TOTALLY GEODESIC IMMERSIONS INTO GRASSMANNIANS (HARMONIC MAPS INTO GRASSMANN MANIFOLDS II)

YASUYUKI NAGATOMO

Abstract. We define a totally geodesic immersion of irreducible type from a symmetric space of compact type into a Grassmannian and classify such immersions. Any totally geodesic immersion is related to a homogeneous vector bundle with the canonical connection and the eigenspaces of the Laplace operator acting on the space of sections of the bundle.

1. Introduction

The main purpose of the present paper is to classify totally geodesic submanifolds of Grassmannians. This subject has been pursued by many authors for a long time. For instance, Chen-Nagano have introduced a new geometric idea ([1] and [2]), and Ikawa-Tasaki have made a detailed study of the corresponding Lie algebras to classify those submanifolds in symmetric spaces [10]. Instead of these methods, we exploit differential geometry of vector bundles.

Since any Grassmannian $Gr_p(W)$ parametrizing p-subspaces of a vector space W is a Riemannian symmetric space, totally geodesic submanifolds are also symmetric spaces, say G/K, and the immersion $f : G/K \to Gr_p(W)$ is G-equivariant. The well-known example is given by a flat torus and such a flat torus is necessarily contained in a maximal torus. Hence we restrict our concern to G/K being a Riemannian symmetric space of compact type. Since the immersion $f : G/K \to Gr_p(W)$ is G-equivariant, the vector space W is considered as a representation space of G. If W is irreducible as G-module, then f is called a totally geodesic submanifold of *irreducible type*. We classify all totally geodesic submanifolds of irreducible type.

To do so, we use a vector bundle and a finite dimensional vector space of sections when we describe a map into a Grassmannian. Such a map is called the *induced map* (Definition 3.2). A famous example of an induced map is the Kodaira embedding from an algebraic manifold into a complex projective space, which is induced by a holomorphic line bundle and the space of holomorphic sections.

As a result, we obtain

Main Theorem. (Theorem 4.21) *Let* $(G = G_1 \times G_2 \times \cdots \times G_\Lambda, K = K_1 \times K_2 \times \cdots \times K_\Lambda)$ *be a symmetric pair of compact type with the standard involution* σ *such that* (G_λ, K_λ) *is an irreducible symmetric pair, where* G_λ

2010 *Mathematics Subject Classification.* 53C35, 53C42.

is a simply-connected compact Lie group and K_λ is a connected subgroup of G_λ for $\lambda = 1, \cdots, \Lambda$.

If $f : G/K \to Gr_p(W)$ is a totally geodesic submanifold of irreducible type into a complex Grassmannian, then,
(i) *in the case when* $\operatorname{rank} G = \operatorname{rank} K$*, W is an irreducible G-module of complete type, or*
(ii) *in the case when* $\operatorname{rank} G > \operatorname{rank} K$*, $W = W_1 \otimes W_2 \otimes \cdots \otimes W_\Lambda$ is an irreducible G-module of complete type such that the irreducible G_λ-module W_λ is self-conjugate when* $\operatorname{rank} G_\lambda > \operatorname{rank} K_\lambda$.

Conversely, let $W = W_1 \otimes W_2 \otimes \cdots \otimes W_\Lambda$ be an irreducible G-module of complete type. When $\operatorname{rank} G_\lambda > \operatorname{rank} K_\lambda$*, suppose further that the irreducible G_λ-module W_λ is self-conjugate. Then W has the unique generalized Cartan decomposition $W = U_0 \oplus V_0$ for (G, K) with $p = \dim U_0$ and $q = \dim V_0$ and we have a totally geodesic submanifold $f : G/K \to Gr_p(W)$ of irreducible type as the mapping induced by $(V = G \times_K V_0 \to G/K, W)$.*

Under these conditions, p and q satisfy

$$\frac{(p-q)^2}{\dim W} = \int_G \chi_\varrho \left(g\sigma(g^{-1}) \right) dg,$$

where χ_ϱ is the character of G-representation (ϱ, W) and dg is the normalized Haar measure on G.

A generalized Cartan decomposition of a representation space of G for a Riemannian symmetric pair (G, K) is defined in Definition 4.4, which plays a significant role in this paper. The corresponding result in the case when the target is a real Grassmannian is obtained in Theorems 4.32. We apply a generalization of Theorem of Tsunero Takahashi (Theorem 2.10) [13] to obtain a classification of totally geodesic immersions of the complex projective line $\mathbf{C}P^1$ into Grassmann manifolds (Theorems 5.4 and 5.5). Theorem 2.10 relates a totally geodesic immersion of a symmetric space M into a Grassmannian to a homogeneous vector bundle $V \to M$ with the canonical connection and the Laplace operator acting on the space of sections of $V \to M$. Every non-trivial irreducible homogeneous vector bundle on $\mathbf{C}P^1$ is of complex rank one and it is easy to describe all eigenspaces of the Laplace operator defined by the canonical connection on each irreducible homogeneous bundle in a similar method to a description of spherical functions (see [14] and [16]). We discuss a totally geodesic immersion of a compact Lie group into a Grassmannian in detail.

Our Main Theorem enables us to generalize Theorem 2.10 in the case when the target is a symmetric space of compact type (Theorem 6.3, see also Theorem 6.1 and Corollary 6.2). This generalization could justify the subtitle of the paper.

The author would like to express his gratitude to Professor M.Takahashi for many valuable comments and discussions. He also wishes to thank the referee for the careful reading of the manuscript and for suggesting various improvements.

2. Preliminaries

We review some standard material, mostly in order to fix our notation in this paper. Throughout this paper, all manifolds are supposed to be connected. For a vector bundle $V \to M$ over a manifold M, $\Gamma(V)$ denotes the space of sections of $V \to M$.

2.1. **A harmonic map.** Let M and N be Riemannian manifolds. We define *the energy density* $e(f) : M \to \mathbf{R}$ of a map $f : M \to N$ as

$$e(f)(x) := |df|^2,$$

where we use both Riemannian metrics on M and N to obtain the Hilbert-Schmidt norm. Then, *the tension field* $\tau(f)$ of f is defined to be

$$\tau(f) := \operatorname{trace} \nabla df = \sum_{i=1}^{m} (\nabla_{e_i} df)(e_i),$$

which is a section of the pull-back bundle $f^*TN \to M$ of the tangent bundle $TN \to N$, where $e_1, \cdots, e_m$ is an orthonormal basis of the tangent space T_xM to M at $x \in M$ and $m = \dim M$.

Definition 2.1. [7] A map $f : M \to N$ is called a *harmonic map* if the tension field vanishes identically ($\tau(f) \equiv 0$).

The symmetric form ∇df with values in $f^*TN \to M$ is called *the second fundamental form*. We say that a map $f : M \to N$ is a *totally geodesic map* if $\nabla df \equiv 0$. By definition, a totally geodesic map is a harmonic map.

If we suppose that $f : M \to N$ is an isometric immersion, then the tension field is a mean curvature vector, the second fundamental form is the same as that in submanifold geometry and a harmonic map is nothing but a minimal immersion.

2.2. **Geometry of Grassmannians.** Though we condense definitions and results in the following two subsections, readers may consult [13] for more details.

Let W be a real vector space with an inner product $(\cdot,\cdot)$ and an orientation or a complex vector space with a Hermitian inner product $(\cdot,\cdot)$. We call $(\cdot,\cdot)$ a *scalar product* for short.

Let $Gr_p(W)$ be a Grassmann manifold of (oriented) p-planes in W. To define a Riemannian metric g_{Gr} on $Gr_p(W)$, let $S \to Gr_p(W)$ be a tautological vector bundle of rank p. We have an exact sequence of vector bundles:

$$0 \to S \xrightarrow{i} \underline{W} \xrightarrow{\pi} Q \to 0,$$

where $\underline{W} \to Gr_p(W)$ is the trivial vector bundle of fiber W, and $Q \to Gr_p(W)$ is the quotient bundle, which is called the *universal quotient bundle*. The scalar product gives a fiber metric on $S \to Gr_p(W)$ denoted by g_S and the orthogonal complementary subbundle of $i(S)$ in $\underline{W}$ can be identified with $Q \to Gr_p(W)$. Hence we also obtain a fiber metric on $Q \to Gr_p(W)$ denoted by g_Q. Consequently, we have two bundle maps $i^* : \underline{W} \to S$ and $\pi^* : Q \to \underline{W}$ as the adjoint bundle maps of the indicated bundle maps. Since the tangent bundle denoted by $T \to Gr_p(W)$ is identified with $S^* \otimes Q$, the

Riemannian metric g_{Gr} is induced as the tensor product of g_{S^*} and g_Q : $g_{Gr} = g_{S^*} \otimes g_Q$, which is called the metric of *Fubini-Study* type.

We can define a connection ∇^Q on $Q \to Gr_p(W)$: if t is a section of $Q \to Gr_p(W)$, then we have

$$\nabla^Q t = \pi d\left(\pi^*(t)\right).$$

In a similar way, we can define a connection ∇^S:

$$\nabla^S s = i^* d\left(i(s)\right), \quad s \in \Gamma(S).$$

In this context, since $S \to Gr_p(W)$ is a subbundle of $\underline{W} \to Gr_p(W)$, it is natural to introduce the second fundamental form H in the sense of Kobayashi [11], which is a 1-form with values in $\mathrm{Hom}(S,Q) \cong S^* \otimes Q$:

$$H(s) = \pi d\left(i(s)\right), \quad \text{for } s \in \Gamma(S).$$

The second fundamental form H gives an explicit identification of T with $S^* \otimes Q$ preserving the metrics and the connections, which yields that the Levi-Civita connection is induced by ∇^S and ∇^Q. Hence we have

Lemma 2.2. *The second fundamental form H can be regarded as the identity transformation of the tangent bundle T.*

Corollary 2.3. *The second fundamental form H is parallel.*

The second fundamental form K is also defined as a 1-form with values in $\mathrm{Hom}(Q,S) \cong Q^* \otimes S$:

$$K(t) = i^* d\left(\pi^*(t)\right), \quad \text{for } t \in \Gamma(Q).$$

For a vector $w \in W$, we have two sections $s = i^*(w)$ and $t = \pi(w)$, each of which is called *the section corresponding to w*. Thus we obtain two linear monomorphisms $W \to \Gamma(S)$ and $W \to \Gamma(Q)$ and W can be regarded as a subspace of $\Gamma(S)$ and $\Gamma(Q)$. From the definition, we have

Proposition 2.4. *If s and t are the sections corresponding to $w \in W$, then*

$$\nabla^S s = -K(t), \quad \nabla^Q t = -H(s).$$

Since $g_Q(H(s),t) = (di(s), \pi^* t) = -\left(i(s), d\pi^*(t)\right) = -g_S(s, K(t))$, we have

Lemma 2.5. *The second fundamental forms H and K satisfy*

$$g_Q(H(s),t) = -g_S(s,K(t)), \quad \textit{for } s \in \Gamma(S) \textit{ and } t \in \Gamma(Q).$$

Lemma 2.6. *The second fundamental form K is also parallel.*

From Lemma 2.2, we obtain

Proposition 2.7. *For arbitrary real tangent vectors X and Y to $Gr_p(W)$, we have*

$$g_{Gr}(X,Y) = -\mathrm{trace}_Q\, H_X K_Y = -\mathrm{trace}_S\, K_X H_Y,$$

in the case where W is a real vector space, and

$$g_{Gr}(X,Y) = -2\mathrm{Re}\left(\mathrm{trace}_Q\, H_X K_Y\right) = -2\mathrm{Re}\left(\mathrm{trace}_S\, K_X H_Y\right),$$

in the case where W is a complex vector space.

A Grassmannian $Gr_p(W)$ with the Fubini-Study metric g_{Gr} is a Riemannian symmetric space and the vector bundles $S \to Gr_p(W)$ and $Q \to Gr_p(W)$ can be regarded as homogeneous vector bundles on $Gr_p(W)$ with invariant fiber metrics and connections.

2.3. **Harmonic maps into Grassmannians.** We introduce some results in [13] which are needed in later chapters.

Let $f : M \to Gr_p(W)$ be a smooth map. Pulling back $Q \to Gr_p(W)$ to M, we obtain a vector bundle $f^*Q \to M$, which is denoted by $V \to M$. Though W also gives sections of $V \to M$, the linear map $W \to \Gamma(V)$ might not be an injection. Even in such a case, W is still called a space of sections.

We fix a scalar product $(\cdot, \cdot)$ on a vector space W. If $f : M \to Gr_p(W)$ is a smooth map, then we also pull back the fiber metric and the connection on $Q \to Gr_p(W)$ to obtain a fiber metric g_V and a connection ∇^V on $V \to M$.

In a similar way, the pull-back bundle $f^*S \to M$ is denoted by $U \to M$.

The second fundamental forms are also pulled back and denoted by the same symbols $H \in \Gamma(f^*T^* \otimes U^* \otimes V)$ and $K \in \Gamma(f^*T^* \otimes V^* \otimes U)$. If we restrict bundle-valued linear forms H and K on the pull-back bundle $f^*T^* \to M$ to linear forms on M, then H and K are nothing but the second fundamental forms of subbundles $U \to \underline{W}$ and $V \to \underline{W}$, respectively, where $\underline{W}$ is a trivial vector bundle $M \times W \to M$.

From now on, we assume that M is a Riemannian manifold. Then, we use the Riemannian structure on M and the pull-back connection on $V \to M$ to define the Laplace operator $\Delta^V = \Delta = \nabla^{V*}\nabla^V = -\sum_{i=1}^m \nabla^V_{e_i}\left(\nabla^V\right)(e_i)$ acting on $\Gamma(V)$ and a bundle endomorphism $A \in \Gamma\left(\operatorname{End} V\right)$ is defined as the trace of the composition of the second fundamental forms H and K:

$$A := \sum_{i=1}^m H_{e_i} K_{e_i},$$

where m is the dimension of M and $e_1, e_2, \dots, e_m$ is an orthonormal basis of the tangent space to M. The bundle endomorphism $A \in \Gamma\left(\operatorname{End} V\right)$ is called the *mean curvature operator* of f.

From Lemma 2.5 and Proposition 2.7, we obtain

Lemma 2.8. *The mean curvature operator A is a negative semi-definite symmetric (or Hermitian) operator.*

Lemma 2.9. *The energy density $e(f)$ is equal to $-\operatorname{trace} A$ in the case when W is a real vector space or $-2\operatorname{trace} A$ in the case when W is a complex vector space.*

We introduce a generalization of Theorem of Tsunero Takahashi [15] which is shown in [13].

Theorem 2.10. *Let M be a Riemannian manifold and $f : M \to Gr_p(W)$ a smooth map. We fix a scalar product $(\cdot, \cdot)$ on W, which gives a Riemannian metric g_{Gr} on $Gr_p(W)$. We regard W as a space of sections of the pull-back bundle $f^*Q \to M$.*

Then, the following two conditions are equivalent.

(1) *$f : M \to Gr_p(W)$ is a harmonic map.*

(2) *There exists a bundle endomorphism* $\tilde{A}$ *of the pull-back of the universal quotient bundle such that* $\Delta t + \tilde{A}t = 0$ *for an arbitrary* $t \in W$.

Under these conditions, $\tilde{A} = A$, *where* A *is the mean curvature operator of* $f : M \to Gr_p(W)$ *and*

$$e(f) = -\mathrm{trace}\, A \ (W \textit{ is real}), \quad e(f) = -2\mathrm{trace}\, A \ (W \textit{ is complex}).$$

3. Induced maps

In this section, we give a way of construction of maps into Grassmannians.

