On a density property of the residual order of $a \pmod{pq}$

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Abstract

We consider a distribution property of the residual order (the multiplicative order) of the residue class $a \pmod{pq}$. It is known that the residual order fluctuates irregularly and increases quite rapidly. We are interested in how the residual orders $a \pmod{pq}$ distribute modulo 4 when we fix $a$ and let $p$ and $q$ vary. In this paper we consider the set $S(x) = \{(p, q); p, q$ are distinct primes, $pq \leq x\}$, and calculate the natural density of the set $\{(p, q) \in S(x); \text{ the residual order of } a \pmod{pq} \equiv l \pmod{4}\}$. We show that, under a simple assumption on $a$, these densities are $\left\{\frac{5}{11}, \frac{1}{11}, \frac{4}{11}\right\}$ for $l = \{0, 1, 2, 3\}$, respectively. For $l = 1, 3$ we need Generalized Riemann Hypothesis.

1 Introduction

Let $a$ be a natural number, $p, q$ be two distinct odd primes, $(\mathbb{Z}/p\mathbb{Z})^*$ and $(\mathbb{Z}/pq\mathbb{Z})^*$ be the set of all invertible residue classes modulo $p$ and modulo $pq$, respectively. $(\mathbb{Z}/p\mathbb{Z})^*$ is a cyclic group of order $p - 1$, but $(\mathbb{Z}/pq\mathbb{Z})^*$ is not a cyclic group.

We define

$$D_a(p) = \text{the order of the residue class } a \pmod{p} \text{ in } (\mathbb{Z}/p\mathbb{Z})^*$$

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$D_a(pq)$ = the order of the residue class $a \pmod{pq}$ in $(\mathbb{Z}/pq\mathbb{Z})^\ast$.

When $D_a(p) = p - 1$, $a \pmod{p}$ is a generator of $(\mathbb{Z}/p\mathbb{Z})^\ast$, and it is conjectured that, for a fixed $a$ which is not a perfect square, there are infinitely many primes $p$ which satisfy $D_a(p) = p - 1$ (Artin’s conjecture for primitive root). This conjecture is not yet solved, but if we assume Generalized Riemann Hypothesis, we have an affirmative answer ([3]). As for the function $D_a(p)$ with an arbitrarily fixed $a$, we can prove that, for any natural number $n$ but only finite number of $n$’s, there exists an odd prime $p$ which satisfies $D_a(p) = n$. So, if we denote by $\mathbb{N}$ and $\mathbb{P}$ the set of all natural numbers and the set of all odd prime numbers respectively, $D_a(p)$ is essentially a surjective function from $\mathbb{P}$ to $\mathbb{N}$. Moreover, since $D_a(pq)$ is the least common multiple of $D_a(p)$ and $D_a(q)$, $D_a(pq)$ is also essentially a surjective function from $\mathbb{P} \times \mathbb{P}$ to $\mathbb{N}$. Since both functions fluctuate quite irregularly and increase very rapidly along with $p$ and $(p, q)$, it is hard to study their properties.

In this paper we consider the condition, for $l = 0, 1, 2, 3$,

$$D_a(pq) \equiv l \pmod{4},$$

and we introduce the set

$$T_a(x; 4, l) = \{(p, q); \ pq \leq x, D_a(pq) \equiv l \pmod{4}\}.$$

We are interested in the natural density of $T_a(x; 4, l)$ with respect to $S(x) = \{(p, q); \ pq \leq x\}$, that is, the limiting value

$$\lim_{x \to \infty} \frac{\#T_a(x; 4, l)}{\#S(x)} = \Theta_a(4, l), \text{ if it exists.} \quad (1)$$

Our result is

**Theorem 1**

Let $a$ be a natural number and we decompose $a$ into $a = a_0^2 \cdot a_1$ with $a_1$ square free. If $a_1$ is odd, then,

(I) for $l = 0, 2$, $T_a(x; 4, 0)$ and $T_a(x; 4, 2)$ have the densities

$$\Theta_a(4, 0) = \frac{5}{9}, \ \Theta_a(4, 2) = \frac{1}{3}.$$

(II) for $l = 1, 3$, if we assume the Generalized Riemann Hypothesis (GRH), then $T_a(x; 4, 1)$ and $T_a(x; 4, 3)$ have the densities

$$\Theta_a(4, 1) = \frac{1}{18}, \ \Theta_a(4, 3) = \frac{1}{18}.$$
More precisely, the GRH we need is the Riemann Hypothesis for the Dedekind zeta function \( \zeta_K(s) \) for the algebraic number field \( K = \mathbb{Q}(a^{1/k}, \exp(2\pi i/m)) \), where \( k, m \in \mathbb{N} \).

