LEFT ORDERABLE SURGERIES OF DOUBLE TWIST KNOTS

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Abstract. A rational number $r$ is called a left orderable slope of a knot $K \subset S^3$ if the $3$-manifold obtained from $S^3$ by $r$-surgery along $K$ has left orderable fundamental group. In this paper we consider the double twist knots $C(k, l)$ in the Conway notation. For any positive integers $m$ and $n$, we show that if $K$ is a double twist knot of the form $C(2m, -2n), C(2m + 1, 2n)$ or $C(2m + 1, -2n)$ then there is an explicit unbounded interval $I$ such that any rational number $r \in I$ is a left orderable slope of $K$.

1. Introduction

The motivation of this paper is the L-space conjecture of Boyer, Gordon and Watson [BGW] which states that an irreducible rational homology $3$-sphere is an L-space if and only if its fundamental group is not left orderable. Here a rational homology $3$-sphere $Y$ is an L-space if its Heegaard Floer homology $\widehat{HF}(Y)$ has rank equal to the order of $H_1(Y; \mathbb{Z})$, and a non-trivial group $G$ is left orderable if it admits a total ordering $<\quad$ such that $g < h$ implies $fg < fh$ for all elements $f, g, h$ in $G$. A knot $K$ in $S^3$ is called an L-space knot if it admits a positive Dehn surgery yielding an L-space. It is known that non-torus alternating knots are not L-space knots, see [OS]. In view of the L-space conjecture, this would imply that any non-trivial Dehn surgery along a non-torus alternating knot produces a 3-manifold with left orderable fundamental group.

A rational number $r$ is called a left orderable slope of a knot $K \subset S^3$ if the 3-manifold obtained from $S^3$ by $r$-surgery along $K$ has left orderable fundamental group. As mentioned above, one would expect that any rational number is a left orderable slope of any non-torus alternating knot. It is known that any rational number $r \in (-4, 4)$ is a left orderable slope of the figure eight knot, and any rational number $r \in [0, 4]$ is a left orderable slope of the hyperbolic twist knot $5_2$, see [BGW] and [HTe2] respectively. Consider the double twist knot $C(k, l)$ in the Conway notation as in Figure 1, where $k, l$ denote the numbers of horizontal half-twists with sign in the boxes. Here the sign of $\binom{k}{l}$ is positive in the box $k$ and is negative in the box $l$. Then the following results were shown in [HTe1, Tr] by using continuous families of hyperbolic $SL_2(\mathbb{R})$-representations of knot groups. If $m, n$ are integers $\geq 1$, any rational number $r \in (-4n, 4m)$ is a left orderable slope of $C(2m, 2n)$. If $m, n$ are integers $\geq 2$ then any rational number $r \in [0, \max\{4m, 4n\})$ is a left orderable slope of $C(2m, -2n)$ and any rational number $r \in [0, 4]$ is a left orderable slope of both
$C(2m, -2)$ and $C(2, -2n)$. Note that $C(2, 2)$ is the figure eight knot and $C(4, -2)$ is the twist knot $5_2$. Moreover $C(2, -2)$ is the trefoil knot, which is the $(2,3)$-torus knot.

$$\begin{array}{ccc}
\text{k} & \text{ } & \text{l} \\
\end{array}$$

Figure 1. The double twist knot/link $C(k, l)$ in the Conway notation.

In this paper, by using continuous families of elliptic SL$_2$($\mathbb{R}$)-representations of knot groups we extend the range of left orderable slopes of $C(2m, -2n)$. Moreover, we also give left orderable slopes of $C(2m + 1, \pm 2n)$.

**Theorem 1.** Suppose $K$ is a double twist knot of the form $C(2m, -2n)$, $C(2m + 1, 2n)$ or $C(2m + 1, -2n)$ in the Conway notation for some positive integers $m$ and $n$. Let

$$\text{LO}_K = \begin{cases} 
(-\infty, 1) & \text{if } K = C(2m, -2n), \\
(-\infty, 2n - 1) & \text{if } K = C(2m + 1, 2n), \\
(3 - 2n, \infty) & \text{if } K = C(2m + 1, -2n) \text{ and } n \geq 2.
\end{cases}$$

Then any rational number $r \in \text{LO}_K$ is a left orderable slope of $K$.

Combining this with results in [HTe1, Tr], we conclude that if $m$ and $n$ are integers $\geq 2$ then any rational number $r \in (-\infty, \max\{4m, 4n\})$ is a left orderable slope of $C(2m, -2n)$ and any rational number $r \in (-\infty, 4]$ is a left orderable slope of both $C(2m, -2)$ and $C(2, -2n)$. In the subsequent paper [KTT] we will use continuous families of hyperbolic SL$_2$($\mathbb{R}$)-representations of knot groups to extend the range of left orderable slopes of $C(2m + 1, -2n)$. More specifically, we will show that any rational number $r \in (-4n, 4m)$ is a left orderable slope of $C(2m + 1, -2n)$ detected by hyperbolic SL$_2$($\mathbb{R}$)-representations of the knot group.

We remark that in the case of $C(2m+1, \pm 2n)$, where $m$ and $n$ are positive integers, Gao [Ga] independently obtains similar results. She proves a weaker result that any rational number $r \in (-\infty, 1)$ is a left orderable slope of $C(2m + 1, 2n)$ and a stronger result that any rational number $r \in (-4n, \infty)$ is a left orderable slope of $C(2m + 1, -2n)$.