3.1. The map induced by a vector bundle and the space of sections.

We refer to [13] for geometric meaning of definitions in this subsection.

Definition 3.1. Let $V \to M$ be a vector bundle over a manifold M and W a subspace of $\Gamma(V)$. An evaluation map $ev : \underline{W} \to V$ is defined as $ev(t)(x) := t(x) \in V_x$ for $t \in W$ and $x \in M$. The vector bundle $V \to M$ is said to be *globally generated by* W if $ev : \underline{W} \to V$ is surjective.

Definition 3.2. Let $V \to M$ be a real or complex vector bundle of rank q which is globally generated by W of dimension N. If the real vector bundle $V \to M$ has an orientation, we also fix an orientation on W. Then we have a map $f : M \to Gr_p(W)$, where $Gr_p(W)$ is a real (oriented) or complex Grassmannian according to the coefficient field of $V \to M$ and $p = N - q$. The map f is defined by

$$f(x) := \mathrm{Ker}\, ev_x = \{t \in W \mid t(x) = 0\}\,,$$

where the orientation of $f(x)$ is induced by those of $V \to M$ and W. We call $f : M \to Gr_p(W)$ the *map induced by* $(V \to M, W)$, or the *map induced by* W, if the vector bundle $V \to M$ is specified.

From the definition of the induced map $f : M \to Gr_p(W)$, the vector bundle $V \to M$ can be identified with $f^*Q \to M$.

Conversely, if $f : M \to Gr_p(W)$ is a smooth map, then we obtain a vector bundle $f^*Q \to M$ which is globally generated by W, where W is regarded as a space of sections of the pull-back bundle. It is easily observed that the map induced by W is the same as the original map $f : M \to Gr_p(W)$. In this way, every map $f : M \to Gr_p(W)$ can be recognized as the map induced by $(f^*Q \to M, W)$.

Definition 3.3. Let $f : M \to Gr_p(W)$ be a map and regard W as a space of sections of $f^*Q \to M$. Then the map $f : M \to Gr_p(W)$ is called a *full map* if the linear map $W \to \Gamma(f^*Q)$ is injective.

Notice that the notion of full map is the same as one in [4], [14] and [17] if the target space is the sphere or the complex projective space.

Definition 3.4. Let f be a full map of M into $Gr_p(W)$ with the Fubini-Study metric.

Then $f : M \to Gr_p(W)$ is called a full map with *trivial summand*, if
(1) the pull-back of the universal quotient bundle is decomposed into $V_0 \oplus V_1 \to M$, where $V_0 \to M$ is a trivial bundle with a flat connection, and
(2) W has a subspace W_0 which consists of parallel sections of $V_0 \to M$ and does not induce any sections of $V_1 \to M$ except the zero section.

We call f a full map with *no trivial summand*, unless f is a full map with trivial summand.

3.2. **Equivariant maps.** Let G be a compact Lie group and K a closed subgroup of G. Let G/K be a compact reductive Riemannian homogeneous space with decomposition of Lie algebra $\mathfrak{g}$ of G: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the corresponding subalgebra of K. We denote by e the unit element of G and by $[e] \in G/K$ the coset represented by e. Thus K is the stabilizer subgroup at $[e]$. By Riemannian homogeneous space, we mean that a G-invariant metric on G/K is fixed.

Let (ϱ, V_0) be an orthogonal or unitary representation of K with a K-invariant scalar product. The representation (ϱ, V_0) is abbreviated by V_0. We can construct a homogeneous vector bundle $V \to G/K$, $V := G \times_K V_0$ with an invariant fiber metric g_V induced by the scalar product on V_0. The restriction of the action of G on $V \to G/K$ to K provides us with an action of K on the fiber $V_{[e]}$ of $V \to G/K$ at $[e]$. Moreover $V \to G/K$ has the canonical connection ∇ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. (This means that the horizontal distribution is defined as $\{L_g \mathfrak{m} \subset TG_g \,|\, g \in G\}$ on the principal fiber bundle $G \to G/K$, where L_g denotes the left translation on G.) A Lie group G naturally acts on the space of sections $\Gamma(V)$ of $V \to G/K$, which has a G-invariant L^2-scalar product.

The next lemma plays an important role in our classification of totally geodesic immersions. Hence we introduce it with a proof [13].

Lemma 3.5. *Let $V = G \times_K V_0$ be a homogeneous vector bundle with an invariant fiber metric and W a G-subspace of $\Gamma(V)$ with the L^2-scalar product. If W globally generates $V \to G/K$, then V_0 can be regarded as a subspace of W.*

Proof. Since the evaluation map $ev : \underline{W} \to V$ is G-equivariant and the scalar product and the fiber metric are G-invariant, the adjoint map $ev^* : V \to \underline{W}$ is also G-equivariant. Then the image of $ev^*_{[e]}$ is a K-module. We identify V_0 with the fiber $V_{[e]}$ of $V \to G/K$ at $[e]$. Since W globally generates $V \to G/K$, we can deduce that the image is a K-module equivalent to V_0. □

We call a map $f : G/K \to Gr_p(W)$ an *equivariant map* if we have an orthogonal or unitary representation (ϱ, W) such that $f(gx) = \varrho(g)f(x)$, where $g \in G$, $x \in G/K$. The image $f(x)$ of $x \in G/K$ represents a subspace of W.

Let $f : G/K \to Gr_p(W)$ be an equivariant map. Then $f^*Q \to G/K$ is a homogeneous vector bundle with an invariant metric and an invariant connection under the action of G. The mean curvature operator is an invariant endomorphism of $f^*Q \to G/K$. The evaluation map $ev : \underline{W} \to V$ is also a G-equivariant bundle map.

Lemma 3.6. *Let $f : G/K \to Gr_p(W)$ be an equivariant map which is not a constant map. If W is an irreducible G-module, then f is a full map with no trivial summand.*

Proof. Suppose that $f : G/K \to Gr_p(W)$ is an equivariant map, which is not full. Then we have a subspace W_0 of W which induces only zero section on $f^*Q \to G/K$ and the restriction of the linear map $W \to \Gamma(f^*Q)$ to the orthogonal complement of W_0 in W is injective. Since $ev : \underline{W} \to G/K$ is G-equivariant, W_0 is a G-module.

Next, we suppose that an equivariant full map $f : G/K \to Gr_p(W)$ has a trivial summand. By definition, the pull-back of the universal quotient bundle has a decomposition $V_0 \oplus V_1 \to G/K$, where $V_0 \to G/K$ is a trivial bundle with a flat connection and W has a subspace W_0 which consists of parallel sections of $V_0 \to G/K$ and does not induce any sections of $V_1 \to G/K$ except the zero section. Moreover we suppose that such a flat vector bundle $V_0 \to G/K$ has a maximal rank. Since f is a full map, we have that $\dim W_0 = \operatorname{rank} V_0$. We take the orthogonal complement denoted by W_1 of W_0 in W. Then the maximality of the rank of V_0 implies that the map induced by (V_1, W_1) is a full map with no trivial summand. If $t \in W_0$ and $g \in G$, then $gt \in W$ is also a parallel section, since the induced connection is an invariant connection. Hence W_0 is a G-submodule.

Since W is irreducible and f is not a constant map, $W_0 = \{0\}$ in both cases. □

Let $V = G \times_K V_0$ be a homogeneous vector bundle of rank q over G/K. Suppose that a G-subspace W of $\Gamma(V)$ globally generates $V \to G/K$. Then we have the map $f_0 : G/K \to Gr_p(W)$ induced by W, where $p = \dim W - q$,

$$f_0([g]) = \{t \in W \,|\, t([g]) = 0\}.$$

Since $V_0 \subset W$ by Lemma 3.5, we have the orthogonal complement of V_0 denoted by U_0, which is also a K-module. Then the induced map $f_0 : G/K \to Gr_p(W)$ is expressed as

$$f_0([g]) = gU_0 \subset W,$$

which is G-equivariant.

Let (ϱ, W) be an orthogonal or a unitary representation of G. We restrict the Lie group homomorphism ϱ to the subgroup K of G to obtain a representation of K. Suppose that V_0 is a K-invariant subspace of W. Then we denote by $\varrho(\mathfrak{m})V_0$ or $\mathfrak{m}V_0$ for short the subspace of W generated by $\mathfrak{m}$ and V_0. With these understood, we have

Lemma 3.7. *Let $f_0 : G/K \to Gr(W)$ be the map induced by (V, W) where W is a G-subspace of $\Gamma(V)$. Then the pull-back connection ∇^V is gauge equivalent to the canonical connection if and only if $\mathfrak{m}V_0 \subset U_0$.*

For a proof, see [13].

4. Totally geodesic submanifolds of Grassmannians

First of all, notice that Corollary 2.3 yields the fundamental equation $\nabla H = H_{\nabla df}$ and we can show the following (see [13]):

Theorem 4.1. *A map $f : M \to Gr_p(W)$ is a totally geodesic map if and only if the second fundamental form H of the pull-back bundles is parallel.*

Corollary 4.2. *If $f : M \to Gr_p(W)$ is a totally geodesic map, then the mean curvature operator of f is parallel.*

Throughout this section, we suppose that (G, K) is a Riemannian symmetric pair of compact type and $M = G/K$ denotes the corresponding symmetric space. The associated standard involutions of G and $\mathfrak{g}$ are denoted by the same symbol σ and the orthogonal decomposition induced by σ is denoted by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

4.1. **A generalized Cartan decomposition.** Let $Gr_p(W)$ be a Grassmannian with the Riemannian metric of Fubini-Study type. For an orthogonal direct sum decomposition of W:$W = U_0 \oplus V_0$ with $p = \dim U_0$ and $q = \dim V_0$, we define an automorphism $I_{p,q}$ of W as

$$I_{p,q}|_{U_0} = Id_{U_0}, \quad \text{and} \quad I_{p,q}|_{V_0} = -Id_{V_0}, \tag{4.1}$$

and a standard involution $\tilde{\sigma}$ of $\operatorname{Aut} W$ as

$$\tilde{\sigma}(S) = I_{p,q} S I_{p,q}, \quad S \in \operatorname{Aut} W.$$

Then we have a Riemannian symmetric pair $(\tilde{G} = \operatorname{Aut} W, \tilde{K})$ of compact type associated with $\tilde{\sigma}$ such that $Gr_p(W) = \tilde{G}/\tilde{K}$ and an orthogonal decomposition: $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{m}}$, which means that

$$\mathfrak{so}(W) = \mathfrak{so}(p) \oplus \mathfrak{so}(q) \oplus \tilde{\mathfrak{m}}, \quad \text{or} \quad \mathfrak{su}(W) = \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathfrak{u}(1) \oplus \tilde{\mathfrak{m}}.$$

Let $f : M \to Gr_p(W)$ be a totally geodesic submanifold, where $Gr_p(W)$ is equipped with the Riemannian metric of Fubini-Study type. According to f, we have an injective Lie algebra homomorphism denoted by $\varrho : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that $\varrho(\mathfrak{m})$ is a subspace of $\tilde{\mathfrak{m}}$ [8, p.224 Theorem 7.2]. The corresponding Lie group homomorphism is denoted by the same symbol. Then f is a G-equivariant mapping and so, G acts on W preserving the scalar product, which is nothing but the representation $\varrho : G \to \operatorname{Aut} W$. Hence the pull-back of the universal quotient bundle denoted by $V \to M$ is also a homogeneous vector bundle. Since f is a totally geodesic immersion, the pull-back of the canonical connection is the canonical one on $V \to M$ (this is also proved by Lemma 3.7). Let V_0 be an associated K-representation with a homogeneous vector bundle $V \to M$, $V := G \times_K V_0$. Lemma 3.5 yields that V_0 can be regarded as a subspace of W. Then we take an orthogonal complement denoted by U_0 of V_0 in W to obtain the direct sum of K-modules :$W = U_0 \oplus V_0$. Since $\varrho(\mathfrak{m}) \subset \tilde{\mathfrak{m}}$, we have that

$$\varrho(\mathfrak{m}) U_0 \subset V_0, \quad \varrho(\mathfrak{m}) V_0 \subset U_0, \quad U_0 \perp V_0, \quad U_0 \neq \{0\}, \quad V_0 \neq \{0\}.$$

Consequently, f can be considered as the map induced by $(V \to M, W)$ and the pull back of the universal quotient bundle with the pull-back connection is isomorphic to $V \to M$ with the canonical connection:$f([g]) = \varrho(g) U_0$. Since f is an immersion, neither U_0 or V_0 is a G-module.

Moreover, we have

Theorem 4.3. *Suppose that $f : M \to Gr_p(W)$ is a totally geodesic submanifold. Let $Q \to Gr_p(W)$ be the universal quotient bundle. Then we have a decomposition $f^*Q = V_1 \oplus \cdots \oplus V_L$ invariant under the action of the holonomy group of the canonical connection, such that W is an eigenspace of the Laplacian of $V_l \to M$ and the mean curvature operator A is a scalar multiplication on $V_l \to M$ for each $l = 1, 2, \cdots, L$.*

Proof. We denote by $V \to M$ the pull-back bundle of $Q \to Gr_p(W)$. Since f is a totally geodesic immersion, $V \to M$ is a homogeneous vector bundle and the pull-back connection on $V \to M$ is the canonical one. It follows from Corollary 4.2 that A is parallel. Hence we have a decomposition of $V \to M$ into eigenbundles of A which is preserved by the canonical connection. Then each eigenbundle has an irreducible decomposition under the action of the holonomy group and we thus have a decomposition: $V = V_1 \oplus \cdots \oplus V_L$ such that A is a scalar on each $V_l \to M$. Since f is a harmonic map, Theorem 2.10 yields that W is an eigenspace of the Laplacian of $V_l \to M$. □

Definition 4.4. Let (ϱ, W) be an orthogonal or a unitary representation of G. Then (ϱ, W) is said to have a *generalized Cartan decomposition* (for the symmetric pair (G, K)) if W has an orthogonal direct sum decomposition: $W = U_0 \oplus V_0$ of two K-modules U_0 and V_0 over the same coefficient field as that of W under the restriction of the homomorphism ϱ to the subgroup K, in such a way that

$$\varrho(\mathfrak{m})U_0 \subset V_0, \quad \varrho(\mathfrak{m})V_0 \subset U_0, \quad U_0 \perp V_0, \quad U_0 \neq \{0\}, \quad V_0 \neq \{0\},$$

and neither U_0 or V_0 is a G-module. The decomposition $W = U_0 \oplus V_0$ is also called a generalized Cartan decomposition, more accurately, a real generalized Cartan decomposition or a complex generalized Cartan decomposition according to the coefficient field of W.

Remark. If the representation W is irreducible, then we do not need the condition that neither U_0 or V_0 is a G-representation in the definition.

Remark. As we have already seen, if $f : G/K \to Gr_p(W)$ is a totally geodesic submanifold, then we have a generalized Cartan decomposition for a symmetric pair (G, K): $W = U_0 \oplus V_0$. In this case, $W = U_0 \oplus V_0$ is called *a generalized Cartan decomposition induced by* f.

For making a description simpler without loss of generality, we suppose that G/K is simply-connected. If M is not simply-connected, then we may take a universal covering of M to obtain such a symmetric pair.

Thus, let (G, K) be a Riemannian symmetric pair of compact type, where G is a *simply-connected compact semi-simple* Lie group and K is a *connected* Lie subgroup of G throughout this section.

