ILL-POSEDNESS FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS UNDER BAROTROPIC CONDITION IN LIMITING BESOV SPACES

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Abstract. We consider the compressible Navier-Stokes system in the critical Besov spaces. It is known that the system is (semi-)well-posed in the scaling semi-invariant spaces of the homogeneous Besov spaces \( B_{p,1}^{n} \times B_{p,1}^{n} \) for all \( 1 < p < 2n \). However, if the data is in a larger scaling invariant class such as \( p > 2n \), then the system is not well-posed. In this paper, we demonstrate that for the critical case \( p = 2n \) the system is ill-posed by showing that a sequence of initial data is constructed to show discontinuity of the solution map in the critical space. Our result indicates that the well-posedness results due to Danchin [10] and Haspot [18] are indeed sharp in the framework of the homogeneous Besov spaces.

1. Introduction

We consider the ill-posedness issue of the Cauchy problem for compressible Navier-Stokes equations under the barotropic condition:

\[
\begin{cases}
\partial_t \tilde{\rho} + \text{div} (\tilde{\rho} u) = 0, & t > 0, x \in \mathbb{R}^n, \\
\tilde{\rho} \partial_t u + \tilde{\rho} (u \cdot \nabla) u + \nabla P(\tilde{\rho}) = L u, & t > 0, x \in \mathbb{R}^n, \\
\tilde{\rho}(0, x) = \tilde{\rho}_0(x), u(0, x) = u_0(x), & x \in \mathbb{R}^n,
\end{cases}
\]

where \( L \) is given by \( L \equiv \mu \Delta + (\mu + \lambda) \nabla \text{div} \) with the Lamé constants \( \lambda, \mu > 0, 2\mu + \lambda > 0 \). The unknown functions are \( \tilde{\rho} = \tilde{\rho}(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R} \) and \( u = u(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \) that denote the density and velocity for the fluid, respectively, and \( P \) is the pressure given by a function of \( \tilde{\rho} \) as \( P = P(\tilde{\rho}) = \tilde{\rho}^\gamma \) with \( \gamma > 1 \). We assume that the density of the fluid \( \tilde{\rho} = \tilde{\rho}(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R} \) satisfies the non-vacuum condition, namely \( \tilde{\rho}(t, x) = 1 + \rho(t, x) \) and \( \rho \) belongs to the critical Lebesgue or Besov spaces and we assume that \( \rho(t, x) \geq \nu > -1 \). Under such setting the barotropist model (1.1) can be reduced to the following Cauchy problem of the compressible Navier-Stokes equations:

\[
\begin{cases}
\partial_t \rho + \text{div} ((1 + \rho) u) = 0, & t > 0, x \in \mathbb{R}^n, \\
\partial_t u - L u + (u \cdot \nabla) u + \nabla P(1 + \rho) \over 1 + \rho = -\rho L u, & t > 0, x \in \mathbb{R}^n, \\
\rho(0, x) = \rho_0(x), u(0, x) = u_0(x), & x \in \mathbb{R}^n.
\end{cases}
\]

Then naturally one can consider the existence theory of the strong solution under the uniformly parabolic setting for the velocity equation.

The existence and the well-posedness for the compressible Navier-Stokes equation have been considered by several researchers (Serrin [28], Nash [26], Itaya [21], Matsumura-Nishida [24], Tani

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by constructing a sequence of initial data that are smooth and belong to the limiting class (1.2) if

\[
\text{strong solution of (1.4). To prove the ill-posedness of the problem (1.2), in particular the strong}
\]

by the elliptic operator

\[
\text{De}.
\]

1.1. convection and viscosity terms. It is noteworthy that we shall show that the norm in

\[
\text{the blow-up but clarifies the optimal regularity of the data between the stability and instability}
\]

the same class at a time near \( t_0 \).

\[
\text{For any} \quad (u, \sigma), \quad \text{we call} \quad (u, \sigma) \quad \text{a mild solution to the Cauchy problem of}
\]

\[
\text{In this paper, we consider all the critical setting} \quad p = 2n \quad \text{with} \quad n \geq 2 \quad \text{in both the density and velocity functions}
\]

\[
\text{spaces, and extract the least regular part from the nonlinear interactions. Indeed, the worst}
\]

\[
\text{term in our setting is essentially different from the known worst term, i.e., the convection term}
\]

\[
(\mathbf{u} \cdot \nabla) \mathbf{u}.
\]

Here the homogeneous Besov spaces are defined by

\[
\dot{B}^{s}_{p,q} := \{ f \in \mathcal{S}'(\mathbb{R}^n) / \mathcal{P}(\mathbb{R}^n) \mid \| f \|_{\dot{B}^{s}_{p,q}} < \infty \}, \quad \| f \|_{\dot{B}^{s}_{p,q}} \equiv \left( \sum_{j \in \mathbb{Z}} 2^{js} \| \phi_j * f \|_p^\sigma \right)^{1/\sigma} \quad (1.3)
\]

where \( 1 \leq p, \sigma \leq \infty, \mathcal{P}(\mathbb{R}^n) \) is the set of all polynomials and \( \{ \phi_j \}_{j \in \mathbb{Z}} \) denotes the Littlewood–Paley dyadic decomposition. It is well-known that if \( s < n/p \) or \( (s, q) = (n/p, 1) \), then the space is characterized subspaces of \( \mathcal{S}'(\mathbb{R}^n) \), namely,

\[
\dot{B}^{s}_{p,q} := \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \| f \|_{\dot{B}^{s}_{p,q}} < \infty, \quad f = \sum_{j \in \mathbb{Z}} \phi_j * f \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \}.
\]

The spaces above are considered as the scaling invariant classes under the following scaling:

For any \( \alpha > 0 \),

\[
\begin{align*}
\rho_\alpha(t, x) &= \rho(\alpha^2 t, \alpha x), \\
u_\alpha(t, x) &= \alpha u(\alpha^2 t, \alpha x).
\end{align*}
\]

We herein consider the strong discontinuity of a strong solution for the Cauchy problem (1.2) in the threshold spaces \( \dot{B}^{n}_{p,1}(\mathbb{R}^n) \times \dot{B}^{n-1}_{p,1}(\mathbb{R}^n) \) with \( p = 2n \) and demonstrate the ill-posedness by constructing a sequence of initial data that are smooth and belong to the limiting class \( \dot{B}^{n}_{p,1}(\mathbb{R}^n) \) with \( p = 2n \) such that their norms decrease while those of the solutions increase in the same class at a time near \( t = 0 \). The result does not answer the question pertaining to the blow-up but clarifies the optimal regularity of the data between the stability and instability of solutions. It is noteworthy that we shall show that the norm inflation stems from both the convection and viscosity terms.

1.1. Mild solution and main result. We introduce the mild solution to the problem (1.2).

**Definition.** Given initial data \( (\rho_0, u_0) \), we call \( (\rho, u) \) a mild solution to the Cauchy problem of (1.2) if

\[
\begin{align*}
\rho(t) &= \rho_0 - \int_0^t \text{div} \left( (1 + \rho(s)) u(s) \right) ds, \quad t > 0, \\
u(t) &= e^{t \mathcal{L}} u_0 - \int_0^t e^{(t-\tau) \mathcal{L}} \left( (u \cdot \nabla) u + \frac{\nabla P(1 + \rho)}{1 + \rho} + \frac{\rho}{1 + \rho} \mathcal{L} u \right) d\tau, \quad t > 0,
\end{align*}
\]

in the Besov space \( C([0, T]; \dot{B}^{n}_{p,\sigma}) \times C([0, T]; \dot{B}^{n-1}_{p,\sigma}) \), where \( e^{t \mathcal{L}} \) denotes the semi-group generated by the elliptic operator \( \mathcal{L} \). We remark that the mild solution \( (\rho, u) \) belonging to this space is a strong solution of (1.4). To prove the ill-posedness of the problem (1.2), in particular the strong
It is well known that for any $t_N \geq 2^{-2N}$ such that the critical Besov norm of the velocity fields $\|u(t_N)\|_{\dot{B}^{-\frac{1}{2}}_{2,1,\sigma}}$ diverges as $N \to \infty$.

**Theorem 1.1.** Let $n \geq 2$. There exist $\{u_{0,N}\}_{N=1}^{\infty} \subset (\dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n))^n$, $\{T_N\}_{N=1}^{\infty}$ with $T_N \to 0$ as $N \to \infty$, and $\{(\rho_N, u_N)\}_{N=1}^{\infty}$, that is a sequence of solutions to (1.2) with the initial data $(\rho_N(0), u_N(0)) = (0, u_{0,N})$ such that

$$\lim_{N \to \infty} \|u_{0,N}\|_{\dot{B}^{-\frac{1}{2}}_{2,1,\sigma}} \to 0,$$

and

$$\lim_{N \to \infty} \|u_N(T_N)\|_{\dot{B}^{-\frac{1}{2}}_{2,1,\sigma}} \to \infty.$$

The main argument showing the theorem above originally goes back to Bourgain-Pavlović [3] (see also a recent paper by Wang [33]). Chen-Miao-Zhang [8] showed that the worst contribution on regularity in a short time stems from the nonlinear interaction of the system, in particular, the convection term $u \cdot \nabla u$. For the limiting case, one may also expect the convection term to provide the lowest regularity and a strong discontinuity may be shown from such an interaction term. However, the threshold case $p = 2n$ is much different from the case $p > 2n$. The quasi-linear dissipation term

$$\frac{\rho}{1 + \rho} \mathcal{L} u$$

appearing in (1.2) also has a comparable order of the singularity in a short time; furthermore, the effect from the convection term and the quasi-linear dissipation term above cancel each other. This can be understood by investigating the high–high frequency interaction to the low frequency of the second iteration, and the crucial point is to obtain better regularity by the cancellation (see Proposition A.1, Proposition 2.2, and Lemma 2.3 with Remarks below). Therefore, a more detailed analysis is required to extract the lowest regularity in the critical scaling space. We employ a sharper example to prove the strong discontinuity with the time sequence $T_N \to 0$ as $N \to \infty$ that was developed in [22]. It is noteworthy that for the drift–diffusion system, a similar limitation to the solvability in the critical Besov space $\dot{B}^{-\frac{2}{n}}_{p,\sigma}(\mathbb{R}^n)$ can be observed in a monopolar case and a bipolar case ([22]).

We first consider the strong solution $u(t)$ by the integral form (1.4). To illustrate our main idea, we introduce the first approximation of the density $\rho$ and the velocity fields $u$ of the strong solution to (1.4), by the following linearized equation: Let $(\rho_1, U)$ solve

$$\begin{cases}
\partial_t \rho_1 + \text{div} \left( (1 + \rho_0)U \right) = 0, & t > 0, x \in \mathbb{R}^n, \\
\partial_t U - \mathcal{L} U = -\frac{\rho_0}{1 + \rho_0} \mathcal{L} U, & t > 0, x \in \mathbb{R}^n, \\
\rho_1(0, x) = \rho_0(x) \equiv 0, \quad U(0, x) = u_0(x), & x \in \mathbb{R}^n.
\end{cases}$$

(1.5)

By using $(\rho_1, U)$, we extract the worst term of the velocity fields that primarily affects the discontinuity to the original solution as $N \to \infty$.

We introduce the Helmholtz decomposition of the vector fields in $L^p(\mathbb{R}^n; \mathbb{R}^n)$.

**Definition.** For any $f \in L^p(\mathbb{R}^n; \mathbb{R}^n)$, let the operators $P_\sigma, P_\pi$ be defined by projections into the closed subspaces:

$$L^p_\sigma(\mathbb{R}^n; \mathbb{R}^n) \equiv \{ f \in L^p(\mathbb{R}^n; \mathbb{R}^n); \text{div} f = 0 \text{ in } D^* \},$$

$$L^p_\pi(\mathbb{R}^n; \mathbb{R}^n) \equiv \{ f \in L^p(\mathbb{R}^n; \mathbb{R}^n); \text{rot} f = 0 \text{ in } D^* \}.$$

It is well known that for any $1 < p < \infty$, the projection operators $P_\sigma, P_\pi$ are bounded from $L^p(\mathbb{R}^n; \mathbb{R}^n)$ into $L^p_\sigma(\mathbb{R}^n; \mathbb{R}^n)$ and $L^p_\pi(\mathbb{R}^n; \mathbb{R}^n)$, respectively, and $L^p(\mathbb{R}^n; \mathbb{R}^n)$ can be decomposed
by the direct sum of those subspaces as
\[ I^p(\mathbb{R}^n; \mathbb{R}^n) = I^p_\sigma(\mathbb{R}^n; \mathbb{R}^n) \oplus I^p_\pi(\mathbb{R}^n; \mathbb{R}^n). \]
Those operators are realized by
\[
\begin{align*}
P_\sigma &\equiv 1 + (-\Delta)^{-1} \nabla \text{div}, \\
P_\pi &\equiv -(\Delta)^{-1} \nabla \text{div}.
\end{align*}
\]

1.2. Worst term from nonlinear interactions. To extract the worst regularity term from the fluid vector \( u \), we examine each term appearing in the right-hand side of the integral equation (1.4). If we assume that the pressure term \( \nabla P \) is smoother compared with the other terms, then the candidates of the worst term are the remaining two terms:
\[
(u \cdot \nabla) u, \quad \frac{\rho}{1 + \rho} \mathcal{L} u.
\]
For the critical space, the regularity difference of the density \( \rho \) and the velocity \( u \) is \(-1\) and they appear as comparable terms. To see it more precisely, we decompose the convection term \( u \cdot \nabla u \) and the quasilinear term \( \frac{\rho}{1 + \rho} \mathcal{L} u \) as follows. We derive the worst term in the convection term by the decomposition.
\[ U = P_\sigma U + P_\pi U, \]
where \( P_\sigma \) and \( P_\pi \) are given in (1.6). On the first term of (1.7), we have
\[
\begin{align*}
u \cdot \nabla u &= U \cdot \nabla U + \text{l.o.t.} \\
&= (P_\sigma U + P_\pi U) \cdot \nabla (P_\sigma U + P_\pi U) + \text{l.o.t.} \\
&= P_\sigma U \cdot \nabla (P_\sigma U + P_\pi U) + P_\pi U \cdot \nabla P_\sigma U + P_\pi U \cdot \nabla P_\pi U + \text{l.o.t.} \\
&= \text{div} \left( P_\sigma U \otimes (P_\sigma U + P_\pi U) \right) + P_\pi U \cdot \nabla P_\sigma U + \frac{1}{2} \nabla |P_\pi U|^2 + \text{l.o.t.,}
\end{align*}
\]
where l.o.t. denotes the lower-order terms. We notice that the \( j \)-th component of the last term on the right-hand side can be represented by
\[
(P_\pi U)_k \cdot \nabla_k (P_\pi U)_j = (\nabla_k (-\Delta)^{-1} \text{div} U) \nabla_j (\nabla_k (-\Delta)^{-1} \text{div} U) = \frac{1}{2} \nabla_j |\nabla (-\Delta)^{-1} \text{div} U|^2.
\]
Hence the first and third terms in (1.8) exhibit divergence form structures and they behave relatively well from the regularity viewpoint. Hence, the worst term is now identified by
\[
u \cdot \nabla u = (P_\pi U \cdot \nabla) P_\sigma U + \text{l.o.t.}
\]
Meanwhile, using the linear approximation \( \rho_1 \) for \( \rho \) defined in (1.5):
\[
\frac{\rho}{1 + \rho} \mathcal{L} u = \frac{\rho_1}{1 + \rho_1} \mathcal{L} u + \frac{\rho_1}{1 + \rho_1} (\mathcal{L} u - \mathcal{L} U) + \left( \frac{\rho}{1 + \rho} - \frac{\rho_1}{1 + \rho_1} \right) \mathcal{L} u.
\]
One can regard that the first term on the right hand side of the identity (1.11) above is the leading term and we apply the Helmholtz decomposition \( U = P_\sigma U + P_\pi U \) and change into the divergence form as follows:
\[
\begin{align*}
\frac{\rho_1}{1 + \rho_1} \mathcal{L} U &= \frac{\rho_1}{1 + \rho_1} \mu \Delta P_\sigma U + \frac{\rho_1}{1 + \rho_1} (2\mu + \lambda) \Delta P_\pi U \\
&= \mu \nabla \cdot \left( \frac{\rho_1}{1 + \rho_1} \nabla P_\sigma U \right) + (2\mu + \lambda) \nabla \cdot \left( \frac{\rho_1}{1 + \rho_1} \nabla P_\pi U \right),
\end{align*}
\]
\[
\begin{align*}
\text{non-divergence term I} &+ (2\mu + \lambda) \nabla \cdot \left( \frac{\rho_1}{1 + \rho_1} \nabla P_\pi U \right) - (2\mu + \lambda) \left( \frac{\nabla \rho_1}{(1 + \rho_1)^2} \cdot \nabla \right) P_\pi U
\end{align*}
\]
\[
\text{“non-divergence term II”}
\]
Apply the first integral equation of (1.5) to the second term on the right-hand side of (1.12); “non-divergence term I” under the condition $\rho_0 = 0$

$$\nabla \rho_1(t) = \nabla \rho_0 - \int_0^t \nabla (\text{div} (1 + \rho_0(s))U(s)) ds$$

$$= - \int_0^t \nabla \text{div} U(s) ds,$$  \hspace{1cm} (1.13)

and then approximate $u$ by $U$ in (1.5) with $\nabla \text{div} U = \Delta P_\pi U$,

$$- \mu \left( \frac{\nabla \rho_1}{(1 + \rho_1)^2} \cdot \nabla \right) P_\pi U$$

$$= - \mu \frac{1}{(1 + \rho_1)^2} \left( - \int_0^t \nabla \text{div} U ds \cdot \nabla \right) P_\pi U$$

$$= \mu \frac{1}{(1 + \rho_1)^2} \left( \int_0^t \Delta P_\pi U ds \cdot \nabla \right) P_\pi U + \text{l.o.t.} \hspace{1cm} (1.14)$$

Here, we have neglected all the terms involving the initial data $\rho_0$ as we assumed that $\rho_0 = 0$. For the fourth term on the right-hand side in (1.12) i.e., ”non-divergence term II”, we can show that

$$-(2\mu + \lambda) \left( \frac{\nabla \rho_1}{(1 + \rho_1)^2} \cdot \nabla \right) P_\pi U$$

$$= (2\mu + \lambda) \frac{1}{(1 + \rho_1)^2} \left( \int_0^t \Delta P_\pi U ds \right) \cdot \nabla P_\pi U + \text{l.o.t.} \hspace{1cm} (1.15)$$

By noting

$$P_\pi \partial_t U(t) = P_\pi \mathcal{L}U(t) = (2\mu + \lambda)\Delta P_\pi U(t), \hspace{1cm} (1.16)$$

substituting (1.16) into (1.15), it follows that

$$-(2\mu + \lambda) \left( \frac{\nabla \rho_1}{(1 + \rho_1)^2} \cdot \nabla \right) P_\pi U$$

$$= \frac{1}{(1 + \rho_1)^2} \left( P_\pi (U(t) - u_0) \right) \cdot \nabla P_\pi U + \text{l.o.t.}$$

$$= \frac{1}{(1 + \rho_1)^2} \left( \frac{1}{2} \nabla |P_\pi U|^2 - P_\pi u_0 \cdot \nabla P_\pi u_0 \right) + \text{l.o.t.} \hspace{1cm} (1.17)$$

where we regard $U(t) \simeq u_0$ if $t \ll 1$. Assuming that $\rho_0 \sim 0$, we neglect all the terms involving $\rho_0$ such that the following term can be regarded as the top term of the quasilinear dissipation:

$$-\frac{\rho}{1 + \rho} \mathcal{L}u = - \frac{\mu}{1 + \rho_1} \int_0^t \Delta P_\pi U ds \cdot \nabla P_\pi U + \text{l.o.t.}$$

$$= - \frac{1}{1 + \rho_1} \int_0^t P_\pi \left( \mathcal{L}U \right) ds \cdot \nabla P_\pi U + \text{l.o.t.} \hspace{1cm} (1.18)$$

$$= - \frac{1}{1 + \rho_1} P_\pi (U(s) - u_0) \cdot \nabla P_\pi U + \text{l.o.t.}$$
From (1.10) and (1.18), the worst term for regularity of approximated solution can be extracted from the term of the convection and the quasilinear dissipation terms as follows:

\[ u(t) = e^{t\mathcal{L}}u_0 - \int_0^t e^{(t-s)\mathcal{L}} \left( u(s) \cdot \nabla u(s) + \frac{\nabla P(1 + \rho(s))}{1 + \rho(s)} + \frac{\rho(s)}{1 + \rho(s)} \mathcal{L} u(s) \right) ds \]

\[ = - \int_0^t e^{(t-s)\mathcal{L}} \left( u(s) \cdot \nabla u(s) + \frac{\rho(s)}{1 + \rho(s)} \mathcal{L} u(s) \right) ds + \text{l.o.t} \]

\[ = - \int_0^t e^{(t-s)\mathcal{L}} \left( (P_\pi U \cdot \nabla)P_\pi U + \frac{\mu}{2\mu + \lambda} (P_\pi (U(s) - u_0) \nabla P_\pi U(s)) \right) ds + \text{l.o.t.} \]  

(1.19)

Since \( U(t) = e^{t\mathcal{L}}u_0 \) and

\[ e^{(t-s)\mathcal{L}}h = e^{\mu(t-s)\Delta}P_\sigma h + e^{(2\mu+\lambda)(t-s)\Delta}P_\pi h, \]

we deduce the right-hand side of (1.19) by

\[ I[u_0] \equiv - \int_0^t e^{(t-s)\mathcal{L}} \left\{ e^{s\mathcal{L}}P_\pi u_0 \cdot \nabla e^{s\mathcal{L}}P_\sigma u_0 + \frac{\mu}{2\mu + \lambda} (e^{s\mathcal{L}}P_\pi u_0 - P_\pi u_0) \nabla e^{s\mathcal{L}}P_\sigma u_0 \right\} ds, \]  

(1.20)

which is the crucial top term for proving the ill-posedness. A striking difference between our critical case \( p = 2n \) and non-critical case \( p > 2n \) is that both the convection and quasilinear viscosity terms share the worst singularity and they are canceled with each other. We show those arguments in a rigorous way in the following section working in the Fourier space and use the cancelling property in Lemma 2.3 below.