In the previous paper [5], we studied a similar problem, as follows. For \( l = 0, 1, 2, 3 \), we define

\[ R_a(x; 4, l) = \{(p, q); \ p \leq x, q \leq x, D_a(pq) \equiv l \pmod{4}\}, \]

and, \( \pi(x) \) being the number of primes \( \leq x \), we considered the limiting value

\[ \lim_{x \to \infty} \frac{\# R_a(x; 4, l)}{\pi(x)^2} = \Delta_a(4, l), \]

if it exists,

and we got the result:

**Theorem A ([5])**

Under the same assumption on \( a_1 \) as Theorem 1,

(I) for \( l = 0, 2 \), \( R_a(x; 4, 0) \) and \( R_a(x; 4, 2) \) have the densities

\[ \Delta_a(4, 0) = \frac{5}{9}, \ \Delta_a(4, 2) = \frac{1}{3}. \]

(II) for \( l = 1, 3 \), if we assume the GRH, then \( R_a(x; 4, 1) \) and \( R_a(x; 4, 3) \) have the densities

\[ \Delta_a(4, 1) = \frac{1}{18}, \ \Delta_a(4, 3) = \frac{1}{18}. \]

The set \( \{(p, q); \ p \leq x, q \leq x\} \) is a square-shaped rather big set which contains approximately \( x^2 \log x \) points, whereas the set \( S(x) = \{(p, q); \ pq \leq x\} \) is a hyperbolic-shaped thin set which contains approximately \( 2x \log x \) points (cf. Proposition 1). The above two theorems show that the set of the points \( (p, q) \) with \( D_a(pq) \equiv l \pmod{4} \) distributes rather uniformly.

In both results, while our statements are unconditional for \( l = 0, 2 \), we need the GRH for \( l = 1, 3 \). This comes from the structure of the additive group \( \mathbb{Z}/4\mathbb{Z} \).

Here we introduce the set of primes which played an important role in previous papers [1] [4]:

\[ Q_a(x; k, l) = \{p \leq x; p \text{ is prime, } D_a(p) \equiv l \pmod{k}\}. \]
The set \( Q_a(x; k, l) \) plays an important role again in the calculation of \( T_a(x; k, l) \).

The group \( \mathbb{Z}/4\mathbb{Z} \) has three subgroups:

\[
\mathbb{Z}/4\mathbb{Z} = \langle 0, 1, 2, 3 \rangle \quad \longleftrightarrow \quad \pi(x)
\]

\[
\quad \downarrow
\]

\[
\langle 0, 2 \rangle \quad \longleftrightarrow \quad \sharp Q_a(x; 2, 0)
\]

\[
\quad \downarrow
\]

\[
\langle 0 \rangle \quad \longleftrightarrow \quad \sharp Q_a(x; 4, 0).
\]

Corresponding to the subgroup \( \langle 0 \rangle \), we have the unconditional asymptotic formula for \( \sharp Q_a(x; 4, 0) \), corresponding to the subgroup \( \langle 0, 2 \rangle \), we have the unconditional asymptotic formula for \( \sharp Q_a(x; 2, 0) \), and the difference of these two gives the unconditional asymptotic formula for \( \sharp Q_a(x; 4, 2) \). In order to separate \( Q_a(x; 4, 1) \) and \( Q_a(x; 4, 3) \), we need the GRH so far. For any \( k \), the additive group \( \mathbb{Z}/k\mathbb{Z} \) always has the trivial subgroup \( \langle 0 \rangle \), and the studies on the set \( Q_a(x; k, 0) \) started earlier (cf. for example [2]).