The de Rham decomposition yields that $G/K = G_1/K_1 \times G_2/K_2 \times \cdots \times G_\Lambda/K_\Lambda$. Here, $G = G_1 \times G_2 \times \cdots \times G_\Lambda$ is a direct product of Lie groups G_λ, $(\lambda = 1, \cdots, \Lambda)$ which are all simply-connected compact simple Lie groups and $K = K_1 \times K_2 \times \cdots \times K_\Lambda$, where each K_λ is a connected Lie subgroup of G_λ. We can regard (G_λ, K_λ) as an irreducible symmetric pair which has an orthogonal decomposition: $\mathfrak{g}_\lambda = \mathfrak{k}_\lambda \oplus \mathfrak{m}_\lambda$. Let e_λ be the unit element of G_λ. We fix $([e_1], \cdots, [e_\Lambda]) \in G/K$ to obtain a totally geodesic submanifold $i_\lambda : G_\lambda/K_\lambda \to G/K$. Then, we obtain $f_\lambda : G_\lambda/K_\lambda \to Gr_p(W)$ as a composition of i_λ and the mapping $f : G/K \to Gr_p(W)$. With this understood,

Lemma 4.5. *If $f : G/K \to Gr_p(W)$ is a totally geodesic submanifold with the induced generalized Cartan decomposition $W = U_0 \oplus V_0$ for (G, K), then $W = U_0 \oplus V_0$ is a generalized Cartan decomposition for each (G_λ, K_λ).*

Proof. Since $f_\lambda : G_\lambda/K_\lambda \to Gr_p(W)$ is a composition of the inclusion and the mapping f, it is a totally geodesic immersion. Then $W = U_0 \oplus V_0$ is a generalized Cartan decomposition for (G_λ, K_λ) induced by f_λ. □

Next, let (ϱ, W) be an orthogonal or unitary representation of G. Suppose that W is decomposed into an orthogonal direct sum of K-modules U_0 and V_0 which are the restriction of ϱ to K: $W = U_0 \oplus V_0$. Since a vector bundle $V := G \times_K V_0 \to M$ is globally generated by W, we obtain a G-equivariant mapping $f : M \to Gr_p(W)$ which is the mapping induced by (V, W). Using the de Rham decomposition, we also obtain $f_\lambda : G_\lambda/K_\lambda \to Gr_p(W)$ as a composite of the inclusion and f. Then the irreducibility of G_λ/K_λ yields that f_λ is an immersion or a constant mapping. By the definition of f_λ, U_0 is a G_λ-representation if and only if f_λ is a constant mapping.

Proposition 4.6. *Let (ϱ, W) be an orthogonal or a unitary representation of G. Suppose that the decomposition $W = U_0 \oplus V_0$ is a common generalized Cartan decomposition of W for each (G_λ, K_λ), where W is regarded as a representation of G_λ under the restriction. Then the map induced by $(G \times_K V_0 \to G/K, W)$ can be regarded as a totally geodesic immersion.*

Proof. We can deduce that $\varrho : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is an injection. Otherwise, G_λ acts trivially on W for some λ. Since neither U_0 or V_0 is a G_λ-representation by definition of a generalized Cartan decomposition, this causes a contradiction.

Since $W = U_0 \oplus V_0$ can also be considered as a generalized Cartan decomposition for (G, K), we see that

$$\begin{cases} \varrho(X) \in \tilde{\mathfrak{k}}, & X \in \mathfrak{k}, \\ \varrho(\xi) \in \tilde{\mathfrak{m}}, & \xi \in \mathfrak{m}, \end{cases}$$

and so,

$$\tilde{\sigma}\varrho(X) = \varrho(X), \quad \text{and} \quad \tilde{\sigma}\varrho(\xi) = -\varrho(\xi).$$

Since

$$\begin{aligned} \varrho\sigma(X) &= \varrho(X) = \tilde{\sigma}\varrho(X) \quad \text{for } X \in \mathfrak{k}, \quad \text{and} \\ \varrho\sigma(\xi) &= -\varrho(\xi) = \tilde{\sigma}\varrho(\xi) \quad \text{for } \xi \in \mathfrak{m}, \end{aligned}$$

we have

$$\varrho\sigma = \tilde{\sigma}\varrho. \tag{4.2}$$

It follows that any geodesic symmetry of G/K can be identified with that of $\tilde{G}/\tilde{K} = Gr_p(W)$ [8, p.224 Theorem 7.2]. □

If (ϱ, W) is an orthogonal or a unitary representation of G, then the composition $\varrho\sigma$ is also a representation of G, since σ is an automorphism of G. Let (φ, W') be another G-representation. When ϱ and φ are equivalent G-representations, we write $\varrho \sim \varphi$ or $W \sim W'$.

Corollary 4.7. *Let (ϱ, W) be an orthogonal or a unitary G-representation. If W has a generalized Cartan decomposition for (G, K), then $\varrho\sigma \sim \varrho$ as G-representation.*

Proof. From the definition of $\tilde{\sigma}$, (4.2) gives us $\varrho\sigma \sim \varrho$ as representation. □

To classify all totally geodesic immersions of G/K into Grassmann manifolds, we need to classify all representations of G which have a generalized Cartan decomposition for (G, K) from Lemma 4.5 and Proposition 4.6.

We now suppose that $f : G/K \to Gr_p(W)$ is a totally geodesic immersion. Let $W = U_0 \oplus V_0$ be the corresponding common generalized Cartan decomposition for each (G_λ, K_λ). It may happen that W can be decomposed into representation spaces of G in such a way that each of them has a common generalized Cartan decomposition for an arbitrary (G_λ, K_λ). Suppose that

$$W = \oplus_{l=1}^{L} W_l$$

is an orthogonal decomposition of W as G-representation such that

$$W_l = U_{l0} \oplus V_{l0}, \quad U_{l0} = W_l \cap U_0,\ V_{l0} = W_l \cap V_0,$$

where $W_l = U_{l0} \oplus V_{l0}$ is a common generalized Cartan decomposition for each (G_λ, K_λ). According to the decomposition, f is decomposed into immersions in an obvious way,

$$f = \left(f^1, \cdots, f^L\right) : G/K \to Gr_{p_1}(W_1) \times \cdots \times Gr_{p_L}(W_L) \to Gr_p(W),$$

where $p = \dim W - \dim V_0$ and $p_l = \dim W_l - \dim V_{l0}$. Since each submanifold $Gr_{p_l}(W_l)$ $(l = 1, \cdots, L)$ of $Gr_p(W)$ is a totally geodesic submanifold, each $f^l : G/K \to Gr_{p_l}(W_l)$ can be regarded as a totally geodesic immersion into $Gr_{p_l}(W_l)$. In this case, f is said to be *decomposable.* If f is not a decomposable mapping, then f is said to be *indecomposable.*

Hence we may focus our attention on full indecomposable totally geodesic immersions with no trivial summand for classification. From Lemma 3.6, we have

Lemma 4.8. *Let $f : G/K \to Gr_p(W)$ be a totally geodesic submanifold. If W is an irreducible G-module, then f is a full indecomposable mapping with no trivial summand.*

However, in general, the eigenspaces of the Laplacian on homogeneous vector bundles over G/K are not irreducible as G-module.

Definition 4.9. If $f : G/K \to Gr_p(W)$ is a totally geodesic submanifold and W is an irreducible G-module, then $f : G/K \to Gr_p(W)$ is called a *totally geodesic submanifold of irreducible type.*

We will classify all totally geodesic submanifolds of irreducible type.

4.2. **The case where the target is a complex Grassmannian.** Let W be an irreducible unitary representation of G. Notice that we have $W = W_1 \otimes W_2 \otimes \cdots \otimes W_\Lambda$, where each W_λ is an irreducible representation of G_λ.

Definition 4.10. Let W_λ be an irreducible unitary representation of G_λ for all $\lambda = 1, \cdots, \Lambda$ and $W = W_1 \otimes W_2 \otimes \cdots \otimes W_\Lambda$ an irreducible unitary representation of $G = G_1 \times \cdots \times G_\Lambda$. Then, W is called of complete type if each W_λ is a non-trivial representation of G_λ.

Lemma 4.11. *If $f : G/K \to Gr_p(W)$ is a totally geodesic submanifold of irreducible type, then W is an irreducible G-module of complete type.*

Proof. From Lemma 4.5, W has a common generalized Cartan decomposition $W = U_0 \oplus V_0$ for each (G_λ, K_λ). If W_λ is a trivial representation of G_λ, then U_0 and V_0 are also G_λ-modules, which contradicts the definition of the generalized Cartan decomposition. □

Lemma 4.12. *Let $(\varrho = \varrho_1 \otimes \cdots \otimes \varrho_\Lambda, W)$ and $(\varrho' = \varrho'_1 \otimes \cdots \otimes \varrho'_\Lambda, W')$ be irreducible representations of G. Then $\varrho \sim \varrho'$ if and only if $\varrho_\lambda \sim \varrho'_\lambda$ for each $\lambda = 1, \cdots, \Lambda$.*

Proof. We may take characters χ_ϱ and $\chi_{\varrho'}$ of the G-representations, since $\chi_\varrho = \chi_{\varrho_1} \cdots \chi_{\varrho_\Lambda}$ and $\chi_{\varrho'} = \chi_{\varrho'_1} \cdots \chi_{\varrho'_\Lambda}$. □

Theorem 4.13. *Let (ϱ, W) be an irreducible unitary representation of G which is a simply-connected compact semi-simple Lie group. If $\varrho\sigma \sim \varrho$, then W has a generalized Cartan decomposition for (G, K).*

Proof. From the hypothesis, we have an automorphism $C \in \mathrm{Aut}(W)$ satisfying $\varrho\sigma = C\varrho C^{-1}$. Since both representations preserve the Hermitian inner product, we may assume that $C^* = C^{-1}$.

If C is a constant multiple of the identity transformation, then we have $\varrho\sigma = \varrho$. It yields that $\varrho(\xi) = 0$ for an arbitrary $\xi \in \mathfrak{m}$. However, since G is semi-simple, the fact that $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$ gives us $\varrho = 0$ and so we get a contradiction.

Since σ is an involution, we have

$$\varrho(g) = \varrho\left(\sigma\sigma(g)\right) = C\left(\varrho\sigma(g)\right)C^{-1} = C^2\varrho(g)C^{-2}.$$

Schur's lemma yields that $C^2 = \mu Id_W$ for some $|\mu| = 1$, since $C^* = C^{-1}$. Hence C is diagonalizable with $\sqrt{\mu}$ and $-\sqrt{\mu}$ as eigenvalues. We denote by U and V the eigenspaces of C with eigenvalues $\sqrt{\mu}$ and $-\sqrt{\mu}$, respectively.

For $u \in U$ and $k \in K$, we have

$$C\varrho(k)u = C\varrho\sigma(k)u = \varrho(k)Cu = \varrho(k)\left(\sqrt{\mu}u\right) = \sqrt{\mu}\varrho(k)u.$$

This shows that $\varrho(k)u \in U$, and so U is a K-invariant subspace of W. In a similar way, we deduce that V is also a K-invariant subspace of W.

Next, since C is a unitary automorphism of W, U is perpendicular to V.

Finally we claim

$$\varrho(\mathfrak{m})U \subset V, \quad \varrho(\mathfrak{m})V \subset U.$$

To do this, notice that $\sigma(\xi) = -\xi$ for $\xi \in \mathfrak{m}$. Then, for $u \in U$, we have

$$\sqrt{\mu}\varrho(\xi)u = \varrho(\xi)Cu = -\varrho\sigma(\xi)Cu = -C\varrho(\xi)u.$$

It follows that $\varrho(\xi)u \in V$. The other claim is also shown in a similar way. □

Lemma 4.14. *If an irreducible unitary representation W of G has a generalized Cartan decomposition for (G, K), then it is unique up to the order.*

Proof. Suppose that W has two generalized Cartan decompositions. Corollary 4.7 and Theorem 4.13 yield that we have two automorphisms C_1 and C_2 of W such that

$$\varrho\sigma = C_1\varrho C_1^{-1} = C_2\varrho C_2^{-1}.$$

It gives

$$C_2^{-1}C_1\varrho = \varrho C_2^{-1}C_1.$$

Schur's lemma yields $C_2 = \lambda C_1$ $(\lambda \in \mathbf{C}\backslash\{0\})$ and so, the eigenspaces of C_1 coincide with those of C_2. □

Corollary 4.15. *Let* (ϱ, W) *be an irreducible unitary representation of* G *satisfying* $\varrho\sigma \sim \varrho$. *Then we have that* $\varrho\sigma = I_{p,q}\varrho I_{p,q}$, *where* $I_{p,q}$ *is defined in* (4.1) *for the corresponding generalized Cartan decomposition* $W = U_0 \oplus V_0$ *for* (G, K).

Lemma 4.16. *Let* $W = W_1 \otimes W_2 \otimes \cdots \otimes W_\Lambda$ *be an irreducible unitary* G*-module of complete type. Then,* $W = U_0 \oplus V_0$ *is the generalized Cartan decomposition for* (G, K) *if and only if for any* $\lambda = 1, 2, \cdots, \Lambda$, W_λ *has the generalized Cartan decomposition for* (G_λ, K_λ).

Proof. Suppose that $W = U_0 \oplus V_0$ is the generalized Cartan decomposition for (G, K). It follows from Corollary 4.7 that $\varrho\sigma \sim \varrho$. We get $\varrho_\lambda\sigma_\lambda \sim \varrho_\lambda$ by lemma 4.12. From the completeness of W, W_λ is not a trivial representation of G_λ. Theorem 4.13 yields that W_λ has the generalized Cartan decomposition.

Conversely, we suppose that W_λ has the generalized Cartan decomposition $W_\lambda = U_\lambda \oplus V_\lambda$ for all $\lambda = 1, 2, \cdots, \Lambda$. For example, in the case when $\Lambda = 2$, we may put $U_0 = U_1 \otimes U_2 \oplus V_1 \otimes V_2$ and $V_0 = U_1 \otimes V_2 \oplus V_1 \otimes U_2$. We can proceed in a successive way. □

Proposition 4.17. *Let* (G, K) *be an irreducible symmetric pair of compact type. If* $\operatorname{rank} G = \operatorname{rank} K$, *then all irreducible unitary representations of* G *have the generalized Cartan decomposition.*

Proof. Let (ϱ, W) be an irreducible unitary representation of G and χ_ϱ be the character of (ϱ, W). From the hypothesis, we can take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ in such a way that $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$. The corresponding maximal torus is denoted by T and is contained in K. Hence all elements of T is fixed by the standard involution σ. Consequently, we obtain

$$\chi_{\varrho\sigma}(t) = \chi_\varrho(t), \quad t \in T.$$

Since a character of a representation is completely determined by the restriction of a maximal torus, it follows that $\varrho\sigma \sim \varrho$. □

Lemma 4.18. *Suppose that* (G, K) *is an irreducible symmetric pair of compact type such that* $\operatorname{rank} G = \operatorname{rank} G/K$. *Then an irreducible unitary representation* ϱ *of* G *has generalized Cartan decomposition if and only if the dual representation of* ϱ *is equivalent to* ϱ *as representation.*

Proof. The assumption allows us to take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ in such a way that $\mathfrak{t} \subset \mathfrak{m} \subset \mathfrak{g}$. On the corresponding maximal torus T, we have $\chi_{\varrho\sigma}(t) = \chi_\varrho(t^{-1})$. In general, $\chi_\varrho(t^{-1}) = \chi_{\varrho^*}(t)$, where ϱ^* is the dual representation of ϱ. Corollary 4.7 and Theorem 4.13 yields the result. □

Remark. The proofs of Proposition 4.17 and Lemma 4.18 show that $\varrho \sim \varrho^*$, if $\operatorname{rank} G = \operatorname{rank} K = \operatorname{rank} G/K$. Such irreducible symmetric spaces of compact type are

$$\mathrm{Sp}(n)/\mathrm{U}(n),\ E_7/\mathrm{SU}(8),\ E_8/\mathrm{SO}(16),\ F_4/\mathrm{Sp}(3)\mathrm{SU}(2),\ G_2/\mathrm{SO}(4).$$

Hence all unitary representations of $\mathrm{Sp}(n)$, E_7, E_8, F_4 and G_2 are self-conjugate, (though it may be well-known).