This paper is organized as follows. In section 2, we consider the discontinuity of the crucial part (1.20) on the initial data contained in the integral equation for the velocity \( u \), and we will provide a counter example of the initial data to the continuity property. In section 3, we prove Theorem 1.1 by decomposing the velocity \( u \) into the top term observed in section 2 and the remainder part.

Hereafter we use the following notation: \( \dot{B}^{s}_{p,\sigma} \) is the homogeneous Besov space defined in (1.3). The intersection space \( \dot{B}^{s}_{[n,\infty],1} := \dot{B}^{s}_{n,1}(\mathbb{R}^n) \cap \dot{B}^{s}_{\infty,1}(\mathbb{R}^n) \) is used in the section 3.

### 2. Crucial bilinear estimate for the ill-posedness

We discuss the bilinear estimates for scalar valued functions in Proposition A.1 and prove the discontinuity of the term (1.20) at the origin in the Besov spaces \( \dot{B}^{-\frac{1}{2}}_{2n,\sigma}(\mathbb{R}^n) \) with \( 1 \leq \sigma < n \) in Proposition 2.1.

#### 2.1. Choice of initial data. The choice of initial data for showing the discontinuity is the most influencing. For the critical homogeneous Besov spaces, \( \dot{B}^{\frac{1}{2}}_{2n,\sigma}(\mathbb{R}^n) \times \dot{B}^{-\frac{1}{2}}_{2n,\sigma}(\mathbb{R}^n) \) (\( \sigma > 2 \)) and a parameter \( N \in \mathbb{N} \) we chose the initial data for the velocity fields as follows:

**Definition.** For some fixed \( 0 < \delta < 1 \) which is determined later and for any \( N \in \mathbb{N} \), we define \( (\rho_0, u_0) = (\rho_{0,N}, u_{0,N}) \) by

\[
\begin{align*}
\rho_0(x) &= 0 \\
u_0(x) &= (f_N(x), g_N(x), 0, \ldots, 0),
\end{align*}
\]

(2.1)
for all $N = 1, 2, \cdots$, where $f_N, g_N$ \footnote{the example here is inspired from the definition of the modulation space} are defined by

\[
\begin{aligned}
    f_N(x) &= R \frac{2^n}{N^\frac{n}{2}} \sum_{-\delta N \leq j \leq 0} 2^{-(n-\frac{1}{2})j} \phi_j(x - 2^j + 2^N e_1) \sin (2^N x_1), \\
    g_N(x) &= R \frac{2^n}{N^\frac{n}{2}} \sum_{-\delta N \leq j \leq 0} 2^{-(n-\frac{1}{2})j} \phi_j(x - 2^j + 2^N e_1) \cos (2^N x_1),
\end{aligned}
\tag{2.2}
\]

$e_1$ is the unit vector for $x_1$ and $R > 0$ is a constant.

It is straightforward that $f_N, g_N \in \dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$, since $f_N, g_N \in S(\mathbb{R}^n)$.

The following proposition is crucial for the proof of the discontinuity of solutions.

**Proposition 2.1.** Let $u_0$ be defined by (2.1). Then there exists $C > 0$ independent of $N$ such that

\[\|u_0\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \leq CR.\]

Furthermore, there exist $c, \varepsilon_0 > 0$ and $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$, $1 \leq \sigma \leq \infty$ and for $T_N = \varepsilon_0 2^{-2N}$

\[\left\| I[u_0](t) \right\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} |_{t=T_N} \geq cN^{\frac{1}{2}-\frac{\sigma}{2}} R^2 - CR^2, \tag{2.3}\]

where $I[u_0]$ is defined by (1.20).

**Remark.** If we take $R = (\log N)^{-1}$ and $\sigma = 1$, the left hand side in (2.3) diverges to infinity while the initial data $u_0$ tends to 0. Therefore one can expect to obtain the ill-posedness result for the compressible Navier-Stokes equations in the Besov space $\dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$.

To prove the estimate (2.3), we decompose $I[u_0]$ into a term with the product of $f_N, g_N$, which cause the discontinuity, and the remainder term. Let us introduce a bilinear form $\widetilde{I}[f, g]$ for scalar valued functions $f$ and $g$, where $f$ is an approximating function of $P_{\pi} u_0$ and $g$ is an approximated function of $P_{\sigma} u_0$, respectively (cf. (1.20)).

**Definition.** Let $A = 2\mu + \lambda$, $B = \mu, c$ be either $2\mu + \lambda$ or $\mu$ and $f, g \in S(\mathbb{R}^n)$. Define the quadratic form $\widetilde{I}[\cdot, \cdot]$ by

\[\widetilde{I}[f, g] = \widetilde{I}[f, g](t) \equiv - \int_0^t e^c(t-s) A \left\{ e^{A s} \nabla e^{B s} g + \frac{B}{A} (e^{A s} f - f) \nabla e^{B s} g \right\} ds, \tag{2.4}\]

We will choose the parameter $R > 0$ such that $f, g$ are arbitrarily small in the critical space $\dot{B}^{-\frac{1}{2}}_{2n,\sigma}$, while $\widetilde{I}[f, g]$ is large. The following proposition shows the discontinuity for $\widetilde{I}[f, g]$.\footnote{independent of $N$}

**Proposition 2.2.** For $R > 0$, let $\{f_N\}_{N \in \mathbb{N}}, \{g_N\}_{N \in \mathbb{N}}$ be defined in (2.2). Then there exists $C_n$ independent of $N$ such that

\[\|f_N\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \leq C_n R, \quad \|g_N\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \leq C_n R. \tag{2.5}\]

Furthermore, there exists $c > 0$ independent of $N$ such that for sufficiently small $\varepsilon_0 > 0$ and $T_N = \varepsilon_0 2^{-2N}$,

\[\left\| \widetilde{I}[f_N, g_N](T_N) \right\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \geq cN^{\frac{1}{2}-\frac{\sigma}{2}} R^2. \tag{2.6}\]
Remark. The condition of $\sigma$ is indeed sharp. When $1 \leq \sigma < n$, the estimates (2.5) and (2.6) with $R = (\log N)^{-1}$ and $N \to \infty$ imply the discontinuity of the map

$$\tilde{B}_{2n, \sigma}^{-\frac{1}{2}} \times \tilde{B}_{2n, \sigma}^{-\frac{1}{2}} \ni (f, g) \mapsto \tilde{I}(f, g) \in C([0, 1]; \tilde{B}_{2n, \sigma}^{-\frac{1}{2}}).$$

On the other hand, the continuity holds when $n \leq \sigma \leq \infty$, which will be discussed in the appendix (see Proposition A.1). Such a situation for the bilinear estimate for the nonlinear terms of the compressible Navier-Stokes system is peculiar since the better regularity provides the bilinear estimate failure and the other case makes the estimate valid.

We prepare some lemmas. Lemma 2.3 gives a decomposition of the Fourier transform of $\tilde{I}[f, g]$ into two parts of divergence and non-divergence form structure. After that we will show inequalities from below in Lemmas 2.4, 2.5, and then we will prove Proposition 2.2, which will be applied to the proof of Proposition 2.1.

Lemma 2.3. Let $\tilde{I}[f, g]$ be defined by (2.4). The Fourier transform of $\tilde{I}[f, g]$ satisfies

$$\tilde{I}[f, g](t) = -\frac{B}{A} \int_{\mathbb{R}^n} \frac{1 - e^{-At|\xi - \eta|^2}}{B|\eta|^2 - c|\xi|^2} e^{-Bt|\eta|^2} i\hat{f}(\xi - \eta)\hat{g}(\eta) d\eta + R[f, g](t),$$

where $R[f, g]$ is defined by

$$R[f, g](t) := \int_{\mathbb{R}^n} \left\{ \frac{(B + c)|\xi|^2 - 2B\xi \cdot \eta}{(A|\xi - \eta|^2 + B|\eta|^2 - c|\xi|^2)(B|\eta|^2 - c|\xi|^2)} \right\} i\hat{f}(\xi - \eta)\hat{g}(\eta) d\eta.$$  

Remark. Let us investigate by focusing on the high–high frequency interaction to the low frequency, namely,

$$|\xi| \simeq 1, \quad \text{supp} \hat{f}, \text{supp} \hat{g} \subset \{ \xi \in \mathbb{R}^n \mid |\xi| \simeq 2^N \}, \quad N \gg 1$$

to compare with the convection term. One can deduce from (2.7), $|\xi| \leq 1, |\xi - \eta| \simeq 2^N, |\eta| \simeq 2^N$ that

$$\tilde{I}[f, g](t) \simeq -\frac{B}{A} \int_{\mathbb{R}^n} \frac{1 - e^{-At2^N}}{B \cdot 2^{2N} - c \cdot 1^2} e^{-Bt2^N} i2^N \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta$$

$$\simeq -\frac{B}{A} \int_{\mathbb{R}^n} \frac{1}{2^{2N} e^{-Bt2^N}} i2^N \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta,$$

if $t \simeq 2^{-2N}$. The term $e^{-Bt2^N}$ above can provide a certain regularity to obtain the estimate for $n \leq \sigma \leq \infty$ in Proposition A.1. Meanwhile, the convection term $(u \cdot \nabla)u$ will exhibit less regularity than that for $(u \cdot \nabla)u + \frac{p}{1 + p}L$. In fact,

$$\mathcal{F} \left[ \int_0^t e^{c(t-s)\Delta} e^{As\Delta} f \nabla e^{Bs\Delta} g ds \right](\xi)$$

$$= e^{-ct|\xi|^2} \int_{\mathbb{R}^n} \frac{1 - e^{-At|\xi - \eta|^2 - Bt|\eta|^2 + ct|\xi|^2}}{A|\xi - \eta|^2 + B|\eta|^2 - c|\xi|^2} i\hat{f}(\xi - \eta)\hat{g}(\eta) d\eta$$

$$\simeq \int_{\mathbb{R}^n} \frac{1}{2^{2N}} i2^N \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta$$

for $2^{-2N} \leq t \leq 1$, and the term $e^{-Bt2^N}$ does not appear. Hence, the observations above suggest that $(u \cdot \nabla)u$ and $\frac{p}{1 + p}L$ would exhibit comparable regularity and some components are canceled out.
Proof of Lemma 2.3. By taking the Fourier transform of $\tilde{I}[f, g]$, it holds that

$$
\tilde{I}[f, g](t) = -\int_0^t e^{-c(t-s)|\xi|^2} \int_{\mathbb{R}^n} \left\{ e^{-As|\xi-\eta|^2} \hat{f}(\xi-\eta) i\eta e^{-Bs|\eta|^2} \hat{g}(\eta) + \frac{B}{A} e^{-As|\xi-\eta|^2} \hat{f}(\xi-\eta) - \hat{f}(\xi-\eta) \right\} i\eta e^{-Bs|\eta|^2} \hat{g}(\eta) \, d\eta \, ds
$$

(2.9)

$$
= -\int_{\mathbb{R}^n} e^{-ct|\xi|^2} \int_0^t e^{cs|\xi|^2} \left\{ e^{-s(A|\xi-\eta|^2 + B|\eta|^2)} + \frac{B}{A} e^{-As|\xi-\eta|^2 - 1} e^{-Bs|\eta|^2} \right\} ds
$$

$$
\times \hat{f}(\xi-\eta) i\eta \hat{g}(\eta) \, d\eta.
$$

Following the integral on variable $s \in (0, t)$,

$$
= \int_0^t e^{cs|\xi|^2} \left\{ \frac{A + B}{A} e^{-s(A|\xi-\eta|^2 + B|\eta|^2-c|\xi|^2)} - \frac{B}{A} e^{-s(B|\eta|^2-c|\xi|^2)} \right\}ds
$$

$$
= \frac{A + B}{A} \cdot \frac{1 - e^{-t(A|\xi-\eta|^2 + B|\eta|^2-c|\xi|^2)}}{A|\xi-\eta|^2 + B|\eta|^2 - c|\xi|^2} - \frac{B}{A} \cdot \frac{1 - e^{-t(B|\eta|^2-c|\xi|^2)}}{B|\eta|^2 - c|\xi|^2}
$$

(2.10)

$$
+ e^{ct|\xi|^2} \left\{ \frac{A + B}{A} \cdot \frac{-e^{-t(A|\xi-\eta|^2 + B|\eta|^2)}}{A|\xi-\eta|^2 + B|\eta|^2 - c|\xi|^2} - \frac{B}{A} \cdot \frac{-e^{-tB|\eta|^2}}{B|\eta|^2 - c|\xi|^2} \right\}
$$

$$
= J_1 + e^{ct|\xi|^2} J_2.
$$

By reducing the two terms of $J_1$ to a common denominator, we have

$$
J_1 = \frac{1}{A} \cdot \frac{(A + B)(B|\eta|^2 - c|\xi|^2)}{(A|\xi-\eta|^2 + B|\eta|^2 - c|\xi|^2)(B|\eta|^2 - c|\xi|^2)} - \frac{B}{A} \cdot \frac{-e^{-t(A|\xi-\eta|^2 + B|\eta|^2)}}{(B|\eta|^2 - c|\xi|^2)}
$$

(2.11)

$$
= \frac{1}{A} \cdot \frac{-A \{(B + c)|\xi|^2 - 2B|\xi| \cdot \eta\}}{(A|\xi-\eta|^2 + B|\eta|^2 - c|\xi|^2)(B|\eta|^2 - c|\xi|^2)}
$$

By considering the difference of $-e^{-t(A|\xi-\eta|^2 + B|\eta|^2)}$ and $-e^{-tB|\eta|^2}$ for the second term of $J_2$, we obtain

$$
J_2 = \left\{ \frac{A + B}{A} \cdot \frac{-e^{-t(A|\xi-\eta|^2 + B|\eta|^2)}}{A|\xi-\eta|^2 + B|\eta|^2 - c|\xi|^2} - \frac{B}{A} \cdot \frac{-e^{-t(A|\xi-\eta|^2 + B|\eta|^2)}}{B|\eta|^2 - c|\xi|^2} \right\}
$$

(2.12)

$$
+ \left\{ \frac{B}{A} \cdot \frac{-e^{-t(A|\xi-\eta|^2 + B|\eta|^2)}}{B|\eta|^2 - c|\xi|^2} - \frac{B}{A} \cdot \frac{-e^{-tB|\eta|^2}}{B|\eta|^2 - c|\xi|^2} \right\}
$$

$$
= -e^{-t(A|\xi-\eta|^2 + B|\eta|^2)} J_1 + \frac{B}{A} \cdot \frac{-e^{-tA|\xi-\eta|^2 + 1}}{B|\eta|^2 - c|\xi|^2} \cdot e^{-tB|\eta|^2}.
$$

Therefore, the equality (2.7) is obtained by (2.9), (2.10), (2.11) and (2.9) because the first term in the right hand side of (2.7) corresponds to the second term in the last right hand side of (2.12), and the remainder term $\tilde{R}[f, g]$ consists of $J_1$ from (2.11) and the first term in the last right hand side of (2.12). \qed

Lemma 2.4. Let $\varphi$ satisfy $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and supp $\hat{\varphi} \subseteq \{ |\xi| \leq 1 \}$, and let $\{f_N\}_N$ and $\{g_N\}_N$ be defined by (2.2). Then there exists $N_0 \in \mathbb{N}$ such that for any $\sigma$ with $1 \leq \sigma \leq \infty$ and $N \in \mathbb{N}$ with
\( N \geq N_0 \)

\[
2^{-2N} \| \varphi * (f_N \partial_{x_1} g_N) \|_{B^{2,0,\sigma}_{2n,\sigma}} \geq c R^2 N^{\frac{1}{2}} - \frac{1}{\pi}.
\]  

(2.13)

**Proof of Lemma 2.4.** Since the norm in the left hand side of (2.13) is restricted in the low frequency part, it follows that

\[
2^{-2N} \| \varphi * (f_N \partial_{x_1} g_N) \|_{B^{2,0,\sigma}_{2n,\sigma}} \geq c 2^{-2N} \left\{ \sum_{-\delta N \leq j \leq 0} \left( 2^{-\frac{1}{2}j} \| \phi_j * (f_N \partial_{x_1} g_N) \|_{L^{2n}} \right)^\sigma \right\}^{\frac{1}{\sigma}}.
\]  

(2.14)

For simplicity, we rewrite the initial of \( f_N \) and \( g_N \) in (2.2) by introducing

\[
\Phi_j := \frac{R2^{\frac{j}{2}}}{N \pi} 2^{(-n+\frac{1}{2})j} \phi_j(x - 2^{j+2N} e_1),
\]  

(2.15)

then \( f_N \) and \( g_N \) are written as

\[
f_N(x) = \sum_{-\delta N \leq k \leq 0} \Phi_k(x) \sin(2^N x_1), \quad g_N(x) = \sum_{-\delta N \leq l \leq 0} \Phi_l(x) \cos(2^N x_1),
\]

and we have the following equality:

\[
f_N \partial_{x_1} g_N = - \sum_{-\delta N \leq k, l \leq 0} 2^N \Phi_k \Phi_l \sin^2(2^N x_1) + \sum_{-\delta N \leq k, l \leq 0} \Phi_k \left( \partial_{x_1} \Phi_l \right) \times \cos(2^N x_1) \sin(2^N x_1)
\]

\[= \sum_{-\delta N \leq k \leq 0} 2^N \Phi_k^2 \sin^2(2^N x_1) + \sum_{-\delta N \leq k, l \leq 0, k \neq l} 2^N \Phi_k \Phi_l \sin^2(2^N x_1)
\]

\[+ \sum_{-\delta N \leq k, l \leq 0} \Phi_k \left( \partial_{x_1} \Phi_l \right) \times \cos(2^N x_1) \sin(2^N x_1)
\]

\[=: F_1 + F_2 + F_3.
\]  

(2.16)

Let us introduce the set \( A_j \) defined by

\[ A_j := \{ x \mid \| x - 2^{j+2N} e_1 \| \leq 2^{-j} \}, \]

which is the main area on the support of \( \Phi_j \). In what follows we estimate \( F_1, F_2 \) and \( F_3 \) one by one.