In this paper \( \varphi(n) \) means Euler’s totient function, \( \mu(n) \) means the Möbius function, \( li(x) = \int_2^x 1/\log t \, dt \), and \((m, n)\) means the G.C.D. of \( m \) and \( n \).

## 2 Preliminaries

We start with the calculation of the denominator of (1).

**Proposition 1**

\[
\sharp S(x) = 2 \frac{x}{\log x} - \log \log x + O\left( \frac{x}{\log x} \right)
\]

**proof**

We have

\[
\sharp S(x) = 2 \left( \sum_{p \leq \sqrt{x}} \sum_{q \leq \frac{x}{p}} 1 \right) - \sum_{p \leq \sqrt{x}} \sum_{q \leq \sqrt{x}} 1.
\]
The second term of the right-hand side is $O(x/\log^2 x)$. The first term is $2 \sum_{p \leq \sqrt{x}} \pi(x/p)$ and the prime number theorem,

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

gives our statement.

We have already in [5]

**Theorem B** ([5])

Let $a$ be a natural number and we decompose $a$ into $a = a_1^2 \cdot a_2$ with $a_1$ square free. We assume the GRH. Then

(I) for any odd natural number $D$ and $l = 1, 3$, we have

$$\sharp Q_a(x; 4D, lD) = \Gamma_a(4D, lD) \log(x) + O\left(\frac{x}{\log x} \log \log x \log^2 D \right),$$

where $\Gamma_a(4D, lD)$ is a computable constant and the constant implied by the $O$-symbol is absolute.

(II) if $a_1$ is odd, then we have

$$\Gamma_a(4, 1) = \Gamma_a(4, 3) = \frac{1}{6}, \quad \Gamma_a(4, 0) = \Gamma_a(4, 2) = \frac{1}{3}.$$  

Here we remark that $\Gamma_a(4D, lD)$ is the natural density of the set $Q_a(x; 4D, lD)$.

### 3 Proof of Theorem 1

First we prove Theorem 1-(II). Similarly to the proof of Proposition 1, we have

$$\sharp T_a(x; 4, 1) = 2 \left( \sum_{p \leq \sqrt{x}} \sum_{q \leq x/p} 1 \right) - \sum_{p \leq \sqrt{x}} \sum_{q \leq \sqrt{x}} 1,$$

(3)

and again the second term of the right-hand side is $O\left(\frac{x}{\log^2 x}\right)$. Now, since $D_a(pq)$ is the L.C.M. of $D_a(p)$ and $D_a(q)$, the condition $D_a(pq) \equiv 1 \pmod{4}$ is divided into
(Case 1)

\[ D_a(p) \equiv 1 \pmod{4} \quad \text{and} \quad \frac{D_a(q)}{(D_a(p), D_a(q))} \equiv 1 \pmod{4} \quad (4) \]

and

(Case 2)

\[ D_a(p) \equiv 3 \pmod{4} \quad \text{and} \quad \frac{D_a(q)}{(D_a(p), D_a(q))} \equiv 3 \pmod{4}. \]

We consider only (Case 1), since (Case 2) proceeds in the same manner.

We put \( Q = (D_a(p), D_a(q)) \), and in order to single out such \( q \)'s satisfying (4), we use the inclusion-exclusion principle:

\[
\sum_{p \leq \sqrt{x}} {\sum_{q \leq x/p \atop D_a(p) \equiv 1 \pmod{4}} \mu(Q')} \sum_{Q < \sqrt{x} \atop D_a(p) \equiv 1 \pmod{4}} \sum_{Q' \leq \sqrt{x/p}} \frac{\mu(Q')}{(D_a(p), D_a(q))} \equiv 1 \pmod{4}
\]

\[
= \sum_{p \leq \sqrt{x} \atop D_a(p) \equiv 1 \pmod{4}} \sum_{Q' | D} \sum_{Q \leq \sqrt{x} \atop D_a(p) \equiv 1 \pmod{4}} \mu(Q') \sum_{q \leq x/p \atop D_a(q) \equiv Q' \pmod{4}} 1
\]

\[
+ \sum_{p \leq \sqrt{x} \atop D_a(p) \equiv 3 \pmod{4}} \sum_{Q' | D} \sum_{Q \leq \sqrt{x} \atop D_a(p) \equiv 3 \pmod{4}} \mu(Q') \sum_{q \leq x/p \atop D_a(q) \equiv Q' \pmod{4}} 1,
\]

where

\[ \bar{Q}' = \begin{cases} 
1 & \text{if } Q' \equiv 1 \pmod{4}, \\
3 & \text{if } Q' \equiv 3 \pmod{4}. 
\end{cases} \quad (5) \]

