On the Positivity of the Dimension of the Global Sections of Adjoint Bundle for Quasi-Polarized Manifold with Numerically Trivial Canonical Bundle

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Abstract

Let \((X, L)\) denote a quasi-polarized manifold of dimension \(n \geq 5\) defined over the field of complex numbers such that the canonical line bundle \(K_X\) of \(X\) is numerically equivalent to zero. In this paper, we consider the dimension of the global sections of \(K_X + mL\) in this case, and we prove that \(h^0(K_X + mL) > 0\) for every positive integer \(m\) with \(m \geq n - 3\). In particular, a Beltrametti-Sommese conjecture is true for quasi-polarized manifolds with numerically trivial canonical divisors.

1 Introduction

Let \(X\) be a smooth projective variety of dimension \(n\) defined over the field of complex numbers, and let \(L\) be an ample (resp. nef and big) line bundle on \(X\). Then, \((X, L)\) is called a polarized (resp. quasi-polarized) manifold. Recently, the positivity of the dimension \(h^0(K_X + mL)\) has been discussed, where \(K_X\) and \(m\) denote the canonical line bundle of \(X\) and a natural number, respectively. For \(m = n - 1\), Beltrametti and Sommese proposed the following conjecture ([3, Conjecture 7.2.7]):

**Conjecture 1** (Beltrametti–Sommese) Let \((X, L)\) be a polarized manifold with \(\dim X = n \geq 3\). Assume that \(K_X + (n - 1)L\) is nef. Then \(h^0(K_X + (n - 1)L) > 0\).

For this conjecture, the following partial results have been obtained:

- In [8, Theorem 2.4] and [11, Theorem 3.1], the author proved that this conjecture is true if \(n \leq 4\). (See also [5] and [6].) Besides, we also note that Andreatti and Fontanari [1] improved the result in [11].

- In [15, 1.2 Theorem], Höring proved that this conjecture is true if \(h^0(L) > 0\).

Moreover, the author has classified \((X, L)\) for the following types in the previously conducted studies:

- Polarized 3-fold \((X, L)\) with \(h^0(K_X + 2L) \leq 2\) ([8], [10]).

- Polarized 4-fold \((X, L)\) with \(h^0(K_X + 3L) \leq 1\) ([11], [12]).
More generally, Ionescu proposed the following conjecture (see [17, Open problems, P.321]).

**Conjecture 2** (Ionescu) Let \((X, L)\) be a polarized manifold. Assume that \(K_X + L\) is nef. Then, \(h^0(K_X + L) > 0\).

It is known that this conjecture is true if \(n \leq 3\) (see [9, Theorem 2.8], [15, 1.5 Theorem]).

Additionally, the author also considered the case where \(m = n + 1\). In [13, Conjecture 2], we proposed the following conjecture.

**Conjecture 3** Let \((X, L)\) be an \(n\)-dimensional polarized manifold with \(n \geq 3\). Then, \(h^0(K_X + mL) \geq \binom{m+1}{n} \) holds for every integer \(m \geq n + 1\). If equality holds for some \(m \geq n + 1\), then \((X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\).

It has been proved that Conjecture 3 is true only in the following cases:

- The case where \(n \leq 4\) ([8, Theorem 2.5] and [13, Theorem 3.1]). (See also [1, Theorem 8] for results concerning with Conjecture 3.)

- The case where \(n \geq 5\) and \(\dim \text{Bs}(L) \leq 1\) ([13, Theorem 3.2 (i)]).

- The case where \(n = 5\) and \(h^0(L) > 0\) ([14, Theorem 3.1]).

In this study, we consider the positivity of \(h^0(K_X + mL)\) when the canonical bundle \(K_X\) is numerically equivalent to zero and \(L\) is nef and big. In this case, Cao and Jiang proved that \(h^0(K_X + L) > 0\) if \(n \leq 4\) (see [7, Remark 5.3]). In this paper, we prove that \(h^0(K_X + mL) > 0\) for every integer \(m \geq n - 3\) for \(n \geq 5\) (Corollary 2.1). In particular, combining this result and the above mentioned result of Cao and Jiang, we deduce that Conjecture 1 is true for quasi-polarized manifolds with numerically trivial canonical divisors (Corollary 2.2).