As in [BGW, HTe1, HTe2, Tr, CD] the proof of Theorem 1 is based on the existence of continuous families of elliptic SL$_2$($\mathbb{R}$)-representations of the knot groups of double twist knots $C(2m, -2n)$ and $C(2m + 1, \pm 2n)$ into SL$_2$($\mathbb{R}$) and the fact that SL$_2$($\mathbb{R}$), which is the universal covering group of SL$_2$($\mathbb{R}$), is a left orderable group.
This paper is organized as follows. In Section 1, we study certain real roots of the Riley polynomial of double twist knots $C(k, -2p)$, whose zero locus describes all non-abelian representations of the knot group into $\text{SL}_2(\mathbb{C})$. In Section 2, we prove Theorem 1.

2. Real roots of the Riley polynomial

For a knot $K$ in $S^3$, let $G(K)$ denote the knot group of $K$ which is the fundamental group of the complement of an open tubular neighborhood of $K$.

Consider the double twist knot/link $C(k, l)$ in the Conway notation as in Figure 1, where $k, l$ are integers such that $|kl| \geq 3$. Note that $C(k, l)$ is the rational knot/link corresponding to continued fraction $k + 1/l$. It is easy to see that $C(k, l)$ is the mirror image of $C(l, k) = C(-k, -l)$. Moreover, $C(k, l)$ is a knot if $kl$ is even and is a two-component link if $kl$ is odd. In this paper, we only consider knots and so we can assume that $k > 0$ and $l = -2p$ is even.

Note that $C(k, -2p)$ is the mirror image of the double twist knot $J(k, 2p)$ in [HS]. Then, by [HS], the knot group of $C(k, -2p)$ has a presentation $G(C(k, -2p)) = \langle a, b \mid aw^p = w^p b \rangle$ where $a, b$ are meridians and

$$w = \begin{cases} (ab^{-1})^m(a^{-1}b)^m & \text{if } k = 2m, \\ (ab^{-1})^mab(a^{-1}b)^m & \text{if } k = 2m + 1. \end{cases}$$

Moreover, the canonical longitude of $C(k, -2p)$ corresponding to the meridian $\mu = a$ is $\lambda = (w^p(w^p)^*a^{-2}\varepsilon)^{-1}$, where $\varepsilon = 0$ if $k = 2m$ and $\varepsilon = 2p$ if $k = 2m + 1$. Here, for a word $u$ in the letters $a, b$ we let $u^*$ be the word obtained by reading $v$ backwards.

Suppose $\rho : G(C(k, -2p)) \to \text{SL}_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2 - y & M^{-1} \end{bmatrix}$$

where $(M, y) \in \mathbb{C}^2$ satisfies the matrix equation $\rho(aw^p) = \rho(w^pb)$. It is known that this matrix equation is equivalent to a single polynomial equation $R_{C(k, -2p)}(x, y) = 0$, where $x = (\text{tr} \rho(a))^2$ and $R_K(x, y)$ is the Riley polynomial of $K$, see [Ri]. This polynomial can be described via the Chebychev polynomials as follows.

Let $\{S_j(v)\}_{j \in \mathbb{Z}}$ be the Chebychev polynomials in the variable $v$ defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_j(v) = vS_{j-1}(v) - S_{j-2}(v)$ for all integers $j$. Note that $S_j(v) = -S_{-j-2}(v)$ and $S_j(\pm 2) = (\pm 1)^j(j + 1)$. Moreover, we have $S_j(v) = (s^{j+1} - s^{-(j+1)})/(s - s^{-1})$ for $v = s + s^{-1} \neq \pm 2$. Using this identity one can prove the following.

Lemma 2.1. For any integer $j$ and any positive integer $n$ we have

1. $S_j^2(v) - vS_j(v)S_{j-1}(v) + S_{j-1}^2(v) = 1$.
2. $S_n(v) - S_{n-1}(v) = \prod_{j=1}^{n} (v - 2 \cos \frac{(2j-1)\pi}{2n+1})$. 


(3) \( S_n(v) + S_{n-1}(v) = \prod_{j=1}^n (v - 2 \cos \frac{2j\pi}{2n+1}). \)

(4) \( S_n(v) = \prod_{j=1}^n (v - 2 \cos \frac{j\pi}{n+1}). \)

The Riley polynomial of \( C(k, -2p) \), whose zero locus describes all non-abelian representations of the knot group of \( C(k, -2p) \) into \( \text{SL}_2(\mathbb{C}) \), is

\[ R_{C(k, -2p)}(x, y) = S_p(t) - zS_{p-1}(t) \]

where

\[ t = \text{tr} \rho(w) = \begin{cases} 2 + (y + 2 - x)(y - 2)S_{m-1}^2(y) & \text{if } k = 2m, \\ 2 - (y + 2 - x)(S_m(y) - S_{m-1}(y))^2 & \text{if } k = 2m + 1, \end{cases} \]

and

\[ z = \begin{cases} 1 + (y + 2 - x)S_{m-1}(y)(S_m(y) - S_{m-1}(y)) & \text{if } k = 2m, \\ 1 - (y + 2 - x)S_m(y)(S_m(y) - S_{m-1}(y)) & \text{if } k = 2m + 1. \end{cases} \]

Moreover, for the representation \( \rho : G(C(k, -2p)) \to \text{SL}_2(\mathbb{C}) \) of the form (2.1) the image of the canonical longitude \( \lambda = (w^p(w^p)^*a^{-2z})^{-1} \) has the form \( \rho(\lambda) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix} \), where

\[ L = -\frac{M^{-1}(S_m(y) - S_{m-1}(y)) - M(S_{m-1}(y) - S_{m-2}(y))}{M(S_m(y) - S_{m-1}(y)) - M^{-1}(S_{m-1}(y) - S_{m-2}(y))} \quad \text{if } k = 2m \]

and

\[ L = -M^{4p}M^{-1}S_m(y) - MS_{m-1}(y) \quad \frac{MS_m(y) - M^{-1}S_{m-1}(y)}{MS_m(y) - M^{-1}S_{m-1}(y)} \quad \text{if } k = 2m + 1. \]

See e.g. [Tr, Pe].