Thus we have
\[ \sum_{p \leq \sqrt{x}} \sum_{q \leq x/p} 1 = \sum_{D < \sqrt{x}} \sum_{Q' | D} \mu(Q') \sum_{p \leq \sqrt{x}} \frac{\sharp Q_a \left( \frac{x}{p}; 4D, \tilde{Q}'D \right)}{p \equiv D \mod 4D} \]

\[ + \sum_{D \equiv 3 \mod 4} \sum_{Q' | D} \mu(Q') \sum_{p \leq \sqrt{x}} \frac{\sharp Q_a \left( \frac{x}{p}; 4D, \tilde{Q}'D \right)}{p \equiv 3D \mod 4D} \]

+ two similar terms coming from (Case 2). \hfill (6)

We put \( X = (\log \log x)^{1/3}, Y = (\log x)^4 \) and we divide the first term of the right-hand side of (6) into three parts:

\[ \sum_{D < \sqrt{x}} \sum_{Q' | D} \mu(Q') \sum_{p \leq \sqrt{x}} \frac{\sharp Q_a \left( \frac{x}{p}; 4D, \tilde{Q}'D \right)}{p \equiv D \mod 4D} \]

\[ = \left( \sum_{D \leq X \mod 4} + \sum_{X < D \leq Y \mod 4} + \sum_{D \equiv 1 \mod 4} \right) \sum_{Q' | D} \mu(Q') \sum_{p < \sqrt{x}} \frac{\sharp Q_a \left( \frac{x}{p}; 4D, \tilde{Q}'D \right)}{p \equiv D \mod 4D} \]

\[ = \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say.} \quad (7) \]

i) Estimate of \( \Sigma_3 \).

For any \( q \in Q_a \left( \frac{x}{p}; 4D, \tilde{Q}'D \right), D_a(q) \equiv \tilde{Q}'D \mod 4D \) and \( D_a(q) | q - 1 \). Thus \( q \equiv 1 \mod D \) and we have an estimate

\[ \sharp Q_a \left( \frac{x}{p}; 4D, \tilde{Q}'D \right) \leq \pi \left( \frac{x}{p}; D, 1 \right) \ll \frac{x}{pD}, \]

where \( \pi(y; k, l) \) is the number of primes \( \leq y \) which are congruent to \( l \) modulo \( k \).

Then we have

\[ \Sigma_3 \ll \sum_{Y \leq D < \sqrt{x}} \sum_{Q' | D \mod D \text{ odd}} |\mu(Q')| \sum_{p \leq \sqrt{x}} \frac{x}{pD} \]

\[ + \sum_{D \equiv 3 \mod 4} \sum_{Q' | D} \mu(Q') \sum_{p \leq \sqrt{x}} \frac{\sharp Q_a \left( \frac{x}{p}; 4D, \tilde{Q}'D \right)}{p \equiv 3D \mod 4D} \]

\[ + \text{ two similar terms coming from (Case 2).} \]
where \( \omega(D) \) is the number of distinct prime factors of \( D \). Since

\[
\sum_{Y < D < \sqrt{x} \atop D \text{ odd}} \frac{2^{\omega(D)}}{D \cdot \varphi(D)} \ll \sum_{Y < D < \sqrt{x} \atop D \text{ odd}} \frac{D^{\log 2}}{D^2} \ll Y^{\log 2 - 1},
\]

we get an estimate

\[
\Sigma_3 \ll x \ (\log \log x)(\log x)^{4(\log 2 - 1)}. \tag{8}
\]

ii) Estimate of \( \Sigma_2 \).

