$ and the principal curvatures must be constant. From (1.2) and (2.14), we have

$$\lambda = H, \quad \langle T, n \rangle = 0.$$
So the nonzero constant vector $T = T^T$ is tangent to $x(M^3)$ at each point of $M^3$. It follows that $x(M^3)$ consists of a family of parallel planes in $\mathbb{R}^4_1$ and thus, up to an isometry of $\mathbb{R}^4_1$, it is a cylinder $\mathbb{H}^1(a_1) \times \mathbb{R}^2$ or $\mathbb{H}^2(a_2) \times \mathbb{R}^1$ for $a_1 > 0$ and $a_2 > 0$, where $\mathbb{H}^1(a_1)$ and $\mathbb{H}^2(a_2)$ are hyperbolic curve and hyperboloid respectively. Besides, the parameters $a_1$ and $a_2$ can be determined by $\lambda$ via the defining equation (1.2). By an easy computation, we have that $\lambda > 0$, $a_1 = \frac{1}{\lambda}$ and $a_2 = \frac{2}{\lambda}$. Theorem 1.1 is proved.

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