Proposition 4.19. *Suppose that (G, K) is an irreducible symmetric pair of compact type such that* $\operatorname{rank} G > \operatorname{rank} K$*. Then an irreducible unitary representation ϱ has the generalized Cartan decomposition if and only if the dual representation of ϱ is equivalent to ϱ as representation.*

Proof. It follows from [8, Theorem 5.6, p.424] that the standard involution σ is an outer automorphism. Then, from [8, Theorem 5.4, p.423] and [8, Theorem 3.29, p.478] with its proof, we see that σ induces a symmetry on the Dynkin diagram with respect to a vertical axis for $\mathfrak{g} = \mathfrak{su}(n)$ or $\mathfrak{e}_6$ and that with respect to a horizontal axis for $\mathfrak{g} = \mathfrak{so}(2n)$ $(n \neq 4)$. In the remaining two cases ($G/K = S^7$ or $Gr_3(\mathbf{R}^8)$), we have $\mathfrak{g} = \mathfrak{so}(8)$. Then the standard representation $\mathbf{C}^8$ has the generalized Cartan decomposition for both. Hence σ also induces a symmetry of the Dynkin diagram with respect to a horizontal axis. Considering the induced action on the set of dominant integral weights, we can deduce that $\varrho\sigma$ is a dual representation of ϱ. □

Remark. When $\operatorname{rank} G = \operatorname{rank} K$, [8, Theorem 5.6, p.424] provides us with another proof of Proposition 4.17. In this case, the standard involution is inner and so, the weights of an irreducible representation are preserved under the action of σ. It follows that $\varrho\sigma$ is equivalent to ϱ.

Theorem 4.20. *Let (ϱ, W) be an irreducible unitary representation of G which has the generalized Cartan decomposition $W = U_0 \oplus V_0$ for (G, K) with $p = \dim U_0$ and $q = \dim V_0$. The character of (ϱ, W) is denoted by χ_ϱ. Then we have*

$$\frac{(p-q)^2}{\dim W} = \int_G \chi_\varrho\left(g\sigma(g^{-1})\right) dg, \tag{4.3}$$

where dg is the normalized Haar measure on G.

Proof. Since W has the generalized Cartan decomposition, Corollaries 4.7 and 4.15 yield that an automorphism $I_{p,q}$ of W satisfies $\varrho\sigma = I_{p,q}\varrho I_{p,q}$.

Schur's lemma yields that we have $\lambda \in \mathbf{C}$ such that

$$\lambda I_W = \int_G \varrho(g) I_{p,q} \varrho(g^{-1}) dg. \tag{4.4}$$

Taking the trace of both sides, we obtain

$$\lambda \dim W = p - q.$$

It follows from $\varrho(g)I_{p,q}\varrho(g^{-1}) = \varrho\left(g\sigma(g^{-1})\right) I_{p,q}$ that

$$\int_G \varrho(g)I_{p,q}\varrho(g^{-1})dg = \int_G \varrho\left(g\sigma(g^{-1})\right) I_{p,q} dg = \int_G \varrho\left(g\sigma(g^{-1})\right) dg I_{p,q},$$

and (4.4) yields that

$$\lambda I_{p,q} = \int_G \varrho\left(g\sigma(g^{-1})\right) dg.$$

Taking the trace again, we obtain the result. □

Remark. The integral in Theorem 4.20 can be described as an integral on a maximal torus T of the symmetric space G/K.

First of all, since the function $\chi_\varrho\left(g\sigma(g^{-1})\right)$ is K-invariant, we have

$$\int_G \chi_\varrho\left(g\sigma(g^{-1})\right) dg = \mathrm{vol}(K) \int_{G/K} i_C^*\chi_\varrho(x)dv,$$

where $i_C : G/K \to G$ is the so-called Cartan embedding $[g] \to g\sigma(g^{-1})$ and dv is the induced volume form on G/K. Since $i_C^*\chi_\varrho(x)$ is invariant under the isotropy action of K on G/K, we obtain

$$\begin{aligned}\int_{G/K} i_C^*\chi_\varrho(x)dv =& \frac{1}{W(G/K)^\sharp}\int_T i_C^*\chi_\varrho(t)D(t)dt \\ =& \frac{1}{W(G/K)^\sharp}\int_T \chi_\varrho(t^2)D(t)dt,\end{aligned}$$

where $W(G/K)$ is the Weyl group of G/K, $D(t)$ is the so-called density function [16, p.124] and dt is the induced volume form on a maximal torus T of G/K.

Theorem 4.21. *Let $(G = G_1 \times G_2 \times \cdots \times G_\Lambda, K = K_1 \times K_2 \times \cdots \times K_\Lambda)$ be a symmetric pair of compact type with the standard involution σ such that (G_λ, K_λ) is an irreducible symmetric pair, where G_λ is a simply-connected compact Lie group and K_λ is a connected subgroup of G_λ for $\lambda = 1, \cdots, \Lambda$.*

If $f : G/K \to Gr_p(W)$ is a totally geodesic submanifold of irreducible type into a complex Grassmannian, then,
(i) *in the case when* $\mathrm{rank}\, G = \mathrm{rank}\, K$*, W is an irreducible G-module of complete type, or*
(ii) *in the case when* $\mathrm{rank}\, G > \mathrm{rank}\, K$*, $W = W_1 \otimes W_2 \otimes \cdots \otimes W_\Lambda$ is an irreducible G-module of complete type such that the irreducible G_λ-module W_λ is self-conjugate when* $\mathrm{rank}\, G_\lambda > \mathrm{rank}\, K_\lambda$.

Conversely, let $W = W_1 \otimes W_2 \otimes \cdots \otimes W_\Lambda$ be an irreducible G-module of complete type. When $\mathrm{rank}\, G_\lambda > \mathrm{rank}\, K_\lambda$*, suppose further that the irreducible G_λ-module W_λ is self-conjugate. Then W has the unique generalized Cartan decomposition $W = U_0 \oplus V_0$ for (G, K) with $p = \dim U_0$ and $q = \dim V_0$ and we have a totally geodesic submanifold $f : G/K \to Gr_p(W)$ of irreducible type as the mapping induced by $(V = G \times_K V_0 \to G/K, W)$.*

Under these conditions, p and q satisfy

$$\frac{(p-q)^2}{\dim W} = \int_G \chi_\varrho\left(g\sigma(g^{-1})\right) dg,$$

where χ_ϱ is the character of (ϱ, W) and dg is the normalized Haar measure on G.

Let $S^k\mathbf{C}^2$ denote the k-th symmetric power of the standard representation $\mathbf{C}^2$ of SU(2) and $\mathbf{C}_l$ an irreducible representation of U(1) with weight l.

Theorem 4.22. *If $f : \mathbf{C}P^1 \to Gr_p(S^k\mathbf{C}^2)$ is a totally geodesic immersion of irreducible type, then we have*

$$|p-q| = \begin{cases} 0, & k : odd \\ 1, & k : even \end{cases}, \quad q := k+1-p.$$

Proof. We consider the corresponding symmetric pair $(\mathrm{SU}(2), \mathrm{U}(1))$ to $\mathbf{C}P^1$. Let

$$S^k\mathbf{C}^2 = \mathbf{C}_k \oplus \mathbf{C}_{k-2} \oplus \cdots \oplus \mathbf{C}_{-(k-2)} \oplus \mathbf{C}_{-k}$$

be a weight decomposition with respect to $\mathrm{U}(1)$. Then $\mathfrak{m}$ acts on $\mathbf{C}_l$ in such a way that $\mathfrak{m}\mathbf{C}_l \subset \mathbf{C}_{l+2} \oplus \mathbf{C}_{l-2}$. To obtain the generalized Cartan decomposition of $S^k\mathbf{C}^2$ for $(\mathrm{SU}(2), \mathrm{U}(1))$, we may put

$$U_0 = \mathbf{C}_k \oplus \mathbf{C}_{k-4} \oplus \cdots, \quad V_0 = \mathbf{C}_{k-2} \oplus \mathbf{C}_{k-6} \oplus \cdots.$$

Theorem 4.21 yields the result. $\square$

4.3. **The case where the target is a real Grassmannian.** In this subsection, suppose that a Riemannian symmetric pair (G, K) is *irreducible*.

Let $W_{\mathbf{C}}$ be a unitary representation of G. We induce a Hermitian inner product on $W_{\mathbf{C}}^*$, which is also a unitary representation of G. Then the direct sum $W_{\mathbf{C}} \oplus W_{\mathbf{C}}^*$ denoted by $\tilde{W}$ has the induced Hermitian inner product by $W_{\mathbf{C}}$ and $W_{\mathbf{C}}^*$ in such a way that $W_{\mathbf{C}}$ is perpendicular to $W_{\mathbf{C}}^*$. Then $\tilde{W} = W_{\mathbf{C}} \oplus W_{\mathbf{C}}^*$ is a unitary representation of G.

Definition 4.23. Let $\tilde{W} = U_0 \oplus V_0$ be a generalized Cartan decomposition. It is called a *decomposition induced* by $W_{\mathbf{C}}$ if $W_{\mathbf{C}}$ has the (complex) generalized Cartan decomposition $W_{\mathbf{C}} = U_0' \oplus V_0'$ and so $W_{\mathbf{C}}^*$ also has the generalized Cartan decomposition $W_{\mathbf{C}}^* = U_0'' \oplus V_0''$ such that

$$U_0 = U_0' \oplus U_0'', \quad V_0 = V_0' \oplus V_0''.$$

Lemma 4.24. *Let $W_{\mathbf{C}}$ be an irreducible unitary representation of G and $W_{\mathbf{C}} \oplus W_{\mathbf{C}}^*$ is denoted by $\tilde{W}$. Suppose that $\tilde{W} = U_0 \oplus V_0$ is a generalized Cartan decomposition for (G, K).*

(i) *If $U_0 \cap W_{\mathbf{C}} \neq \{0\}$, then it is a decomposition induced by $W_{\mathbf{C}}$.*

(ii) *If $\tilde{W} = U_0 \oplus V_0$ is not a decomposition induced by $W_{\mathbf{C}}$, then, U_0, V_0, $W_{\mathbf{C}}$ and $W_{\mathbf{C}}^*$ are equivalent K-representations.*

Proof. (i) Let $U_1 := U_0 \cap W_{\mathbf{C}} \neq \{0\}$. Since both U_0 and $W_{\mathbf{C}}$ are K-modules, so is U_1. Since $W_{\mathbf{C}}$ is an irreducible G-representation, $\mathfrak{m}U_1 \neq \{0\} \subset W_{\mathbf{C}}$. By the definition of generalized Cartan decomposition, $\mathfrak{m}U_1 \subset V_0$. Thus we have $\mathfrak{m}U_1 \subset W_{\mathbf{C}} \cap V_0$ and $V_1 := W_{\mathbf{C}} \cap V_0 \neq \{0\}$. For the same reason, $\mathfrak{m}^2U_1 \subset W_{\mathbf{C}} \cap U_0 = U_1$. Eventually we have $\mathfrak{m}^{2l}U_1 \subset U_1$ and $\mathfrak{m}^{2l+1}U_1 \subset V_1$ ($l \in \mathbf{Z}_{\geqq 0}$) and can deduce that $U_1 \oplus V_1$ is a G-representation. It follows from the irreducibility of $W_{\mathbf{C}}$ that $W_{\mathbf{C}}$ has a generalized Cartan decomposition $W_{\mathbf{C}} = U_1 \oplus V_1$.

Next we take the orthogonal complements $U_1^{\perp}$ of U_1 in U_0 and $V_1^{\perp}$ of V_1 in V_0. Since $U_1^{\perp} (\subset U_0) \perp V_0$, we have $U_1^{\perp} \perp V_1$. It follows from $U_1^{\perp} \perp U_1 \oplus V_1$ that $U_1^{\perp} \subset W_{\mathbf{C}}^*$. In a similar way, $V_1^{\perp} \subset W_{\mathbf{C}}^*$. Consequently, $W_{\mathbf{C}}^*$ also has a generalized Cartan decomposition $W_{\mathbf{C}}^* = U_1^{\perp} \oplus V_1^{\perp}$.

Since $U_0 = U_1 \oplus U_1^{\perp}$ and $V_0 = V_1 \oplus V_1^{\perp}$, $\tilde{W} = U_0 \oplus V_0$ is an induced decomposition.

(ii) Suppose that $\tilde{W} = U_0 \oplus V_0$ is not a decomposition induced by $W_{\mathbf{C}}$. From (i), we have that

$$U_0 \cap W_{\mathbf{C}} = \{0\}, \ U_0 \cap W_{\mathbf{C}}^* = \{0\}, \ V_0 \cap W_{\mathbf{C}} = \{0\}, \ V_0 \cap W_{\mathbf{C}}^* = \{0\}.$$

Let $\pi_1 : \tilde{W} \to W_{\mathbf{C}}$ and $\pi_2 : \tilde{W} \to W^*_{\mathbf{C}}$ be the orthogonal projections, respectively. Notice that $\pi_i (i = 1, 2)$ are G-equivariant homomorphisms. Hence, $\pi_i|_{U_0} : U_0 \to W_{\mathbf{C}}$ or $W^*_{\mathbf{C}}$ and $\pi_i|_{V_0} : V_0 \to W_{\mathbf{C}}$ or $W^*_{\mathbf{C}}$ are injective homomorphisms. In particular, $\dim U_0 \leqq \dim W_{\mathbf{C}}$ and $\dim V_0 \leqq \dim W_{\mathbf{C}}$. However, $\dim U_0 + \dim V_0 = 2 \dim W_{\mathbf{C}}$ by definition and the equalities hold. Consequently, $\pi_1|_{U_0} : U_0 \to W_{\mathbf{C}}$ and $\pi_2|_{U_0} : U_0 \to W^*_{\mathbf{C}}$ are K-equivariant isomorphisms. □

From now on, we assume that W is an *orthogonal* G-module. The complexification of a real vector space W is denoted by $W^{\mathbf{C}}$. Obviously, we have

Lemma 4.25. *If $W = U_0 \oplus V_0$ is a real generalized Cartan decomposition, then $W^{\mathbf{C}}$ has a complex generalized Cartan decomposition $W^{\mathbf{C}} = U_0^{\mathbf{C}} \oplus V_0^{\mathbf{C}}$.*

Lemma 4.26. *Let W be an irreducible orthogonal G-representation. If $W^{\mathbf{C}}$ is an irreducible unitary G-module, then*

(i) *we have a totally geodesic and totally real submanifold $Gr_p(W)$ of a complex Grassmannian $Gr_p(W^{\mathbf{C}})$ and a totally geodesic immersion $f : G/K \to Gr_p(W)$, where p satisfies (4.3) or*

(ii) *we have a totally geodesic submanifold $f : G/K \to Gr_N(W^{\mathbf{C}})$, where $\dim W^{\mathbf{C}} = 2N$ and the image of f is not contained in any totally real submanifold $Gr_N(W)$ of $Gr_N(W^{\mathbf{C}})$. Moreover, W has a K-invariant complex structure and $W^{\mathbf{C}} = W_{1,0} \oplus W_{0,1}$ is the generalized Cartan decomposition induced by f.*

Proof. Since $W^{\mathbf{C}}$ is the complexification of W, $W^{\mathbf{C}}$ has a G-invariant real structure denoted by r and we have a totally geodesic and totally real submanifold $Gr_p(W)$ of a complex Grassmannian $Gr_p(W^{\mathbf{C}})$ for an arbitrary p such that $1 \leqq p \leqq \dim W$. The real structure gives $W^{\mathbf{C}} \sim W^{\mathbf{C}^*}$ as G-representation. It follows from Propositions 4.17 and 4.19 that $W^{\mathbf{C}}$ has the complex generalized Cartan decomposition $W^{\mathbf{C}} = U_0^{\mathbf{C}} \oplus V_0^{\mathbf{C}}$ and $\dim U_0^{\mathbf{C}}$ can be computed by the dimension formula (4.3).