On the estimate of \( \| \phi_j * F_1 \|_{L^{2n}} \), we restrict the \( L^{2n} \)-norm in \( A_j \) and expect that the case \( k = j \) in the sum in the definition of \( F_1 \) is the largest, and obtain

\[
\| \phi_j * F_1 \|_{L^{2n}(A_j)} \geq \| \phi_j * F_1 \|_{L^{2n}(A_j)} \\
\geq 2^N \| \phi_j * (\Phi_j^2 \sin^2(2^N x_1)) \|_{L^{2n}(A_j)} \\
- \sum_{-\delta N \leq k \leq 0, k \neq j} 2^N \| \phi_j * (\Phi_k^2 \sin^2(2^N x_1)) \|_{L^{2n}(A_j)}.
\]

(2.17)

On the first term in the last right hand side of (2.17), it holds that for \( j \) with \( -N \leq j \leq 0 \)

\[
2^N \| \phi_j * (\Phi_j^2 \sin^2(2^N x_1)) \|_{L^{2n}(A_j)} = 2^N \left( R2^{\frac{j}{2}} N \pi \right)^2 \\
\times 2^{3nj} \left\| \int_{\mathbb{R}^n} \phi_0(2^j (x - y)) \left( \phi_0(2^j (y - 2^{j+2N} e_1)) \right)^2 \sin^2(2^N y_1) dy \right\|_{L^{2n}(A_j)}
\]
Since \( \sin^2(2^Ny_1) = (1 - \cos(2^{N+1}y_1))/2 \), it follows from the triangle inequality that

\[
\left\| \int_{\mathbb{R}^n} \phi_0(2^j(x-y)) \left( \phi_0(2^j(y-2^|j|+2^N\epsilon_1)) \right)^2 \sin^2(2^Ny_1) dy \right\|_{L^2(A_j)} \\
\geq \frac{1}{2} \left\| \int_{\mathbb{R}^n} \phi_0(2^j(x-y)) \left( \phi_0(2^j(y-2^|j|+2^N\epsilon_1)) \right)^2 dy \right\|_{L^2(A_j)} \\
- \frac{1}{2} \left\| \int_{\mathbb{R}^n} \phi_0(2^j(x-y)) \left( \phi_0(2^j(y-2^|j|+2^N\epsilon_1)) \right)^2 \cos(2^{N+1}y_1) dy \right\|_{L^2(A_j)}.
\]

By change of variables, we write the first term in the right hand side

\[
\frac{1}{2} \left\| \int_{\mathbb{R}^n} \phi_0(2^j(x-y)) \left( \phi_0(2^j(y-2^|j|+2^N\epsilon_1)) \right)^2 dy \right\|_{L^2(A_j)} = \frac{1}{2} \cdot 2^{-nj} \cdot 2^{-\frac{j}{2}} \left\| \int_{\mathbb{R}^n} \phi_0(x-y) \left( \phi_0(y) \right)^2 dy \right\|_{L^2(A_j(\{|x| \leq 1\})}.
\]

The second term is dealt with the integration by parts, and we have that

\[
\frac{1}{2} \left\| \int_{\mathbb{R}^n} \phi_0(2^j(x-y)) \left( \phi_0(2^j(y-2^|j|+2^N\epsilon_1)) \right)^2 \cos(2^{N+1}y_1) dy \right\|_{L^2(A_j)} = \frac{1}{2} \left\| \int_{\mathbb{R}^n} \partial_1 \left( \phi_0(2^j(x-y)) \left( \phi_0(2^j(y-2^|j|+2^N\epsilon_1)) \right)^2 \right) \sin(2^{N+1}y_1) dy \right\|_{L^2(A_j)} \\
\leq C \frac{2^j}{2^N+1} \times 2^{-nj} \cdot 2^{-\frac{j}{2}}.
\]

The three inequalities above imply that

\[
2^N \left\| \phi_j * (\Phi_k^2 \sin^2(2^N x_1)) \right\|_{L^2(A_j)} \\
\geq c2^N R^2 2^{(-2n+1)j} N^{-\frac{1}{n}} \times 2^{3nj} \cdot 2^{-2^j} \cdot 2^{-\frac{j}{2}} \left( 1 - \frac{C2^j}{2^N} \right) (2.18)
\]

On the second term in the last right hand of (2.17), let \( \alpha \) satisfy \( \alpha > n \), and it follows from \( \phi_0 \in \mathcal{S}(\mathbb{R}^n) \) that

\[
2^N \left\| \phi_j * (\Phi_k^2 \sin^2(2^N x_1)) \right\|_{L^2(A_j)} \\
\leq 2^N \left( R 2^{\frac{1}{2}n} 2^{(-n+\frac{1}{2})k} N^{-\frac{1}{n}} \right)^2 \times 2^{2nk+nj} \int_{\mathbb{R}^n} \phi_0(2^j(x-y)) \left( \phi_0(2^k(y-2^|k|+2^N\epsilon_1)) \right)^2 \frac{1}{1+2^j|x-y|} dy \left\|_{L^2(A_j)} \\
\leq C2^N R^2 2^{(-2n+1)k} N^{-\frac{1}{n}} \times 2^{2nk+nj} \left\| \frac{1}{1+2^j|x-y|} \right\|_{L^2(A_j)}.
\]

\[
(2.19)
\]

Since

\[
2^k|x - 2^k|+2^N\epsilon_1| \geq c2^k|2^j|+2^N - 2^|k|+2N| \geq c2^k \cdot 2^N \quad \text{for any } j, k \text{ with } j \neq k \text{ and } x \in A_j,
\]

we have from (2.19) that

\[
2^N \left\| \phi_j * (\Phi_k^2 \sin^2(2^N x_1)) \right\|_{L^2(A_j)} \leq C2^N R^2 2^{(-2n+1)k} N^{-\frac{1}{n}} \times 2^{2nk+nj} \left\| \frac{2^{-nj}}{2^{k}2^N} \right\|_{L^2(A_j)} (2.20)
\]

\[
\leq CR^2 2^{-\frac{j}{2}}.
\]
Therefore in the right hand side of (2.14), the term consisting of $F_1$ is estimated by (2.17), (2.18) and (2.20) as follows
\[
2^{-2N} \left\{ \sum_{-\delta N \leq j \leq 0} \left( 2^{-\frac{1}{2}j} \| \phi_j * F_1 \|_{L^{2n}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} \\
\geq c 2^{-2N} \left\{ \sum_{-\delta N \leq j \leq 0} \left( 2^{-\frac{1}{2}j} 2^{2N} R^2 2^{\frac{3}{2}j} N^{-\frac{1}{n}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} - C 2^{-2N} \left\{ \sum_{-\delta N \leq j \leq 0} \left( 2^{-\frac{1}{2}j} R^2 2^{-\frac{3}{2}j} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} (2.21) \\
\geq c R^2 N^{\frac{1}{2} - \frac{1}{n}} - C R^2 2^{-N}.
\]

On the term $F_2$, by the embedding $\dot{B}_{0,\sigma}^0(\mathbb{R}^n) \hookrightarrow \dot{B}_{2n,\sigma}^{-\frac{1}{2}}(\mathbb{R}^n)$,
\[
2^{-2N} \left\{ \sum_{-\delta N \leq j \leq 0} \left( 2^{-\frac{1}{2}j} \| \phi_j * F_2 \|_{L^{2n}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} \leq C 2^{-2N} \left\{ \sum_{-\delta N \leq j \leq 0} \| \phi_j * F_2 \|_{L^{n}} \right\}^{\frac{1}{\sigma}}. (2.22)
\]

In order to estimate $\| \phi_j * F_2 \|_{L^{n}}$, let $\alpha$ be defined by $\alpha = \frac{n}{2} + \frac{1}{2}$, and it follows from Young’s inequality and $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ that
\[
\| \phi_j * F_2 \|_{L^{n}} \leq \| \phi_j \|_{L^1} \| F_2 \|_{L^{n}} \leq \| \phi_0 \|_{L^1} \sum_{-\delta N \leq k,l \leq 0, k \neq l} 2^N \| \Phi_k \Phi_l \|_{L^{n}}
\]
\[
\leq C \sum_{-\delta N \leq k,l \leq 0, k \neq l} 2^N \left( \frac{R^2 2^{\frac{3}{2}N}}{N^{\frac{1}{n}}} \right)^2 2^{(-n + \frac{1}{2})(k+l)} \| \phi_k (\cdot - 2^{|k|+2N} e_1) \phi_l (\cdot - 2^{|l|+2N} e_1) \|_{L^{n}} (2.23)
\]
\[
\leq C \sum_{-\delta N \leq k,l \leq 0, k \neq l} 2^{2N} R^2 N^{-\frac{1}{n}} 2^{(-n + \frac{1}{2})(k+l)} \cdot 2^{-2\alpha N}.
\]

Then we obtain by (2.22) and (2.23)
\[
2^{-2N} \left\{ \sum_{-\delta N \leq j \leq 0} \left( 2^{-\frac{1}{2}j} \| \phi_j * F_2 \|_{L^{2n}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} \\
\leq C 2^{-2N} \left\{ \sum_{-\delta N \leq j \leq 0} \left( \sum_{-\delta N \leq k,l \leq 0, k \neq l} 2^{2N} R^2 N^{-\frac{1}{n}} 2^{(-n + \frac{1}{2})(k+l)} \cdot 2^{n(k+l)} 2^{-2\alpha N} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} (2.24) \\
\leq C R^2 N^{\frac{1}{2} - \frac{1}{n}} 2^{-2\alpha N} \\
\leq C R^2 N^{\frac{1}{2} - \frac{1}{n}} 2^{-2\alpha N},
\]
where $C$ depends on $\delta$.

The term $F_3$ can be treated as in the similar way for $F_2$, we estimate the right hand side of (2.22) for $F_3$ instead of $F_2$ by Young’s inequality and Hölder’s inequality that
\[
\| \phi_j * F_3 \|_{L^{n}} \leq \| \phi_j \|_{L^1} \| F_3 \|_{L^n} \leq \| \phi_0 \|_{L^1} \sum_{-\delta N \leq k,l \leq 0} \left\| \Phi_k \left( \partial_{x_1} \Phi_l \right) \right\|_{L^n}
\]
\[
\leq C \sum_{-\delta N \leq k,l \leq 0} \| \Phi_k \|_{L^{2n}} \| \partial_{x_1} \Phi_l \|_{L^{2n}}
\]
\[
\leq C \sum_{-\delta N \leq k,l \leq 0} \left( \frac{R^2 2^{\frac{3}{2}N}}{N^{\frac{1}{n}}} \right)^2 2^{(-n + \frac{1}{2})(k+l)} \| \phi_k \|_{L^{2n}} 2^l \| \phi_l \|_{L^{2n}} (2.25)
\]
\[
\leq C R^2 2^{2N} N^{-\frac{1}{n}} \sum_{-\delta N \leq k,l \leq 0} 2^l \\
\leq C R^2 2^{2N} N^{1 - \frac{1}{n}}.
\]
Then we have from the corresponding estimate for $F_3$ of (2.22) and (2.25)
\[
2^{-2N}\left\{ \sum_{-\delta N \leq j \leq 0} \left( 2^{-\frac{j}{2}} \| \phi_j * I_{11} \|_{L^2} \right)^2 \right\}^{\frac{1}{2}} \leq C 2^{-2N}\left\{ \sum_{-\delta N \leq j \leq 0} \left( R^2 2^N N^{1-\frac{1}{n}} \right)^2 \right\}^{\frac{1}{2}} \\
\leq C 2^{-N} N^{1-\frac{1}{n} + \frac{1}{2}} R^2.
\] (2.26)

Therefore we obtain by (2.14), (2.16), (2.21), (2.24) and (2.26)
\[
2^{-2N} \| \varphi \ast (f \partial_{x_1} g) \|_{B_{2n,\sigma}}^{-\frac{1}{2}} \geq c R^2 N^{\frac{1}{2} - \frac{1}{n}} - CR^2 2^{-N} - CR^2 N^{\frac{1}{2} + \frac{1}{2}} 2^{-2nN} - C 2^{-N} N^{1-\frac{1}{n} + \frac{1}{2}} R^2 \\
= R^2 \left( c N^{\frac{1}{2} - \frac{1}{n}} - C N^{\frac{1}{2} + \frac{1}{2}} 2^{-2nN} - C 2^{-N} N^{1-\frac{1}{n} + \frac{1}{2}} \right).
\]

The proof of the estimate (2.13) for sufficiently large $N \in \mathbb{N}$ is completed.

Next, we show a lemma on the bilinear estimate, which is concerned with the top term of (2.7).

**Lemma 2.5.** Let $f_N, g_N$ be defined by (2.2). For sufficiently small $\varepsilon_0 > 0$ and $t = \varepsilon_0 2^{-2N}$, it holds that for any large $N \in \mathbb{N}$
\[
\left\| \mathcal{F}^{-1} \left[ \frac{B}{A} \int_{\mathbb{R}^n} \frac{1 - e^{-At|\xi-\eta|^2}}{B|\eta|^2 - c|\xi|^2} e^{-Bt|\eta|^2} \, \sinf \widehat{f}_N(\xi - \eta) \, \widehat{g}_N(\eta) \, d\eta \right] \right\|_{B_{2n,\sigma}}^{-\frac{1}{2}} \geq c \varepsilon_0 N^{\frac{1}{2} - \frac{1}{n} - \frac{1}{2}} R^2.
\] (2.27)

**Proof of Lemma 2.5.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\text{supp} \varphi \subset \{ \xi \mid |\xi| \leq 1 \}$ and we consider the restriction to low frequency part:
\[
\left\| \mathcal{F}^{-1} \left[ \frac{B}{A} \int_{\mathbb{R}^n} \frac{1 - e^{-At|\xi-\eta|^2}}{B|\eta|^2 - c|\xi|^2} e^{-Bt|\eta|^2} \, \sinf \widehat{f}_N(\xi - \eta) \, \widehat{g}_N(\eta) \, d\eta \right] \right\|_{B_{2n,\sigma}}^{-\frac{1}{2}} \geq c \varepsilon_0 N^{\frac{1}{2} - \frac{1}{n}} R^2.
\] (2.28)

Noting our frequency $|\xi| \leq 1$, $|\xi - \eta| \simeq 2^N$, $|\eta| \simeq 2^N$ and $t = \varepsilon_0 2^{-2N}$ we consider the expansion
\[
1 - e^{-At|\xi-\eta|^2} = - \sum_{m_1=1}^{\infty} \frac{(-At|\xi-\eta|^2)^{m_1}}{m_1!} = A|\eta|^2 + At(-2\xi \cdot \eta + |\xi|^2) - \sum_{m_1=2}^{\infty} \frac{(-At|\xi-\eta|^2)^{m_1}}{m_1!} = At|\eta|^2 + O\left(t 2^N\right) + O\left(\varepsilon_0^2\right),
\]
\[
\frac{1}{B|\eta|^2 - c|\xi|^2} = \frac{1}{B|\eta|^2} \left(1 - \frac{c|\xi|^2}{B|\eta|^2}\right) = \frac{1}{B|\eta|^2} \left(1 + \sum_{m_2=1}^{\infty} \left(\frac{c|\xi|^2}{B|\eta|^2}\right)^{m_2}\right) = \frac{1}{B|\eta|^2} (1 + O(2^{-2N})),
\]
\[
e^{-Bt|\eta|^2} = 1 + \sum_{m_3=1}^{\infty} \frac{(-Bt|\eta|^2)^{m_3}}{m_3!} = 1 + O(\varepsilon_0),
\]
and the leading term should be $\frac{A}{B} t$, since
\[
\frac{1 - e^{-At|\xi-\eta|^2}}{B|\eta|^2 - c|\xi|^2} e^{-Bt|\eta|^2} = \frac{A}{B} t + O\left(\frac{\varepsilon_0^2}{2^{2N}}\right), \quad N \to \infty
\] (2.29)
for sufficiently large $N$. On the leading term, the inequality (2.13) and $t = \varepsilon_0 2^{-2N}$ imply

$$
\| F^{-1} \left[ \hat{\varphi}(\xi) \int_{\mathbb{R}^n} \left( \frac{1}{B|\eta|^2} - c |\xi|^2 \right) i\eta \hat{f}_N(\xi - \eta) \hat{g}_N(\eta) d\eta \right] \|_{\mathcal{B}_{2n, \sigma}^{\frac{1}{2}}} \geq c\varepsilon_0 N^{\frac{1}{\sigma} - \frac{1}{2}} R^2.
$$

The remainder can be estimated in a similar manner to the proof of Lemma 2.4, where the main part there was the estimate from below but one can prove the bounds from above. In fact, it follows by the boundedness of the Fourier transform in Lebesgue spaces and (2.29) that

$$
\| F^{-1} \left[ \hat{\varphi}(\xi) \int_{\mathbb{R}^n} \left( \frac{1}{B|\eta|^2} - c |\xi|^2 \right) i\eta \hat{f}_N(\xi - \eta) \hat{g}_N(\eta) d\eta \right] \|_{\mathcal{B}_{2n, \sigma}^{\frac{1}{2}}}
\leq C \left\{ \sum_{j \leq 0} \left( 2^{-\frac{j}{2}} \| \hat{\varphi}_j F \left[ f_N \partial_x g_N \right] \|_{L^{2n}_{\sigma}} \right)^{\frac{1}{\sigma}} \right\}
\leq C \left\{ \sum_{j \leq 0} \left( 2^{-\frac{j}{2}} \| \hat{\varphi}_j F \left[ F_1 + F_2 + F_3 \right] \|_{L^{2n}_{\sigma}} \right)^{\frac{1}{\sigma}} \right\}
\leq C \left\{ \sum_{j \leq 0} \left( 2^{-\frac{j}{2}} \| \hat{\varphi}_j F \left[ F_1 \right] \|_{L^{2n}_{\sigma}} \right)^{\frac{1}{\sigma}} \right\}
\leq C 2^{-\frac{j}{2}} \varepsilon_0^2 \sum_{\max\{j - 3, -\delta N\} \leq k \leq 0} \frac{R^2}{N^{\frac{1}{n}} 2^{(-n+\frac{1}{2})k}} \| \hat{\varphi}_j \|_{L^{2n}_{\sigma}}
\leq C \sum_{\max\{j - 3, -\delta N\} \leq k \leq 0} \frac{R^2}{N^{\frac{1}{n}} 2^{(-n+\frac{1}{2})k} 2^{nk}}
\leq C \sum_{\max\{j - 3, -\delta N\} \leq k \leq 0} \frac{R^2}{N^{\frac{1}{n}} 2^{(-n+\frac{1}{2})k} 2^{(n-1)j}},
$$

where we employ the decomposition (2.16). We then focus on the worst term $F_1$ which is given by

$$
2^{-\frac{j}{2}} \varepsilon_0^2 \left\| \hat{\varphi}_j F \left[ F_1 \right] \right\|_{L^{2n}_{\sigma}}
\leq C \sum_{\max\{j - 3, -\delta N\} \leq k \leq 0} \frac{R^2}{N^{\frac{1}{n}} 2^{(-n+\frac{1}{2})k} 2^{nk}}
\leq C \sum_{\max\{j - 3, -\delta N\} \leq k \leq 0} \frac{R^2}{N^{\frac{1}{n}} 2^{(-n+\frac{1}{2})k} 2^{(n-1)j}},
$$