Throughout this paper, we work over the field of complex numbers, and we use the customary canonical divisors (Corollary 2.2).

## 2 Main Result

First, we note that we can prove the following two propositions by the same method as in the proof of [7, Theorems 5.1 and 5.2].

**Proposition 2.1** Let \(X\) be a smooth projective variety of dimension \(2k + 1\) with \(k \geq 1\), and let \(L\) be a nef and big divisor on \(X\). Assume that \(K_X\) is numerically equivalent to zero. Then, the following holds: For every positive integers \(\alpha_1, \ldots, \alpha_k\) with \(1 \leq \alpha_1 < \ldots < \alpha_k\), there exists \(i\) such that \(h^0(K_X + \alpha_iL) > 0\).

**Proposition 2.2** Let \(X\) be a smooth projective variety of dimension \(4k + 2\) or \(4k + 4\) with \(k \geq 0\), and let \(L\) be a nef and big divisor on \(X\). Assume that \(K_X\) is numerically equivalent to zero. Then, the following holds: For every positive integers \(\alpha_1, \ldots, \alpha_{2k+1}\) with \(1 \leq \alpha_1 < \ldots < \alpha_{2k+1}\), there exists \(i\) such that \(h^0(K_X + \alpha_iL) > 0\).

**Remark 2.1** Let \(X\) be a smooth projective variety of dimension \(n\). Assume that the canonical bundle \(K_X\) of \(X\) is numerically equivalent to zero. Then, for any line bundle \(L\) on \(X\), we have \(\chi(tL) = (-1)^n \chi(-L + K_X) = (-1)^n \chi(-L)\). In particular, \(\chi(tL)\) is an even (resp. odd) function of integers if \(n\) is even (resp. odd), and we also note that \(\chi(O_X) = 0\) if \(n\) is odd. Furthermore, we see that if \(\chi(aL) = 0\) for some \(a \in \mathbb{N}\), then \(\chi(-aL) = 0\).
Theorem 2.1 Let $X$ be a smooth projective variety of dimension $n$, and let $L$ be a nef and big divisor on $X$. Assume that $K_X$ is numerically equivalent to zero. Then the following hold.

(a) If $n$ is odd with $n \geq 5$, then $h^0(K_X + mL) > 0$ for every integer $m$ with $m \geq n - 3$.

(b) If $n$ is odd with $n \geq 7$, then $h^0(K_X + (n-4)L) > 0$.

(c) If $n = 4k + 4$ with $1 \leq k \in \mathbb{Z}$, then $h^0(K_X + mL) > 0$ for every integer $m$ with $m \geq n - 5$.

(d) If $n = 4k + 2$ with $1 \leq k \in \mathbb{Z}$, then $h^0(K_X + mL) > 0$ for every integer $m$ with $m \geq n - 3$.

Proof. First of all, we note that we have $h^i(K_X + mL) = h^i(mL) = 0$ for every positive integers $i$ and $m$ by the Kawamata-Viehweg vanishing theorem because $K_X$ is numerically equivalent to zero. Hence $h^0(K_X + mL) = \chi(K_X + mL) = \chi(mL) = h^0(mL)$ for every positive integer $m$.

(I) First, we study the case (a) in Theorem 2.1. Here we set $n = 2k + 1$, where $2 \leq k \in \mathbb{Z}$.

(I.1) If $k = 2$, then we consider the pair $(h^0(L), h^0(mL))$. We remark that $m \geq n - 3 = 2$. Then we see from Proposition 2.1 that $h^0(L) > 0$ or $h^0(mL) > 0$. If $h^0(mL) > 0$, then we are done. If $h^0(L) > 0$, then it follows $h^0(mL) > 0$.

(I.2) We assume that $k \geq 3$ (i.e. $n \geq 7$). Then, we take the following string of $k - 2$ pairs

$$
(1) \quad (h^0(L), h^0((m-1)L)), (h^0(2L), h^0((m-2)L)), \ldots, (h^0((k-2)L), h^0((m-k+2)L)).
$$

Here we note that there are no overlaps among $iL$’s in the string (1) if and only if $m-k+2 > k-2$, that is, $m \geq 2k-3 = n-4$. We also remark the following.