We use the estimate

\[
\pi\left(\frac{x}{p}; 4D, \tilde{Q}'D\right) \ll \pi\left(\frac{x}{p}; D, 1\right).
\]

When \( D \) is in the range \( X < D \leq Y \), we can apply Siegel-Walfisz Theorem:

\[
\pi(x; k, l) = \frac{1}{\varphi(k)} \left( \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{x}{\log^3 x} \right) + O\left( \frac{x}{\log^4 x} \right). \tag{9}
\]

The part of \( \Sigma_2 \) coming from the first term of the right-hand side of (9) satisfies

\[
\ll \sum_{X < D \leq Y \atop D \text{ odd}} \sum_{Q' \mid D} |\mu(Q')| \sum_{p \leq \sqrt{x} \atop p \equiv 1 \mod D} \frac{1}{\varphi(D) \log \left( \frac{x}{p} \right)}
\]

\[
\ll \sum_{X < D \leq Y} \frac{\sum_{Q' \mid D} |\mu(Q')|}{\varphi(D)} \frac{x}{\log x} \sum_{p \leq \sqrt{x} \atop D(p) \equiv D \mod 4D} \frac{1}{p}
\]

\[
\ll \frac{x \log \log x}{\log x} \sum_{X < D \leq Y} \frac{D^{\log 2} (\log \log D)^2}{D^2}
\]

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\[ \ll \frac{x}{\log x} (\log \log x)^{2+\log \frac{2}{3}} (\log \log \log x)^2. \]

We can estimate similarly the parts of \( \Sigma_2 \) coming from the second and the third term of the right-hand side of (9). The fourth term of the right-hand side of (9) does not have the factor \( \frac{1}{\varphi(D)} \). The part of \( \Sigma_2 \) coming from this term satisfies

\[ \ll \sum_{X < D \leq Y \atop D \text{ odd}} \left| \mu(Q') \right| \sum_{p \leq \sqrt{x}} \frac{x/p}{\log^4 x}, \]

\[ \ll \frac{x}{\log^4 x} (\log \log x) \sum_{X < D \leq Y \atop D \text{ odd}} \frac{1}{\varphi(D)} \sum_{Q'D=D|D} |\mu(Q')|, \]

\[ \ll \frac{x}{\log x} \log \log x \left( (\log x)^{\log 2 - 3} (\log \log \log x) \right), \]

and \( 4 \log 2 - 3 < 0. \)

In total we have

\[ \Sigma_2 \ll \frac{x}{\log x} (\log \log x)^{2+\log \frac{2}{3}} (\log \log \log x)). \tag{10} \]

### iii) Calculation of \( \Sigma_1 \).

We substitute the right-hand side of the formula in Theorem B-(I) for \( \#Q_a(x, p; 4D, \bar{Q}'D) \) and first we calculate the sum of the main term:

\[ \sum_{D \leq X \atop D \equiv 1 \mod 4} \sum_{Q'|D} \mu(Q') \sum_{p \leq \sqrt{x}} \gamma_a(4D, \bar{Q}'D) \operatorname{li} \left( \frac{x}{p} \right). \]

We use an estimate

\[ \operatorname{li} \left( \frac{x}{p} \right) = \frac{x/p}{\log \left( \frac{x}{p} \right)} + O \left( \frac{x/p}{\log^2 \left( \frac{x}{p} \right)} \right) \]

\[ = \frac{1}{p \log x} + \frac{1}{p \log x} \sum_{k=1}^{\infty} \frac{\log p}{\log x} k + O \left( \frac{x/p}{\log^2 \left( \frac{x}{p} \right)} \right) \]

\[ = \frac{1}{p \log x} + O \left( \frac{x}{\log^2 x} \frac{\log p}{p} \right). \tag{11} \]
The part of \( \Sigma_1 \) coming from the first term of (11) is equal to

\[
\frac{x}{\log x} \sum_{D \leq X \atop D \equiv 1 \mod 4} \sum_{Q' | D} \mu(Q') \Gamma_a(4D, \tilde{Q}'D) \sum_{p \leq \sqrt{x} \atop D_a(p) \equiv D \mod 4D} 1 \cdot \frac{1}{p}.
\]