Lemmas (2.2)–(2.4) below describe continuous families of real roots of the Riley polynomials of the double twist knots \( C(2m, -2n) \), \( C(2m + 1, 2n) \) and \( C(2m + 1, -2n) \) respectively, where \( m \) and \( n \) are positive integers.

**Lemma 2.2.** There exists a continuous real function \( y : [4 - 1/(mn), 4] \to [2, \infty) \) in the variable \( x \) such that

- \( y(4 - 1/(mn)) = 2 \) and
- \( R_{C(2m, -2n)}(x, y(x)) = 0 \) for all \( x \in [4 - 1/(mn), 4] \).

**Proof.** Let \( K = C(2m, -2n) \). We have \( R_K(x, y) = S_n(t) - zS_{n-1}(t) \) where

\[ t = 2 + (y + 2 - x)(y - 2)S_{m-1}^2(y), \]

\[ z = 1 + (y + 2 - x)S_{m-1}(y)(S_m(y) - S_{m-1}(y)). \]

Consider real numbers \( x \in [4 - 1/(mn), 4] \) and \( y \in [2, \infty) \). Since \( y \geq 2 \geq x - 2 \), we have \( t \geq 2 \) and \( z \geq 1 \). This implies that \( zS_{n-1}(t) - S_{n-2}(t) \geq S_{n-1}(t) - S_{n-2}(t) \geq 0 \), by Lemma 2.1. The equation \( R_K(x, y) = 0 \) is then equivalent to

\[ (S_n(t) - zS_{n-1}(t))(S_{n-2}(t) - zS_{n-1}(t)) = 0. \]
Let $P(x, y)$ denote the left hand side of equation (2.2). By Lemma 2.1, we have $S_n^2(t) - tS_n(t)S_{n-1}(t) + S_{n-1}^2(t) = 1$. This can be written as $S_n(t)S_{n-2}(t) = S_{n-1}^2(t) - 1$. From this and $S_n(t) + S_{n-2}(t) = tS_{n-1}(t)$ we get

$$P(x, y) = (z^2 - tz + 1)S_{n-1}^2(t) - 1.$$  

By a direct calculation, using $S_m^2(y) + S_{m-1}^2(y) - yS_m(y)S_{m-1}(y) = 1$, we have

$$z^2 - tz + 1$$

$$= (z - 1)^2 - (t - 2)z$$

$$= (y + 2 - x)^2S_{m-1}^2(y)(S_m(y) - S_{m-1}(y))^2$$

$$- (y + 2 - x)(y - 2)S_{m-1}^2(y)[1 + (y + 2 - x)S_m(y)(S_m(y) - S_{m-1}(y))]$$

$$= (y + 2 - x)S_{m-1}^2(y)[4 - x + (y + 2 - x)(y - 2)S_{m-1}^2(y)]$$

$$= (y + 2 - x)S_{m-1}^2(y)(t + 2 - x).$$

Hence $P(x, y) = (y + 2 - x)S_{m-1}^2(y)(t + 2 - x)S_{n-1}^2(t) - 1$.

By Lemma 2.1(4), for any positive integer $l$ the Chebychev polynomial $S_l(v) = \prod_{j=1}^{l}(v - 2 \cos \frac{\pi}{l+1})$ is a strictly increasing function in $v \in [2, \infty)$. This implies that, for a fixed real number $x \in [4 - 1/(mn), 4]$, the polynomials $t = 2 + (y + 2 - x)(y - 2)S_{m-1}^2(y) \geq 2$ and $P(x, y) = (y + 2 - x)S_{m-1}^2(y)(t + 2 - x)S_{n-1}^2(t) - 1$ are strictly increasing functions in $y \in [2, \infty)$. Note that $\lim_{y \to \infty} P(x, y) = \infty$ and

$$\lim_{y \to 2^+} P(x, y) = P(x, 2) = (4 - x)^2m^2n^2 - 1 \leq 0.$$  

Hence there exists a unique real number $y(x) \in [2, \infty)$ such that $P(x, y(x)) = 0$. Since $P(4 - \frac{1}{mn}, 2) = 0$ we have $y(4 - \frac{1}{mn}) = 2$. Finally, by the implicit function theorem $y = y(x)$ is a continuous function in $x \in [4 - 1/(mn), 4]$. \hfill $\Box$

**Lemma 2.3.** There exists a continuous real function $x : [2, \infty) \to (4 \cos^2 \frac{(2n-1)\pi}{4n+2}, \infty)$ in the variable $y$ such that

- $x(2) < 4 \cos^2 \frac{(2n-2)\pi}{4n+2}$,
- $\lim_{y \to \infty} x(y) = \infty$ and
- $R_{C(2m+1,2n)}(x(y), y) = 0$ for all $y \in [2, \infty)$.

**Proof.** Let $K = C(2m+1, 2n)$. We have $R_K(x, y) = S_{-n}(t) - zS_{-n-1}(t)$ where

$$t = 2 - (y + 2 - x)(S_m(y) - S_{m-1}(y))^2$$

$$z = 1 - (y + 2 - x)S_m(y)(S_m(y) - S_{m-1}(y)).$$

Note that $R_K(x, y) = (t - z)S_{-n-1}(t) - S_{-n-2}(t) = S_n(t) - (t - z)S_{n-1}(t)$.

