A virtual knot whose virtual unknotting number equals one and a sequence of $n$-writhes

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Abstract. Satoh and Taniguchi introduced the $n$-writhe $J_n$ for each non-zero integer $n$, which is an integer invariant for virtual knots. The sequence of $n$-writhes $\{J_n\}_{n \neq 0}$ of a virtual knot $K$ satisfies $\sum_{n \neq 0} n J_n(K) = 0$. They showed that for any sequence of integers $\{c_n\}_{n \neq 0}$ with $\sum_{n \neq 0} nc_n = 0$, there exists a virtual knot $K$ with $J_n(K) = c_n$ for any $n \neq 0$. It is obvious that the virtualization of a real crossing is an unknotting operation for virtual knots. The unknotting number by the virtualization is called the virtual unknotting number and is denoted by $u^v$. In this paper, we show that if $\{c_n\}_{n \neq 0}$ is a sequence of integers with $\sum_{n \neq 0} nc_n = 0$, then there exists a virtual knot $K$ such that $u^v(K) = 1$ and $J_n(K) = c_n$ for any $n \neq 0$.

1. Introduction

Kauffman [3] defined the notation of a virtual knot which is a generalization of that of a classical knot.

Satoh and Taniguchi [8] defined a virtual knot invariant $J_n$, called the $n$-writhe, for any non-zero integer $n$. The $n$-writhes give the coefficients of some polynomial invariants for virtual knots. They gave a necessary and sufficient condition for a sequence of integers to be that of the $n$-writhes of a virtual knot as follows.

**Theorem 1 ([8]).** The sequence of $n$-writhes $\{J_n(K)\}_{n \neq 0}$ of a virtual knot $K$ satisfies $\sum_{n \neq 0} n J_n(K) = 0$. Conversely, for any sequence of integers $\{c_n\}_{n \neq 0}$ with $\sum_{n \neq 0} nc_n = 0$, there is a virtual knot $K$ such that $J_n(K) = c_n$ for any $n \neq 0$.

We note that there are at most finitely many non-zero integers in the set $\{c_n\}_{n \neq 0}$ in Theorem 1. It is well known that local moves in Figure 1 are unknotting operations for classical knots or virtual knots. The crossing change and the Delta-move are unknotting operations for classical knots, and the virtualization and the forbidden moves are unknotting operations for virtual knots. The unknotting number by the virtualization is called the virtual unknotting number and is denoted by $u^v$. Local moves and unknotting numbers are defined in Section 2.

There are some studies on unknotting operations and $n$-writhes, where the changes of the values of $n$-writhes by some unknotting operations are calculated. For a crossing change and a Delta move see [8], for forbidden moves see [7].

Recently, we have shown the following theorem for the virtualization.

**Theorem 2 ([6]).** For any given non-zero integer $n$ and any given integer $N$, there exists a virtual knot $K$ such that $u^v(K) = 1$ and $J_n(K) = N$.

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In this paper, we obtain a more powerful result.

**Theorem 3.** Let \( \{c_n\}_{n \neq 0} \) be a sequence of integers. If \( \sum_{n \neq 0} nc_n = 0 \), then there is a virtual knot \( K \) such that

\[
u^v(K) = 1 \text{ and } J_n(K) = c_n\]

for any \( n \neq 0 \).

The virtual unknotting number cannot be estimated from below by a sequence of \( n \)-writhe.

In Section 2, we recall the basic definitions and explain the virtual knot invariant called \( n \)-writhe. In Section 3, we prove Theorem 3.

## 2. Preliminaries

### 2.1. Virtual knots and unknotting operations

Let \( S^1 \) be a unit circle. Suppose that \( f : S^1 \to \mathbb{R}^3 \) is a smoothing embedding. The image \( f(S^1) \) is called a knot. Let \( K \) be a knot and \( p \) a regular projection \( \mathbb{R}^3 \to \mathbb{R}^2 \), i.e., \( p(K) \) is a generic immersed plane closed curve. Every self-intersection of the image \( p(K) \) is a transverse double point. Then, the image \( p(K) \) with over/under information of each double point is called a knot diagram of \( K \).

A virtual knot diagram is a generalization of a knot diagram and it has virtual crossings as well as real crossings in Figure 2. Especially, the virtual knot diagram which has no real and virtual crossings is called the trivial knot diagram. We say that two virtual knot diagrams are equivalent if one can be obtained from the other by a finite sequence of generalized Reidemeister moves in Figure 3 [3]. A virtual knot is an equivalence class of virtual knot diagrams modulo the generalized Reidemeister moves.