By partial summation, from Theorem B-(I) we can derive

\[
\sum_{p \leq \sqrt{x} \atop D_a(p) \equiv D \mod 4D} \frac{1}{p} = \Gamma_a(4D, D) \log \log x + O(D^2 \log D \log \log \log x).
\]

With the estimate

\[
\sum_{p \leq \sqrt{x} \atop D_a(p) \equiv D \mod 4D} \log p \frac{1}{p} \ll \log x,
\]

we get the formula

\[
\Sigma_1 = \frac{x}{\log x} \log \log x \sum_{D \leq X \atop D \equiv 1 \mod 4} \sum_{Q' | D} \mu(Q') \Gamma_a(4D, \tilde{Q}'D) \Gamma_a(4D, D)
\]

\[
+ \frac{x}{\log x} \log \log \log x \sum_{D \leq X \atop D \equiv 1 \mod 4} \sum_{Q' | D} |\mu(Q')| O(D^2 \log D)
\]

\[
+ \sum_{D \leq X \atop D \equiv 1 \mod 4} \sum_{Q' | D} |\mu(Q')| \sum_{p \leq \sqrt{x} \atop D_a(p) \equiv D \mod 4D} O\left(\frac{x}{p \log \frac{x}{p} \log \log \frac{x}{p}} D^2 \log D\right)
\]

\[
= \Sigma_1^{(1)} + \Sigma_1^{(2)} + \Sigma_1^{(3)}, \text{ say.}
\]

We can estimate the last error term as follows:

\[
\Sigma_1^{(3)} \ll \frac{x}{\log x} \log x \log \log x \sum_{D \leq X \atop D \equiv 1 \mod 4} \sum_{Q' | D} |\mu(Q')| \sum_{p \leq \sqrt{x} \atop D_a(p) \equiv D \mod 4D} \frac{1}{p} D^2 \log D
\]

\[
\ll \frac{x}{\log x} X^{2+\log^2 \log \log x}
\]

\[
\ll \frac{x}{\log x} \left( \log \log x \right)^{\frac{2+\log^2 \log \log x}{4} + \varepsilon}, \text{ for any } \varepsilon > 0.
\]
We can estimate $\Sigma_1^{(2)}$ in a same manner, and consequently, we complete the calculation of the first term of the right-hand side of (6) as follows:

$$
\sum_{D<\sqrt{X}} \sum_{Q'|D} \mu(Q') \sum_{p \leq \sqrt{X}} \frac{\#Q_a(x/p; 4D, Q'D)}{D \equiv 1 \mod 4} \sum_{D \equiv 0 \mod 4D} P_a \left( \frac{x}{p} ; 4D, Q'D \right)
$$

$$
= \frac{x}{\log x} \log \log x \sum_{D \leq X} \sum_{Q'|D} \mu(Q') \frac{\Gamma_a(4D, Q'D) \Gamma_a(4D, D)}{D \equiv 1 \mod 4}
$$

$$
\left( \sum_{D \leq X} \sum_{Q'|D} \mu(Q') \frac{\Gamma_a(4D, Q'D) \Gamma_a(4D, 3D)}{D \equiv 3 \mod 4} \right)
$$

$$
+ O\left( \frac{x}{\log x} \log \log x \left( \log \log x \right)^{\frac{2+\log 2}{3}+\varepsilon} \right).
$$

Similarly we have

$$
\sum_{D<\sqrt{X}} \sum_{Q'|D} \mu(Q') \sum_{p \leq \sqrt{X}} \frac{\#Q_a(x/p; 4D, Q'D)}{D \equiv 3 \mod 4} \sum_{D \equiv 3 \mod 4D} P_a \left( \frac{x}{p} ; 4D, Q'D \right)
$$

$$
= \frac{x}{\log x} \log \log x \sum_{D \leq X} \sum_{Q'|D} \mu(Q') \frac{\Gamma_a(4D, Q'D) \Gamma_a(4D, 3D)}{D \equiv 3 \mod 4}
$$

$$
\left( \sum_{D \leq X} \sum_{Q'|D} \mu(Q') \frac{\Gamma_a(4D, Q'D) \Gamma_a(4D, 3D)}{D \equiv 3 \mod 4} \right)
$$

$$
+ O\left( \frac{x}{\log x} \log \log x \left( \log \log x \right)^{\frac{2+\log 2}{3}+\varepsilon} \right).
$$