By Lemma 2.1 we have

$$S_n(t) - S_{n-1}(t) = \prod_{j=1}^{n}(t - 2 \cos \frac{(2j - 1)\pi}{2n+1}),$$

and
\[ S_n(t) + S_{n-1}(t) = \prod_{j=1}^{n} \left( t - 2 \cos \frac{2j\pi}{2n+1} \right). \]

Let \( t_j = 2 \cos \frac{j\pi}{2n+1} \) for \( j = 1, \ldots, 2n \). By writing \( t_{2j-1} = e^{i\theta} + e^{-i\theta} \) where \( \theta = \frac{(2j-1)\pi}{2n+1} \), we have

\[
S_n(t_{2j-1}) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin \frac{(2j-1)(n+1)\pi}{2n+1}}{\sin \frac{(2j-1)\pi}{2n+1}}
\]

This implies that \((-1)^{j-1}S_n(t_{2j-1}) > 0\). Similarly, \((-1)^jS_n(t_{2j}) > 0\).

Fix a real number \( y \geq 2 \). Let \( s_j(y) = y + 2 - \frac{2-t_j}{(S_m(y)-S_{m-1}(y))^2} \) for \( j = 1, \ldots, 2n \). We also let \( s_0 = y+2 \). Since \(-2 < t_{2n} < \cdots < t_1 \) we have \( s_{2n}(y) < \cdots < s_1(y) < y+2 = s_0(y) \).

At \( x = s_{2j-1}(y) \) we have \( t = t_{2j-1} \) and so \( S_n(t) = S_{n-1}(t) \). This implies that

\[
R_K(s_{2j-1}(y), y) = (1 - (t - z))S_n(t_{2j-1}) = -(y + 2 - s_{2j-1}(y))S_m(y) - S_{m-1}(y))S_n(t_{2j-1}).
\]

Since \( y \geq 2 \), by Lemma 2.1 we have \( S_m(y) - S_{m-1}(y) \geq S_m(2) - S_{m-1}(2) = 1 \) and \( S_{m-1}(y) \geq S_{m-1}(2) = m \). Hence \((-1)^jR_K(s_{2j-1}(y), y) > 0\).

Similarly, for \( 1 \leq j \leq n \) we have

\[
R_K(s_{2j}(y), y) = (1 + t - z)S_n(t_{2j}) = \left[ 2 + (y + 2 - s_{2j-1}(y))S_m(y) - S_{m-1}(y) \right] S_n(t_{2j}),
\]

which implies that \((-1)^jR_K(s_{2j}(y), y) > 0\).

For each \( 1 \leq j \leq n-1 \), since

\[
R_K(s_{2j+1}(y), y)R_K(s_{2j}(y), y) < 0
\]

there exists \( x_j(y) \in (s_{2j+1}(y), s_{2j}(y)) \) such that \( R_K(x_j(y), y) = 0 \). Since

\[
R_K(s_0(y), y) = R_K(y + 2, y) = 1
\]

and \( R_K(s_1(y), y) < 0 \) there exists \( x_0(y) \in (s_1(y), s_0(y)) \) such that \( R_K(x_0(y), y) = 0 \).

Since \( R_K(x, y) = zS_{n-1}(t) = S_{n-2}(t) \), we see that \( R_K(x, y) \) is a polynomial of degree \( n \) in \( x \) for each fixed real number \( y \geq 2 \). This polynomial has exactly \( n \) simple real roots \( x_0(y), \ldots, x_{n-1}(y) \) satisfying \( x_{n-1}(y) < \cdots < x_0(y) < y + 2 \), hence the implicit function theorem implies that each \( x_j(y) \) is a continuous function in \( y \geq 2 \).

By letting \( x(y) = x_{n-1}(y) \) for \( y \geq 2 \), we have \( R_K(x(y), y) = 0 \). Moreover, since

\[
x(y) > s_{2n-1}(y) = y + 2 - \frac{2 - 2 \cos \frac{(2n-1)\pi}{2n+1}}{(S_m(y) - S_{m-1}(y))^2}
\]

we have \( \lim_{y \to \infty} x(y) = \infty \) and \( x(y) > 4 - \left( 2 - 2 \cos \frac{(2n-1)\pi}{2n+1} \right) = 4 \cos^2 \frac{(2n-1)\pi}{4n+2} \) for \( y \geq 2 \).
Finally, since \( x(y) < s_{2n-2}(y) \) for all \( y \geq 2 \) we have \( x(2) < s_{2n-2}(2) = 4 \cos^2 \frac{(2n-2)\pi}{4n+2} \). \qed

**Lemma 2.4.** Suppose \( n \geq 2 \). Then there exists a continuous real function \( x : [2, \infty) \to (4 \cos^2 \frac{(2n-3)\pi}{4n+2}, \infty) \) in the variable \( y \) such that

- \( x(2) < 4 \cos^2 \frac{(2n-3)\pi}{4n+2} \),
- \( \lim_{y \to \infty} x(y) = \infty \) and
- \( R_{C(2m+1,-2n)}(x(y), y) = 0 \) for all \( y \in [2, \infty) \).

**Proof.** Let \( K = C(2m+1,-2n) \). We have \( R_K(x,y) = S_n(t) - zS_{n-1}(t) \) where

\[
t = 2 - (y + 2 - x)(S_m(y) - S_{m-1}(y))^2,
\]

\[
z = 1 - (y + 2 - x)S_m(y)(S_m(y) - S_{m-1}(y)).
\]

Fix a real number \( y \geq 2 \). Choose \( t_j \) and \( s_j(y) \) for \( 1 \leq j \leq 2n \) as in Lemma 2.3. Since

\[
R_K(s_{2j-1}(y), y) = (1 - z)S_n(t_{2j-1}) = (y + 2 - s_{2j-1}(y))S_m(y)(S_m(y) - S_{m-1}(y))S_n(t_{2j-1}),
\]

we have \((-1)^{j-1}R_K(s_{2j-1}(y), y) > 0\). Hence, there exists \( x_j(y) \in (s_{2j+1}(y), s_{2j-1}(y)) \) such that \( R_K(x_j(y), y) = 0 \) for each \( 1 \leq j \leq n - 1 \).