![Crossing types](image2.png)
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Let $K$ be a virtual knot and $D$ a virtual knot diagram of $K$. Then, $D$ is regarded as the image of a generic immersion $S^1 \to \mathbb{R}^2$. A Gauss diagram for $D$ is the preimage of $D$ with chords, each of which connects the preimages of each real crossing. We specify over/under information of each real crossing on the corresponding chord by directing the chord toward the under path and assigning each chord with the sign of the crossing in Figure 4. We suppose that virtual knot diagrams are oriented.

It is well-known that there exists a bijection from the set of virtual knots to the set of equivalence classes of their Gauss diagrams modulo the generalized Reidemeister moves of Gauss diagrams as shown in Figure 5. We identify a virtual knot with an equivalence class of Gauss diagrams.

A local modification on a virtual knot diagram is called a local move. Generalized
Reidemeister moves are local moves. Let us fix a local move. If any virtual knot diagram is transformed into a trivial knot diagram by a finite sequence of the local moves and generalized Reidemeister moves, then the local move is called an unknotting operation for virtual knots. The local move on virtual knot diagrams in Figure 6 is called a virtualization. A virtualization is an unknotting operation.

Figure 6. Virtualization.

For an unknotting operation, the minimum number of the operations needed to transform a diagram of a virtual knot \( K \) into a trivial knot diagram is called the unknotting number of \( K \) by the unknotting operation. Here, when we operate the local move, we are allowed to do generalized Reidemeister moves before or after the operation. The unknotting number of a virtual knot \( K \) by the virtualization is called the virtual unknotting number and is denoted by \( u_v(K) \). We identify a virtual knot \( K \) and the Gauss diagram \( G \) associated with \( K \). When we consider the Gauss diagram, we use the notation \( u_v(G) \) instead of \( u_v(K) \). Namely, \( u_v(G) \) means \( u_v(K) \) where \( K \) is a virtual knot whose Gauss diagram is \( G \).

2.2. Invariants for virtual knots

We define some invariants for virtual knots.

**Definition 1** ([8]). A Gauss diagram \( G \) consists of an oriented circle \( S^1 \) together with signed, oriented \( m \) chords \((m \geq 0)\) connecting \( 2m \) points on \( S^1 \). Let \( \gamma = \overrightarrow{PQ} \) be a chord in \( G \) with sign \( \varepsilon(\gamma) \) where \( \gamma \) is oriented from \( P \) to \( Q \). We give the signs to the endpoints \( P \) and \( Q \), denoted by \( \varepsilon(P) \) and \( \varepsilon(Q) \), respectively, such that \( \varepsilon(P) = -\varepsilon(\gamma) \) and \( \varepsilon(Q) = \varepsilon(\gamma) \).

For a chord \( \gamma = \overrightarrow{PQ} \) in a Gauss diagram \( G \), the specified arc \( \text{arc}(\gamma) \) of \( \gamma \) is the arc in \( S^1 \) with endpoints \( P \) and \( Q \) oriented from \( P \) to \( Q \) along the orientation of \( S^1 \). See Figure 7. The index of \( \gamma \) is the sum of the signs of all the points on \( \text{arc}(\gamma) \) except \( P \) and \( Q \). We denote it by \( i(\gamma) \). For an integer \( n \), the \( n \)-writhe of \( G \) is the sum of the signs of all the chords with index \( n \) and is denoted by

\[
J_n(G) = \sum_{i(\gamma)=n} \varepsilon(\gamma).
\]

If \( G \) is a Gauss diagram associated with a virtual knot diagram of \( K \), then this number \( J_n(G) \) defines an invariant of \( K \) for \( n \neq 0 \). It is called the \( n \)-writhe of \( K \) and is denoted by \( J_n(K) \).

**Definition 2.** The index polynomial \( p_t(K) \) ([2]), the odd writhe polynomial \( f_K(t) \) ([1]), and the affine index polynomial \( P_K(t) \) ([4]) for a virtual knot \( K \) are defined as follows.
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\[ Q \varepsilon(\gamma) \]
\[ \text{arc}(\gamma) \]
\[ P \varepsilon(\gamma) \]

Figure 7. The specified arc in the circle of a Gauss diagram.

1. \( p_t(K) = \sum_{\gamma} \varepsilon(\gamma) (t^{|i(\gamma)|} - 1) \),

2. \( f_K(t) = \sum_{i(\gamma) \text{ is odd}} \varepsilon(\gamma) t^{1-i(\gamma)} \), and

3. \( P_K = \sum_{\gamma} \varepsilon(\gamma) (t^{-i(\gamma)} - 1) \).