(2.31)
where \( \Phi_k \) is given by (2.15) as
\[
\Phi_k := \frac{R_2}{N^{\frac{3}{2}}} 2^{-(n+\frac{1}{2})k} \phi_k(x - 2^{[k]+2N} e_1)
\]
and the range of \( k \) is restricted by the support of \( \tilde{\phi}_j \) as \( j - 3 \leq k \). We then have that
\[
\left\{ \sum_{j \leq 0} \left( 2^{-\frac{1}{2}j} \frac{\epsilon_0^2}{2N} \left\| \tilde{\phi}_j \mathcal{F} [F_1] \right\|_{L^2_{2\pi^{-1}}} \right)^{\frac{1}{\sigma}} \right\}^{\frac{\sigma}{2}} 
\leq \left\{ \sum_{j \leq -\delta N} \left( 2^{-\frac{1}{2}j} \frac{\epsilon_0^2}{2N} \left\| \tilde{\phi}_j \mathcal{F} [F_1] \right\|_{L^2_{2\pi^{-1}}} \right)^{\frac{1}{\sigma}} \right\}^{\frac{\sigma}{2}}
\]
\[
+ \left\{ \sum_{-\delta N \leq j \leq 0} \left( 2^{-\frac{1}{2}j} \frac{\epsilon_0^2}{2N} \left\| \tilde{\phi}_j \mathcal{F} [F_1] \right\|_{L^2_{2\pi^{-1}}} \right)^{\frac{1}{\sigma}} \right\}^{\frac{\sigma}{2}}
\]
\[
\leq C \epsilon_0^2 \frac{R^2}{N^{\frac{1}{2}}} \left\{ \sum_{j \leq -\delta N} \left( 2^{(n-1)j} \sum_{-\delta N \leq k \leq 0} 2^{(-n+1)k} \right)^{\frac{\sigma}{2}} \right\}^{\frac{1}{\sigma}}
\]
\[
+ C \epsilon_0^2 \frac{R^2}{N^{\frac{1}{2}}} \left\{ \sum_{-\delta N \leq j \leq 0} \left( 2^{(n-1)j} \sum_{j-3 \leq k \leq 0} 2^{(-n+1)k} \right)^{\frac{\sigma}{2}} \right\}^{\frac{1}{\sigma}}
\]
\[
\leq C \epsilon_0^2 \frac{R^2}{N^{\frac{1}{2}}} (1 + N^{\frac{1}{\sigma}}),
\]
while the other terms involving \( F_2 + F_3 \) in the right hand side of (2.30) can be estimated by (2.24) and (2.26) which are lower order terms when \( N \to \infty \). Besides all the other terms involving the derivatives \( f_N \partial_{x_n} g_N, \cdots, f_N \partial_{x_n} g_N \) corresponding to (2.30) have an extra order \( 2^{-N} \). Hence we conclude that
\[
\left\| \mathcal{F}^{-1} \left[ \tilde{\phi}(\xi) \int_{\mathbb{R}^n} \frac{1}{B|\eta|^2 - c|\xi|^2} e^{-B|\eta|^2} - A l \right] \eta \tilde{f}_N(\xi - \eta) \tilde{g}_N(\eta) d\eta \right\|_{\dot{B}_{2^{-\frac{1}{2}}, \sigma}} \leq C \epsilon_0^2 N^{\frac{1}{2}} \cdot \frac{1}{2} R^2.
\]
Therefore, we obtained (2.27). \( \square \)

**Proof of Proposition 2.2.** We firstly prove the estimate (2.5) and we only treat \( f_N \) since \( g_N \) can be treated analogously. Since the support of \( \tilde{f}_N \) satisfies
\[
\text{supp} \tilde{f}_N \subset \{ \xi \in \mathbb{R}^n ; 2^N - 2 \leq |\xi| \leq 2^N + 2 \},
\]
we have
\[
\|f_N\|_{\dot{B}_{2^{-\frac{1}{2}}, \sigma}} \leq C 2^{-\frac{1}{2}N} \|f_N\|_{L^{2n}}.
\]
(2.33)

By \( |\cos(2^N x_1)| \leq 1 \),
\[
2^{-\frac{1}{2}N} \|f_N\|_{L^{2n}} \leq \left( \int_{\mathbb{R}^n} \left| \sum_{-\delta N \leq j \leq 0} R N^{-\frac{1}{2n}} 2^{-(n+\frac{1}{2})j} \phi_j(x - 2^{[j]+2N} e_1) \right|^{2n} dx \right)^{\frac{1}{2n}}.
\]
(2.34)

For \( j = j_1, j_2 \) with \( j_1 \neq j_2 \), the main area of the support of the function \( \phi_j(x - 2^{[j]+2N} e_1) = 2^{n-j} \phi_0 \left( 2^{j} (x - 2^{[j]+2N} e_1) \right) \) is the set \( A_j := \{ x \mid |x - 2^{[j]+2N} e_1| \leq 2^{-j} \} \), and \( A_{j_1} \) and \( A_{j_2} \) are away sufficiently if \( j_1 \neq j_2 \), therefore one can expect that the main part of \( |\sum_j 2^{-(n+\frac{1}{2})j} \phi_j|^{2n} \) is \( \sum_j |2^{-(n+\frac{1}{2})j} \phi_j|^{2n} \). For simplicity, let \( \Phi_j \) be defined by \( \Phi_j(x) := 2^{\frac{1}{2}j} \phi_0 \left( 2^{j} (x - 2^{[j]+2N} e_1) \right) \), and
we have
\[ \left| \sum_{-\delta N \leq j \leq 0} R_{\frac{2n}{N}} \frac{1}{2^{(n+\frac{1}{2})j}} \phi_j(x - 2^{[j+2N]}e_1) \right|^{2n} \leq \frac{R_{2n}^2 N}{N} \sum_{-\delta N \leq j \leq 0} |\Phi_j(x)|^{2n} + \frac{R_{2n}^2 N}{N} \sum_{(j_1, \cdots, j_{2n}) \in J} \left| \Phi_{j_1}(x) \cdots \Phi_{j_{2n}}(x) \Phi_{j_{n+1}}(x) \cdots \Phi_{j_{2n}}(x) \right|, \quad (2.35) \]
where \( J \) is defined by
\[ J := \{ (j_1, j_2, \cdots, j_{2n}) \mid \text{there exist } j_1, j_2 \text{ such that } j_1 \neq j_2, -\delta N \leq j \leq 0 \ (1 \leq l \leq 2n) \}. \]

On the integral of the first term in the right hand side of (2.35), it holds that
\[ \int_{\mathbb{R}^n} \frac{R_{2n}^2 N}{N} \sum_{-\delta N \leq j \leq 0} |\Phi_j(x)|^{2n} dx = \frac{R_{2n}^2 N}{N} \sum_{-\delta N \leq j \leq 0} \int_{\mathbb{R}^n} |2^{\frac{j}{2}} \phi_j(2^{j}x)|^{2n} dx = C \frac{R_{2n}^2 N}{N} \cdot N = C R_{2n}^2. \quad (2.36) \]

On the integral of the second term in the right hand side of (2.35), we utilize the following estimate: Let \( \alpha \) be a positive constant satisfying \( \alpha > n \), it follows from \( j_1 \neq j_2 \) and \( |\Phi_j(x)| \leq C 2^{\frac{j}{2}} (1 + 2^{[j+2N]}e_1)^{-\alpha} \) that
\[ \int_{\mathbb{R}^n} |\Phi_{j_1}(x)\Phi_{j_2}(x)\Phi_{j_3}(x) \cdots \Phi_{j_{2n}}(x)| dx \leq C 2^{\frac{j}{2}} (j_1 + j_2 + j_3 + \cdots + j_{2n}). \]

By the above estimate,
\[ \frac{R_{2n}^2 N}{N} \sum_{(j_1, \cdots, j_{2n}) \in J} \int_{\mathbb{R}^n} |\Phi_{j_1}(x)\Phi_{j_2}(x)\Phi_{j_3}(x) \cdots \Phi_{j_{2n}}(x)| dx \leq C \frac{R_{2n}^2 N}{N}. \quad (2.37) \]

Therefore, we obtain (2.5) by (2.33), (2.34), (2.36) and (2.37).

We secondly show the crucial estimate (2.6). By the equality (2.7) and the triangle inequality, it holds that
\[ \left\| I[f_N, g_N](T_N) \right\|_{B_{2n, \sigma}^{-\frac{1}{2}}} \geq \left\| \mathcal{F}^{-1} \left[ \mathcal{B} \int_{\mathbb{R}^n} \frac{1 - e^{-A|\xi| - \eta|2^2} - c|\xi|^2 e^{-B|\eta|2^2} i\eta f_N(\xi - \eta) \tilde{g}_N(\eta) d\eta} B_{2n, \sigma}^{-\frac{1}{2}} \right] \right\|_{B_{2n, \sigma}^{-\frac{1}{2}}}, \quad (2.38) \]

In order to estimate the first term in the right hand side of (2.38), we apply Lemma 2.5 (2.27) and then obtain that
\[ \left\| \mathcal{F}^{-1} \left[ \mathcal{B} \int_{\mathbb{R}^n} \frac{1 - e^{-A|\xi| - \eta|2^2} - c|\xi|^2 e^{-B|\eta|2^2} i\eta f_N(\xi - \eta) \tilde{g}_N(\eta) d\eta} B_{2n, \sigma}^{-\frac{1}{2}} \right] \right\|_{B_{2n, \sigma}^{-\frac{1}{2}}} \geq c N^{\frac{1}{2} - \frac{1}{2}} R^2. \quad (2.39) \]

We next consider the estimate of the remainder term \( R[f_N, g_N] \) of (2.38). Taking advantage of the divergence form structure due to (2.8) analogously to (A.10) and (A.11), we have from the embedding \( B_{2n, \sigma}^{-\frac{1}{2}}(\mathbb{R}^n) \hookrightarrow B_{2n, 2\sigma}^{-\frac{1}{2}}(\mathbb{R}^n) \) and (2.5)
\[ \left\| R[f_N, g_N] \right\|_{B_{2n, \sigma}^{-\frac{1}{2}}} \leq C \left\| f_N \right\|_{B_{2n, 2\sigma}^{-\frac{1}{2}}} \left\| g_N \right\|_{B_{2n, 2\sigma}^{-\frac{1}{2}}} \leq C \left\| f_N \right\|_{B_{2n, \sigma}^{-\frac{1}{2}}} \left\| g_N \right\|_{B_{2n, \sigma}^{-\frac{1}{2}}} \leq C R^2. \quad (2.40) \]

Therefore, we obtain (2.6) by (2.38), (2.39) and (2.40), and the proof of Proposition 2.2 is completed.

Before proving Proposition 2.1, we recall a Fourier multiplier theorem.
Lemma 2.6. ([5]) Let \( m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C} \) be a smooth function such that
\[
|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \leq C_\alpha (|\xi| + |\eta|)^{-|\alpha|}
\]
for all multi-index \( \alpha \) and some positive constant \( C_\alpha \) depending on \( \alpha \). Suppose that \( 1 < p, p_1, p_2 < \infty \) and \( 1/p = 1/p_1 + 1/p_2 \). Then
\[
\left\| F^{-1} \left[ \int_{\mathbb{R}^n} m(\xi - \eta, \eta) \widehat{h}_1(\xi - \eta) \widehat{h}_2(\eta) d\eta \right] \right\|_{L^p} \leq C \| h_1 \|_{L^{p_1}} \| h_2 \|_{L^{p_2}}.
\]

Proof of Lemma 2.6. This is a multiplier Fourier multiplier theorem for bilinear case. We refer to a paper by Coifman-Meyer [5] for its proof.

Proof of Proposition 2.1. The estimate of the initial data \( \| u_0 \|_{B^{1/2}_{2n, \sigma}} \leq CR \) is obtained by (2.5), and it suffices to show (2.3).

We introduce an approximation of \( u_0 \) so that
\[
P_\pi u_0 \simeq u_{0, \pi} := (f_N, 0, 0), \quad P_\sigma u_0 \simeq u_{0, \sigma} := (0, g_N, 0),
\]
where \( f_N \) and \( g_N \) are defined by (2.2). We can see that
\[
\| P_\sigma u_0 - u_{0, \sigma} \|_{L^n} + \| P_\pi u_0 - u_{0, \pi} \|_{L^n} \leq CR 2^{(-\frac{1}{2} + \frac{1}{2} \frac{\lambda}{n})} N, \tag{2.41}
\]
\[
\| \nabla (P_\sigma u_0 - u_{0, \sigma}) \|_{L^n} + \| \nabla (P_\pi u_0 - u_{0, \pi}) \|_{L^n} \leq CR 2^{(\frac{1}{2} + \frac{1}{2} \frac{\lambda}{n})} N, \tag{2.42}
\]
so that leading terms of \( P_\sigma u_0, P_\pi u_0 \) can be \( u_{0, \sigma}, u_{0, \pi} \), respectively. We also use the following estimates:
\[
\| \nabla P_\sigma u_0 \|_{L^{2n}} \leq CR 2^{\frac{3}{2} N}, \quad \| u_{0, \pi} \|_{L^{2n}} \leq CR 2^{\frac{1}{2} N}, \tag{2.43}
\]
which are obtained by (2.5) and \( \text{supp} \widehat{u}_0 \subset \{ \xi \in \mathbb{R}^n | 2^{N-1} \leq |\xi| \leq 2^{N+1} \} \). We write
\[
I[u_0] = - \int_0^t e^{(t-s)\mathcal{L}} \left\{ (e^{s(\lambda+2\mu) \Delta} u_{0, \pi} \cdot \nabla) e^{s\mu \Delta} u_{0, \sigma} \right. \\
- \frac{\mu}{\lambda + 2\mu} \left( (e^{s(\lambda+2\mu) \Delta} u_{0, \pi} - u_{0, \pi}) \cdot \nabla \right) e^{s\mu \Delta} u_{0, \sigma} \} ds
\]
\[
- \int_0^t e^{(t-s)\mathcal{L}} \left\{ (e^{s(\lambda+2\mu) \Delta} P_\pi u_0 \cdot \nabla) e^{s\mu \Delta} P_\sigma u_0 - (e^{s(\lambda+2\mu) \Delta} u_{0, \pi} \cdot \nabla) e^{s\mu \Delta} u_{0, \sigma} \right\} ds
\]
\[
- \int_0^t e^{(t-s)\mathcal{L}} \frac{\mu}{\lambda + 2\mu} \left\{ \left( (e^{s(\lambda+2\mu) \Delta} P_\pi u_0 - P_\pi u_0) \cdot \nabla \right) e^{s\mu \Delta} P_\sigma u_0 \\
- \left( (e^{s(\lambda+2\mu) \Delta} u_{0, \pi} - u_{0, \pi}) \cdot \nabla \right) e^{s\mu \Delta} u_{0, \sigma} \right\} ds
\]
\[
=: K_1 + K_2 + K_3.
\]
By taking \( A = \lambda + 2\mu, B = \mu, \) and \( c = \lambda + 2\mu, \mu, \) it follows from (2.6) in Proposition 2.2 that for \( t = 2^{-2N} \)
\[
\| K_1 \|_{B^{1/2}_{2n, \sigma}} \geq c N^{-\frac{1}{2} - \frac{1}{n}} R^2. \tag{2.44}
\]
On the estimate of $K_2$, we have from the embedding $\hat{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^n) \hookrightarrow \hat{B}^{\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$, the linear estimate of $e^{(t-s)L}$ from $L^{\frac{2n}{n+1}}(\mathbb{R}^n)$ to $\hat{B}^{\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$, the Hölder inequality, (2.41), (2.42) and (2.43)

$$\|K_2\|_{\hat{B}^{\frac{1}{2}}_{2n,1}} \leq C \|K_2\|_{\hat{B}^{\frac{1}{2}}_{2n,1}},$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2}} \left\{ \left\| (e^{s(\lambda+2\mu)}\Delta(P_{\pi}u_0) - u_0) \cdot \nabla \right\|_{L^{\frac{2n}{n+1}}} + \left\| (e^{s(\lambda+2\mu)}\Delta u_0) \cdot \nabla \right\|_{L^{\frac{2n}{n+1}}} \right\} ds$$

$$\leq C t^{-\frac{3}{2}} \left\{ \left\| P_{\pi}u_0 - u_0 \right\|_{L^n} \left\| \nabla P_{\pi}u_0 \right\|_{L^{2n}} + \left\| u_0 \right\|_{L^{2n}} \right\}$$

$$\leq CR^2 (2^{-N}) \left( 2^{N-\frac{1}{2}} + 2^{\frac{N}{2}} \right) N$$

The estimate for $K_3$ follows analogously to $K_2$. Therefore we obtain by (2.44)

$$\|I[u_0](t)\|_{\hat{B}^{\frac{1}{2}}_{2n,\sigma}} \geq cN^{-\frac{1}{2} - \frac{1}{n}} R^2 \quad \text{for} \quad t = 2^{-2N}, \tag{2.45}$$

for sufficiently large $N$.

At the end of this section, we consider the difference of $I[u_0]$ and the sum of the second iterate for the convection term and the quasi-linear term to show that they are small.

**Lemma 2.7.** Let $u_0$ be defined by (2.1). If $t \leq 2^{-2N}$, then

$$\left\| I[u_0] - \left\{ - \int_0^t e^{(t-s)L} (e^{sL}u_0 \cdot \nabla) e^{sL}u_0 ds + \int_0^t e^{(t-s)L} \left( \int_0^s \nabla e^{sL}u_0 ds \right) L e^{sL}u_0 ds \right\} \right\|_{\hat{B}^{\frac{1}{2}}_{2n,1}} \leq CR^2, \tag{2.46}$$

where $I[u_0]$ is defined by (1.20).