(*) If there exists an integer $i$ with $1 \leq i \leq k-2$ such that $h^0(iL) > 0$ and $h^0((m-i)L) > 0$, then we have $h^0(mL) \geq h^0(iL) + h^0((m-i)L) - 1 > 0$ by [16, 15.6.2 Lemma]. So we may assume that $h^0(iL) = 0$ or $h^0((m-i)L) = 0$ for every integer $i = 1, 2, \ldots, k-2$.

So we may assume that we can pick up $k-2$ integers $1 \leq \beta_1 < \ldots < \beta_{k-2} \leq m - 1$ such that $h^0(\beta_iL) = 0$ for every $i$ with $1 \leq i \leq k-2$.

Now we consider the set

$$
\mathcal{A} = \{ t \in \mathbb{Z} \mid k - 1 \leq t \leq m - k + 1 \}.
$$

We note that $h^0(tL)$ is not contained in the string (1) for every $t \in \mathcal{A}$. Then, there is at most one $p \in \mathcal{A}$ such that $h^0(pL) = 0$ by Proposition 2.1 since we have already $k-2$ zeros of $h^0(tL)$ in the string (1).

(I.2.1) Assume that $m \geq n - 2 = 2k - 1$. Here we note that $k - 1 \neq m - k + 1$. We consider the pair $(h^0((k-1)L), h^0((m-k+1)L))$. This is not contained in the string (1). So by Proposition 2.1, at least one of them is positive. If $h^0((k-1)L) > 0$ and $h^0((m-k+1)L) > 0$, then we get $h^0(mL) > 0$ by the same argument as (*) above. If $h^0((k-1)L) = 0$ (resp. $h^0((m-k+1)L) = 0$), then by applying Proposition 2.1 to $\{ \beta_1, \ldots, \beta_{k-2}, k-1, m \}$ (resp. $\{ \beta_1, \ldots, \beta_{k-2}, m-k+1, m \}$), we also get $h^0(mL) > 0$.

(I.2.2) Assume that $m = n - 3 = 2k - 2$. In this case, $\mathcal{A} = \{ k - 1 \}$. So by applying Proposition 2.1 to $\{ \beta_1, \ldots, \beta_{k-2}, k-1, m \}$, we see that $h^0((k-1)L) > 0$ or $h^0(mL) > 0$. If $h^0(mL) > 0$, then we are done. So we may assume that $h^0((k-1)L) > 0$. But then we get $h^0(mL) > 0$ because $mL = 2(k-1)L$.

(II) Next, we study the case (b) in Theorem 2.1. We set $n = 2k + 1$, where $3 \leq k \in \mathbb{Z}$. We
remark that \( n - 4 = 2k - 3 \) in this case. Then, we take \( k - 2 \) pairs \((h^0(L), h^0((2k-4)L))\), \ldots, \((h^0((k-2)L), h^0((k-1)L))\). By the same argument as (\*) in (I.2), we may assume that \( h^0(iL) = 0 \) or \( h^0((2k-3-i)L) = 0 \) for every \( i = 1, 2, \ldots, k-2 \). Let \( a_1, \ldots, a_{k-2} \) be integers such that \( 1 \leq a_1 < \ldots < a_{k-2} \leq 2k-4 \) and \( h^0(a_iL) = 0 \) for every \( i \) with \( 1 \leq i \leq k-2 \).

If \( h^0(L) > 0 \), then we have \( 0 < h^0((2k-3)L) = h^0((n-4)L) \) and we are done. So we may assume that \( h^0(L) = 0 \). Then, we note that we can take \( a_1 = 1 \). If \( h^0((2k-4)L) = 0 \), then by applying Proposition 2.1 to \( \{a_1, \ldots, a_{k-2}, 2k-4, 2k-3\} \), we have \( h^0((2k-3)L) > 0 \) holds. Hence we may also assume that \( h^0((2k-4)L) > 0 \), that is, \( a_{k-2} \leq 2k-5 = n-6 \).