**Proposition 1.** The coefficients of polynomial invariants \( p_t \), \( f(t) \), and \( P \) are expressed by the \( n \)-writhes, as the following formulas ([8] and [7]):

1. \( p_t(K) = \sum_{n>0} \{ J_n(K) + J_{-n}(K) \} (t^n - 1) \),

2. \( f_K(t) = \sum_{n \in \mathbb{Z}} J_{1-2n}(K) t^{2n} \), and

3. \( P_K = \sum_{n \in \mathbb{Z}} J_n(K) (t^{-n} - 1) \).

Satoh and Taniguchi have shown that the \( n \)-writhes have the following property.

**Theorem 4 ([8]).** A sequence of \( n \)-writhes \( \{ J_n(K) \}_{n \neq 0} \) of a virtual knot \( K \) satisfies \( \sum_{n \neq 0} n J_n(K) = 0 \). Conversely, for any sequence of integers \( \{ c_n \}_{n \neq 0} \) with \( \sum_{n \neq 0} nc_n = 0 \), there is a virtual knot \( K \) such that \( J_n(K) = c_n \) for any \( n \neq 0 \).

3. **Proof of Theorem 3.**

**Notation 1.** Let \( G_1 \) and \( G_2 \) be Gauss diagrams with chords \( \gamma_1 \) and \( \gamma_2 \) respectively. If \( \gamma_1 \) and \( \gamma_2 \) have the same sign, then the **vertex connected sum** \( G_1 \# G_2 \) with respect to \( \gamma_1 \) and \( \gamma_2 \) is the Gauss diagram obtained by removing the interiors of regular neighborhoods of the head of \( \gamma_1 \) and the tail of \( \gamma_2 \) from the diagrams and connecting them as shown in Figure 8.
Lemma 1. If \( i(\gamma_1) = i(\gamma_2) = 0 \), then it holds that \( J_n(G_1 \natural G_2) = J_n(G_1) + J_n(G_2) \) for any \( n \neq 0 \).

Proof. Since the sum of the signs of two end points of a chord on \( G_i \) is equal to zero, the index of any chord in \( G_1 \natural G_2 \) is equal to that in \( G_1 \) or \( G_2 \).

Lemma 2. For any given integers \( n \) and \( N \) with \( n \neq 0, 1 \) and \( N \neq 0 \), there exists a Gauss diagram \( G = G(n, N) \) which satisfies the conditions

\[
J_n(G) = N, \\
J_k(G) = 0 \ (k \neq n, 0, 1) \text{ and} \\
u(G) = 1.
\]

Proof. First, we prove Lemma 2 in the case that both \( n \) and \( N \) are positive.

Case 1: \( n > 0 \) and \( N > 0 \). For \( \ell = 2, 3, \ldots, n \), let \( G_{\ell N} \) be the Gauss diagram as shown in Figure 9. If \( \ell = 2 \), then calculating the indices of the chords of the Gauss diagram \( G_{2N} \), we have

\[
i(\gamma_0) = 0, \\
i(\gamma_{2s-1}) = 2, \\
i(\gamma_{2s}) = 1 \text{ and} \\
i(\gamma_{2N+s}) = 1
\]

for \( s = 1, 2, \ldots, N \). From these indices, we obtain the values of the \( n \)-writhes of the Gauss diagram \( G_{2N}^N \):

\[
J_2(G_{2N}) = N \text{ and} \\
J_m(G_{2N}) = 0 \ (m \neq 0, 1, 2).
\]

If \( \ell \geq 3 \), then calculating the indices of the chords of the Gauss diagram \( G_{\ell N}^N \), we have
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\[
i(\gamma_0) = 0, \\
i(\gamma_{(s-1)\ell+1}) = \ell, \\
i(\gamma_{(s-1)\ell+2}) = \ell - 1, \\
i(\gamma_{(s-1)\ell+t}) = 1 \text{ and} \\
i(\gamma_{N\ell+u}) = 1
\]

for \( s = 1, 2, \ldots, N, \ t = 3, 4, \ldots, \ell \) and \( u = 1, 2, \ldots, (\ell - 1)N \). From these indices, we obtain the values of the \( n \)-writhes of the Gauss diagram \( G^\ell_N \) (\( \ell \geq 3 \)):

\[
J_\ell(G^\ell_N) = N, \\
J_{\ell-1}(G^\ell_N) = -N \text{ and} \\
J_m(G^\ell_N) = 0 \ (m \neq 0, 1, \ell - 1, \ell).
\]

