Representation type of surfaces in $\mathbb{P}^3$

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Abstract. The goal of this article is to prove that every surface with a regular point in the three-dimensional projective space of degree at least four, is of wild representation type under the condition that either $X$ is integral or $\text{Pic}(X) \cong \langle \mathcal{O}_X(1) \rangle$; we construct families of arbitrarily large dimension of indecomposable pairwise non-isomorphic aCM vector bundles. On the other hand, we prove that every non-integral aCM scheme of arbitrary dimension at least two, is also very wild in a sense that there exist arbitrarily large dimensional families of pairwise non-isomorphic aCM non-locally free sheaves of rank one.

1. Introduction

An arithmetically Cohen-Macaulay (for short, aCM) sheaf on a projective scheme $X$ is a coherent sheaf supporting $X$, which has trivial intermediate cohomology and the stalk at each point whose depth equals the dimension of $X$. ACM vector bundles correspond to maximal Cohen-Macaulay modules over the associated graded ring and they reflect the properties of the graded ring. It is believed that the category generated by aCM sheaves on $X$ measures the complexity of $X$. Indeed, a classification of aCM varieties was proposed as finite, tame or wild representation type according to the complexity of this category in [10] and there are several contributions to this trichotomy such as [11, 4, 8, 13]. It is only recent when such a representation type is determined for any aCM reduced scheme; see [14].

In this article, we pay our attention to the representation type of surfaces in three-dimensional projective space. Since the aCM vector bundles on smooth surfaces of degree at most two are completely classified due to the work by Horrocks and [17, 18], we may focus on surfaces of degree at least three. The case of cubic surfaces is dealt in [5, 12] and the case of quartic surfaces is from [20]. Our main result is the following, which implies that the surfaces in Theorem 1.1 are of wild representation type.

**Theorem 1.1.** Let $X \subset \mathbb{P}^3$ be a surface, defined as the zero set of a homogeneous polynomial in four variables of degree at least four with $X_{\text{reg}} \neq \emptyset$. Assume further that either $\text{Pic}(X) = \mathbb{Z}(\mathcal{O}_X(1))$ or that $X$ is integral. For every even and positive integer $r$, there exists a family $\{E_\lambda\}_{\lambda \in \Lambda}$ of indecomposable aCM vector bundles of rank $r$ such that $\Lambda$ is an integral quasi-projective variety with $\dim \Lambda = r$ and $E_\lambda \not\cong E_{\lambda'}$ for all $\lambda \neq \lambda'$ in $\Lambda$.

It has to be noticed that although the result in [14] is more general than the implication of Theorem 1.1 regarding the wildness of the representation type, Theorem 1.1 provides a concrete way of constructing families of indecomposable aCM ‘vector bundles’ with prescribed rank, even on singular surfaces.

On the other hand, every non-integral aCM projective scheme of arbitrary dimension at least two, whose associated reduced scheme contains at least one aCM irreducible component, is of ‘very wild’ representation type, in a sense that there exist arbitrarily large dimensional families of pairwise non-isomorphic aCM non-locally free sheaves of rank one; see Proposition 5.3.

Here we summarize the structure of this article. In Section 2 we collect several definitions and basic results that are used throughout the article. In Section 3 we state the main result in Theorem 3.10 which would automatically imply Theorem 1.1. We also give a proof of Theorem 3.10 in special case and suggest a number of its variation to construct aCM vector bundles. Then we spend the whole Section 4 for the proof of Theorem 3.10 basically we use induction on rank and the main ingredient for the proof is Lemma 4.5 and the use of a monodromy argument. Then we show in Section 5 the wildness of any aCM projective scheme of dimension at least two by investigating non-locally free ideal sheaves.