**Proof of Lemma 2.7.** By the equalities from (1.8) to (1.9), the terms in the left hand side of (2.46) is decomposed as follows:

$$I[u_0] = \left\{ - \int_0^t e^{(t-s)L} (e^{sL}u_0 \cdot \nabla) e^{sL}u_0 ds + \int_0^t e^{(t-s)L} \left( \int_0^s \nabla e^{sL}u_0 ds \right) L e^{sL}u_0 ds \right\}$$

$$= \widetilde{K}_1 + \widetilde{K}_2$$

where

$$\widetilde{K}_1 := \int_0^t e^{(t-s)L} \left\{ \nabla \cdot (e^{sL}u_0 \otimes e^{sL}u_0) + \frac{1}{2} \nabla e^{s(\lambda+2\mu)\Delta P_{\pi}u_0} \right\} ds$$

$$- \int_0^t e^{(t-s)L} \nabla \cdot \left\{ \left( \frac{-\Delta}{2\mu + \lambda} \text{div} \left( e^{s(\lambda+2\mu)\Delta P_{\pi}u_0} \right) \right) \right. \nabla \left( e^{s(\lambda+2\mu)\Delta P_{\pi}u_0} \right) ds,$$

$$\widetilde{K}_2 := \int_0^t e^{(t-s)L} \left( e^{s(\lambda+2\mu)\Delta P_{\pi}u_0} - P_{\pi}u_0 \right) \cdot \nabla e^{s(\lambda+2\mu)\Delta P_{\pi}u_0} ds.$$
We apply the embedding $\dot{B}^0_{n,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$, the boundedness of $e^{(t-s)\Delta}L$ from $\dot{B}^{-1}_{n,\infty}(\mathbb{R}^n)$ to $\dot{B}^0_{n,1}(\mathbb{R}^n)$, the embedding $L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{n,\infty}(\mathbb{R}^n)$, and the Hölder inequality to $\tilde{K}_1$ to obtain

$$\|\tilde{K}_1\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \leq C \int_0^t (t-s)^{-\frac{1}{2}} \left\| \nabla \cdot (e^{s\mu \Delta}P_\sigma u_0 \otimes e^{s\xi}u_0) + \frac{1}{2} \nabla |e^{s(\lambda+2\mu)\Delta}P_\pi u_0|^2 \right\|_{\dot{B}^{-\frac{1}{2}}_{n,\infty}} ds
$$

$$+ C \int_0^t (t-s)^{-\frac{1}{2}} \left\| \nabla \cdot \left\{ \left( -\Delta \right)^{-1} \text{div} \left( \frac{(\nabla e^{s(\lambda+2\mu)\Delta}P_\pi u_0 - \nabla e^{s(\lambda+2\mu)\Delta}P_\pi u_0)}{2\mu + \lambda} \right) \right\} \times \nabla (\mu e^{s\mu \Delta}P_\pi u_0 + (\lambda + 2\mu)e^{s(\lambda+2\mu)\Delta}P_\pi u_0) \right\|_{\dot{B}^{-\frac{1}{2}}_{n,\infty}} ds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2}} \left\{ \|e^{s\mu \Delta}P_\sigma u_0 \otimes e^{s\xi}u_0\|_{L^n} + \|e^{s(\lambda+2\mu)\Delta}P_\pi u_0\|_{L^n}^2 \right\} ds$$

$$+ C \int_0^t (t-s)^{-\frac{1}{2}} \left\{ \left\| \left( -\Delta \right)^{-1} \text{div} \left( e^{s(\lambda+2\mu)\Delta}P_\pi u_0 - e^{s(\lambda+2\mu)\Delta}P_\pi u_0 \right) \right\| \times \nabla (\mu e^{s\mu \Delta}P_\pi u_0 + (\lambda + 2\mu)e^{s(\lambda+2\mu)\Delta}P_\pi u_0) \right\|_{L^n} ds$$

$$\leq Ct^\frac{1}{2} \|u_0\|_{L^{2n}} + Ct^\frac{1}{2} \left\| \left( -\Delta \right)^{-1} \text{div} u_0 \right\|_{L^{2n}} \|\nabla u_0\|_{L^{2n}}$$

$$\leq C(2^{-2N})^\frac{1}{2} R^{2N}$$

$$= CR^2,$$

where the estimate $\|\nabla^\alpha u_0\|_{L^{2n}} \leq CR^2(\alpha+\frac{1}{2})N$ ($\alpha = -1, 0, 1$) is obtained by (2.5) and supp $\widehat{u_0} \subset \{ \xi \in \mathbb{R}^n \mid 2^{N-1} \leq |\xi| \leq 2^{N+1} \}$.

We consider the estimate of $\tilde{K}_2$. Although the term $(P_\pi u_0 \cdot \nabla)e^{s(\lambda+2\mu)\Delta}P_\pi u_0$ can not be written by the divergence form directly, it is possible to show the smallness of $(P_\pi u_0 \cdot \nabla)e^{s(\lambda+2\mu)\Delta}P_\pi u_0$ by means of

$$(e^{s(\lambda+2\mu)\Delta}P_\pi u_0 \cdot \nabla)e^{s(\lambda+2\mu)\Delta}P_\pi u_0 = \frac{1}{2} \nabla |e^{s(\lambda+2\mu)\Delta}P_\pi u_0|^2.$$

In fact, we have

$$\tilde{K}_2 = -\int_0^t (t-s)^{-\frac{1}{2}} e^{s\xi} \nabla (e^{s(\lambda+2\mu)\Delta}P_\pi u_0) ds$$

$$+ \int_0^t (t-s)^{-\frac{1}{2}} \left\{ (P_\pi u_0 \cdot \nabla)e^{s(\lambda+2\mu)\Delta}P_\pi u_0 - (e^{\frac{1}{2}s(\lambda+2\mu)\Delta}P_\pi u_0 \cdot \nabla)e^{\frac{1}{2}s(\lambda+2\mu)\Delta}P_\pi u_0 \right\} ds$$

$$+ \int_0^t (t-s)^{-\frac{1}{2}} \frac{1}{2} \nabla |e^{\frac{1}{2}s(\lambda+2\mu)\Delta}P_\pi u_0|^2 ds$$

$$=: \tilde{K}_{21} + \tilde{K}_{22} + \tilde{K}_{23}.$$

Since $\tilde{K}_{21}$ and $\tilde{K}_{23}$ have the divergence form structure, it follows from the same method as (2.47) that

$$\|\tilde{K}_{21}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} + \|\tilde{K}_{23}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \leq CR^2.$$

On the estimate of $\tilde{K}_{22}$, we consider the following Fourier transform and have from $e^{s(\lambda+2\mu)\Delta} = e^{\frac{1}{2}s(\lambda+2\mu)\Delta}e^{\frac{1}{2}s(\lambda+2\mu)\Delta}$

$$\mathcal{F} \left[ (P_\pi u_0 \cdot \nabla)e^{s(\lambda+2\mu)\Delta}P_\pi u_0 - (e^{\frac{1}{2}s(\lambda+2\mu)\Delta}P_\pi u_0 \cdot \nabla)e^{\frac{1}{2}s(\lambda+2\mu)\Delta}P_\pi u_0 \right] (\xi)$$

$$= \int_{\mathbb{R}^n} \left( e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} - e^{-\frac{1}{2}s(2\mu+\lambda)|\xi-\eta|^2} \right) \left( \widehat{P_\pi u_0}(\xi-\eta) \cdot i\eta \right) e^{\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \widehat{P_\pi u_0}(\eta) d\eta.$$

$$= 2^{\frac{1}{2}} \int_{\mathbb{R}^n} \left( e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} - e^{-\frac{1}{2}s(2\mu+\lambda)|\xi-\eta|^2} \right) \left( \widehat{P_\pi u_0}(\xi-\eta) \cdot i\eta \right) e^{\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \widehat{P_\pi u_0}(\eta) d\eta.$$
On the term $e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} - e^{-\frac{1}{2}s(2\mu+\lambda)|\xi-\eta|^2}$, it holds that
\[
e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} - e^{-\frac{1}{2}s(2\mu+\lambda)|\xi-\eta|^2} = \int_0^1 \partial_\theta e^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} d\theta \\
= (2\mu + \lambda) \xi \cdot \int_0^1 (\eta - \theta\xi) se^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} d\theta.
\]
Therefore (2.49) can be regarded as the term with the divergence form structure and we have from the embedding $\dot{B}^0_{n,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\frac{3}{2}}_{2n,1}(\mathbb{R}^n)$, the linear estimate of $e^{(t-s)L}$ from $\dot{B}^{-1}_{n,\infty}(\mathbb{R}^n)$ to $\dot{B}^0_{n,1}(\mathbb{R}^n)$, (2.49), the above equality,
\[
\|\bar{K}_{22}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \leq \|\bar{K}_{22}\|_{\dot{B}^0_{n,1}} \\
\leq C \int_0^t (t-s)^{-\frac{1}{2}} \left\| \nabla \cdot F^{-1} \left[ \int_{\mathbb{R}^n} (\eta - \theta\xi) se^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} d\theta \\
\times \left( \tilde{P}_\pi u_0(\xi - \eta) \cdot i\eta \right) e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \tilde{P}_\pi u_0(\eta) d\eta \right] \right\|_{\dot{B}^{-1}_{n,\infty}} ds \\
\leq C \int_0^1 \int_0^t (t-s)^{-\frac{1}{2}} \left\| F^{-1} \left[ \int_{\mathbb{R}^n} (\eta - \theta\xi) se^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} \\
\times \left( \tilde{P}_\pi u_0(\xi - \eta) \cdot i\eta \right) e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \tilde{P}_\pi u_0(\eta) d\eta \right] \right\|_{L^n} ds d\theta.
\]
Here let us consider the Fourier multiplier
\[
m(\xi - \eta, \eta) = \frac{(\eta - \theta\xi)}{2N} se^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} i\eta e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \tilde{\Phi}_N(\xi - \eta) \tilde{\Phi}_N(\eta),
\]
where $\Phi_N := \Phi_{n+1} + \Phi_N + \Phi_{-N+1}$. Noting that $P_\pi u_0 = \Phi_N * P_\pi u_0$, we apply Lemma 2.6 and then have that
\[
\left\| F^{-1} \left[ \int_{\mathbb{R}^n} (\eta - \theta\xi) se^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} \left( \tilde{P}_\pi u_0(\xi - \eta) \cdot i\eta \right) e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \tilde{P}_\pi u_0(\eta) d\eta \right] \right\|_{L^n} \\
\leq C 2^{2N} \| P_\pi u_0 \|_{L^{2n}}^2,
\]
which proves that
\[
\|\bar{K}_{22}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \leq C t^2 2^{2N} \| P_\pi u_0 \|_{L^{2n}}^2 \leq CR^2,
\]
where the estimate $\| u_0 \|_{L^{2n}} \leq CR^{\frac{1}{2}N}$ is obtained by (2.5) and $\text{supp} \hat{u}_0 \subset \{ \xi \in \mathbb{R}^n | 2^{N-1} \leq |\xi| \leq 2^{N+1} \}$. Then it follows from (2.48) and the above inequality that
\[
\|\bar{K}_{22}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \leq CR^2.
\]
Therefore we obtain (2.46). \qed

3. PROOF OF THEOREM 1.1

Let the initial data $(\rho_0, u_0)$ be defined by
\[
\rho_0(x) = 0, \quad u_0(x) = (f_N(x), g_N(x), 0, \cdots, 0), \quad (3.1)
\]
where $f_N$ and $g_N$ are defined by (2.2). To prove the ill-posedness, we first prove the existence of a local solution of the compressible Navier-Stokes equations for the initial data (3.1).

**Proposition 3.1** ([14]). Let $1 < p \leq n$. For $\rho_0 \in \dot{B}^\frac{n}{p,1}_p(\mathbb{R}^n)$ and $u_0 \in \dot{B}^{\frac{p}{p}}_{p,1}(\mathbb{R}^n)$ with
\[
\|\rho_0\|_{\dot{B}^\frac{n}{p,1}_p} \leq \eta,
\]

\[
\|u_0\|_{\dot{B}^{\frac{p}{p}}_{p,1}} \leq \eta,
\]

...
for certain $\eta > 0$, there exists $T > 0$ such that the equations (1.1) have a unique local-in-time solution

$$\rho \in C([0,T]; \dot{B}^\frac{n}{p}_p(\mathbb{R}^n)), \quad u \in C([0,T]; \dot{B}^{-\frac{n}{p}+1}_p(\mathbb{R}^n)) \cap L^1(0,T; \dot{B}^{\frac{n}{p}+1}_p(\mathbb{R}^n)). \quad (3.2)$$

We refer to the paper by Danchin [14] for the proof.

Since our initial data $\rho_0 = 0$ and $u_0 \in S(\mathbb{R}^n)$ satisfy the assumption of Proposition 3.1, the solution to (1.1) exists on $[0,T]$ by Proposition 3.1. Here we need to show $T \geq T_N \equiv \varepsilon_0 2^{-2N}$ with $\varepsilon_0 < 1$ (cf. Proposition 2.1). To this end, we show the following proposition.

**Proposition 3.2.** Let the initial data $u_0$ be defined by (3.1). For some small $\delta < 1/n$, there exists $N_0$ such that for every $N \geq N_0$ there exists a local solution $(\rho, u)$ of (1.2) such that

$$\rho \in C([0,T]; \dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1}) \cap L^\infty([0,T], \dot{B}^1_{2,1} \cap \dot{B}^1_{\infty,1}),$$

$$u \in C([0,T]; \dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1}) \cap L^1(0,T; \dot{B}^2_{2,1} \cap \dot{B}^2_{\infty,1}), \quad (3.3)$$

where $T = 2^{-2N}$, and they satisfy the following:

$$\sup_{t \in [0,T]} \|\rho(t)\|_{\dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1}} \leq C 2^\left(-\frac{1}{2} + \frac{\delta}{n}\right) N, \quad (3.4)$$

$$\sup_{t \in [0,T]} \|\rho(t)\|_{\dot{B}^1_{2,1} \cap \dot{B}^1_{\infty,1}} \leq C 2^\left(\frac{1}{2} + \frac{\delta}{n}\right) N, \quad (3.5)$$

$$\sup_{t \in [0,T]} \|u(t)\|_{\dot{B}^2_{2,1} \cap \dot{B}^0_{\infty,1}} + \|u\|_{L^1(0,T; \dot{B}^2_{2,1} \cap \dot{B}^2_{\infty,1})} \leq C 2^\left(\frac{1}{2} + \frac{\delta}{n}\right) N. \quad (3.6)$$

For the proof of Proposition 3.2, we will use maximal regularity estimates to control the $L^\infty$-norm of density and the quasi-linear term.

**Lemma 3.3 ([27]).** Assume that $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$ and $f \in L^1(0,\infty; \dot{B}^0_{p,1}(\mathbb{R}^n))$ with $1 \leq p \leq \infty$ and $u$ satisfies

$$\partial_t u - \mathcal{L} u = f, \quad u(0) = u_0.$$ 

Then there exists a positive number $C$ independent of $u_0, f$ such that

$$\|\partial_t u\|_{L^1(0,\infty; \dot{B}^0_{p,1})} + \|\mathcal{L} u\|_{L^1(0,\infty; \dot{B}^0_{p,1})} \leq C \|u_0\|_{\dot{B}^0_{p,1}} + C \|f\|_{L^1(0,\infty; \dot{B}^0_{p,1})}.$$ 

We refer to several papers [10, 11, 14, 27] for maximal regularity and its application to compressible Navier-Stokes equations.

We also use the bilinear estimates below.

**Lemma 3.4.** Let $\dot{B}^\alpha_{[2,\infty],1} := \dot{B}^\alpha_{2,1}(\mathbb{R}^n) \cap \dot{B}^\alpha_{\infty,1}(\mathbb{R}^n)$. For $\alpha$ and $\varepsilon$ with $\alpha > 0$ and $0 < \varepsilon < 1/8$, we have

$$\|fg\|_{\dot{B}^\alpha_{[2,\infty],1}} \leq C \min \{\|f\|_{\dot{B}^\alpha_{[2,\infty],1}} \|g\|_{\dot{B}^\alpha_{[2,\infty],1}}, \|f\|_{\dot{B}^\alpha_{[2,\infty],1}} \|g\|_{\dot{B}^\alpha_{\infty,1} \cap \dot{B}^\varepsilon_{[2,\infty],1}}\}, \quad (3.7)$$

$$\|fg\|_{\dot{B}^\alpha_{[2,\infty],1}} \leq C \{\|f\|_{\dot{B}^\alpha_{[2,\infty],1}} \|g\|_{\dot{B}^\alpha_{[2,\infty],1}}, \|f\|_{\dot{B}^\alpha_{[2,\infty],1}} \|g\|_{\dot{B}^\alpha_{\infty,1}} + \|f\|_{\dot{B}^\alpha_{[2,\infty],1}} \|g\|_{\dot{B}^\alpha_{[2,\infty],1}}\}. \quad (3.8)$$

The proof will be given in Appendix C. We note that $\dot{B}^0_{[2,\infty],1} \cap \dot{B}^\alpha_{[2,\infty],1}$ is a Banach algebra.

**Proof of Proposition 3.2.** Since $u_0 \in S(\mathbb{R}^n)$, we know from the existing result Proposition 3.1 that there exist $T_0 > 0$ and a unique local solution such that

$$\rho \in C([0,T_0]; \dot{B}^\frac{n}{p}_p(\mathbb{R}^n)), \quad u \in C([0,T_0]; \dot{B}^{-\frac{n}{p}+1}_p(\mathbb{R}^n)) \cap L^1(0,T_0; \dot{B}^{\frac{n}{p}+1}_p(\mathbb{R}^n)), \quad (3.2)$$

where $1 < p \leq n$. We notice that the solution may also be in our solution space (3.3) since the data is in the Schwartz class. More precisely, following Proposition 3.1, one can construct solution
\((\rho, u)\) and can deduce that it belongs to not only the scaling critical class but also smoother class (3.2)
\[
\rho \in C([0, T_0], \dot{B}_{2,1}^0 \cap \dot{B}_{\infty,1}^0) \cap L^\infty([0, T_0], \dot{B}_{2,1}^1 \cap \dot{B}_{\infty,1}^1),
\]
\[
u \in C([0, T_0], \dot{B}_{2,1}^0 \cap \dot{B}_{\infty,1}^0) \cap L^1(0, T_0; \dot{B}_{2,1}^2 \cap \dot{B}_{\infty,1}^2).
\]
We also note that \(T_0\) can be taken as in the case when \(p = n\) in Proposition 3.1, since \(\dot{B}_{2,1}^0 \cap \dot{B}_{\infty,1}^0 \hookrightarrow \dot{B}_{n,1}^0\) and the existence time can be chosen from the argument in the scaling critical case. To obtain the solution on the time interval \([0, 2^{-N}]\), we will derive a priori estimates (3.4), (3.5) and (3.6).

The following estimate is obtained by Danchin [14] and we utilized it for our purpose. Let \(\alpha = 0, 1\). It holds that
\[
\|\rho(t)\|_{\dot{B}_{2,1}^\alpha} \leq e^{C_\alpha \int_0^t \|\nabla u(\mu\rho)\|_{\dot{B}_{\infty,1}^\alpha} d\mu} \left\{ \|\rho_0\|_{\dot{B}_{2,1}^\alpha} + \int_0^t e^{-C_\alpha \int_0^s \|\nabla u(\mu)\|_{\dot{B}_{\infty,1}^\alpha} d\mu} \|(\rho + 1) \text{div} u\|_{\dot{B}_{2,1}^\alpha} ds \right\}.
\]
We apply the bilinear estimates (3.7) and (3.8) in Lemma 3.4 to the second term of the right hand side of (3.9) to obtain
\[
\|\rho(t)\|_{\dot{B}_{2,1}^\alpha} \leq C \int_0^t e^{C_\alpha \int_0^s \|\nabla u(\mu)\|_{\dot{B}_{\infty,1}^\alpha} d\mu} \left\{ \|\rho\|_{\dot{B}_{2,1}^\alpha} + \|u\|_{\dot{B}_{2,1}^\alpha} \right\} ds,
\]
\[
\|\rho(t)\|_{\dot{B}_{2,1}^\alpha} \leq C \int_0^t e^{C_\alpha \int_0^s \|\nabla u(\mu)\|_{\dot{B}_{\infty,1}^\alpha} d\mu} \left\{ \|\rho\|_{\dot{B}_{2,1}^\alpha} + \|u\|_{\dot{B}_{2,1}^\alpha} + (1 + \|\rho\|_{\dot{B}_{2,1}^\alpha}) \right\} ds.
\]
Then the Gronwall inequality (see Lemma B.1) implies
\[
\left\{ \begin{array}{ll}
\|\rho(t)\|_{\dot{B}_{2,1}^\alpha} \leq C \int_0^t e^{C_\alpha \int_0^s \|\nabla u(\mu)\|_{\dot{B}_{\infty,1}^\alpha} d\mu} \right\} ds,
\|\rho(t)\|_{\dot{B}_{2,1}^\alpha} \leq C \int_0^t e^{C_\alpha \int_0^s \|\nabla u(\mu)\|_{\dot{B}_{\infty,1}^\alpha} d\mu} (1 + \|\rho\|_{\dot{B}_{2,1}^\alpha}) \right\} ds.
\]
On the other hand, the velocity can be treated by applying Maximal regularity estimate in Lemma 3.3 and we have
\[
\|\partial_t e^{tL} u_0\|_{L^1(0, T; \dot{B}_{2,1}^0)} + \|\nabla^2 e^{tL} u_0\|_{L^1(0, T; \dot{B}_{2,1}^0)} \leq C(\|u_0\|_{\dot{B}_{2,1}^0} + \|u_0\|_{\dot{B}_{\infty,1}^0}) \leq C2^{1+\frac{n}{2}+\frac{3}{2}}.
\]
Note that the nonlinear term of the velocity equation is given by
\[
-(u \cdot \nabla) u - \frac{\nabla P(1 + \rho)}{1 + \rho} - \frac{\rho}{1 + \rho} \text{div} u.
\]
Hence it is sufficient to estimate the nonlinear terms in (3.11) by the norm in \(L^1(0, T; \dot{B}_{2,1}^0(\mathbb{R}^n))\).