By writing \( R_K(x, y) = (t - z)S_{n-1}(t) - S_{n-2}(t) \) and noting that

\[
t - z = 1 + (y + 2 - x)(S_m(y) - S_{m-1}(y))S_{m-1}(y),
\]

we see that \( R_K(x, y) \) is a polynomial of degree \( n \) in \( x \) with negative highest coefficient for each fixed real number \( y \geq 2 \). Since \( \lim_{x \to \infty} R_K(x, y) = -\infty \) and \( R_K(y+2, y) = 1 \), there exists \( x_0(y) \in (y + 2, \infty) \) such that \( R_K(x_0(y), y) = 0 \). For a fixed real number \( y \geq 2 \), the polynomial \( R_K(x, y) \) of degree \( n \) in \( x \) has exactly \( n \) simple real roots \( x_0(y), \ldots, x_{n-1}(y) \) satisfying \( x_{n-1}(y) < \cdots < x_1(y) < y + 2 < x_0(y) \), hence the implicit function theorem implies that each \( x_j(y) \) is a continuous function in \( y \geq 2 \).

By letting \( x(y) = x_{n-1}(y) \) for \( y \geq 2 \), we have \( R_K(x(y), y) = 0 \). Moreover, since

\[
x(y) > s_{2n-1}(y) = y + 2 - \frac{2 - 2 \cos \frac{(2n-1)\pi}{2n+1}}{(S_m(y) - S_{m-1}(y))^2}
\]

we have \( \lim_{y \to \infty} x(y) = \infty \) and \( x(y) > 4 - (2 - 2 \cos \frac{(2n-1)\pi}{2n+1}) = 4 \cos^2 \frac{(2n-1)\pi}{4n+2} \) for \( y \geq 2 \).

Finally, since \( x(y) < s_{2n-3}(y) \) for all \( y \geq 2 \) we have \( x(2) < s_{2n-3}(2) = 4 \cos^2 \frac{(2n-3)\pi}{4n+2} \). \qed

3. **Proof of Theorem 1**

Suppose \( K \) is a double twist knot of the form \( C(2m, -2n) \), \( C(2m+1, 2n) \) or \( C(2m+1, -2n) \) in the Conway notation for some positive integers \( m \) and \( n \). Let \( X \) be the complement of an open tubular neighborhood of \( K \) in \( S^3 \), and \( X_r \) the 3-manifold obtained
from $S^3$ by $r$-surgery along $K$. Recall that

$$LO_K = \begin{cases} 
(-\infty, 1) & \text{if } K = C(2m, -2n), \\
(-\infty, 2n - 1) & \text{if } K = C(2m + 1, 2n), \\
(3 - 2n, \infty) & \text{if } K = C(2m + 1, -2n) \text{ and } n \geq 2.
\end{cases}$$

An element of $\text{SL}_2(\mathbb{R})$ is called elliptic if its trace is a real number in $(-2, 2)$. A representation $\rho : \mathbb{Z}^2 \to \text{SL}_2(\mathbb{R})$ is called elliptic if the image group $\rho(\mathbb{Z}^2)$ contains an elliptic element of $\text{SL}_2(\mathbb{R})$. In which case, since $\mathbb{Z}^2$ is an abelian group every non-trivial element of $\rho(\mathbb{Z}^2)$ must also be elliptic.

Using Lemmas 2.2–2.4 we first prove the following.

**Proposition 3.1.** For each rational number $r \in LO_K \setminus \{0\}$ there exists a representation $\rho : \pi_1(X_r) \to \text{SL}_2(\mathbb{R})$ such that $\rho|_{\pi_1(\partial X)} : \pi_1(\partial X) \cong \mathbb{Z}^2 \to \text{SL}_2(\mathbb{R})$ is an elliptic representation.

**Proof.** We first consider the case $K = C(2m, -2n)$. Let $\theta_0 = \arccos 1/(4mn)$. For $\theta \in (0, \theta_0) \cup (\pi - \theta_0, \pi)$ we let $x = 4 \cos^2 \theta$. Then $x \in (4 - 1/(mn), 4)$. Consider the continuous real function

$$y : [4 - 1/(mn), 4] \to [2, \infty)$$

in Lemma 2.2. Let $M = e^{i\theta}$. Then $x = 4 \cos^2 \theta = (M + M^{-1})^2$. Since $R_K(x, y(x)) = 0$ there exists a non-abelian representation $\rho : \pi_1(X) \to \text{SL}_2(\mathbb{C})$ such that

$$\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2 - y(x) & M^{-1} \end{bmatrix}.$$ 

Note that $x$ is the square of the trace of a meridian. Moreover, the image of the canonical longitude $\lambda$ corresponding to the meridian $\mu = a$ has the form $\rho(\lambda) = L = \begin{bmatrix} * \\ L \end{bmatrix}$, where

$$L = -\frac{M^{-1} \alpha - M \beta}{M \alpha - M^{-1} \beta}$$

and $\alpha = S_m(y(x)) - S_{m-1}(y(x))$, $\beta = S_{m-1}(y(x)) - S_{m-2}(y(x))$. Note that $\alpha > \beta > 0$, since $y(x) > 2$.