Here we assume by contradiction that \( h^0((n-4)L) = h^0((2k-3)L) = 0 \). Since \( n \) is odd, we can describe \( \chi(tL) \) as follows by Remark 2.1.

\[
\chi(tL) = \alpha(t^2 - a_1^2) \cdots (t^2 - a_{k-2}^2)(t^2 - (2k-3)^2)(t^2 - \beta)t,
\]

where \( \alpha, \beta \in \mathbb{R} \). We note that \( \alpha \) is positive, and \( a_i \leq 2k - 5 \) for every \( i \) with \( 1 \leq i \leq k-2 \). Moreover, \((2k-4)^2 - \beta < 0 \) holds because \( h^0((2k-4)L) > 0 \), \( \alpha > 0 \), \((2k-4)^2 - (2k-3)^2 < 0 \) and \( a_i^2 < (2k-4)^2 \) for every \( i \). Namely, \( \beta > (2k-4)^2 > 0 \). Here, we note that the coefficient of \( t^{2k-1} \) in the right hand side of (2) is

\[
-\alpha \left( \beta + \sum_{i=1}^{k-2} a_i^2 + (2k-3)^2 \right).
\]

On the other hand, by employing the Hirzebruch-Riemann-Roch formula for \( \chi(tL) \), we have

\[
-\alpha \left( \beta + \sum_{i=1}^{k-2} a_i^2 + (2k-3)^2 \right) = \frac{L^{n-2}c_2(X)}{12(n-2)!}
\]

and the right hand side of (3) is non-negative by [18, Theorem 6.6]. But the left hand side of (3) is negative because \( \alpha > 0 \) and \( \beta > 0 \). Hence, this is a contradiction. Therefore, \( h^0((n-4)L) > 0 \) holds.

(III) Here we consider the cases (c) and (d) in Theorem 2.1. Then, we take the following string of \( 2k-1 \) pairs

\[
(h^0(L), h^0((m-1)L)), (h^0(2L), h^0((m-2)L)), \ldots, (h^0((2k-1)L), h^0((m-2k+1)L)).
\]

First, we note the following.

\[
\begin{align*}
\begin{cases}
\text{In (c), we consider the case that } n = 4k + 4 \text{ and } m \geq n - 5 = 4k - 1. \\
\text{In (d), we consider the case that } n = 4k + 2 \text{ and } m \geq n - 3 = 4k - 1. 
\end{cases}
\end{align*}
\]

We also note that there are no overlaps among \( iL \)'s in the string (4) in these cases. By an argument similar to (\*) in (I.2), we obtain that there are at least \( 2k-1 \) zeros of \( h^0(tL) \) in the string (4).

Here, we consider the set

\[
\mathcal{B} = \{ t \in \mathbb{Z} \mid 2k \leq t \leq m-2k \}.
\]

We remark that \( h^0(tL) \) is not contained in the string (4) for every \( t \in \mathcal{B} \). We see from Proposition 2.2 that there is at most one \( p \in \mathcal{B} \) such that \( h^0(pL) = 0 \).

(III.1) We assume that \( m \geq 4k \). Then, we remark that \( \mathcal{B} \) is nonempty.

(III.1.1) If \( m > 4k \) (namely, \( m \geq n - 1 \) (resp. \( m \geq n - 3 \) if \( n = 4k + 2 \) (resp. \( n = 4k + 4 \))), then \( 2k \neq m-2k \). Moreover, we remark that \( 2k + (m-2k) = m \). Since \( 2k \) and \( m-2k \) are elements of \( \mathcal{B} \), we can conclude that \( h^0(mL) > 0 \) by using Proposition 2.2 in a similar way to the case (I.2.1) above.
(III.1.2) If \( m = 4k \) (namely, \( m = n - 2 \) (resp. \( m = n - 4 \)) if \( n = 4k + 2 \) (resp. \( n = 4k + 4 \)), then \( \mathcal{B} = (2k) \). Then we can conclude that \( h^0(mL) = h^0(4kL) > 0 \) by using Proposition 2.2 in a similar way to the case (I.2.2) above.

(III.2) Finally, we consider the case that \( m = 4k - 1 \).

In the string (4), we consider the pair \( (h^0(L), h^0((4k - 2)L)) \). If \( h^0(L) > 0 \), then we can see that \( h^0(mL) > 0 \). So we may assume that \( h^0(L) = 0 \). If \( h^0((4k - 2)L) = 0 \), then there are at least 2k zeros of \( h^2(tL) \) in the string (4). Hence, we see from Proposition 2.2 that \( h^0(mL) > 0 \). Therefore, we may assume that \( h^0(L) = 0 \) and \( h^0((4k - 2)L) > 0 \).