Since the pressure given by the monomial \(P(1 + \rho) = (1 + \rho)^\gamma\) \((\gamma > 0)\) is analytic for small \(\rho\), we have by the Taylor expansion
\[
P(1 + \rho) = \sum_{m=0}^{\infty} a_m \rho^m
\]
with a bounded sequence \(\{a_m\}_{m=1}^{\infty}\) and
\[
\frac{1}{1 + \rho} = \sum_{m=0}^{\infty} (-\rho)^m.
\]
which give that
\[
\frac{\nabla P(1+\rho)}{1+\rho} = \nabla \sum_{m=0}^{\infty} \tilde{a}_m \rho^m = \nabla \sum_{m=1}^{\infty} \tilde{a}_m \rho^m,
\]
where \(\{\tilde{a}_m\}_{m=0}^\infty\) is also a bounded sequence. We then have that
\[
\left\| \frac{\nabla P(1+\rho)}{1+\rho} \right\|_{L^1(0,T;\dot{B}^0_{2,\infty},1)} \leq C \sum_{m=1}^{\infty} |\tilde{a}_m| \left\| \rho^m \right\|_{L^1(0,T;\dot{B}^0_{2,\infty},1)}
\leq C \left( \sum_{m=1}^{\infty} m|\tilde{a}_m| \left\| \rho \right\|_{L^\infty(0,T;\dot{B}^0_{2,\infty},1)}^{m-1} \right) \left\| \rho \right\|_{L^1(0,T;\dot{B}^1_{2,\infty},1)}
\leq CT \left( \sum_{m=1}^{\infty} m|\tilde{a}_m| \left\| \rho \right\|_{L^\infty(0,T;\dot{B}^0_{2,\infty},1)}^{m-1} \right) \left\| \rho \right\|_{L^\infty(0,T;\dot{B}^1_{2,\infty},1)}.
\]
On the quasi-linear term, applying the expansion
\[
\frac{\rho}{1+\rho} = \sum_{m=0}^{\infty} (-1)^m \rho^{m+1},
\]
and the bilinear estimate (3.7), we have
\[
\left\| \frac{\rho}{1+\rho} \mathcal{L} u \right\|_{L^1(0,T;\dot{B}^1_{2,\infty},1)} \leq C \int_0^T \left\| \frac{\rho}{1+\rho} \right\|_{L^\infty(0,T;\dot{B}^0_{2,\infty},1)} \left\| \mathcal{L} u \right\|_{\dot{B}^0_{2,\infty},1} \, ds
\leq C \left( \sum_{m=0}^{\infty} m \left\| \rho \right\|_{L^\infty(0,T;\dot{B}^0_{2,\infty},1)}^{m} \right) \left\| \rho \right\|_{L^\infty(0,T;\dot{B}^1_{2,\infty},1)} \left\| u \right\|_{L^1(0,T;\dot{B}^2_{2,\infty},1)}.
\]
Therefore we obtain that
\[
\left\| u \right\|_{L^\infty(0,T;\dot{B}^0_{2,\infty},1)} \left\| u \right\|_{L^1(0,T;\dot{B}^2_{2,\infty},1)}
\leq C 2^{(\frac{1}{2}+\frac{n}{4})N} + C (T^{\frac{1}{2}} + T^{\frac{1}{2}+\frac{n}{4}}) \left\| u \right\|_{L^\infty(0,T;\dot{B}^0_{2,\infty},1)}^2 \left\| u \right\|_{L^1(0,T;\dot{B}^2_{2,\infty},1)}
+ CT \left( \sum_{m=1}^{\infty} m |a_m| \left\| \rho \right\|_{L^\infty(0,T;\dot{B}^0_{2,\infty},1)}^{m-1} \right) \left\| \rho \right\|_{L^\infty(0,T;\dot{B}^1_{2,\infty},1)}
+ C \left( \sum_{m=0}^{\infty} m \left\| \rho \right\|_{L^\infty(0,T;\dot{B}^0_{2,\infty},1)}^{m} \right) \left\| u \right\|_{L^1(0,T;\dot{B}^2_{2,\infty},1)}.
\]
(3.12)

We then derive a priori estimates (3.4)-(3.6) for \( t \in [0,2^{-N}] \) by the continuity argument. Since our initial data \( \rho_0, u_0 \) satisfy
\[
\left\| \rho_0 \right\|_{\dot{B}^0_{2,\infty},1} = 0, \quad \sup_{t > 0} \left\| e^{t \mathcal{L}} u_0 \right\|_{\dot{B}^0_{2,\infty},1} + \left\| e^{t \mathcal{L}} u_0 \right\|_{L^1(0,\infty;\dot{B}^2_{2,\infty},1)} \leq 2C_0 2^{(\frac{1}{2}+\frac{n}{4})N},
\]
with some \( C_0 > 0 \), letting \( C_1 \) be the biggest constant among \( 2C_0 \) and those in (3.10) and (3.12), we find that for some \( T_1 > 0 \), the solution satisfy for \( t \in (0,T_1] \)
\[
\sup_{\tau \in [0,t]} \left\| \rho(\tau) \right\|_{\dot{B}^0_{2,\infty},1} \leq 2C_1 2^{(\frac{1}{2}+\frac{n}{4})N}, \quad (3.13)
\sup_{\tau \in [0,t]} \left\| \rho(\tau) \right\|_{\dot{B}^1_{2,\infty},1} \leq 2C_1 2^{(\frac{1}{2}+\frac{n}{4})N}, \quad (3.14)
\sup_{\tau \in [0,t]} \left\| u(\tau) \right\|_{L^1(0,t;\dot{B}^2_{2,\infty},1)} \leq 2C_1 2^{(\frac{1}{2}+\frac{n}{4})N}. \quad (3.15)
\]
If $T_1 \geq 2^{-2N}$ then the proof is done. If otherwise, let

$$T^* := \sup \left\{ t > 0 \mid (3.13), (3.14) \text{ and } (3.15) \text{ hold on the time interval } [0,t] \right\}$$

and assume that $T^* < 2^{-2N}$. Then by the continuity of obtained solution $(\rho(t), u(t))$ in the solution space (3.3), for any small $0 < \eta < 1$, there exists $0 < T_\varepsilon \ll 1$ with $T^* + T_\varepsilon < 2^{-2N}$ such that one of the following estimates hold: For $t \in (T^*, T^* + T_\varepsilon)$,

$$\sup_{\tau \in [0,t]} \|\rho(\tau)\|_{\dot{B}^{2}_{[2,\infty],1}} < 2(1 + \eta)C_12^{2(-\frac{1}{2} + \frac{n\delta}{2})^N}, \quad (3.16)$$

$$\sup_{\tau \in [0,t]} \|\rho(\tau)\|_{\dot{B}^1_{[2,\infty],1}} < 2(1 + \eta)C_1^22^{(\frac{1}{2} + \frac{n\delta}{2})^N}, \quad (3.17)$$

$$\sup_{\tau \in [0,t]} \|u(\tau)\|_{\dot{B}^{0}_{[2,\infty],1}} + \|u\|_{L^1(0,t;\dot{B}^2_{[2,\infty],1})} < (1 + \eta)C_12^{(\frac{1}{2} + \frac{n\delta}{2})^N}. \quad (3.18)$$

In either case, all the estimates (3.16)-(3.18) are valid, we then apply (3.16) and (3.18) to (3.10) to obtain that

$$\|\rho(t)\|_{\dot{B}^0_{[2,\infty],1}} \leq C_1 \exp \left( C_1 \left( t^{\frac{1}{2}} + t^{\frac{1}{2} + \frac{n\delta}{2}} \right) \|u\|_{L^\infty(0,t;\dot{B}^0_{[2,\infty],1})} \cap L^1(0,t;\dot{B}^2_{[2,\infty],1}) \right) \left( \frac{1}{2} \|\rho\|_{L^\infty(0,t;\dot{B}^0_{[2,\infty],1})} \cap L^1(0,t;\dot{B}^2_{[2,\infty],1}) \right)$$

$$\leq C_1 \exp \left( C_1(2^{-N} + 2^{-(1-\varepsilon)N}) \cdot (1 + \eta)C_12^{\frac{1}{2} + \frac{n\delta}{2})^N \right) \cdot 2^{-N} \cdot (1 + \eta)C_12^{\frac{1}{2} + \frac{n\delta}{2})^N$$

$$\leq (1 + \eta)C_1^2 \exp \left( (1 + \eta)C_1^22^{(-\frac{1}{2} + \frac{n\delta}{2})^N} \right) 2^{(-\frac{1}{2} + \frac{n\delta}{2})^N}$$

$$\leq 2C_1^22^{(-\frac{1}{2} + \frac{n\delta}{2})^N}$$

and

$$\|\rho(t)\|_{\dot{B}^1_{[2,\infty],1}} \leq C_1 \exp \left( C_1 T^{\frac{1}{2}} \|u\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1})} \cap L^1(0,T;\dot{B}^2_{[2,\infty],1}) \right) \left( 1 + \|\rho\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1})} \cap L^1(0,T;\dot{B}^2_{[2,\infty],1}) \right)$$

$$\times \|u\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1})} \cap L^1(0,T;\dot{B}^2_{[2,\infty],1})$$

$$\leq (1 + \eta)C_1^2 \exp \left( (1 + \eta)C_1^22^{\frac{1}{2} + \frac{n\delta}{2})^N} \right) \left( 1 + \|\rho\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1})} \right) 2^{\frac{1}{2} + \frac{n\delta}{2})^N$$

$$\leq 2C_1^22^{\frac{1}{2} + \frac{n\delta}{2})^N},$$
provided that $N$ is sufficiently large. And applying (3.16)-(3.18) to (3.12), we also obtain that

\[
\|u\|_{L^\infty(0,T;\hat{B}^0_{[2,\infty],1})} + L^1(0,T;\hat{B}^2_{[2,\infty],1}) \\
\leq C_0 2^{\frac{1}{2} + \frac{\alpha}{2}} N + C(T^\frac{1}{2} + T^{\frac{1}{4}}) \|u\|_{L^\infty(0,T;\hat{B}^0_{[2,\infty],1})} + L^1(0,T;\hat{B}^2_{[2,\infty],1}) \\
+ CT \left( \sum_{m=1}^{\infty} m |a_m| \|\rho\| \right)^{m-1} \|\rho\|_{L^\infty(0,T;\hat{B}^0_{[2,\infty],1})} \\
+ C \left( \sum_{m=0}^{\infty} m |a_m| \|\rho\| \right)^{m} \|\rho\|_{L^\infty(0,T;\hat{B}^0_{[2,\infty],1})} \\
\leq \frac{C_1}{2} 2^{\frac{1}{2} + \frac{\alpha}{2}} N + C(2^{-N} + 2^{-(1-\varepsilon)N}((1 + \eta)C_1 2^{\frac{1}{2} + \frac{\alpha}{2}} N)^2 \\
+ C2^{-2N} \left( \sum_{m=1}^{\infty} m |a_m| \right) (2(1 + \eta)C_1^2 2^{(-\frac{1}{2} + \frac{\alpha}{2}) N})^{m-1} \right) 2(1 + \eta)C_1^2 2^{\frac{1}{2} + \frac{\alpha}{2}} N \\
+ C \left( \sum_{m=0}^{\infty} m (2(1 + \eta)C_1^2 2^{(-\frac{1}{2} + \frac{\alpha}{2}) N})^{m} \right) \times 2(1 + \eta)C_1^2 2^{(-\frac{1}{2} + \frac{\alpha}{2}) N} \\
\leq C_1 2^{\frac{1}{2} + \frac{\alpha}{2}} N.
\]

Therefore we obtain the three bounds (3.13), (3.14) and (3.15), which contradict to the definition of $T^*$, and we conclude that $T^* \geq 2^{-2N}$. \hfill \Box

In the next three lemmas below, we consider several approximation. The first lemma concerns with the approximation of the solution by the linear part.

**Lemma 3.5.** Let $\varepsilon$ satisfy $0 < \varepsilon < 1/8$, and $T := 2^{-2N}$. Then the solution $(\rho, u)$ obtained in Proposition 3.2 satisfies the following:

\[
\sup_{t \in [0,T]} \left\| \rho(t) + \int_0^t \text{div} e^s \mathcal{L} u_0 ds \right\|_{\hat{B}^0_{2,1} \cap \hat{B}^0_{\infty,1}} \leq C2^{(1+\varepsilon+n\delta)N}, \tag{3.20}
\]

\[
\sup_{t \in [0,T]} \left\| u(t) - e^{t \mathcal{L}} u_0 \right\|_{\hat{B}^0_{2,1} \cap \hat{B}^0_{\infty,1}} + \left\| u - e^{t \mathcal{L}} u_0 \right\|_{L^1(0,T;\hat{B}^2_{[2,\infty],1})} \leq C2^{(\varepsilon+n\delta)N}, \tag{3.21}
\]

**Proof of Lemma 3.5.** We put $\hat{B}^0_{[2,\infty],1} := \hat{B}^0_{2,1} \cap \hat{B}^0_{\infty,1}$ for simplicity. To show (3.21), we apply a similar argument to (3.19) without the linear term and then obtain

\[
\sup_{t \in [0,T]} \left\| u(t) - e^{t \mathcal{L}} u_0 \right\|_{\hat{B}^0_{[2,\infty],1}} + \left\| u - e^{t \mathcal{L}} u_0 \right\|_{L^1(0,T;\hat{B}^2_{[2,\infty],1})} \\
\leq C \left\| \int_0^t e^{(t-s) \mathcal{L}} \left( (u \cdot \nabla) u + \frac{\nabla P(\rho)}{1+\rho} + \frac{\rho}{1+\rho} \mathcal{L} u \right) ds \right\|_{L^\infty(0,T;\hat{B}^0_{[2,\infty],1})} \\
\leq C2^{(\varepsilon+n\delta)N}.
\]

Then (3.21) is proved.

To prove (3.20), we use the following equality on the integral equation of $\rho$: \[\rho(t) + \int_0^t \text{div} e^{s \mathcal{L}} u_0 ds = \int_0^t \left( \text{div} (e^{s \mathcal{L}} u_0 - u) \right) ds - \int_0^t \text{div} (\rho u) ds.\]
It follows from the interpolation inequality, Hölder inequality, (3.8), (3.4), (3.5), (3.6) and (3.21) that

\[
\left\| \rho(t) + \int_0^t \text{div} e^{s \mathcal{L}} u_0 ds \right\|_{\dot{B}^0_{2, \infty}, 1} \\
\leq C \int_0^T \left\| u - e^{s \mathcal{L}} u_0 \right\|_{\dot{B}^0_{2, \infty}, 1}^{1/2} \left\| u - e^{s \mathcal{L}} u_0 \right\|_{\dot{B}^2_{2, \infty}, 1}^{1/2} ds \\
+ C \int_0^t \left( \| \rho \|_{\dot{B}^1_{2, \infty}} \| u \|_{\dot{B}^0_{2, \infty}, 1} + \| \rho \|_{\dot{B}^0_{2, \infty}, 1} \| u \|_{\dot{B}^1_{2, \infty}} \right) ds \\
\leq C(2^{-2N})^{1/2} 2^{(\varepsilon+n\delta)N} + C2^{-2N} \cdot 2^{(1/2 + n\delta)N} \cdot 2^{(1/2 + n\delta)N} + C(2^{-2N})^{1/2} \cdot 2^{(-1/2 + n\delta)N} \cdot 2^{(1/2 + n\delta)N} \\
\leq C2\left(-1 + \varepsilon + n\delta \right)N.
\]

Therefore (3.20) is obtained, and we complete the proof. \hfill \Box

In the next lemma, we approximate the nonlinear part of the solution by the second iterate of the convection term and the quasi linear term.