Since r is an invariant real structure, we get a complex generalized Cartan decomposition $r(W^{\mathbf{C}}) = W^{\mathbf{C}} = r(U_0^{\mathbf{C}}) \oplus r(V_0^{\mathbf{C}})$. From Lemma 4.14, the complex generalized Cartan decomposition of $W^{\mathbf{C}}$ is unique up to the order. The uniqueness yields that $r(U_0^{\mathbf{C}}) = U_0^{\mathbf{C}}$ or $r(U_0^{\mathbf{C}}) = V_0^{\mathbf{C}}$.

If $r(U_0^{\mathbf{C}}) = U_0^{\mathbf{C}}$ and so, $r(V_0^{\mathbf{C}}) = V_0^{\mathbf{C}}$, then the real structure gives us a real generalized Cartan decomposition $W = U_0 \oplus V_0$. The real generalized Cartan decomposition $W = U_0 \oplus V_0$ yields a totally geodesic immersion $f : G/K \to Gr_p(W)$ as the induced mapping from Proposition 4.6, where $p = \dim U_0$.

If $r(U_0^{\mathbf{C}}) = V_0^{\mathbf{C}}$, then $\dim U_0^{\mathbf{C}} = \dim V_0^{\mathbf{C}}$ and we have $W = \{u + r(u) | u \in U_0^{\mathbf{C}}\} = \{v + r(v) | v \in V_0^{\mathbf{C}}\}$. Since r respects the Hermitian inner product h (which means that $h(r(w_1), r(w_2)) = \overline{h(w_1, w_2)}$), W is perpendicular to the set $\sqrt{-1}W = \{u - r(u) | u \in U_0^{\mathbf{C}}\} = \{v - r(v) | v \in V_0^{\mathbf{C}}\}$ with respect to the inner product $\mathrm{Re}\, h$ on $W^{\mathbf{C}}$. Consequently, the real isomorphism $U_0^{\mathbf{C}} \to W$ given by $u \mapsto u + r(u)$ provides us with a K-invarinat complex structure of W, and thus $U_0^{\mathbf{C}} = W_{1,0}$. The complex generalized Cartan decomposition of $W^{\mathbf{C}}$ yields a totally geodesic immersion $f : G/K \to Gr_N(W^{\mathbf{C}})$, where

$N = \dim U_0^{\mathbf{C}}$. If W has a real generalized Cartan decomposition, then the complexification gives a complex generalized Cartan decomposition of $W^{\mathbf{C}}$, which is $W^{\mathbf{C}} = U_0^{\mathbf{C}} \oplus V_0^{\mathbf{C}}$ by the uniqueness of the complex generalized Cartan decomposition (Lemma 4.14). However we have already seen that $U_0^{\mathbf{C}} \cap W = \{0\}$, which is a contradiction. Hence $f(G/K)$ is not contained in any totally real submanifold $Gr_N(W)$. □

Suppose that W is an *irreducible* orthogonal G-module and $W^{\mathbf{C}}$ is *not* irreducible. This means that W has a G-invariant complex structure J and $W^{\mathbf{C}} = W_{1,0} \oplus W_{0,1}$ is a G-irreducible decomposition.

Lemma 4.27. *Let W be an irreducible orthogonal G-module. We suppose that $W^{\mathbf{C}}$ is not irreducible. If W has a real generalized Cartan decomposition $W = U_0 \oplus V_0$, then U_0 is a complex subspace of W or $U_0 \cap JU_0 = \{0\}$.*

Proof. Let $U_1 = U_0 \cap JU_0$. Since the complex structure J is also K-invariant, U_1 is a complex K-module. By the definition of a generalized Cartan decomposition, $V_1 = \mathfrak{m}U_1$ is contained in V_0. Since J is G-invariant and U_1 is a complex subspace, V_1 is also a complex subspace.

We claim that $\mathfrak{m}V_1 \subset U_1$. Otherwise, the generating subspace over $\mathbf{R}$ by U_1 and $\mathfrak{m}V_1$ is again a complex subspace and sits in U_0. It contradicts the definition of U_1.

Hence, $U_1 \oplus V_1$ is a G-module, and the irreducibility of W yields that $U_1 \oplus V_1 = W$ or $U_1 \oplus V_1 = \{0\}$, in other words, $U_1 = U_0$ or $U_1 = \{0\}$. □

Lemma 4.28. *Let $W = U_0 \oplus V_0 = U_0' \oplus V_0'$ be two real generalized Cartan decompositions of an irreducible orthogonal G-module W. Then U_0' is equivalent to U_0 or V_0 and V_0' is equivalent to U_0 or V_0 as K-modules.*

Proof. We put $U_1 = U_0 \cap U_0'$ which is a K-representation. Let $V_1 = \mathfrak{m}U_1$. It follows from the definition of a generalized Cartan decomposition that $V_1 \subset V_0 \cap V_0'$. In a similar way, we obtain $\mathfrak{m}V_1 \subset U_0 \cap U_0'$, and so, $\mathfrak{m}V_1 \subset U_1$. This yields that $U_1 \oplus V_1$ is a G-representation. The irreducibility of W gives $U_1 \oplus V_1 = \{0\}$ or $U_1 \oplus V_1 = W$.

If $U_1 \oplus V_1 = W$, then $U_1 = U_0 = U_0'$ and $V_1 = V_0 = V_0'$.

We can change the roles of U_0 and V_0 to get $(U_0 \cap V_0') \oplus (V_0 \cap U_0') = \{0\}$ or $(U_0 \cap V_0') \oplus (V_0 \cap U_0') = W$. The latter condition yields that $U_0 = V_0'$ and $V_0 = U_0'$.

From now on, we suppose that

$$U_0 \cap U_0' = \{0\},\ U_0 \cap V_0' = \{0\},\ V_0 \cap U_0' = \{0\},\ V_0 \cap V_0' = \{0\}. \tag{4.5}$$

In addition, assume that $\dim U_0 \leqq \dim V_0$. Let $\pi_1 : W \to U_0$ and $\pi_2 : W \to V_0$ be the orthogonal projections, which are K-equivariant. From (4.5), we have that $\pi_1|_{U_0'} : U_0' \to U_0$ and $\pi_1|_{V_0'} : V_0' \to U_0$ are injective K-equivariant homomorphisms. Then a dimension count gives $\dim U_0 = \dim V_0 = \dim U_0' = \dim V_0'$, and so, $\pi_1|_{U_0'}$ and $\pi_1|_{V_0'}$ are K-equivariant isomorphisms.

A similar method yields that $\pi_2|_{U_0'}$ and $\pi_2|_{V_0'}$ are K-equivariant isomorphisms and thereby our claim is proved. □

For a complex vector space W, $W^{\mathbf{R}}$ denotes the underlying real vector space of W.

Lemma 4.29. *Let W be an irreducible unitary G-module with no invariant real structure. Then W has a complex generalized Cartan decomposition if and only if $W^{\mathbf{R}}$ has a real generalized Cartan decomposition $W^{\mathbf{R}} = U_0 \oplus V_0$ with U_0 being a complex subspace of $(W^{\mathbf{R}}, J)$. Under these conditions, it is a unique real generalized Cartan decomposition of $W^{\mathbf{R}}$ up to isomorphism.*

Proof. Let $W = U_0 \oplus V_0$ be a complex generalized Cartan decomposition. We can regard it as a real generalized Cartan decomposition. Since $W^{\mathbf{R}}$ is irreducible as *real* G-module from the assumption, Lemma 4.28 yields that it is a unique real generalized Cartan decomposition up to isomorphism.

Conversely, if $W^{\mathbf{R}} = U_0 \oplus V_0$ is a real generalized Cartan decomposition with U_0 being a complex subspace of (W, J), then V_0 is also a complex subspace. Since the complex structure is invariant, $W = U_0 \oplus V_0$ can be regarded as a complex generalized Cartan decomposition. □

Lemma 4.30. *We suppose that W is an irreducible unitary G-module with no invariant real structure. Then W has a complex generalized Cartan decomposition $W = U_0 \oplus V_0$ with $p = \dim U_0$ if and only if we have a natural inclusion of $Gr_p(W)$ into a real Grassmannian $Gr_{2p}(W^{\mathbf{R}})$ and a totally geodesic immersion $f : G/K \to Gr_p(W) \to Gr_{2p}(W^{\mathbf{R}})$. Under the conditions, p satisfies the dimension formula (4.3).*

Proof. The complex generalized Cartan decomposition can also be regarded as the unique real generalized Cartan decomposition (Lemma 4.29). Lemma 4.5 and Proposition 4.6 yield the result. The uniqueness of a complex generalized Cartan decomposition (Lemma 4.14) gives the value of p. □

Remark. A totally geodesic immersion into a real Grassmann manifold $f : G/K \to Gr_p(W) \to Gr_{2p}(W^{\mathbf{R}})$ in Lemma 4.30 is called a *trivial extension* of a totally geodesic immersion into a complex Grassmannian $G/K \to Gr_p(W)$ (to a real Grassmannian).

It follows from Proposition 4.17 that every irreducible unitary G-module has a unique complex generalized Cartan decomposition, if $\operatorname{rank} G = \operatorname{rank} K$.

In the case when $\operatorname{rank} G > \operatorname{rank} K$, an irreducible unitary representation W has a generalized Cartan decomposition if and only if $W \sim W^*$ as representation, in other words, W has a real structure or a quaternion structure (Proposition 4.19).

In these cases, Lemmas 4.26, 4.28 and 4.30 yield that we have no essentially new totally geodesic immersion, when we regard a unitary representation as orthogonal one.

However, we need to take account of the case where a complex irreducible representation W is not equivalent to W^* as representation, when $\operatorname{rank} G > \operatorname{rank} K$.

Lemma 4.31. *Let (G, K) be a symmetric pair of compact type satisfying $\operatorname{rank} G > \operatorname{rank} K$ and W an irreducible unitary representation of G with an invariant complex structure J such that $W \not\sim W^*$ as G-module. We denote by r the induced invariant real structure on the complexification $W^{\mathbf{C}}$ of W.*

Then W has a real generalized Cartan decomposition $W = U_0 \oplus V_0$ or $W^{\mathbf{C}}$ has a complex generalized Cartan decomposition $W^{\mathbf{C}} = U_0 \oplus V_0$ satisfying $JU_0 = V_0$, $JV_0 = U_0$, $rU_0 = V_0$ and $rV_0 = U_0$, if and only if W has *a K-invariant real or quaternion structure compatible with the Hermitian inner product on W. Moreover, under these conditions, we have* $\dim_{\mathbf{C}} W = \dim_{\mathbf{R}} U_0 = \dim_{\mathbf{R}} V_0$ *in the case when* $W = U_0 \oplus V_0$*, or* $\dim_{\mathbf{C}} W = \dim_{\mathbf{C}} U_0 = \dim_{\mathbf{C}} V_0$ *in the case when* $W^{\mathbf{C}} = U_0 \oplus V_0$.

Proof. Suppose that W has a real generalized Cartan decomposition $W = U_0 \oplus V_0$ and hence the complexification gives a complex generalized Cartan decomposition $W^{\mathbf{C}} = U_0^{\mathbf{C}} \oplus V_0^{\mathbf{C}}$. The hypothesis $W \not\sim W^*$ and Proposition 4.19 yield that W has no complex generalized Cartan decomposition. Thus the complex generalized Cartan decomposition of $W^{\mathbf{C}}$ is not the decomposition induced by W. Since W is a unitary representation, $W^{\mathbf{C}} = W \oplus W^*$ as G-module. Then it follows from Lemma 4.24 that W, W^*, $U_0^{\mathbf{C}}$ and $V_0^{\mathbf{C}}$ are all equivalent unitary representations of K. Since $U_0^{\mathbf{C}}$ is a complexification of U_0, $W \sim U_0^{\mathbf{C}}$ provides us with a K-invariant real structure on W.

Next, suppose that $W^{\mathbf{C}}$ has a complex generalized Cartan decomposition $W^{\mathbf{C}} = U_0 \oplus V_0$ satisfying $JU_0 = V_0$, $JV_0 = U_0$, $rU_0 = V_0$ and $rV_0 = U_0$. In a similar way, we conclude that $W^{\mathbf{C}} = U_0 \oplus V_0$ is not the induced decomposition and W, W^*, U_0 and V_0 are all equivalent unitary representations of K. We define a complex linear automorphism $j : W^{\mathbf{C}} \to W^{\mathbf{C}}$ as $j|_{U_0} = \sqrt{-1} Id_{U_0}$ and $j|_{V_0} = -\sqrt{-1} Id_{V_0}$. We immediately have $j^2 = -1$ and j is K-invariant.

If $u \in U_0$, then $Ju \in V_0$ from $JU_0 = V_0$, and we see that

$$jJu = -\sqrt{-1}Ju = -J(\sqrt{-1}u) = -Jju.$$

We also have that $jJv = -Jjv$ for $v \in V_0$ and hence $jJ = -Jj$.

Under the same notation, it follows from $rU_0 = V_0$ and $rV_0 = U_0$ that

$$jr(u) = -\sqrt{-1}r(u) = r(\sqrt{-1}u) = rj(u)$$

and $jr(v) = rj(v)$. Consequently, j can be restricted to W as a K-invariant quaternion structure on W.

Conversely, suppose that W has a K-invariant real structure r_K compatible with the Hermitian inner product on W. We have already seen that $\varrho\sigma$ is equivalent to ϱ^*, where $\varrho : G \to \operatorname{Aut} W$ is a representation (see a proof of Proposition 4.19). Let h be a G-invariant Hermitian inner product on W. Using h and r_K, we identify W with W^* as $w \mapsto \phi_w = h(\cdot, r_K(w)) \in W^*$ for $w \in W$, which provides us with $W \sim W^*$ as K-modules. Then we can construct an equivalent representation of G on W^* to $(\varrho\sigma, W)$ as $\varrho\sigma(g)\phi_w = \phi_{\varrho\sigma(g)w} = h(\cdot, r_K\varrho\sigma(g)w)$. On the other hand, it follows from the definition of contravariant representation that $\varrho^*(g)\phi_w = \phi_w(\varrho(g^{-1})\cdot) = h(\varrho(g^{-1})\cdot, r_K w) = h(\cdot, \varrho(g) r_K w)$. Since $\varrho\sigma \sim \varrho^*$, there may exist a unitary transformation $C \in \mathrm{U}(W)$ such that $r_K \varrho\sigma r_K = C\varrho C^{-1}$ without loss of generality. If $k \in K$, then $C\varrho(k)C^{-1} = r_K\varrho\sigma(k) r_K = r_K \varrho(k) r_K = \varrho(k)$ and thus C is K-equivariant. It follows from $\sigma^2 = 1$ that $r_K\varrho r_K = C\varrho\sigma C^{-1}$ and

$$(r_K C)^2 \varrho (r_K C)^{-2} = r_K C r_K C \varrho C^{-1} r_K C^{-1} r_K = r_K C \varrho\sigma C^{-1} r_K = \varrho.$$

Schur's lemma yields that $(r_K C)^2 = \mu Id$ for some constant $\mu \in \mathbf{C}$.