Let \( G(n, N) \) be the vertex connected sum of the Gauss diagrams \( G^\ell_N \) (\( \ell = n, \ldots, 3, 2 \)) with respect to \( \gamma_0 \)'s, that is \( G(n, N) = G^0_N \# \cdots \# G^3_N \# G^2_N \). From Lemma 1, we have

\[
J_n(G(n, N)) = N \text{ and} \\
J_k(G(n, N)) = 0 \ (k \neq n, 0, 1).
\]
By removing the chord $\gamma_0$, the Gauss diagram $G(n, N)$ can be transformed into a Gauss diagram of the trivial knot using generalized Reidemeister moves I and II. Thus, $w^v(G(n, N)) = 1$.

Next, we prove Lemma 2 in the case that $n$ is positive and $N$ is negative.

**Case 2:** $n > 0$ and $N < 0$. For $\ell = 2, 3, \ldots, n$, let $G_{\ell}^N$ be the Gauss diagram as shown in Figure 10. Calculating the indices of the chords of the Gauss diagram $G_{\ell}^N$, we have

$$
\begin{align*}
i(\gamma_0) &= 0, \\
i(\gamma_{s-1}(\ell+2)+1) &= \ell, \\
i(\gamma_{s-1}(\ell+2)+2) &= \ell - 1, \\
i(\gamma_{s-1}(\ell+2)+\ell) &= 1 \\
i(\gamma-N(\ell+2)+u) &= 1
\end{align*}
$$

for $s = 1, 2, \ldots, -N$, $t = 3, 4, \ldots, \ell + 2$ and $u = 1, 2, \ldots, -(\ell - 1)N$. From these indices, we obtain the values of the $n$-writhes of the Gauss diagram $G_{\ell}^N$:

$$
\begin{align*}
J_{\ell}(G_{\ell}^N) &= N, \\
J_{\ell-1}(G_{\ell}^N) &= -N \text{ and} \\
J_m(G_{\ell}^N) &= 0 \quad (m \neq 0, 1, \ell - 1, \ell).
\end{align*}
$$

Figure 10. The Gauss diagram $G_{\ell}^N$ for $\ell \geq 2$ and $N < 0$.

Let $G(n, N)$ be the vertex connected sum of the Gauss diagrams $G_{\ell}^N$ ($\ell = n, \ldots, 3, 2$).
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with respect to $\gamma_0$'s, that is $G(n,N) = G_N^0 \cdots G_N^1 \cdots G_N^n$. In a way similar to the case where $n > 0$ and $N > 0$, we have $J_n(G(n,N)) = N$, $J_k(G(n,N)) = 0$ ($k \neq n, 0, 1$), $w^v(G(n,N)) = 1$.

Then, we prove Lemma 2 in the case that $n$ is negative and $N$ is positive.

**Case 3:** $n < 0$ and $N > 0$. For $\ell = -1, -2, \ldots, n$, let $G_N^\ell$ be the Gauss diagram as shown in Figure 11. Calculating the indices of the chords of the Gauss diagram $G_N^\ell$, we have

\[
\begin{align*}
i(\gamma_0) &= 0, \\
i(\gamma_{(s-1)(-\ell+2)+1}) &= \ell, \\
i(\gamma_{(s-1)(-\ell+2)+2}) &= \ell + 1, \\
i(\gamma_{(s-1)(-\ell+2)+t}) &= 1 \text{ and} \\
i(\gamma_{N(-\ell+2)+u}) &= 1
\end{align*}
\]

for $s = 1, 2, \ldots, N$, $t = 3, 4, \ldots, -\ell + 2$ and $u = 1, 2, \ldots, (-\ell - 1)N$. From these indices, we obtain the values of the $n$-writhes of the Gauss diagram $G_N^\ell$:

\[
\begin{align*}
J_\ell(G_N^\ell) &= N, \\
J_{\ell+1}(G_N^\ell) &= -N \text{ and} \\
J_m(G_N^\ell) &= 0 \quad (m \neq 0, 1, \ell, \ell + 1).
\end{align*}
\]

Figure 11. The Gauss diagram $G_N^\ell$ for $\ell \leq -1$ and $N > 0$. 
Let $G(n,N)$ be the vertex connected sum of the Gauss diagrams $G^\ell_N$ ($\ell = n, \ldots, -3, -2, -1$) with respect to $\gamma_0$’s, that is $G(n,N) = G^\ell_N \# G^\ell_{N-1} \# \cdots \# G^\ell_1 \# G^\ell_0$. In a way similar to the case where $n > 0$ and $N > 0$, we have $J_n(G(n,N)) = N$, $J_k(G(n,N)) = 0$ ($k \neq n, 0, 1$), and $u^\ell(G(n,N)) = 1$.