**Lemma 3.6.** Let \( \varepsilon \) be such that \( 0 < \varepsilon < 1/8 \) and let \( \rho_1 \) be defined by

\[
\rho_1(t) = -\int_0^t \text{div} e^{s \mathcal{L}} u_0 ds.
\]

Then the solution \( u \) obtained in Proposition 3.2 satisfies the following:

\[
\left\| \left( u(t) - e^{t \mathcal{L}} u_0 \right) - \left\{ - \int_0^t e^{(t-s) \mathcal{L}} (e^{s \mathcal{L}} u_0 \cdot \nabla) e^{s \mathcal{L}} u_0 ds - \int_0^t e^{(t-s) \mathcal{L}} \frac{\rho_1}{1 + \rho_1} \mathcal{L} e^{s \mathcal{L}} u_0 ds \right\} \right\|_{\dot{B}^0_{2, \infty}, 1} \\
\leq C2\left(-\frac{1}{2} + \frac{3\delta}{2} + 2\varepsilon \right)N \quad \text{for } t \in [0, 2^{-2N}].
\]  

(3.22)

**Proof of Lemma 3.6.** For simplicity, we put \( \dot{B}^0_{2, \infty}, 1 := \dot{B}^0_{2, 1} \cap \dot{B}^0_{\infty, 1} \). From the integral equation for the velocity \( u \), we have the following:

\[
\left\| \left( u(t) - e^{t \mathcal{L}} u_0 \right) - \left\{ - \int_0^t e^{(t-s) \mathcal{L}} (e^{s \mathcal{L}} u_0 \cdot \nabla) e^{s \mathcal{L}} u_0 ds - \int_0^t e^{(t-s) \mathcal{L}} \frac{\rho_1}{1 + \rho_1} \mathcal{L} e^{s \mathcal{L}} u_0 ds \right\} \right\|_{\dot{B}^0_{2, \infty}, 1} \\
\leq \left\| \int_0^t e^{(t-s) \mathcal{L}} (u \cdot \nabla) u_0 ds - \int_0^t e^{(t-s) \mathcal{L}} (e^{s \mathcal{L}} u_0 \cdot \nabla) e^{s \mathcal{L}} u_0 ds \right\|_{\dot{B}^0_{2, \infty}, 1} \\
+ \left\| \int_0^t e^{(t-s) \mathcal{L}} \frac{\rho \mathcal{P}(\rho)}{1 + \rho} ds \right\|_{\dot{B}^0_{2, \infty}, 1} \\
+ \left\| \int_0^t e^{(t-s) \mathcal{L}} \frac{\rho \mathcal{L} u_0 ds - \int_0^t e^{(t-s) \mathcal{L}} \frac{\rho_1}{1 + \rho_1} \mathcal{L} e^{s \mathcal{L}} u_0 ds \right\|_{\dot{B}^0_{2, \infty}, 1} \\
=: L_1 + L_2 + L_3,
\]

(3.23)

and we estimate \( L_1, L_2 \) and \( L_3 \) one by one.
For the estimate of $L_1$, we consider the difference of $u$ and $e^{s\mathcal{L}}u_0$, apply the bilinear estimate (3.7) to obtain
\[
L_1 \leq C \int_0^t \left( \| (u - e^{s\mathcal{L}}u_0) \cdot \nabla u \|_{\dot{B}^{0}_{2,\infty},1} + \| e^{s\mathcal{L}}u_0 \cdot \nabla (u - e^{s\mathcal{L}}u_0) \|_{\dot{B}^{0}_{2,\infty},1} \right) ds \\
\leq C \int_0^t \| u - e^{s\mathcal{L}}u_0 \|_{\dot{B}^{0}_{2,\infty},1} \| \nabla u \|_{\dot{B}^{0}_{2,\infty},1} ds \\
+ C \int_0^t \| e^{s\mathcal{L}}u_0 \|_{\dot{B}^{0}_{2,\infty},1} \| \nabla (u - e^{s\mathcal{L}}u_0) \|_{\dot{B}^{0}_{2,\infty},1} ds \\
=: L_{11} + L_{12}.
\]
It follows on $L_{11}$ from the Hölder inequality, $T = 2^{-2N}$, (3.6) and (3.21) that
\[
L_{11} \leq C \int_0^t \| u - e^{s\mathcal{L}}u_0 \|_{\dot{B}^{0}_{2,\infty},1} \left( \| u \|_{\dot{B}^{1}_{2,\infty}} + \| u \|_{\dot{B}^{1+\varepsilon}_{2,\infty}} \right) ds \\
\leq C \| u - e^{s\mathcal{L}}u_0 \|_{L^\infty(0,T;\dot{B}^{0}_{2,\infty},1)} \left( T^{\frac{1}{2}} \| u \|_{L^\infty(0,T;\dot{B}^{0}_{2,\infty},1),1} \| u \|_{L^1(0,T;\dot{B}^{2}_{2,\infty},1)}^{\frac{1}{2}} \\
+ T^{\frac{1}{2} - \varepsilon} \| u \|_{L^\infty(0,T;\dot{B}^{0}_{2,\infty},1)}^{\frac{1}{2} + \varepsilon} \| u \|_{L^1(0,T;\dot{B}^{2}_{2,\infty},1)}^{\frac{1}{2} + \varepsilon} \right) \\
\leq C 2^{(\varepsilon + \gamma\delta)N} \times \left( (2^{-2N})^{\frac{1}{2}} \cdot 2^{(n\varepsilon + \frac{\delta}{2})} + (2^{-2N})^{\frac{1}{2} + \varepsilon} \cdot 2^{(2(n\varepsilon + \frac{\delta}{2}))} \right) \\
\leq C 2^{(-\frac{1}{2} + \frac{n\varepsilon}{2} + 2\varepsilon)N}.
\]
We also have on $L_{12}$ in the similar estimate to (3.25)
\[
L_{12} \leq C \| e^{s\mathcal{L}}u_0 \|_{L^\infty(0,T;\dot{B}^{0}_{2,\infty},1)} \left( T^{\frac{1}{2}} \| u - e^{s\mathcal{L}}u_0 \|_{L^\infty(0,T;\dot{B}^{0}_{2,\infty},1),1}^{\frac{1}{2}} \| u - e^{s\mathcal{L}}u_0 \|_{L^1(0,T;\dot{B}^{2}_{2,\infty},1)}^{\frac{1}{2}} \\
+ T^{\frac{1}{2} - \varepsilon} \| u - e^{s\mathcal{L}}u_0 \|_{L^\infty(0,T;\dot{B}^{0}_{2,\infty},1)}^{\frac{1}{2} + \varepsilon} \| u - e^{s\mathcal{L}}u_0 \|_{L^1(0,T;\dot{B}^{2}_{2,\infty},1)}^{\frac{1}{2} + \varepsilon} \right) \\
\leq C 2^{(\frac{1}{2} + \frac{n\varepsilon}{2})N} \times \left( (2^{-2N})^{\frac{1}{2}} \cdot 2^{(n\varepsilon + \varepsilon)N} + (2^{-2N})^{\frac{1}{2} + \varepsilon} \cdot 2^{(2(n\varepsilon + \frac{\delta}{2}))} \right) \\
\leq C 2^{(-\frac{1}{2} + \frac{n\varepsilon}{2} + 2\varepsilon)N}.
\]
Then we obtain by (3.24), (3.25) and (3.26)
\[
L_1 \leq C 2^{(-\frac{1}{2} + \frac{n\varepsilon}{2} + 2\varepsilon)N}.
\]
On the estimate of $L_2$, it follows from the same estimate as the pressure estimate in (3.19) (see also the third term of the right hand side of (3.12)) that
\[
L_2 \leq C 2^{(-\frac{1}{2} + \frac{n\varepsilon}{2})N}.
\]
On the estimate of $L_3$, we consider the differences $\rho - \rho_1$ and $u - e^{s\mathcal{L}}u_0$ to obtain
\[
L_3 \leq \left\| \int_0^t e^{(t-s)\mathcal{L}} \left( \frac{\rho_1}{1 + \rho_1} - \frac{\rho_1}{1 + \rho_1} \right) \mathcal{L} u ds \right\|_{\dot{B}^{0}_{2,\infty},1} \\
+ \left\| \int_0^t e^{(t-s)\mathcal{L}} \frac{\rho_1}{1 + \rho_1} \mathcal{L} (u - e^{s\mathcal{L}}u_0) ds \right\|_{\dot{B}^{0}_{2,\infty},1} \\
=: L_{31} + L_{32}.
\]
By applying (3.7) to $L_{31}$, it holds that
\[
L_{31} \leq C \int_0^t \left\| \frac{\rho}{1 + \rho} - \frac{\rho_1}{1 + \rho_1} \right\|_{\dot{B}^{0}_{\infty,1} \cap \dot{B}^{\varepsilon}_{2,\infty},1} \| \mathcal{L} u \|_{\dot{B}^{0}_{2,\infty},1} ds \\
\leq C \sup_{t \in [0,T]} \left\| \frac{\rho}{1 + \rho} - \frac{\rho_1}{1 + \rho_1} \right\|_{\dot{B}^{0}_{\infty,1} \cap \dot{B}^{\varepsilon}_{2,\infty},1} \| u \|_{L^1(0,T;\dot{B}^{2}_{2,\infty},1)}.
\]
On the norm with $\rho$ and $\rho_1$, we have from the embedding:

$$\tilde{B}^{\varepsilon}_{[2,\infty]} = \tilde{B}^{\varepsilon}_{2,1} \cap \tilde{B}^{\varepsilon}_{\infty,1} \subset \tilde{B}^{0}_{\infty,1} \cap \tilde{B}^{\varepsilon}_{\infty,1}. \tag{3.31}$$

that

$$\left\| \frac{\rho}{1 + \rho} - \frac{\rho_1}{1 + \rho_1} \right\|_{\tilde{B}^{\varepsilon}_{\infty,1} \cap \tilde{B}^{0}_{\infty,1}} \leq C \left\| \frac{\rho}{1 + \rho} - \frac{\rho_1}{1 + \rho_1} \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}}. \tag{3.32}$$

Here we estimate by following geometric series:

$$\frac{\rho}{1 + \rho} - \frac{\rho_1}{1 + \rho_1} = \sum_{m=1}^{\infty} (-1)^{m-1} (\rho^m - \rho_1^m) = \rho - \rho_1 + \sum_{m=2}^{\infty} (-1)^{m-1} (\rho^m - \rho_1^m).$$

It follows from (3.20), (3.7), the embedding $\tilde{B}^{\varepsilon}_{[2,\infty]}(\mathbb{R}^n) \hookrightarrow \tilde{B}^{0}_{\infty,1}(\mathbb{R}^n)$ obtained by (3.31), (3.4) that

$$\left\| \frac{\rho}{1 + \rho} - \frac{\rho_1}{1 + \rho_1} \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} \leq \left\| \rho - \rho_1 \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} + \sum_{m=2}^{\infty} \left( \left\| \rho^m \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} + \left\| \rho_1^m \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} \right)$$

$$\leq C 2^{-1+n\delta+\varepsilon}N + \sum_{m=2}^{\infty} C^{m-1} \left( \left\| \rho \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}}^{m-1} \left\| \rho \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} + \left\| \rho_1 \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}}^{m-1} \left\| \rho_1 \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} \right) \tag{3.33}$$

$$\leq C 2^{-1+n\delta+\varepsilon}N + \sum_{m=2}^{\infty} C^{m} \left( \frac{1}{2} + \frac{n\delta}{2} \right) N^{m-1} \cdot 2^{-1+n\delta+\varepsilon+N}$$

$$\leq C 2^{-1+n\delta+\varepsilon}N.$$ 

Therefore we obtain the following estimate for $L_{31}$ by (3.30), (3.32), (3.33) and (3.6)

$$L_{31} \leq C 2^{(-1+\varepsilon+n\delta)}N \cdot 2^{(-\frac{1}{2}+\frac{n\delta}{2})N} = C 2^{(-\frac{1}{2}+\varepsilon+n\delta)N}. \tag{3.34}$$

For the estimate of $L_{32}$, we apply (3.7) and (3.31) to see that

$$L_{32} \leq C \int_{0}^{t} \left\| \frac{\rho_1}{1 + \rho_1} \right\|_{\tilde{B}^{0}_{\infty,1} \cap \tilde{B}^{\varepsilon}_{\infty,1}} \| \mathcal{L}(u - e^{s\mathcal{L}}u_0) \|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} \, ds$$

$$\leq C \sup_{t \in [0,T]} \left\| \frac{\rho_1}{1 + \rho_1} \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} \| u - e^{s\mathcal{L}}u_0 \|_{L^1(0,T;\tilde{B}^{\varepsilon}_{[2,\infty],1})}. \tag{3.35}$$

On the norm with $\rho_1$, we again utilize the geometric series, and apply (3.7), the embedding $\tilde{B}^{\varepsilon}_{[n,\infty]}(\mathbb{R}^n) \hookrightarrow \tilde{B}^{0}_{\infty,1}(\mathbb{R}^n)$ obtained by (3.31), (3.4) to obtain

$$\left\| \frac{\rho_1}{1 + \rho_1} \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} \leq \sum_{m=1}^{\infty} \left\| \left( -1 \right)^{m-1} \rho_1^m \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} \leq \sum_{m=1}^{\infty} C^m \left\| \rho_1 \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}}^{m-1} \left\| \rho_1 \right\|_{\tilde{B}^{\varepsilon}_{[2,\infty],1}} \tag{3.36}$$

$$\leq C 2^{(-\frac{1}{2}+\varepsilon+n\delta)N}.$$ 

It follows from (3.35), the above estimate and (3.21) that

$$L_{32} \leq C 2^{(-\frac{1}{2}+\varepsilon+n\delta)N} \cdot 2^{(n\delta+\varepsilon)N} = C 2^{(-\frac{1}{2}+2\varepsilon+n\delta)N}. \tag{3.37}$$

We have the following estimate from (3.29), (3.34) and (3.37)

$$L_3 \leq C 2^{(-\frac{1}{2}+2\varepsilon+n\delta)N}. \tag{3.38}$$

Therefore, (3.22) is obtained by by (3.23), (3.27), (3.28) and the above estimate. $\square$
The last lemma justifies the approximation
\[
\frac{\rho}{1 + \rho} \mathcal{L} u \simeq \rho \mathcal{L} u
\]
for the second iterate of the quasi-linear term.

**Lemma 3.7.** On the last term in the left hand side of (3.22) in Lemma 3.6, it holds that
\[
\left\| \int_0^t e^{(t-s)\mathcal{L}} \frac{\rho_1}{1 + \rho_1} \mathcal{L} e^{s\mathcal{L}} u_0 ds - \int_0^t e^{(t-s)\mathcal{L}} \rho_1 \mathcal{L} e^{s\mathcal{L}} u_0 ds \right\|_{\dot{B}^{0}_{2,1} \cap \dot{B}^{0}_{\infty,1}}
\leq C 2^{(-\frac{1}{2} + \frac{n}{2} + \epsilon)N}
\text{for } t \in [0, 2^{-2N}].
\]

**Proof of Lemma 3.7.** For simplicity, we put \( \dot{B}^0_{[2,\infty],1} = \dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1} \) and \( T = 2^{-2N} \). It follows from \( \frac{\rho_1}{1 + \rho_1} - \rho_1 = -\frac{\rho_1^2}{1 + \rho_1} \), the same estimate as (3.30), and (3.31) that
\[
\left\| \int_0^t e^{(t-s)\mathcal{L}} \frac{\rho_1}{1 + \rho_1} \mathcal{L} e^{s\mathcal{L}} u_0 ds - \int_0^t e^{(t-s)\mathcal{L}} \rho_1 \mathcal{L} e^{s\mathcal{L}} u_0 ds \right\|_{\dot{B}^{0}_{[2,\infty],1}}
= \left\| \int_0^t e^{(t-s)\mathcal{L}} \frac{\rho_1^2}{1 + \rho_1} \mathcal{L} e^{s\mathcal{L}} u_0 ds \right\|_{\dot{B}^{0}_{[2,\infty],1}}
\leq C \sup_{t \in [0,T]} \left\| \frac{\rho_1^2}{1 + \rho_1} \right\|_{\dot{B}^{0}_{[2,\infty],1}} \left\| u \right\|_{L^1(0,T; \dot{B}^{0}_{[2,\infty],1})},
\]
On the term with \( \rho_1 \), we utilize the geometric series and estimate similarly to (3.36) to see that
\[
\left\| \frac{\rho_1^2}{1 + \rho_1} \right\|_{\dot{B}^{0}_{[2,\infty],1}} \leq \sum_{m=1}^{\infty} \left\| -(1)^{m-1} \rho_1^{m+1} \right\|_{\dot{B}^{0}_{[2,\infty],1}} \leq \sum_{m=1}^{\infty} C^m (2^{(-\frac{1}{2} + \frac{n}{2})N}) m 2^{(-\frac{1}{2} + \frac{n}{2} + \epsilon)N}
\leq C 2^{(-1+n\delta+\epsilon)N}.
\]
Then we have from (3.39), (3.40) and (3.6) that
\[
\left\| \int_0^t e^{(t-s)\mathcal{L}} \frac{\rho_1}{1 + \rho_1} \mathcal{L} e^{s\mathcal{L}} u_0 ds - \int_0^t e^{(t-s)\mathcal{L}} \rho_1 \mathcal{L} e^{s\mathcal{L}} u_0 ds \right\|_{\dot{B}^{0}_{[2,\infty],1}}
\leq C 2^{(-1+n\delta+\epsilon)N} 2 \left( \frac{1}{2} + \frac{\epsilon}{2} \right) N = 2^{(-\frac{1}{2} + \frac{3n}{2} + \epsilon)N},
\]
and (3.38) is verified. \( \square \)

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let the initial data \((\rho_0, u_0)\) be defined by (3.1) and set \( R = (\log N)^{-1} \). By Proposition 3.2, we have a solution \((\rho, u)\) on the time interval \([0, 2^{-2N}]\). It follows from Proposition 2.1 that
\[
\left\| u_0 \right\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leq CR = C(\log N)^{-1} \to 0
\]
\[
\left\| I[u_0] \right\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \left\|_{t=t_0} \to 0 \right\|_{t=2^{-2N}} \geq C N^{1-\frac{n}{2}} (\log N)^{-2} - C (\log N)^{-2} \to \infty,
\]
as \( N \to \infty \). Then we need to prove that
\[
\sup_{0 < t < 2^{-2N}} \left\| u(t) - I[u_0] \right\|_{\dot{B}^{\frac{1}{2}}_{2,1}} < \infty.
\]
To this end, we write
\[ u(t) - e^{t\mathcal{L}}u_0 - I[u_0] \]
\[ = \left[ u(t) - \left\{ - \int_0^t e^{(t-s)\mathcal{L}}(e^{s\mathcal{L}}u_0 \cdot \nabla)e^{s\mathcal{L}}u_0 ds - \int_0^t e^{(t-s)\mathcal{L}} \frac{\rho_1}{1+\rho_1} \mathcal{L}e^{s\mathcal{L}}u_0 ds \right\} \right] \]
\[ - \left[ \int_0^t e^{(t-s)\mathcal{L}} \frac{\rho_1}{1+\rho_1} \mathcal{L}e^{s\mathcal{L}}u_0 ds - \int_0^t e^{(t-s)\mathcal{L}} \rho_1 \mathcal{L}e^{s\mathcal{L}}u_0 ds \right] \]
\[ + \left[ - \int_0^t e^{(t-s)\mathcal{L}}(e^{s\mathcal{L}}u_0 \cdot \nabla)e^{s\mathcal{L}}u_0 ds + \int_0^t e^{(t-s)\mathcal{L}} \left( \int_0^s \text{div} e^\mathcal{L}u_0 ds \right) \mathcal{L}e^{s\mathcal{L}}u_0 ds - I[u_0] \right] \]
\[ =: R_1 + R_2 + R_3. \]

If \( t \leq 2^{-2N} \), it follows that by Lemmas 3.6 and 3.7
\[ \|R_1\|_{\bar{B}_{2n,1}}^{\frac{1}{2}} + \|R_2\|_{\bar{B}_{2n,1}}^{\frac{1}{2}} \leq C2^{-\frac{1}{2} + \frac{n}{2} + 2\varepsilon}N, \]
and by Lemma 2.7
\[ \|R_3\|_{\bar{B}_{2n,1}}^{\frac{1}{2}} \leq C(\log N)^{-\frac{1}{2}}, \]
which prove (3.41). Therefore, we have obtained a sequence of initial data such that it verifies the discontinuity of data-to-solution map.

\[ \square \]

**APPENDIX A**

We discuss the sharpness in Remark after Proposition 2.2.

**Proposition A.1.** For positive constants \( A, B, \) and \( c \), let \( \tilde{I}[f, g] \) be defined by (2.4). For \( \sigma \) with \( n \leq \sigma \leq \infty \), there exists \( C = C_{n,\sigma} > 0 \) such that
\[ \sup_{t > 0} \|\tilde{I}[f, g](t)\|_{\bar{B}_{2n,\sigma}}^{\frac{1}{2}} \leq C\|f\|_{\bar{B}_{2n,\sigma}}^{\frac{1}{2}} \|g\|_{\bar{B}_{2n,\sigma}}^{\frac{1}{2}} \quad (A.1) \]
for all \( f, g \in \bar{B}_{2n,\sigma}(\mathbb{R}^n) \).

**Remark.** If we consider only the convection term \((u \cdot \nabla)u\) without the quasilinear dissipation term, the corresponding estimate does not hold for any \( \sigma \) with \( 1 \leq \sigma \leq \infty \). This indicates that cancellation occurs for the sum of the convection term and the quasi linear term.

We prepare a lemma to prove Proposition A.1.