Since r_K respects a Hermitian inner product, we have

$$h\left((r_K C)^2 w_1, (r_K C)^2 w_2\right) = \overline{h\left(C r_K C w_1, C r_K C w_2\right)}$$
$$= \overline{h\left(r_K C w_1, r_K C w_2\right)} = h\left(C w_1, C w_2\right) = h(w_1, w_2).$$

We get $|\mu|^2 = 1$.

We use $r_K C = \mu C^{-1} r_K$ to obtain

$$\mu Id = (r_K C)(\mu C^{-1} r_K) = \overline{\mu} r_K C C^{-1} r_K = \overline{\mu} Id.$$

Consequently, we have

$$(r_K C)^2 = \pm Id.$$

If $(r_K C)^2 = Id$, and so $r_K C$ is a K-invariant real structure on W, then we can put

$$U_0 := \{w \in W | r_K C(w) = w\}, \quad V_0 := \{w \in W | r_K C(w) = -w\},$$

because $r_K C \neq Id$, otherwise we get a contradiction $\varrho\sigma = \varrho$. Since an inner product on W is given as the real part of h and r_K respects h, U_0 is perpendicular to V_0. If $\xi \in \mathfrak{m}$ and $u \in U_0$, then

$$\varrho(\xi)u = -\varrho(\sigma(\xi)) r_K C(u) = -r_K C \varrho(\xi)u,$$

and so, $\varrho(\xi)u \in V_0$. In a similar way, we obtain $\varrho(\xi)v \in U_0$ for arbitrary $\xi \in \mathfrak{m}$ and $v \in V_0$. Consequently, $W = U_0 \oplus V_0$ is a real generalized Cartan decomposition.

If $(r_K C)^2 = -Id$, in other words, $j = r_K C$ defines a quaternion structure on W, then j can be extended as complex linear transformation on $W^{\mathbf{C}}$. We can put

$$U_0 := \left\{w \in W^{\mathbf{C}} | j(w) = \sqrt{-1}w\right\}, \quad V_0 := \left\{w \in W^{\mathbf{C}} | j(w) = -\sqrt{-1}w\right\}.$$

Since h can be extended to obtain a Hermitian inner product on $W^{\mathbf{C}}$ and r_K respects h, U_0 is perpendicular to V_0. If $\xi \in \mathfrak{m}$ and $u \in U_0$, then

$$j\varrho(\xi)u = -\varrho(\xi) r_K C(u) = -\varrho(\xi)\sqrt{-1}u = -\sqrt{-1}\varrho(\xi)u,$$

and so, $\varrho(\xi)u \in V_0$. In a similar way, we obtain $\varrho(\xi)v \in U_0$ for arbitrary $\xi \in \mathfrak{m}$ and $v \in V_0$. Consequently, $W^{\mathbf{C}} = U_0 \oplus V_0$ is a complex generalized Cartan decomposition, which is not an induced decomposition. It is easily shown that $JU_0 = V_0$, $JV_0 = U_0$, $rU_0 = V_0$, and $rU_0 = V_0$.

Finally suppose that W has a K-invariant quaternion structure j compatible with the Hermitian inner product on W. In a similar way, we have $j\varrho\sigma j^{-1} = C\varrho C^{-1}$ for some $C \in \mathrm{U}(W)$ and $(j^{-1}C)^2 = Id$ or $(j^{-1}C)^2 = -Id$.

The proof goes through word-for-word, if we replace r_K by j^{-1}. If $j^{-1}C$ defines a K-invariant real structure on W, then we obtain a real generalized Cartan decomposition of W.

If $j^{-1}C$ defines a K-invariant quaternion structure on W, then we obtain a complex generalized Cartan decomposition of $W^{\mathbf{C}}$, which is not an induced decomposition. It is now clear that $JU_0 = V_0$, $JV_0 = U_0$, $rU_0 = V_0$, and $rU_0 = V_0$. □

Theorem 4.32. *Let (G, K) be an irreducible Riemannian symmetric pair of compact type and W an irreducible orthogonal representation of G.*

(i) *If W has no G-invariant complex structure, then*

(i-a) *we have a totally geodesic and totally real submanifold $Gr_p(W)$ of $Gr_p(W^{\mathbf{C}})$ and a totally geodesic immersion $f : G/K \to Gr_p(W)$, where p satisfies the dimension formula (4.3) for $W^{\mathbf{C}}$, or* (i-b) *we have a totally geodesic submanifold* $f : G/K \to Gr_N(W^{\mathbf{C}})$, *where* $\dim W^{\mathbf{C}} = 2N$ *and the image of* f *is not contained in any totally real submanifold* $Gr_N(W)$ *of* $Gr_N(W^{\mathbf{C}})$. *In this case,* W *has a* K*-invariant complex structure.*

(ii) *Suppose that* W *has a* G*-invariant complex structure* J *and so,* (W, J) *is a unitary* N*-dimensional representation of* G.

(ii-a) *If* $\operatorname{rank} G = \operatorname{rank} K$, *or* $W \sim W^*$ *in the case where* $\operatorname{rank} G > \operatorname{rank} K$, *then we have a totally geodesic immersion* $f : G/K \to Gr_{2p}(W)$ *which is a trivial extension of the totally geodesic immersion induced by the complex generalized Cartan decomposition of* (W, J) *to a real Grassmannian* $Gr_{2p}(W)$, *where* p *satisfies the dimension formula (4.3) for* (W, J). *When* $\operatorname{rank} G > \operatorname{rank} K$, W *has a* G*-invariant quaternion structure,*

(ii-b) *If* $\operatorname{rank} G > \operatorname{rank} K$ *and* $W \not\sim W^*$, *then* W *has a* K*-invariant real or quaternion structure compatible with the Hermitian inner product on* W. *Moreover,*

(ii-b-1) *we have a totally geodesic immersion* $f : G/K \to Gr_N(W)$, *or*

(ii-b-2) *we have a trivial extension of a totally geodesic immersion* $f : G/K \to Gr_N(W^{\mathbf{C}})$ *induced by the complex generalized Cartan decomposition* $W^{\mathbf{C}} = W_{1,0} \oplus W_{0,1}$ *to a real Grassmannian* $Gr_{2N}(W \oplus W)$.

Conversely, let $f : G/K \to Gr_p(W)$ *be a totally geodesic immersion into a real Grassmannian of irreducible type.*

If $\operatorname{rank} G = \operatorname{rank} K$, *then it is one of the two cases* (i-a) *and* (ii-a) *up to isometry of a real Grassmannian.*

If $\operatorname{rank} G > \operatorname{rank} K$, *then it is one of the three cases* (i-a), (ii-a) *and* (ii-b-1) *up to isometry of a real Grassmannian.*

Proof. In the case (i), we may apply Lemma 4.26. For (ii), Lemma 4.30 yield the result (ii-a). When W has a G-invariant complex structure J and $W \sim W^*$ in the case where $\operatorname{rank} G > \operatorname{rank} K$, W has a G-invariant quaternion structure, since W is an irreducible orthogonal representation. If $W \not\sim W^*$, then Lemma 4.31 yields the result. Lemmas 4.14 and 4.28 assures the uniqueness. □

5. Examples

In this section, we suppose that (G, K) is an *irreducible* Riemannian symmetric pair of compact type, where G is a simply-connected compact *simple* Lie group and K is a connected subgroup of G.

Theorem 5.1. *Suppose that* W *is an irreducible unitary representation of* G *such that* $W = U_0 \oplus V_0$ *as unitary* K*-module, where both* U_0 *and* V_0 *are irreducible* K*-representations.*

Then $W = U_0 \oplus V_0$ *is the complex generalized Cartan decomposition, if* (i) $\operatorname{rank} G = \operatorname{rank} K$ *or* (ii) $\operatorname{rank} G \neq \operatorname{rank} K$ *and* $W \sim W^*$ *as* G*-module.*

Proof. In both cases, W has a generalized Cartan decomposition from Propositions 4.17 and 4.19. The uniqueness of the decomposition (Lemma 4.14) gives the result. □

Theorem 5.2. *Suppose that* W *is an irreducible orthogonal representation of* G *such that* $W = U_0 \oplus V_0$ *as orthogonal* K*-module, where both* U_0 *and* V_0 *are irreducible* K*-modules.*

Then $W = U_0 \oplus V_0$ is a real generalized Cartan decomposition, if
(i) *the complexification $W^{\mathbf{C}}$ is an irreducible G-module and $U_0^{\mathbf{C}}$ and $V_0^{\mathbf{C}}$ are irreducible K-modules,*
(ii) $\operatorname{rank} G = \operatorname{rank} K$ *and W has a G-invariant complex structure,*
(iii) $\operatorname{rank} G \neq \operatorname{rank} K$, *$W$ has a G-invariant complex structure and $W \sim W^*$ as unitary G-representation, or*
(iv) $\operatorname{rank} G \neq \operatorname{rank} K$, *$W$ has a G-invariant complex structure such that $W \not\sim W^*$ as unitary G-representation, and W has a K-invariant real structure and no K-invariant quaternion structure.*

Proof. In case of (i), we have that $W^{\mathbf{C}} \sim W^{\mathbf{C}^*}$ as G-module. It follows from Propositions 4.17 and 4.19 that $W^{\mathbf{C}}$ has a complex generalized Cartan decomposition. The uniqueness (Lemma 4.14) yields the result.

In cases of (ii) and (iii), W has a unique complex generalized Cartan decomposition $W = U_1 \oplus V_1$. If U_1 or V_1 has a K-invariant real structure or is not irreducible as complex K-module, then W has at least 3 irreducible real K-modules, which is a contradiction. Hence we have $U_1 = U_0$ and $V_1 = V_0$.

In the final case, Lemma 4.31 and its proof imply that W has a real generalized Cartan decomposition $W = U_1 \oplus V_1$ such that $\dim_{\mathbf{C}} W = \dim_{\mathbf{R}} U_1 = \dim_{\mathbf{R}} V_1$. Lemma 4.28 gives the result. □

Example. We take a quaternion projective space $\mathbf{H}P^n = \mathrm{Sp}(n+1)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ and a complex irreducible representation space $\mathbf{C}^{2n+2}$ of $\mathrm{Sp}(n+1)$. As $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$-representations, we have $\mathbf{C}^{2n+2} = \mathbf{C}^2 \oplus \mathbf{C}^{2n}$ or $\mathbf{R}^{4n+4} = \mathbf{R}^4 \oplus \mathbf{R}^{4n}$. These are generalized Cartan decompositions from Theorems 5.1 and 5.2. Theorems 4.21 and 4.32 imply that $\mathbf{H}P^n \to Gr_2(\mathbf{C}^{2n+2}) \to Gr_4(\mathbf{R}^{4n+4})$ is a totally geodesic embedding.

This example can be generalized to compact quaternion symmetric spaces. In this context, $\mathbf{C}^{2n+2}$ can be considered as a space of twistor sections of an associated vector bundle with $\mathbf{C}^2$ (see [12]).

For example, we take a compact quaternion symmetric space $G_2/\mathrm{SO}(4)$ and a real irreducible representation $\mathbf{R}^7$ of G_2. There exists a decomposition $\mathbf{R}^7 = \mathbf{R}^3 \oplus \mathbf{R}^4$. We can use Theorem 5.2 to deduce that it is a generalized Cartan decomposition. We obtain a totally geodesic submanifold $G_2/\mathrm{SO}(4) \to Gr_4(\mathbf{R}^7)$ (see also [12]).

Example. Let us consider a Hermitian symmetric space $\mathrm{Sp}(n)/\mathrm{U}(n)$. We pick an irreducible representation $\mathbf{C}^{2n}$ of $\mathrm{Sp}(n)$. We have a decomposition $\mathbf{C}^{2n} = \mathbf{C}^n \oplus \mathbf{C}^{n^*}$ as a $\mathrm{U}(n)$-module. Theorems 5.1 and 5.2 yields that it is a generalized Cartan decomposition. We obtain a totally geodesic submanifold $\mathrm{Sp}(n)/\mathrm{U}(n) \to Gr_n(\mathbf{C}^{2n}) \to Gr_{2n}(\mathbf{R}^{4n})$.

Example. We take a compact symmetric space $\mathrm{SU}(n)/\mathrm{SO}(n)$ and an irreducible representation $\mathbf{C}^n$ of $\mathrm{SU}(n)$. We have a decomposition $\mathbf{C}^n = \mathbf{R}^n \oplus \mathbf{R}^n$ as real $\mathrm{SO}(n)$-module. Since $\mathbf{C}^n$ has an $\mathrm{SO}(n)$-invariant real structure and no $\mathrm{SO}(n)$-invariant quaternion structure, Theorem 5.2 implies that $\mathbf{C}^n = \mathbf{R}^n \oplus \mathbf{R}^n$ is a real generalized Cartan decomposition. Hence we obtain a totally geodesic submanifold $\mathrm{SU}(n)/\mathrm{SO}(n) \to Gr_n(\mathbf{R}^{2n})$.

Finally, we give a totally geodesic submanifold of non-irreducible type, which is indecomposable.

Example. We take a compact symmetric space $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ and irreducible representations $\mathbf{C}^{2n}$ and $\mathbf{C}^{2n^*}$ of $\mathrm{SU}(2n)$. We put $W = \mathbf{C}^{2n} \oplus \mathbf{C}^{2n^*}$ with the induced Hermitian inner product. As $\mathrm{Sp}(n)$-module, $\mathbf{C}^{2n}$ is equivalent to $\mathbf{C}^{2n^*}$, because of the symplectic form ω on $\mathbf{C}^{2n}$.

To be more precise, let h be an invariant Hermitian product and $j : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ the quaternion structure such that $\omega(u,v) = -h(u,jv)$. As usual, $\mathfrak{su}(2n) = \mathfrak{sp}(n) \oplus \mathfrak{m}$ denotes the orthogonal decomposition. Since the standard involution is given by $\sigma(g) = jgj^{-1}$ for $g \in \mathrm{SU}(2n)$, we have $j\xi = -\xi j$ for an arbitrary $\xi \in \mathfrak{m}$.

We define

$$\begin{aligned} U_0 &= \left\{ (u, \omega(\cdot, u)) \in W \mid u \in \mathbf{C}^{2n} \right\} \\ V_0 &= \left\{ (u, -\omega(\cdot, u)) \in W \mid u \in \mathbf{C}^{2n} \right\}. \end{aligned}$$

Then it is clear that $U_0 \perp V_0$.

We claim that $W = U_0 \oplus V_0$ is a complex generalized Cartan decomposition. Indeed, for an arbitrary $\xi \in \mathfrak{m}$, we have

$$\begin{aligned} \xi\left(u, \omega(\cdot, u)\right) &= (\xi u, -h(-\xi \cdot, ju)) = (\xi u, -h(\cdot, \xi j u)) \\ &= (\xi u, h(\cdot, j\xi u)) = (\xi u, -\omega(\cdot, \xi u)), \end{aligned}$$

which shows that $\mathfrak{m}U_0 \subset V_0$. In the same way, we have $\mathfrak{m}V_0 \subset U_0$. Consequently, we get a totally geodesic submanifold $\mathrm{SU}(2n)/\mathrm{Sp}(n) \to Gr_{2n}(\mathbf{C}^{4n})$.