It is easy to see that $|L| = \sqrt{LL} = 1$, where $\bar{L}$ denotes the complex conjugate of $L$. Moreover, by a direct calculation, we have

$$\Re(L) = \frac{(2\alpha \beta - (\alpha^2 + \beta^2) \cos 2\theta)}{|M \alpha - M^{-1} \beta|^2},$$

$$\Im(L) = \frac{(\alpha^2 - \beta^2) \sin 2\theta}{|M \alpha - M^{-1} \beta|^2}.$$ 

Note that $\Im(L) > 0$ if $\theta \in (0, \theta_0)$ and $\Im(L) < 0$ if $\theta \in (\pi - \theta_0, \pi)$. Let

$$\varphi(\theta) = \begin{cases} 
\arccos \frac{(2\alpha \beta - (\alpha^2 + \beta^2) \cos 2\theta)}{|e^{i\theta} \alpha - e^{-i\theta} \beta|^2} & \text{if } \theta \in (0, \theta_0), \\
-\arccos \frac{(2\alpha \beta - (\alpha^2 + \beta^2) \cos 2\theta)}{|e^{i\theta} \alpha - e^{-i\theta} \beta|^2} & \text{if } \theta \in (\pi - \theta_0, \pi).
\end{cases}$$

Then $L = e^{i\varphi(\theta)}$. Note that $\varphi(\theta) \in (0, \pi)$ if $\theta \in (0, \theta_0)$ and $\varphi(\theta) \in (-\pi, 0)$ if $\theta \in (\pi - \theta_0, \pi)$.
The function \( f(\theta) := -\frac{\varphi(\theta)}{\theta} \) is a continuous function on each of the intervals \((0, \theta_0)\) and \((\pi - \theta_0, \pi)\). As \( \theta \to 0^+ \) we have \( M \to 1 \) and \( L = -\frac{M^{-1}\alpha - M\beta}{M\alpha - M^{-1}\beta} \to -1 \), so \( \varphi(\theta) \to \pi \). As \( \theta \to \theta_0^+ \) we have \( x \to 4 - 1/(mn) \), \( y(x) \to 2 \) and \( \alpha, \beta \to 1 \), so \( L = -\frac{M^{-1}\alpha - M\beta}{M\alpha - M^{-1}\beta} \to 1 \) and \( \varphi(\theta) \to 0 \). This implies that

\[
\lim_{\theta \to 0^+} \frac{-\varphi(\theta)}{\theta} = -\infty \quad \text{and} \quad \lim_{\theta \to \theta_0^+} \frac{-\varphi(\theta)}{\theta} = 0.
\]

Hence the image of \( f(\theta) \) on the interval \((0, \theta_0)\) contains the interval \((-\infty, 0)\).

Similarly, since

\[
\lim_{\theta \to (\pi-\theta_0)^+} \frac{-\varphi(\theta)}{\theta} = 0 \quad \text{and} \quad \lim_{\theta \to \pi^+} \frac{-\varphi(\theta)}{\theta} = 1,
\]

the image of \( f(\theta) \) on the interval \((\pi - \theta_0, \pi)\) contains the interval \((0, 1)\).

Suppose \( r = \frac{p}{q} \) is a rational number such that \( r \in (-\infty, 0) \cup (0, 1) \). Then \( r = f(\theta) = -\frac{\varphi(\theta)}{\theta} \) for some \( \theta \in (0, \theta_0) \cup (\pi - \theta_0, \pi) \). Since \( M^p L^q = e^{i(p\varphi + q\varphi(\theta))} = 1 \), we have \( \rho(M^p L^q) = L \). This means that the non-abelian representation \( \rho : \pi_1(X) \to \text{SL}_2(\mathbb{C}) \) extends to a representation \( \rho : \pi_1(X_r) \to \text{SL}_2(\mathbb{C}) \). Finally, since \( 2 - y(x) < 0 \), a result in [Kh, page 786] implies that \( \rho \) can be conjugated to an \( \text{SL}_2(\mathbb{R}) \)-representation. Note that the restriction of this representation to the peripheral subgroup \( \pi_1(\partial X) \) of the knot group is an elliptic representation. This completes the proof of Proposition 3.1 for \( K = C(2m, -2n) \).

We now consider the case \( K = C(2m + 1, 2n) \). Consider the continuous real function

\[
x : [2, \infty) \to \left( 4 \cos^2 \left( \frac{(2n-1)\pi}{4n+2} \right), \infty \right)
\]
in Lemma 2.3. Since \( x(2) < 4 \cos^2 \left( \frac{(2n-2)\pi}{4n+2} \right) \) and \( \lim_{y \to \infty} x(y) = \infty \), there exists \( y^* > 2 \) such that \( x(y^*) = 4 \) and \( 4 \cos^2 \left( \frac{(2n-1)\pi}{4n+2} \right) < x(y) < 4 \) for all \( y \in [2, y^*) \).

For each \( y \in [2, y^*) \) we let \( \theta(y) = \arccos(\sqrt{x(y)/2}) \). Then \( \theta(2) > \frac{(2n-2)\pi}{4n+2} \), and for \( y \in [2, y^*) \) we have \( 0 < \theta(y) < \frac{(2n-1)\pi}{4n+2} \) and \( x(y) = 4 \cos^2 \theta(y) \). Since \( R_K(x(y), y) = 0 \) there exists a non-abelian representation \( \rho : \pi_1(X) \to \text{SL}_2(\mathbb{C}) \) such that

\[
\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2 - y & M^{-1} \end{bmatrix},
\]

where \( M = e^{i\theta(y)} \). Moreover, the image of the canonical longitude \( \lambda \) corresponding to the meridian \( \mu = a \) has the form \( \rho(\lambda) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix} \), where

\[
L = -M^{-4n} \frac{M^{-1}\gamma - M\delta}{M\gamma - M^{-1}\delta}
\]

and \( \gamma = S_m(y) \), \( \delta = S_{m-1}(y) \). Note that \( \gamma > \delta > 0 \), since \( y > 2 \).

As in the previous case, we write \( L = e^{i\varphi(y)} \) where

\[
\varphi(y) = (2n - 2)\pi - 4n\theta(y) + \arccos \left[ (2\gamma\delta - (\gamma^2 + \delta^2) \cos 2\theta(y)) / |e^{i\theta(y)}\gamma - e^{-i\theta(y)}\delta| \right].
\]
Since \( \frac{(2n-2)\pi}{4n+2} < \theta(2) < \frac{(2n-1)\pi}{4n+2} \) we have \(-\frac{2\pi}{2n+1} < \varphi(2) < 2\pi - \frac{3\pi}{2n+1} \).