(III.2.1) We consider the case that \( (n, m) = (4k + 2, 4k - 1) \). Assume by contradiction that \( h^0(mL) = 0 \). Then, \( \chi(tL) \) has 2k - 1 zeros \( 1 = a_1 < \ldots < a_{2k-1} \leq 4k - 3 = m - 2 \) and another zero \( m = 4k - 1 \). Therefore, we see from Remark 2.1 that \( \chi(tL) \) has 4k zeros \( \pm a_1, \ldots, \pm a_{2k-1}, \pm (4k - 1) \) and we may write
\[
\chi(tL) = \alpha(t^2 - a_1^2) \cdots (t^2 - a_{2k-1}^2)(t^2 - (4k - 1)^2)(t^2 - \beta),
\]
where \( \alpha, \beta \in \mathbb{R} \). But then, we can get a contradiction by an argument similar to (II) above. Therefore, \( 0 < h^0(mL) = h^0((n - 3)L) \) holds.

(III.2.2) We consider the case that \( (n, m) = (4k + 4, 4k - 1) \). Assume by contradiction that \( h^0(mL) = 0 \). Then, \( \chi(tL) = 0 \) has 2k - 1 zeros
\[
1 = d_1 < \ldots < d_{2k-1} \leq 4k - 3
\]
and another zero \( m = 4k - 1 \). Hence, we see from Remark 2.1 that \( \chi(tL) \) has 4k zeros \( \pm d_1, \ldots, \pm d_{2k-1}, \pm (4k - 1) \) and we may write
\[
\chi(tL) = \alpha(t^2 - d_1^2) \cdots (t^2 - d_{2k-1}^2)(t^2 - (4k - 1)^2)(t^4 - pt^2 + q),
\]
where \( \alpha, p, q \in \mathbb{R} \). We note that \( \alpha > 0 \).

Assume that \( p \geq 0 \). Then, the coefficient of \( t^{4k+2} \) in the right hand side of (6) is
\[
-\alpha \left( p + \sum_{i=1}^{2k-1} d_i^2 + (4k - 1)^2 \right)
\]
and this is negative. On the other hand, by employing the Hirzebruch-Riemann-Roch formula for \( \chi(tL) \), we have
\[
-\alpha \left( p + \sum_{i=1}^{2k-1} d_i^2 + (4k - 1)^2 \right) = \frac{L^{n-2}c_2(X)}{12(n-2)!}
\]
and the right hand side of (7) is non-negative by [18, Theorem 6.6]. Hence we get a contradiction and we see that
\[
p < 0.
\]

By (6), we have
\[
0 < h^0((4k - 2)L) = \chi((4k - 2)L) = \alpha((4k - 2)^2 - d_1^2) \cdots ((4k - 2)^2 - d_{2k-1}^2)((4k - 2)^2 - (4k - 1)^2)((4k - 2)^4 - p(4k - 2)^2 + q).
\]
Hence, we see from (5) and (8) that \( q < 0 \) holds, and we get
\[
\chi(\mathcal{O}_X) = \alpha(-d_1^2) \cdots (-d_{2k-1}^2)(- (4k - 1)^2)q < 0.
\]
But, this is impossible because $\chi(O_X) \geq 0$ by the Beauville-Bogomolov decomposition (see [2], [4], [7, Theorem 2.1]).

Therefore we get $0 < h^0(mL) = h^0((n - 5)L)$.

By Theorem 2.1 and [7, Remark 5.3], we obtain the following corollaries.

**Corollary 2.1** Let $X$ be a smooth projective variety of dimension $n \geq 5$, and let $L$ be a nef and big divisor on $X$. Assume that $K_X$ is numerically equivalent to zero. Then, $h^0(K_X + mL) > 0$ for every integer $m \geq n - 3$.

**Corollary 2.2** Let $(X, L)$ be a quasi-polarized manifold. Assume that $K_X$ is numerically equivalent to zero. Then, Conjecture 1 is true.

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