When $a_1$ is odd, it is proved in Theorem 6.1 of [5] that $\Delta_a(4D, D) = \Delta_a(4D, 3D)$, so by the definition (5),

$$
\Gamma_a(4D, Q'D) = \Gamma_a(4D, D) = \Gamma_a(4D, 3D).
$$

Therefore, the leading coefficient of (6) is

$$
2 \left( \sum_{D \leq X} \sum_{Q'|D} \mu(Q') \Gamma_a(4D, D)^2 \right) + 2 \left( \sum_{D \leq X} \sum_{Q'|D} \mu(Q') \Gamma_a(4D, D)^2 \right).
$$

Taking into account the relations

$$
\sum_{Q'|D} \mu(Q') = \begin{cases} 
1 & \text{if } D = 1 \\
0 & \text{if } D > 1
\end{cases}
$$

and $\Gamma_a(4, 1) = \frac{1}{6}$,

the leading coefficient of (6) is $2 \Gamma_a(4, 1)^2 = \frac{1}{18}$. 

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We finally proved
\[ \sharp T_a(x; 4, 1) = \frac{1}{18} x \frac{\log \log x}{\log x} + o\left(\frac{x}{\log x \log \log x}\right), \]
\[ \sharp T_a(x; 4, 3) = \frac{1}{18} x \frac{\log \log x}{\log x} + o\left(\frac{x}{\log x \log \log x}\right), \]
and combining with Proposition 1, we obtain Theorem 1-(II).

The proof of Theorem 1-(I) is much easier. We note that
\[ D_a(pq) = \text{the L.C.M. of } D_a(p) \text{ and } D_a(q). \]

So
\[ D_a(pq) \equiv 0 \pmod{4} \iff D_a(p) \equiv 0 \pmod{4} \text{ or } D_a(q) \equiv 0 \pmod{4}, \]
and
\[ \sharp T_a(x; 4, 0) = 2 \left( \sum_{\substack{p \leq \sqrt{x} \atop D_a(p) \equiv 0 \pmod{4}}} \sum_{q \leq x/p} 1 \right) - \left( \sum_{\substack{p \leq \sqrt{x} \atop D_a(q) \equiv 0 \pmod{4}}} \sum_{q \leq x/p} 1 \right) + O\left(\frac{x}{\log^2 x}\right). \]

Similar calculations give
\[ \sharp T_a(x; 4, 0) = \left(2 \Gamma_a(4, 0) - \Gamma_a(4, 0)^2\right) x \frac{\log \log x}{\log x} + o\left(\frac{x}{\log x \log \log x}\right) \]
\[ = \frac{5}{9} x \frac{\log \log x}{\log x} + o\left(\frac{x}{\log x \log \log x}\right), \]
and similarly
\[ \sharp T_a(x; 4, 2) = \frac{1}{3} x \frac{\log \log x}{\log x} + o\left(\frac{x}{\log x \log \log x}\right), \]
which prove Theorem 1-(I).

In this paper we cannot obtain the density \( \Theta_a(4, 1) \) for \( a_1 \equiv 2 \pmod{4} \).
When \( a_1 \) is odd, we know \( \Theta_a(4, 1) + \Theta_a(4, 3) = \frac{1}{9} \) unconditionally, and our Theorem 1-(II) shows that, under GRH, this \( \frac{1}{9} \) is divided equally:
\[ \Theta_a(4, 1) = \Theta_a(4, 3) = \frac{1}{18}. \]
When $a_1 \equiv 2 \pmod{4}$, we know that $\Theta_2(4,0) = \frac{95}{144}$, $\Theta_2(4,2) = \frac{49}{192}$ and $\Theta_2(4,1) + \Theta_2(4,3) = \frac{49}{576}$.

But $\Delta_a(4D,D) \neq \Delta_a(4D,3D)$ happens rather frequently, and most probably $\Theta_a(4,1) \neq \Theta_a(4,3)$.

References


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