As \( y \to 2^+ \), \( \rho \) approaches a reducible representation and so \( L \to 1 \), \( \varphi(y) \to \varphi(2) = k2\pi \) for some integer \( k \). Since \(-\frac{2\pi}{2n+1} < \varphi(2) < 2\pi - \frac{3\pi}{2n+1} \), we must have \( \varphi(2) = 0 \). As \( y \to (y)^- \), we have \( x(y) \to 4 \), \( M \to 1 \), \( L = -M^{-\frac{4n}{M\gamma-M\delta}} \to -1 \) and hence \( \theta(y) \to 0^+ \), \( \varphi(y) \to (2l-1)\pi \) for some integer \( l \). Since

\[
(2l-1)\pi = \lim_{y \to (y)^-} (2n-2)\pi - 4n\theta(y)
+ \arccos \left( \frac{2\gamma\delta - (\gamma^2 + \delta^2) \cos 2\theta(y)}{|e^{i\theta(y)} - e^{-i\theta(y)}\delta|^2} \right)
\]

we have \((2n-2)\pi \leq (2l-1)\pi \leq (2n-1)\pi \). This implies that \( 2l - 1 = 2n - 1 \) and \( \varphi(y) \to (2n-1)\pi \) as \( y \to (y)^- \). Hence the image of \( g(y) := -\frac{\varphi(y)}{\theta(y)} \) on the interval \((2, y^*)\) contains the interval \((-\infty, 0)\).

Similarly, with \( \theta_1(y) = \pi - \theta(y) \) we have \( x(y) = 4\cos^2(\theta_1(y)) \) and hence for each \( y \in [2, y^*) \) there exists a non-abelian representation \( \rho_1 : \pi_1(X) \to \text{SL}_2(\mathbb{C}) \) such that

\[
\rho_1(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho_1(b) = \begin{bmatrix} M & 0 \\ 2 - y & M^{-1} \end{bmatrix},
\]

where \( M = e^{i\theta_1(y)} \). Moreover, the image of the canonical longitude \( \lambda \) corresponding to the meridian \( \mu = a \) has the form \( \rho_1(\lambda) = \begin{bmatrix} L & \star \\ 0 & L^{-1} \end{bmatrix} \), where \( L = e^{i\varphi_1(y)} \) and

\[
\varphi_1(y) = -(2n-2)\pi + 4n\pi - 4n\theta_1(y)
- \arccos \left( \frac{2\gamma\delta - (\gamma^2 + \delta^2) \cos 2\theta_1(y)}{|e^{i\theta_1(y)} - e^{-i\theta_1(y)}\delta|^2} \right)
\]

Since \( \frac{(2n-2)\pi}{4n+2} < \theta(2) < \frac{(2n-1)\pi}{4n+2} \) we have \(-2\pi + \frac{3\pi}{2n+1} < \varphi_1(2) < \frac{2\pi}{2n+1} \).

As \( y \to 2^+ \), \( \rho_1 \) approaches a reducible representation and so \( L \to 1 \), \( \varphi_1(1) \to 0 \). As \( y \to (y)^- \), we have \( x(y) \to 4 \), \( M \to -1 \), \( L = -M^{-\frac{4n}{M\gamma-M\delta}} \to -1 \) and hence \( \theta_1(y) \to \pi \), \( \varphi_1(y) \to -(2n-1)\pi \). This implies that the image of \( g_1(y) := -\frac{\varphi_1(y)}{\theta_1(y)} \) on the interval \((2, y^*)\) contains the interval \((0, 2n-1)\).

The rest of the proof of Proposition 3.1 for \( C(2m+1, 2n) \) is similar to that for \( C(2m, -2n) \).

Lastly, we consider the case \( K = C(2m+1, -2n) \) and \( n \geq 2 \). Consider the continuous real function

\[
x : [2, \infty) \to \left( 4\cos^2\left( \frac{(2n-1)\pi}{4n+2} \right), \infty \right)
\]
in Lemma 2.4. Since \( x(2) < 4 \cos^2 \left( \frac{(2n-3)\pi}{4n+2} \right) \) and \( \lim_{y \to \infty} x(y) = \infty \), there exists \( y^* > 2 \) such that \( x(y^*) = 4 \) and \( 4 \cos^2 \left( \frac{(2n-1)\pi}{4n+2} \right) < x(y) < 4 \) for all \( y \in (2, y^*) \).

For each \( y \in [2, y^*) \) we let \( \theta(y) = \arccos(\sqrt{x(y)}/2) \). Then \( \theta(2) > \frac{(2n-3)\pi}{4n+2} \), and for \( y \in [2, y^*) \) we have \( 0 < \theta(y) < \frac{(2n-1)\pi}{4n+2} \) and \( x(y) = 4 \cos^2 \theta(y) \). Since \( R_K(x(y), y) = 0 \) there exists a non-abelian representation \( \rho : \pi_1(X) \to \text{SL}_2(\mathbb{C}) \) such that

\[
\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2 - y & M^{-1} \end{bmatrix},
\]

where \( M = e^{i\varphi(y)} \). Moreover, the image of the canonical longitude \( \lambda \) corresponding to the meridian \( \mu = a \) has the form \( \rho(\lambda) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix} \), where

\[
L = -M^{4n}M^{-\gamma} - M\delta \quad \frac{M\gamma}{M\gamma - M^{-1}\delta},
\]

and \( \gamma = S_n(y), \delta = S_{n-1}(y) \). Note that \( \gamma > \delta > 0, \) since \( y > 2 \).