Since $W = (\mathbf{C}^{2n})^{\mathbf{C}}$ and $\mathbf{C}^{2n}$ has an $\mathrm{Sp}(n)$-invariant quaternion structure, this example is also interpreted by Theorem 4.32.

5.1. **The complex projective line.** We use the same notation as in Theorem 4.22. Let $\mathcal{O}(k) = \mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathbf{C}_{-k}$ be a homogeneous line bundle associated with $\mathbf{C}_{-k}$, $k \in \mathbf{Z}$. Since an irreducible unitary representation space $S^{2k}\mathbf{C}^2$ has an invariant real structure, we can take an irreducible orthogonal representation as the invariant real subspace of $S^{2k}\mathbf{C}^2$ denoted by $S^k\mathbf{C}^2_{\mathbf{R}}$.

Then from [14] and [16], we have

Theorem 5.3. *We have a decomposition of* $\Gamma(\mathcal{O}(k))$ *in the* L^2*-sense*:

$$\Gamma(\mathcal{O}(k)) = \sum_{l=0}^{\infty} S^{|k|+2l}\mathbf{C}^2.$$

Moreover, $S^{|k|+2l}\mathbf{C}^2$ *is an eigenspace of the Laplacian induced by the canonical connection with an eigenvalue* $|k| + 2l(|k| + l + 1)$. *In particular, each eigenspace of the Laplacian is an irreducible* $SU(2)$*-module.*

Theorem 5.4. *If* $f : \mathbf{C}P^1 \to Gr_p(W)$ *is a full indecomposable totally geodesic submanifold with no trivial summand into a complex Grassmann manifold, then we have* $W = S^k\mathbf{C}^2$ *for some* $k \in \mathbf{Z}_{\geqq 1}$ *and*

$$p = \begin{cases} l, & \text{if } k = 2l - 1 \\ l \text{ or } l+1, & \text{if } k = 2l. \end{cases} \tag{5.1}$$

Moreover, if k *is even, say* $2l$, *then we have a totally real submanifold* $Gr_p(S^{2l}\mathbf{C}^2_{\mathbf{R}})$ *of* $Gr_p(S^{2l}\mathbf{C}^2)$ *and* f *can be considered as a full indecomposable* *totally geodesic submanifold with no trivial summand into a real Grassmannian* $Gr_p(S^{2l}\mathbf{C}^2_{\mathbf{R}})$.

Conversely, for any irreducible unitary representation $S^k\mathbf{C}^2$ *of* SU(2)*, we can construct a totally geodesic submanifold of irreducible type* $f : \mathbf{C}P^1 \to Gr_p(S^k\mathbf{C}^2)$*, where* p *is determined by* (5.1).

Proof. Let $f : \mathbf{C}P^1 \to Gr_p(W)$ be a full indecomposable totally geodesic submanifold with no trivial summand into a complex Grassmannian manifold. Theorem 2.10 yields that the pull-back of the universal quotient bundle is decomposed into an orthogonal direct sum of line bundles with the canonical connections: $f^*Q = \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_q)$. From Lemma 3.5, the corresponding K-module denoted by $V_0 = \mathbf{C}_{-k_1} \oplus \cdots \oplus \mathbf{C}_{-k_q}$ can be regarded as a K-submodule of W. Let $\mathfrak{su}(2) = \mathfrak{u}(1) \oplus \mathfrak{m}$ be the orthogonal decomposition of the corresponding Riemannian symmetric pair $(\mathrm{SU}(2), \mathrm{U}(1))$. We denote by W_1 the SU(2)-module generated by $\mathbf{C}_{-k_1}$ in W. Since f is a full map with no trivial summand, W_1 is not a trivial representation of SU(2). From the definition of generalized Cartan decomposition, $\oplus_s \mathfrak{m}^{2s}\mathbf{C}_{-k_1}$ is a subspace of V_0. Suppose that $\oplus_s \mathfrak{m}^{2s}\mathbf{C}_{-k_1}$ is a *proper* subspace of V_0. This means that there exists $j = 1, \cdots, q$ such that $\mathbf{C}_{-k_j} \perp \oplus_s \mathfrak{m}^{2s}\mathbf{C}_{-k_1}$. Thus the SU(2)-submodule generated by $\mathbf{C}_{-k_j}$ is perpendicular to W_1. Then we deduce that f is decomposable or have a trivial summand, which is a contradiction. Hence we have that $W = W_1$.

Next, using Theorem 2.10 again, we deduce that W is an eigenspace of the Laplacian acting on $\Gamma(\mathcal{O}(k_1))$. Theorem 5.3 yields that each irreducible representation $S^{|k_1|+2l}\mathbf{C}^2$ ($l \in \mathbf{Z}_{\geqq 0}$) appears exactly once in the spectral decomposition of $\Gamma(\mathcal{O}(k_1))$. Consequently, we can deduce that $W = W_1 = S^k\mathbf{C}^2$ for a suitable $k > 0$.

Since $S^{2l}\mathbf{C}^2_{\mathbf{R}}$ has no U(1)-invariant complex structure, the result follows from Theorems 4.21, 4.22 and 4.32. □

When we regard a unitary representation $S^k\mathbf{C}^2$ as an orthogonal representation, it is denoted by $S^k\mathbf{C}^{2\mathbf{R}}$.

Theorem 5.5. *If* $f : \mathbf{C}P^1 \to Gr_p(W)$ *is a full indecomposable totally geodesic submanifold with no trivial summand into a real Grassmann manifold, then we have*
(i) $W = S^{2k}\mathbf{C}^2_{\mathbf{R}}$ *for some* $k \in \mathbf{Z}_{\geqq 1}$ *and* $p = k$ *or* $k+1$*, or*
(ii) $W = S^{2k-1}\mathbf{C}^{2\mathbf{R}}$, $p = 2k$ *and* f *is a trivial extension of* $\mathbf{C}P^1 \to Gr_k(S^{2k-1}\mathbf{C}^2)$ *to a real Grassmannian* $Gr_{2k}(S^{2k-1}\mathbf{C}^{2\mathbf{R}})$.

Conversely, for any irreducible orthogonal representation $S^{2k}\mathbf{C}^2_{\mathbf{R}}$ *of* SU(2)*, we can construct a totally geodesic submanifold of irreducible type* $f : \mathbf{C}P^1 \to Gr_p(S^{2k}\mathbf{C}^2_{\mathbf{R}})$*, where* $p = k$ *or* $k+1$.

Proof. Suppose that $f : \mathbf{C}P^1 \to Gr_p(W)$ is a full indecomposable totally geodesic submanifold with no trivial summand into a real Grassmann manifold. We consider a composition of f and a totally real submanifold $i : Gr_p(W) \to Gr_p(W^{\mathbf{C}})$, which is denoted by $\tilde{f} = i \circ f : \mathbf{C}P^1 \to Gr_p(W^{\mathbf{C}})$.

Then $\tilde{f}$ is a full totally geodesic submanifold with no trivial summand of $Gr_p(W^{\mathbf{C}})$. Let $\tilde{f} = (f_1, \cdots, f_L) : \mathbf{C}P^1 \to Gr_{p_1}(W_1) \times \cdots \times Gr_{p_L}(W_L)$ be a decomposition into indecomposable mappings. Theorem 5.4 implies that each G-representation W_l is an irreducible submodule of $W^{\mathbf{C}}$. Since the real structure r of $W^{\mathbf{C}}$ with respect to W is G-invariant, $r(W_1)$ is also an irreducible G-module. Hence we can assume that $r(W_1) = W_1$ or $r(W_1) = W_2$. The indecomposability of f yields that $W^{\mathbf{C}} = W_1$ if $r(W_1) = W_1$ or $W^{\mathbf{C}} = W_1 \oplus W_2$ if $r(W_1) = W_2$.

If $W^{\mathbf{C}} = W_1$, then $W^{\mathbf{C}}$ has an invariant real structure and so, $W = S^{2k}\mathbf{C}^2_{\mathbf{R}}$. Since $S^{2k}\mathbf{C}^2_{\mathbf{R}}$ has no K-invariant complex structure, Theorems 4.22 and 4.32 imply that $p = k$ or $k+1$.

When $W^{\mathbf{C}} = W_1 \oplus W_2$ and $r(W_1) = W_2$, we can conclude that W has an invariant complex structure such that $W_{1,0} = W_1$. Then W has a unique complex generalized Cartan decomposition from Proposition 4.17 and so, $f_i : \mathbf{C}P^1 \to Gr_{p_i}(W_i)$, $(i = 1, 2)$ is uniquely determined. Since $i \circ f = (f_1, f_2)$, $f_2(x) = rf_1(x)$ for any $x \in \mathbf{C}P^1$, where $f_i(x)$ is now regarded as a subspace of W_i. This means that we can identify $f : \mathbf{C}P^1 \to Gr_p(W)$ with $f_1 : \mathbf{C}P^1 \to Gr_{p_1}(W_1)$. Consequently, f is a trivial extension of f_1 and it follows that $p = 2p_1$, where p_1 is determined in Theorem 4.22.

But in the case when $W = S^{2k}\mathbf{C}^2$, we have already seen that the image of $f_1 : \mathbf{C}P^1 \to Gr_k(S^{2k}\mathbf{C}^2)$ is in a totally real Grassmannian $Gr_k(S^{2k}\mathbf{C}^2_{\mathbf{R}})$ of $Gr_k(S^{2k}\mathbf{C}^2)$. Hence f is considered as a composition:

$$\mathbf{C}P^1 \to Gr_k(S^{2k}\mathbf{C}^2_{\mathbf{R}}) \to Gr_k(S^{2k}\mathbf{C}^2) \to Gr_{2k}(S^{2k}\mathbf{C}^{2^{\mathbf{R}}}).$$

However, the imaginary part $\sqrt{-1}S^{2k}\mathbf{C}^2_{\mathbf{R}}$ of $S^{2k}\mathbf{C}^{2^{\mathbf{R}}}$ gives only a zero section, which contradicts the assumption that f is a full map. It follows that $W = S^{2k-1}\mathbf{C}^2$.

Now the converse implication follows from Theorem 4.32. □

5.2. **Compact Lie groups.** We assume that G is a *simply-connected compact simple* Lie group in this subsection. For G-representations (ϱ_i, W_i),$(i = 1, 2)$, we define a representation $(\varrho, W_1 \otimes W_2)$ of $G \times G$ as

$$\varrho(g,h)(w_1 \otimes w_2) = (\varrho_1(g)w_1) \otimes (\varrho_2(h)w_2), \quad g, h \in G, \ w_i \in W_i.$$

The representation $(\varrho, W_1 \otimes W_2)$ is denoted by $(\varrho_1 \boxtimes \varrho_2, W_1 \boxtimes W_2)$.

First of all, we consider a totally geodesic immersion into a complex Grassmannian of irreducible type.

Lemma 5.6. *We regard G as a symmetric space with a symmetric pair $(G \times G, G)$. Let $\varrho = \varrho_1 \boxtimes \varrho_2$ be an irreducible unitary representation of $G \times G$, where ϱ_1 and ϱ_2 are irreducible unitary representations of G. Then ϱ has a generalized Cartan decomposition if and only if $\varrho_1 \sim \varrho_2$.*

Proof. In this case, $\sigma(g,h) = (h,g)$, where σ is the corresponding standard involution of $G \times G$. Hence we have $\varrho\sigma(g,h) = \varrho_1(h) \boxtimes \varrho_2(g)$. The result follows from Corollary 4.7 and Theorem 4.13. □

Theorem 5.7. *If W is an irreducible unitary representation of G, then $W \boxtimes W = S^2W \oplus \wedge^2 W$ is the generalized Cartan decomposition for $(G \times G, G)$.*

Proof. Let $\varrho : G \times G \to \mathrm{Aut}\ (W \boxtimes W)$ be a representation. For $w_1, w_2 \in W$ and $X \in \mathfrak{g}$, we have

$$\begin{aligned}2\varrho(X, -X)(w_1 \cdot w_2) &= \varrho(X, -X)(w_1 \otimes w_2 + w_2 \otimes w_1)\\ &= \varrho(X)w_1 \otimes w_2 - w_1 \otimes \varrho(X)w_2 + \varrho(X)w_2 \otimes w_1 - w_2 \otimes \varrho(X)w_1\\ &= 2\left\{\varrho(X)w_1 \wedge w_2 - w_1 \wedge \varrho(X)w_2\right\},\end{aligned}$$

and so, $\varrho(\mathfrak{m})\left(S^2W\right) \subset \wedge^2 W$. In a similar way, it can be shown that $\varrho(\mathfrak{m})\left(\wedge^2 W\right) \subset S^2W$. □

Theorem 5.8. *A map $f : G \to Gr_p(\tilde{W})$ is a totally geodesic immersion into a complex Grassmannian of irreducible type if and only if there exists an irreducible unitary representation W of G such that $\tilde{W} = W \boxtimes W$ and $p = \dim S^2W$ or $p = \dim\ \wedge^2 W$*

Remark. Theorem 5.8 is also obtained by Rawnsley (unpublished, see also [6]).

Next, we consider a totally geodesic immersion into a real Grassmannian of irreducible type.

Lemma 5.9. *Let W be an irreducible orthogonal representation of $G \times G$ and $W^{\mathbf{C}}$ the complexification of W. Suppose that $W^{\mathbf{C}} = W_1 \boxtimes W_2$ is an irreducible unitary representation of $G \times G$ and so, has an $G \times G$-invariant real structure r, where (ϱ_i, W_i) $(i = 1, 2)$ are irreducible unitary representations of G. Then W has a real generalized Cartan decomposition if and only if we have $\varrho_1 \sim \varrho_2$. Under these conditions, $W = (S^2W_1)^{\mathbf{R}} \oplus (\wedge^2 W_1)^{\mathbf{R}}$ is the unique real generalized Cartan decomposition.*

Proof. We suppose that W has a real generalized Cartan decomposition. Lemma 4.25 yields that $W^{\mathbf{C}}$ has a complex generalized Cartan decomposition. Then Lemma 5.6 gives us $\varrho_1 \sim \varrho_2$.

Conversely, if $\varrho_1 \sim \varrho_2$, then we have a complex generalized Cartan decomposition $W^{\mathbf{C}} = S^2W_1 \oplus \wedge^2 W_1$. Since $r(W^{\mathbf{C}}) = r(S^2W_1) \oplus r(\wedge^2 W_1)$ is also a complex generalized Cartan decomposition and $\dim S^2W_1 \neq \dim\ \wedge^2 W_1$, the uniqueness of a complex generalized Cartan decomposition (Lemma 4.14) yields that $S^2W_1 = r(S^2W_1)$ and $\wedge^2 W_1 = r(\wedge^2 W_1)$. Then $W = (S^2W_1)^{\mathbf{R}} \oplus (\wedge^2 W_1)^{\mathbf{R}}$ is a real generalized Cartan decomposition.

Now Lemma 4.14 yields the uniqueness of the statement. □

Suppose again that W is an irreducible orthogonal representation of $G \times G$. If the complexification $W^{\mathbf{C}}$ is not irreducible, then W itself has a $G \times G$-invariant complex structure and so, W can be regarded as an irreducible unitary representation of $G \times G$. Hence, we may assume that $W = W_1 \boxtimes W_2$, where W_i are irreducible unitary G-modules and W has no $G \times G$-invariant real structure.

Lemma 5.10. *An irreducible unitary $G \times G$-representation $W = W_1 \boxtimes W_2$ has a $G \times G$-invariant real structure if and only if both W_1 and W_2 have G-invariant real structures or have G-invariant quaternion structures.*

Proof. If $W = W_1 \boxtimes W_2$ has an invariant real structure, then we have $W_1 \boxtimes W_2 \sim W_1^* \boxtimes W_2^*$ as representation. Then we get $W_1 \sim W_1^*$ and $W_2 \sim W_2^*$

as G-modules. Since W_1 and W_2 are irreducible modules, W_1 and W_2 have real structures or both have quaternion structures.