Finally, we prove Lemma 2 in the case that both $n$ and $N$ are negative.

**Case 4:** $n < 0$ and $N < 0$. For $\ell = -1, -2, \ldots, n$, let $G^\ell_N$ be the Gauss diagram as shown in Figure 12. If $\ell = -1$, then calculating the indices of the chords of the Gauss diagram $G^{-1}_N$, we have

$$i(\gamma_0) = 0,$$

$$i(\gamma_s) = -1$$

for $s = 1, 2, \ldots, -N$. From these indices, we obtain the values of the $n$-writhes of the Gauss diagram $G^{-1}_N$:

$$J_{-1}(G^{-1}_N) = N$$

$$J_m(G^{-1}_N) = 0 \quad (m \neq -1, 0, 1).$$

If $\ell = -2$, then calculating the indices of the chords of the Gauss diagram $G^{-2}_N$, we have

$$i(\gamma_0) = 0,$$

$$i(\gamma_{2s-1}) = -2,$$

$$i(\gamma_{2s}) = -1$$

for $s = 1, 2, \ldots, -N$. From these indices, we obtain the values of the $n$-writhes of the Gauss diagram $G^{-2}_N$:

$$J_{-2}(G^{-2}_N) = N,$$

$$J_{-1}(G^{-1}_N) = -N$$

$$J_m(G^\ell_N) = 0 \quad (m \neq -2, -1, 0, 1).$$

If $\ell = -3, -4, \ldots, n$, then calculating the indices of the chords of the Gauss diagram $G^\ell_N$, we have

$$i(\gamma_0) = 0,$$

$$i(\gamma_{-(s+1)\ell+1}) = \ell,$$

$$i(\gamma_{-(s+1)\ell+2}) = \ell + 1,$$

$$i(\gamma_{-(s+1)\ell+1}) = 1$$
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\[ i(\gamma_{Nt+u}) = 1 \]

for \( s = 1, 2, \ldots, -N, \ t = 3, 4, \ldots, -\ell \) and \( u = 1, 2, \ldots, (\ell + 1)N \). From these indices, we obtain the values of the \( n \)-writhes of the Gauss diagram \( G^t_N \) (\( \ell \leq -3 \)):

\[
J_{\ell}(G^t_N) = N,
J_{\ell+1}(G^t_N) = -N \text{ and}
J_{m}(G^t_N) = 0 \quad (m \neq 0, 1, \ell, \ell + 1).
\]

Let \( G(n, N) \) be the vertex connected sum of the Gauss diagrams \( G^t_N \) (\( \ell = n, \ldots, -3, -2, -1 \)) with respect to \( \gamma_0 \)'s, that is \( G(n, N) = G^n_N \# \cdots \# G^{-3}_N \# G^{-2}_N \# G^{-1}_N \). In a way similar to the case where \( n > 0 \) and \( N > 0 \), we have \( J_0(G(n, N)) = N, \ J_k(G(n, N)) = 0 \ (k \neq n, 0, 1) \), and \( u^v(G(n, N)) = 1 \).

Therefore \( K \) is a virtual knot such that \( u^v(K) = 1 \) and \( J_n(K) = 0 \) for any \( n \neq 0 \).

Hereinafter, it is assumed that at least one integer in \( \{c_n\} \) is non-zero and \( \sum_{n \neq 0} nc_n = 0 \). For such a sequence of integers \( \{c_n\} \), we prepare the Gauss diagram \( G(i, c_i) \) for each nonzero integer \( c_i \) and let \( G \) be a vertex connected sum of these \( G(i, c_i) \)'s with respect to \( \gamma_0 \)'s. Let \( K \) be the virtual knot with the Gauss diagram \( G \). We have \( u^v(K) = 1 \).

By Lemmas 1 and 2, we have \( J_n(K) = J_n(G) = c_n \ (n \neq 0, 1) \). Moreover, it holds that

\[
J_1(K) = J_1(G) = - \sum_{n \neq 0, 1} nc_n = c_1
\]

by the assumption \( \sum_{n \neq 0} nc_n = 0 \). This completes the proof of Theorem 3.

\[ \square \]
Figure 12. The Gauss diagram $G_N^\ell$ for $\ell \leq -1$ and $N < 0$.

References

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\[ K \]

Figure 13. Kishino’s knot \( K \) and its Gauss diagram \( G \).


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