As above, we write \( L = e^{i\varphi(y)} \) where

\[
\varphi(y) = -(2n-2)\pi + 4n\theta(y) + \arccos \left[ \frac{\left( 2\gamma\delta - (\gamma^2 + \delta^2) \cos 2\theta(y) \right)}{|e^{i\theta(y)}\gamma - e^{-i\theta(y)}\delta|^2} \right].
\]

Since \( \frac{(2n-3)\pi}{4n+2} < \theta(2) < \frac{(2n-1)\pi}{4n+2} \) we have \(-2\pi + \frac{4\pi}{2n+1} < \varphi(2) < 2\pi - \frac{(2n-1)\pi}{2n+1} \).

As \( y \to 2^+ \), \( \rho \) approaches a reducible representation and so \( L \to 1 \), \( \varphi(y) \to \varphi(2) = k2\pi \) for some integer \( k \). Since \( -2\pi + \frac{4\pi}{2n+1} < \varphi(2) < 2\pi - \frac{(2n-1)\pi}{2n+1} \), we must have \( \varphi(2) = 0 \).

As \( y \to (y^*)^- \), we have \( x(y) \to 4, M \to 1 \), \( L = -M^{4n}M^{-\gamma} - M\delta \to -1 \) and hence \( \theta(y) \to 0^+ \), \( \varphi(y) \to (2l-1)\pi \) for some integer \( l \). Since

\[
(2l-1)\pi = \lim_{y \to (y^*)^-} -(2n-2)\pi + 4n\theta(y) + \arccos \left[ \frac{\left( 2\gamma\delta - (\gamma^2 + \delta^2) \cos 2\theta(y) \right)}{|e^{i\theta(y)}\gamma - e^{-i\theta(y)}\delta|^2} \right]
\]

we have \(-(2n-2)\pi \leq (2l-1)\pi \leq -(2n-3)\pi \). This implies that \( 2l-1 = -(2n-3) \) and \( \varphi(y) \to -(2n-3)\pi \) as \( y \to (y^*)^- \). Hence the image of \( h(y) := -\frac{\varphi(y)}{\theta(y)} \) on the interval \((2, y^*)\) contains the interval \((0, \infty)\).

Similarly, with \( \theta_1(y) = \pi - \theta(y) \) we have \( x(y) = 4 \cos^2(\theta_1(y)) \) and hence for each \( y \in [2, y^*) \) there exists a non-abelian representation \( \rho_1 : \pi_1(X) \to \text{SL}_2(\mathbb{C}) \) such that

\[
\rho_1(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho_1(b) = \begin{bmatrix} M & 0 \\ 2 - y & M^{-1} \end{bmatrix},
\]

where \( M = e^{i\varphi_1(y)} \). Moreover, the image of the canonical longitude \( \lambda \) corresponding to the meridian \( \mu = a \) has the form \( \rho_1(\lambda) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix} \), where \( L = e^{i\varphi_1(y)} \) and

\[
\varphi_1(y) = (2n-2)\pi - 4n\pi + 4n\theta_1(y)
\]
\[ - \arccos \left[ \frac{(2 \gamma \delta - (\gamma^2 + \delta^2) \cos 2 \theta_1(y))}{|e^{i \theta_1(y)} \gamma - e^{-i \theta_1(y)} \delta|^2} \right] = (2n - 2) \pi - 4n \theta(y) \]
\[ - \arccos \left[ \frac{(2 \gamma \delta - (\gamma^2 + \delta^2) \cos 2 \theta_1(y))}{|e^{i \theta_1(y)} \gamma - e^{-i \theta_1(y)} \delta|^2} \right]. \]

Since \( \frac{2(2n-3)\pi}{4n+2} < \theta(2) < \frac{2(n-1)\pi}{4n+2} \), we have \(-2\pi + \frac{2(n-1)\pi}{2n+1} < \varphi_1(2) < 2\pi - \frac{4\pi}{2n+1} \).

As \( y \to 2^+ \), \( \rho_1 \) approaches a reducible representation and so \( L \to 1 \). \( \varphi_1(y) \to \varphi_1(2) = 0 \).

As \( y \to (y^*)^{-} \), we have \( x(y) \to 4 \), \( M \to -1 \), \( L = -M^{4n} M^{-\gamma_2} \rightarrow -1 \) and hence \( \theta_1(y) \to \pi \), \( \varphi_1(y) \to (2n - 3)\pi \). This implies that the image of \( h_1(y) := -\frac{\varphi_1(y)}{\varphi_1(y)} \) on the interval \((2, y^*)\) contains the interval \((-2n - 3, 0)\).

The rest of the proof of Proposition 3.1 for \( C(2m + 1, -2n) \) is similar to that for \( C(2m, -2n) \). \( \square \)

We now finish the proof of Theorem 1. Suppose \( r \) is a rational number such that \( r \in \text{LO}_K \). If \( r \neq 0 \), by Proposition 3.1, there exists a representation \( \rho : \pi_1(X_r) \to \text{SL}_2(\mathbb{R}) \) such that \( \rho|_{\pi_1(\partial X)} \) is an elliptic representation. This representation lifts to a representation \( \hat{\rho} : \pi_1(X_r) \to \text{SL}_2(\mathbb{R}) \), where \( \text{SL}_2(\mathbb{R}) \) is the universal covering group of \( \text{SL}_2(\mathbb{R}) \). See e.g. [CD, Sec. 3.5] and [Va, Sec. 2.2]. Note that \( X_r \) is an irreducible 3-manifold (by [HTh]) and \( \text{SL}_2(\mathbb{R}) \) is a left orderable group (by [Be]). Hence, by [BRW], \( \pi_1(X_r) \) is a left orderable group. Finally, 0-surgery along a knot always produces a prime manifold whose first Betti number is 1, and by [BRW] such manifold has left orderable fundamental group.

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References


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