The converse is trivial. $\square$

Lemma 5.11. *Let $W = W_1 \boxtimes W_2$ be an irreducible unitary representation of $G \times G$, where W_i are irreducible unitary G-modules and suppose that W has no $G \times G$-invariant real structure. We regard W as an irreducible orthogonal $G \times G$-module.*

Then W has a real generalized Cartan decomposition $W = U_0 \oplus V_0$ and U_0 and V_0 can also be regarded as a complex subspace if and only if $W_1 = W_2$. Then $W = S^2 W_1 \oplus \wedge^2 W_1$ is the unique real generalized Cartan decomposition.

Proof. From the proof of Lemma 4.28, the complex generalized Cartan decomposition of W is the unique real generalized Cartan decomposition of W when we regard W as an orthogonal $G \times G$-module. Then Theorem 5.7 yields the result. $\square$

Lemma 5.12. *Let $W = W_1 \boxtimes W_2$ be an irreducible unitary representation with an invariant complex structure J of $G \times G$, where (ϱ_i, W_i) $(i = 1, 2)$ are irreducible unitary G-modules and suppose that W has no $G \times G$-invariant real structure. We regard W as an irreducible orthogonal $G \times G$-module. Suppose that $\varrho_1 \not\sim \varrho_2$. We denote by r the induced invariant real structure on the complexification $W^{\mathbf{C}}$ of W.*

Then W has a real generalized Cartan decomposition $W = U_0 \oplus V_0$ or the complexification $W^{\mathbf{C}}$ has a complex generalized Cartan decomposition $W^{\mathbf{C}} = U_0 \oplus V_0$ satisfying $JU_0 = V_0$, $JV_0 = U_0$, $rU_0 = V_0$ and $rV_0 = U_0$, if and only if W has a G-invariant real structure or a G-invariant quaternion structure compatible with the Hermitian inner product on W.

Moreover, under these conditions, we have $\dim_{\mathbf{C}} W = \dim_{\mathbf{R}} U_0 = \dim_{\mathbf{R}} V_0$ in the case when $W = U_0 \oplus V_0$, or $\dim_{\mathbf{C}} W = \dim_{\mathbf{C}} U_0 = \dim_{\mathbf{C}} V_0$ in the case when $W^{\mathbf{C}} = U_0 \oplus V_0$.

Proof. It follows from$\varrho_1 \not\sim \varrho_2$ that $W \not\sim W^*$ as $G \times G$-module. Then we apply Lemma 4.31 to obtain the result. $\square$

Corollary 5.13. *Let W_1 be an irreducible unitary representation of G which is not self-conjugate and $W = W_1 \boxtimes W_1^*$ a unitary representation of $G \times G$. We denote by $\mathrm{H}(W_1)$ the set of Hermitian endomorphisms of W_1 and by $\mathrm{SH}(W_1)$ the set of skew-Hermitian endomorphisms of W_1.*

If W is regarded as an orthogonal representation, then $W = \mathrm{H}(W_1) \oplus \mathrm{SH}(W_1)$ is a unique real generalized Cartan decomposition for $(G \times G, G)$.

Proof. The action on W of $G \times G$ is given by $\varrho(g, h)A = gAh^{-1}$ for $(g, h) \in G \times G$ and $A \in \mathrm{End}\, W_1$. Since, for $X \in \mathfrak{g}$, $H \in \mathrm{H}(W_1)$, we have

$$\varrho(X, -X)H = XH + HX,$$

$$(XH + HX)^* = -HX - XH = -(XH + HX),$$

we obtain $\varrho(\mathfrak{m})\mathrm{H}(W_1) \subset \mathrm{SH}(W_1)$. A similar method gives $\varrho(\mathfrak{m})\mathrm{SH}(W_1) \subset \mathrm{H}(W_1)$. Thus $W = \mathrm{H}(W_1) \oplus \mathrm{SH}(W_1)$ is a real generalized Cartan decomposition.

Since W_1 is not self-conjugate, neither is W. This yields that W is irreducible as an orthogonal representation of $G \times G$. Lemma 4.28 yields $W = \mathrm{H}(W_1) \oplus \mathrm{SH}(W_1)$ is the unique real generalized Cartan decomposition up to isomorphisms. □

Remark. Notice that the statement of Corollary 5.13 is still valid except uniqueness if W_1 is self-conjugate. In this case, W is *not* irreducible as an orthogonal representation of $G \times G$.

As a result, we obtain a classification of a totally geodesic immersion of a compact Lie group into a real Grassmannian of irreducible type, which is almost the same as in Theorem 4.32. The main difference is that we can see the exact value of p and the generalized Cartan decomposition explicitly. Instead, we apply our results to SU(2).

Theorem 5.14. *Let $S^k\mathbf{C}^2$ be the k-th symmetric power of the standard representation $\mathbf{C}^2$ of* SU(2)*. Then we have a totally geodesic immersion of irreducible type of* SU(2) *into a complex Grassmannian $Gr_p(\mathbf{C}^N)$ if and only if $\mathbf{C}^N = S^k\mathbf{C}^2 \otimes S^k\mathbf{C}^2$ $(N = (k+1)^2)$ and*

$$p = \frac{(k+1)(k+2)}{2}, \quad or \quad \frac{k(k+1)}{2}.$$

Moreover each totally geodesic immersion of irreducible type into a complex Grassmannian $Gr_p(\mathbf{C}^N)$ can factor through a totally geodesic immersion of irreducible type of SU(2) *into a real Grassmannian $Gr_p(\mathbf{R}^N)$ which is a totally real submanifold of $Gr_p(\mathbf{C}^N)$. The real subspace $\mathbf{R}^N$ can be obtained by the invariant real structure of $S^k\mathbf{C}^2 \otimes S^k\mathbf{C}^2$.*

Theorem 5.15. *Let $S^k\mathbf{C}^2$ be the k-th symmetric power of the standard representation $\mathbf{C}^2$ of* SU(2)*. Then we have a full totally geodesic submanifold with no trivial summand of* SU(2) *into a real Grassmann manifold $Gr_{(k+1)^2}(S^k\mathbf{C}^2 \otimes S^k\mathbf{C}^2)$, where $S^k\mathbf{C}^2 \otimes S^k\mathbf{C}^2$ is regarded as an orthogonal representation of* $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

Proof. The result follows from the Remark after Corollary 5.13. □

6. A generalization of Theorem 2.10

First of all, we give an example of real generalized Cartan decomposition. Let (G, K) be a Riemannian symmetric pair of compact type with the orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We denote by m the dimension of G/K.

Example. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a real generalized Cartan decomposition of $\mathfrak{g}$. We have a totally geodesic immersion $i : G/K \to Gr_m(\mathfrak{g})$, which is the map induced by a vector bundle $G \times_K \mathfrak{k} \to G/K$ and $\mathfrak{g}$.

This example yields a generalization of Theorem 2.10.

Theorem 6.1. *Let (G, K) be a Riemannian symmetric pair of compact type and f a mapping of a Riemannian manifold M into G/K.*

Then, the following two conditions are equivalent.

(1) *$f : M \to G/K$ is a harmonic map.*

(2) *There exists a bundle endomorphism A of the pull-back bundle with the pull-back connection of a homogeneous vector bundle $G \times_K \mathfrak{k} \to G/K$ with the canonical connection, such that $\Delta t + At = 0$ for an arbitrary $t \in \mathfrak{g}$.*

Under these conditions, A is the mean curvature operator of $i \circ f$.

Proof. We can consider a composition $i \circ f : M \to G/K \to Gr_m(\mathfrak{g})$. Since i is a totally geodesic immersion, (1) is equivalent to the condition that $i \circ f$ is a harmonic map. On the other hand, the pull-back of the universal quotient bundle by i is the homogeneous bundle $G \times_K \mathfrak{k} \to G/K$. Then Theorem 2.10 implies the result. □

Of course, we can replace the role of $\mathfrak{k}$ by $\mathfrak{m}$. Then the vector bundle $G \times_K \mathfrak{m} \to G/K$ is nothing but the tangent bundle $T(G/K) \to G/K$ of G/K and $\mathfrak{g}$ is the space of the Killing vector fields on G/K.

Corollary 6.2. *Let (G, K) be a Riemannian symmetric pair of compact type and f a mapping of a Riemannian manifold M into G/K.*

Then, the following two conditions are equivalent.

(1) *$f : M \to G/K$ is a harmonic map.*

(2) *There exists a bundle endomorphism A of the pull-back bundle with the pull-back connection of $T(G/K) \to G/K$ with the Levi-Civita connection such that $\Delta t + At = 0$ for an arbitrary pull-back section $t \in \mathfrak{g}$ of Killing vector field on G/K.*

Under these conditions, A is the mean curvature operator of the composite of f and the totally geodesic immersion of G/K into a real Grassmann manifold induced by $(T(G/K) \to G/K, \mathfrak{g})$.

It is now obvious that we have a more abstract generalization of Theorem 2.10. To do so, let $W = U_0 \oplus V_0$ be a generalized Cartan decomposition of G-representation space W for (G, K). We denote by $i : G/K \to Gr_p(W)$ the induced totally geodesic immersion, where $p = \dim U_0$. Hence the pull-back of the universal quotient bundle with the pull-back connection is isomorphic to the homogeneous vector bundle $G \times_K V_0 \to G/K$ with the canonical connection.

Theorem 6.3. *Let (G, K) be a Riemannian symmetric pair of compact type and f a mapping of a Riemannian manifold M into G/K.*

Then, the following three conditions are equivalent.

(1) *$f : M \to G/K$ is a harmonic map.*

(2) *For any orthogonal or unitary G-representation W with a generalized Cartan decomposition $W = U_0 \oplus V_0$, there exists a bundle endomorphism A of the pull-back bundle with the pull-back connection of a homogeneous vector bundle $G \times_K V_0 \to G/K$ with the canonical connection such that $\Delta t + At = 0$ for an arbitrary $t \in W$.*

(3) *There exists an orthogonal or a unitary G-representation W with a generalized Cartan decomposition $W = U_0 \oplus V_0$ and a bundle endomorphism A of the pull-back bundle with the pull-back connection of a homogeneous vector bundle $G \times_K V_0 \to G/K$ with the canonical connection such that $\Delta t + At = 0$ for an arbitrary $t \in W$.*

Under these conditions, A is the mean curvature operator of $i \circ f$, where i is the map induced by $(G \times_K V_0 \to G/K, W)$.

Example. We pick a symmetric pair $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$. Let $(\varrho, \mathbf{C}^2)$ be the standard representation of SU(2). Then W denotes the direct sum of two copies of $\mathbf{C}^2$: $W = \mathbf{C}^2 \oplus \mathbf{C}^2$. We define a representation (φ, W) of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ as

$$\varphi(g, h)(u, v) = (\varrho(g)u, \varrho(h)v)\,.$$

Then we have a generalized Cartan decomposition of $W = U_0 \oplus V_0$ for $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$, where

$$U_0 = \left\{(u, u) \in W \mid u \in \mathbf{C}^2\right\}, \quad V_0 = \left\{(v, -v) \in W \mid v \in \mathbf{C}^2\right\}.$$

The associated homogeneous vector bundles with U_0 and V_0 are isomorphic to the spin bundle $\mathbf{H} \to S^3$. Thus we have a direct sum of vector bundles: $\underline{W} = \mathbf{H} \oplus \mathbf{H}$. We denote by H the second fundamental form of $\mathbf{H}$ in $\underline{W}$ [11] which can be regarded as a 1-form with values in $\mathrm{End}\,\mathbf{H}$.

Let f be a harmonic map of a Riemann surface M into S^3. In this case, f is a harmonic map if and only if $(\nabla_Z df)(\overline{Z}) = \left(\nabla_{\overline{Z}} df\right)(Z) = 0$ for $Z \in T_{1,0}M$ (see, for example, [5]). The pull-back of the second fundamental form H is decomposed according to the bidegree: $f^*H = \Phi + \Psi$, where $\Phi \in \Omega^{1,0}(f^*\mathrm{End}\,\mathbf{H})$ and $\Psi \in \Omega^{0,1}(f^*\mathrm{End}\,\mathbf{H})$. Since $\nabla H = H_{\nabla df}$ [13], f is a harmonic map if and only if Φ is a holomorphic 1-form [3]. The Gauss equation of vector bundles ([11] or see also [13]) yields that

$$R(Z, \overline{Z}) = \Phi_Z \Phi^*_{\overline{Z}} - \Phi^*_{\overline{Z}} \Phi_Z,$$

where R is the curvature of the pull-back bundle of $\mathbf{H} \to S^3$. The equations

$$\overline{\partial}\Phi = 0, \quad R = [\Phi, \Phi^*]$$

are deeply considered in Hitchin [9] which are obtained as a dimensional reduction of the self-dual Yang-Mills equation in $\mathbf{R}^4$.

References

[1] B.Y.Chen and T.Nagano, *Totally geodesic submanifolds of symmetric spaces, I*, Duke. Math. J. **44** (1977), 745–755.

[2] B.Y.Chen and T.Nagano, *Totally geodesic submanifolds of symmetric spaces, II*, Duke. Math. J. **45** (1978), 405–425.

[3] S.S.Chern and J.G.Wolfson, *Harmonic maps of the two-sphere into a complex Grassmann manifold II*, Ann.Math. **125** (1987), 301–335.

[4] M.P.do Carmo and N.R.Wallach, *Minimal immersions of spheres into spheres*, Ann.Math. **93** (1971), 43–62.

[5] J.Eells and L.Lemaire, *A report on Harmonic maps*, Bull. London. Math. Soc. **10** (1978), 1–68.

[6] J.Eells and L.Lemaire, *Another report on Harmonic maps*, Bull. London Math. Soc. **20** (1988), 385–524.

[7] J.Eells and J.H.Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109-160.

[8] S.Helgason, “Differential Geometry, Lie groups and symmetric spaces” Academic Press, Cambridge (1978).

[9] N.J.Hitchin, *Harmonic maps from a 2-torus to the 3-sphere*, J.Differential Geometry **31** (1990), 627–710.

[10] O. Ikawa and H.Tasaki, *Totally geodesic submanifolds of maximal rank in symmetric spaces*, Japan. J. Math. **26** (2000), 1–29.

[11] S.Kobayashi, "Differential Geometry of Complex Vector Bundles", Iwanami Shoten and Princeton University, Tokyo (1987).
[12] Y.Nagatomo, *Twistor sections on the Wolf spaces*, Trans. Amer. Math. Soc. **360**, (2008), 4497-4517.
[13] Y.Nagatomo, *Harmonic maps into Grassmann manifolds*, to appear in J. Math. Soc. Japan.
[14] Y.Ohnita, *Homogeneous Harmonic Maps into Complex Projective Spaces*, Tokyo Journal of Mathematics **13** (1990), 87–116.
[15] T.Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380-385.
[16] M.Takeuchi, "Modern Spherical Functions" Translation of Mathematical Monographs, Vol.135 American Mathematical Society, Providence (1994).
[17] G.Toth, *Moduli Spaces of Polynomial Minimal Immersions between Complex Projective Spaces*, Michigan Math.J. **37** (1990), 385–396.

Department of Mathematics, MEIJI UNIVERSITY, Higashi-Mita, Tama-ku, Kawasaki-shi, Kanagawa 214-8571, JAPAN
Email address: yasunaga@meiji.ac.jp