**Lemma A.2.** Let \( N_0 \geq 10 \) be a fixed natural number. Suppose that \( F_k, G_l \in L^{2n}(\mathbb{R}^n) \) be such that
\[ \text{supp } \hat{F}_k \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1} \}, \quad \text{supp } \hat{G}_l \subset \{2^{l-1} \leq |\xi| \leq 2^{l+1} \}. \]
If \( |k - l| \leq 2 \), then
\[ \left\| \sum_{j \leq k-N_0} \phi_j \ast \left\{ (\mathcal{F}^{-1} \left[ \int_{\mathbb{R}^n} \frac{1}{B|\eta|^2} - c|\xi|^2 \hat{F}_k(\xi - \eta)\hat{G}_l(\eta)d\eta \right] ) \right\} \right\|_{\bar{B}_{2n,\sigma}}^{\frac{1}{2}} \]
\[ \leq C\|F_k\|_{L^{2n}} 2^{-2\sigma} \|G_l\|_{L^{2n}}. \]

**Proof of Lemma A.2.** By the embedding \( L^{n}(\mathbb{R}^n) \hookrightarrow \bar{B}_{2n,\sigma}(\mathbb{R}^n) \), it is sufficient to estimate the \( L^{n}(\mathbb{R}^n) \) norm. Since the supp of the Fourier transform of \( F_k, G_k \) is restricted, we write
\[ \hat{F}_k = \hat{\Psi}_k \hat{F}_k, \quad \hat{G}_l = \hat{\Psi}_l \hat{G}_l, \]
where \( \Psi_k := \phi_{k-1} + \phi_k + \phi_{k+1} \). Let us choose \( \varphi, \varphi_j \in \mathcal{S}(\mathbb{R}^n) \) such that
\[ \varphi(\xi) = \begin{cases} \frac{1}{\phi_0(\xi)} & \text{if } |\xi| \leq 1, \\ \phi_{k-1}(\xi) & \text{if } |\xi| \geq 1, \end{cases} \quad \varphi_j(x) = 2^{nj}\varphi(2^jx). \]
Then we can apply Lemma 2.6 to the Fourier multiplier
\[ m(\xi - \eta, \eta) = \varphi_{k-N_0}(\xi) \frac{\hat{\psi}_k(\xi - \eta) \cdot 2^{2l} \hat{\psi}_l(\eta)}{B|\eta|^2 - c|\xi|^2}, \]
and then obtain
\[ \left\| \varphi_{k-N_0} \ast \left\{ F^{-1} \left[ \int_{\mathbb{R}^n} \frac{\hat{\psi}_k(\xi - \eta) \cdot 2^{2l} \hat{\psi}_l(\eta) \hat{F}_k(\xi - \eta)2^{-2l} \hat{G}_l(\eta) d\eta} {B|\eta|^2 - c|\xi|^2} \right] \right\} \right\|_{L^2} \leq C\|F_k\|_{L^{2n}} 2^{-2l}\|G_l\|_{L^{2n}}. \]
We notice that the function in the right hand side above belongs to \( L^n(\mathbb{R}^n) \) and
\[ \sum_{j \leq k-N_0} \phi_j \ast \left\{ F^{-1} \left[ \int_{\mathbb{R}^n} \frac{1}{B|\eta|^2 - c|\xi|^2} \hat{F}_k(\xi - \eta)\hat{G}_l(\eta) d\eta \right] \right\} = \varphi_{k-N_0} \ast \left\{ F^{-1} \left[ \int_{\mathbb{R}^n} \frac{1}{B|\eta|^2 - c|\xi|^2} \hat{F}_k(\xi - \eta)\hat{G}_l(\eta) d\eta \right] \right\} \text{ in } S'(\mathbb{R}^n), \]
which leads to the desired inequality.

**Proof of Proposition A.1.** For simplicity, let \( f_k \) and \( g_l \) be defined by \( f_k := \phi_k \ast f \) and \( g_l := \phi_l \ast g \). By Bony’s paraproduc formula [2], it holds that
\[ \tilde{T}[f, g](t) = \left( \sum_{k \geq l+3} + \sum_{l \geq k+3} + \sum_{|k-l| \leq 2} \right) I[f_k, g_l] =: I_{1} + I_{2} + I_{3}. \quad (A.2) \]
The high–low and low–high interaction parts \( I_{1} \) and \( I_{2} \), respectively, can be treated easily. We show the estimate for \( I_{1} \) and omit the estimate for \( I_{2} \). It follows from the linear estimate
\[ e^{(t-s)\Delta} \hat{B}_{2n,\sigma}^{-\frac{3}{2}}(\mathbb{R}^n) \to \hat{B}_{2n,\sigma}^{-\frac{3}{2}}(\mathbb{R}^n) \]
that
\[ \|I_{1}\|_{\hat{B}_{2n,\sigma}^{-\frac{3}{2}}} \leq C \int_{0}^{t} (t-s)^{-\frac{1}{2}} \left\| \sum_{l \geq k+3} \left( e^{As\Delta} f_k \right) \nabla e^{B_k\Delta} g_l + \frac{B}{A} \left( e^{A_k\Delta} f_k - f_k \right) \nabla e^{B_k\Delta} g_l \right\|_{\hat{B}_{2n,\sigma}^{-\frac{3}{2}}} \, ds \]
\[ \leq C \int_{0}^{t} (t-s)^{-\frac{1}{2}} \left\{ \left\| \sum_{l \geq k+3} \left( e^{A_k\Delta} f_k \right) \nabla e^{B_k\Delta} g_l \right\|_{\hat{B}_{2n,\sigma}^{-\frac{3}{2}}} + \left\| \sum_{l \geq k+3} f_k \nabla e^{B_k\Delta} g_l \right\|_{\hat{B}_{2n,\sigma}^{-\frac{3}{2}}} \right\} \, ds \]
\[ =: I_{1,1} + I_{1,2}. \quad (A.3) \]
Here, to take the above norm in the Besov spaces, we use the equality \( \phi_j \ast \left\{ \left( e^{A_k\Delta} f_k \right) \nabla e^{B_k\Delta} g_l \right\} = 0 \) for \( j, k, l \) with \(|j-k| \geq 2 \) and \( k \geq l+3 \) because the support of Fourier transform of the functions satisfy
\[ \text{supp } F[\phi_j] \subset \{ \xi \in \mathbb{R}^n \mid 2^{j-1} \leq |\xi| \leq 2^{j+1} \}, \quad (A.4) \]
\[ \text{supp } F[\left( e^{A_k\Delta} f_k \right) \nabla e^{B_k\Delta} g_l] \subset \{ \xi \in \mathbb{R}^n \mid 2^{k-2} \leq |\xi| \leq 2^{k+2} \}. \quad (A.5) \]
Subsequently, on the sum over \( k \geq l+3 \) in the definition of \( I_{1,1} \) and \( I_{1,2} \), it is sufficient to consider \( k \) with \(|k-j| \leq 3 \) for each \( j \). We have from Young’s inequality, \( \|\phi_j\|_{L^1} = \|\phi_0\|_{L^1} \), Hölder’s inequality, the boundedness of \( e^{A_k\Delta} \) in \( L^{2n}(\mathbb{R}^n) \), and the linear estimate of \( e^{B_k\Delta} : \hat{B}_{2n,\sigma}^{-\frac{3}{2}}(\mathbb{R}^n) \to \hat{B}_{2n,\sigma}^{-\frac{3}{2}}(\mathbb{R}^n) \) and the embedding \( \hat{B}_{2n,\sigma}^{-\frac{3}{2}}(\mathbb{R}^n) \hookrightarrow \hat{B}_{2n,\sigma}^{-\frac{3}{2}}(\mathbb{R}^n) \)
\[ \left\| \sum_{l \geq k+3} \left( e^{A_k\Delta} f_k \right) \nabla e^{B_k\Delta} g_l \right\|_{\hat{B}_{2n,\sigma}^{-\frac{3}{2}}} \]
\[ \leq \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{-\frac{7}{2}j} \|\phi_0\|_{L^1} \sum_{|k-j| \leq 3} \|f_k\|_{L^{2n}} \left( \sum_{l \leq k-3} \|\nabla e^{B_k\Delta} g_l\|_{L^\infty} \right)^{\sigma} \right)^{\frac{1}{\sigma}} \right\} \]
\[ \leq C \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{-\frac{7}{2}j} \sum_{|k-j| \leq 3} \|f_k\|_{L^{2n}} 2^{-k^2} \right)^{\frac{1}{2}} \|e^{B_k\Delta} g\|_{\hat{B}_{2n,\sigma}^{-\frac{3}{2}}} \leq C s^{-\frac{1}{2}} \left\| f \right\|_{\hat{B}_{2n,\sigma}^{-\frac{3}{2}}} \left\| g \right\|_{\hat{B}_{2n,\sigma}^{-\frac{3}{2}}}. \quad (A.6) \]
Next, it follows from (A.6) that
\[
II_{1,1} \leq C \int_{0}^{t} (t-s)^{-\frac{2}{3}} s^{-\frac{1}{2}} ds \|f\|_{B_{2n,\sigma}^{-\frac{1}{2}}} \|g\|_{B_{2n,\sigma}^{-\frac{1}{2}}} \leq C \|f\|_{B_{2n,\sigma}^{-\frac{1}{2}}} \|g\|_{B_{2n,\sigma}^{-\frac{1}{2}}}. \tag{A.7}
\]

The same estimate as (A.7) holds for \(II_{1,2}\) as we only replaced the propagator \(e^{As\Delta}\) with the identity operator. We then obtain the bilinear estimates for \(II_1\).

Next, we show the estimate for \(II_3\) on high-high frequency interaction. The following equality \(\phi_j * I[f_k, g_l] = 0\) holds for \(j, k, l\) with \(|k - l| \leq 2\) and \(k < j - 5\) which is obtained by (A.4) and (A.5). We decompose the sum appearing in \(\phi_j * II_3\) into the sum over \(|k - j| \leq N_0\) and \(k \geq j + N_0\):
\[
\sum_{j \in \mathbb{Z}} \phi_j * II_3 = \sum_{j \in \mathbb{Z}} \phi_j * \left( \sum_{|k-j| \leq N_0} \sum_{|l-k| \leq 2} I[f_k, g_l] \right) + \sum_{j \in \mathbb{Z}} \phi_j * \left( \sum_{k \geq j + N_0} \sum_{|l-k| \leq 2} I[f_k, g_l] \right) \tag{A.8}
\]

where \(N_0 > 0\) is a sufficiently large number that will be taken later. It is possible to estimate \(II_{3,1}\) analogously to the estimate of \(II_1\) with the correspondences \(|k - j| \leq 2, l \leq k - 3\) of the summation in (A.6) and \(|k - j| \leq N_0, |l - k| \leq 2\) of the summation in the definition of \(II_{3,1}\), respectively; subsequently, one can obtain
\[
\|II_{3,1}\|_{B_{2n,\sigma}^{-\frac{1}{2}}} \leq C \|f\|_{B_{2n,\sigma}^{-\frac{1}{2}}} \|g\|_{B_{2n,\sigma}^{-\frac{1}{2}}}. \tag{A.9}
\]

We consider the estimate of \(II_{3,2}\) using the equality (2.7). We apply Lemma A.2 for sufficiently large \(N_0 \in \mathbb{N}\) and the Fourier multiplier theorem, and then it holds that
\[
\|II_{3,2}\|_{B_{2n,\sigma}^{-\frac{1}{2}}} \leq \sum_{|l-k| \leq 2} \left\| \sum_{j \leq k - N_0} \phi_j * \left\{ \mathcal{F}^{-1} \left[ \frac{A}{B} \int_{\mathbb{R}^n} \frac{(1 - e^{-A|\xi-k\eta|^2}) \hat{f}_k(\xi - \eta) e^{-B|\eta|^2} \eta \hat{g}_l(\eta) d\eta}{|\eta|^2 - c|\xi|^2} \right] \right\} \right\|_{B_{2n,\sigma}^{-\frac{1}{2}}}
\]
\[
\leq C \sum_{|l-k| \leq 2} \left\| (1 - e^{At\Delta}) f_k \right\|_{L^{2n}} e^{2B|\Delta|} \|g_l\|_{L^{2n}}
\]
\[
\leq C \sum_{|l-k| \leq 2} (1 - e^{C^{-1}At^{2k}}) \|f_k\|_{L^{2n}} \frac{2^l e^{-C^{-1}Bt^{2l}}}{2^{2l}} \|g_l\|_{L^{2n}}.
\]

An elementary calculation gives that
\[
\|II_{3,2}\|_{B_{2n,\sigma}^{-\frac{1}{2}}} \leq C \sum_{k \in \mathbb{Z}} 2^{2k} e^{-C^{-1}t^{2k}} \|f\|_{B_{2n,\infty}} \|g\|_{B_{2n,\infty}} \leq C \|f\|_{B_{2n,\infty}} \|g\|_{B_{2n,\infty}}. \tag{A.10}
\]

We next consider the estimate of the remainder term \(R\) defined by (2.8). Since \(R[f_k, g_l]\) exhibits some divergence structures due to (2.8), we have from the embedding \(B_{n,\sigma}(\mathbb{R}^n) \hookrightarrow B_{2n,\sigma}(\mathbb{R}^n)\), Hölder’s inequality
\[
\left\| \sum_{j \in \mathbb{Z}} \phi_j * \left( \sum_{k \geq j + N_0} \sum_{|l-k| \leq 2} R[f_k, g_l] \right) \right\|_{B_{2n,\sigma}^{-\frac{1}{2}}}
\]
\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{k \geq j + N_0} \sum_{|l-k| \leq 2} 2^l \|f_k\|_{L^{2n}} \|g_l\|_{L^{2n}} \right) \frac{(B + c)^{2^j} + \frac{1}{2} 2^{2j} \left( e^{-C^{-1}dt^{2j}} + e^{-C^{-1}At^{2k} - C^{-1}Bt^{2l}} \right)}{(A2^{2k} + B2^{2l} - c2^{2j})(Bc^{2l} - c2^{2j})} \right\}^{\frac{1}{2}}
\]
\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{k \geq j + N_0} \sum_{|l-k| \leq 2} 2^l 2^{-2l} \|f_k\|_{L^{2n}} \|g_l\|_{L^{2n}} \right)^{\frac{1}{2}} \right\}^{\frac{1}{2}}.
\]
By introducing the index \( r \) by \( r = k - j \) (\( r \geq N_0 \)) and applying Hölder’s inequality, it holds that

\[
\left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{k \geq j + N_0} \sum_{|l-k| \leq 2} 2^j 2^{-2l} \| f_k \|_{L^2} \| g_l \|_{L^2} \right) \right\}^{\frac{1}{\sigma}}
\]

\[
= \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{r \geq N_0} 2^j 2^{-2l} \| f_{j+r} \|_{L^2} \sum_{|l-j-r| \leq 2} \| g_l \|_{L^2} \right) \right\}^{\frac{1}{\sigma}} \leq \sum_{r \geq N_0} 2^{-r} \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{j+r} 2^{-2l} \| f_{j+r} \|_{L^2} \sum_{|l-j-r| \leq 2} \| g_l \|_{L^2} \right) \right\}^{\frac{1}{\sigma}} \leq C \| f \|_{B^{\frac{1}{2}}_{2,2r}} \| g \|_{B^{\frac{1}{2}}_{2,2r}}.
\] (A.11)

Therefore, the estimate for \( II_3 \) is obtained by (A.8), (A.10) and (A.11), and the proof is completed.

\[ \square \]

**APPENDIX B**

We introduce an inequality of Gronwall type.

**Lemma B.1.** Let \( f, g_1, g_2, g_3 \) be continuous function such that

\[
f, g_2, g_3 \geq 0, \quad f(t) \leq \int_0^t e^{f_s} g_1(s) \, ds + g_2(s) f(s) \, ds, \quad t > 0.
\]

Then

\[
f(t) \leq \int_0^t e^{f_s} (g_1 + g_3) \, ds.
\]

**Proof.** Let \( \Psi \) be the right hand side of the assumption.

\[
\Psi(t) := \int_0^t e^{f_s} (g_1 + g_3) \, ds.
\]

Then

\[
\Psi' = g_1 \Psi + g_2 + g_3 f \leq g_1 \Psi + g_2 + g_3 \Psi,
\]

\[
(e^{f_0} (g_1 + g_3))' \leq e^{f_0} (g_1 + g_3) g_2.
\]

Integrating over \([0, t]\) gives the desired inequality.

\[ \square \]

**APPENDIX C**

We give proof of the bilinear estimates in Lemma 3.4. The first estimate (3.7) is obtained by Bony’s paraproduct formula and bilinear estimates. Indeed, we write

\[
f g = \left( \sum_{k \geq l+2} + \sum_{l \geq k+2} + \sum_{|k-l| \leq 2} \right) f_k g_l,
\]

where \( f_k := \phi_k \ast f, g_l := \phi_l \ast g \). When \( k > l + 2 \), it follows that

\[
\left\| \sum_{k \geq l+2} f_k g_l \right\|_{\dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1}} \leq C \| f \|_{\dot{B}^0_{\infty,1}} \| g \|_{L^2} \leq C \| f \|_{\dot{B}^0_{\infty,1} \cap \dot{B}^0_{[2,\infty],1}} \| g \|_{\dot{B}^0_{[2,\infty],1}},
\]

and we also have

\[
\left\| \sum_{k \geq l+2} f_k g_l \right\|_{\dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1}} \leq C \| f \|_{\dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1}} \| g \|_{L^\infty} \leq C \| f \|_{\dot{B}^0_{[2,\infty],1}} \| g \|_{\dot{B}^0_{\infty,1} \cap \dot{B}^0_{[2,\infty],1}},
\]

which proves that

\[
\left\| \sum_{k \geq l+2} f_k g_l \right\|_{\dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1}} \leq C \min \left\{ \| f \|_{\dot{B}^0_{\infty,1} \cap \dot{B}^0_{[2,\infty],1}}, \| g \|_{\dot{B}^0_{\infty,1} \cap \dot{B}^0_{[2,\infty],1}}, \| f \|_{\dot{B}^0_{[2,\infty],1}}, \| g \|_{\dot{B}^0_{\infty,1} \cap \dot{B}^0_{[2,\infty],1}} \right\}.
\]

The second case when \( l > k + 2 \) follows analogously. When \(|k-l| \leq 2\), we apply the embedding

\[
\dot{B}^\varepsilon_{p_1,1}(\mathbb{R}^n) \cap \dot{B}^\varepsilon_{p_2,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{2,1}(\mathbb{R}^n) \cap \dot{B}^0_{\infty,1}(\mathbb{R}^n), \quad p_1 := \frac{2}{1 + \frac{2\varepsilon}{n}}, \quad p_2 := \frac{n}{\varepsilon}
\]

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to obtain that
\[ \left\| \sum_{|k-l|\leq 2} f_k g_l \right\|_{\dot{B}^0_{r,1}(\mathbb{R}^d)} \leq C \left\| \sum_{|k-l|\leq 2} f_k g_l \right\|_{\dot{B}^0_{r,1}(\mathbb{R}^d)} \]
\[ \leq C \min \left\{ \|f\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^d)}, \|g\|_{\dot{B}^0_{2,\infty,1}(\mathbb{R}^d)}, \|f\|_{\dot{B}^0_{[2,\infty],1}}, \|g\|_{\dot{B}^0_{[2,\infty],1}} \right\}. \]

The second estimate (3.8) follows by the standard bilinear estimate
\[ \|fg\|_{\dot{B}^s_{r,1}} \leq C \left( \|f\|_{\dot{B}^s_{r,1}}, \|g\|_{L^{s/2}}, \|f\|_{L^{\infty}}, \|g\|_{L^{\infty}} \right), \quad s > 0, \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}. \]

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