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2. Preliminary

Throughout the article our base field $k$ is algebraically closed of characteristic 0. We always assume that our projective schemes $X \subset \mathbb{P}^N$ are arithmetically Cohen-Macaulay, namely, $h^i(\mathcal{I}_{X,\mathbb{P}^N}(t)) = 0$ for all $t \in \mathbb{Z}$ and $h^i(\mathcal{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$ and all $i = 1, \ldots, \dim X - 1$, of pure dimension at least two. Then by [25, Théorème 1 in page 268] all local rings $\mathcal{O}_{X,x}$ are Cohen-Macaulay of dimension $\dim X$. From $h^1(\mathcal{I}_{X,\mathbb{P}^N}) = 0$ we see that $X_{\text{red}}$ is connected. Since in all our main result we have $N = \dim X + 1 = 3$, the reader can always assume that $X$ is a surface in $\mathbb{P}^3$, although there are several statements that hold in more general situations. By a surface of degree $m \geq 1$ in $\mathbb{P}^3$, we always mean the zero locus of a homogeneous polynomial of degree $m$ in four variables. For a vector bundle $\mathcal{E}$ of rank $r \in \mathbb{Z}$ on $X$, we say that $\mathcal{E}$ splits if all its indecomposable factors are $\mathcal{O}_X(t)$ for some $t \in \mathbb{Z}$; $\mathcal{E} \cong \oplus_{i=1}^r \mathcal{O}_X(t_i)$ for some $t_i \in \mathbb{Z}$ with $i = 1, \ldots, r$.

We always fix the embedding $X \subset \mathbb{P}^N$ and the associated polarization $\mathcal{O}_X(1)$. For a coherent sheaf $\mathcal{E}$ on a closed subscheme $X$ of a fixed projective space, we denote $\mathcal{E} \otimes \mathcal{O}_X(t)$ by $\mathcal{E}(t)$ for $t \in \mathbb{Z}$. For another coherent sheaf $\mathcal{G}$, we denote by $\text{hom}_X(\mathcal{F}, \mathcal{G})$ the dimension of $\text{Hom}_X(\mathcal{F}, \mathcal{G})$, and by $\text{ext}^1_X(\mathcal{F}, \mathcal{G})$ the dimension of $\text{Ext}^1_X(\mathcal{F}, \mathcal{G})$. Finally we denote the canonical sheaf of $X$ by $\omega_X$.

**Definition 2.1.** A coherent sheaf $\mathcal{E}$ on $X$ is called arithmetically Cohen-Macaulay (for short, aCM) if the following conditions hold:

(i) $\mathcal{E}$ is locally Cohen-Macaulay, that is, the stalk $\mathcal{E}_x$ has depth equal to $\dim \mathcal{O}_{X,x}$ for any $x \in X$;

(ii) $H^i(\mathcal{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $i = 1, \ldots, \dim(X) - 1$.

**Remark 2.2.** In the condition (i) of Definition 2.1 we may only require that the stalk $\mathcal{E}_x$ has positive depth for any point $x \in X$; see [2] Remark 2.2 and [25 Théorème 1 in page 268].

If $\mathcal{E}$ is a coherent sheaf on a closed subscheme $X$ of a fixed projective space, then we may consider its Hilbert polynomial $P_{\mathcal{E}}(t) \in \mathbb{Q}[t]$ with the leading coefficient $\mu(\mathcal{E})/d!$, where $d$ is the dimension of $\text{Supp}(\mathcal{E})$ and $\mu = \mu(\mathcal{E})$ is called the multiplicity of $\mathcal{E}$. The normalized Hilbert polynomial $p_{\mathcal{E}}(t)$ of $\mathcal{E}$ is defined to be the Hilbert polynomial of $\mathcal{E}$ divided by $\mu(\mathcal{E})$.

**Definition 2.3.** If $\dim \text{Supp}(\mathcal{E}) = \dim(X)$, then the rank of $\mathcal{E}$ is defined to be

$$\text{rank}(\mathcal{E}) = \frac{\mu(\mathcal{E})}{\mu(\mathcal{O}_X)}.$$

Otherwise it is defined to be zero.

For an integral scheme $X$, the rank of $\mathcal{E}$ is the dimension of the stalk $\mathcal{E}_x$ at the generic point $x \in X$. But in general rank($\mathcal{E}$) needs not be integer.

Now the following construction of a coherent sheaf with higher rank and almost the same cohomological data as the starting coherent sheaf in Lemma 2.4 is due to [3]. In case of some surfaces in $\mathbb{P}^3$ of degree at least two, the construction provides an indecomposable aCM vector bundles of rank three; see Proposition 3.3.

**Lemma 2.4.** Let $(X, \mathcal{O}_X(1))$ be an aCM projective scheme of dimension $n \geq 2$. For a fixed coherent sheaf $\mathcal{G}$ with pure depth $n$ on $X$, assume the existence of $t_0 \in \mathbb{Z}$ such that $s := h^1(\mathcal{G}(t_0)) > 0$. Then the vector space $W := H^1(\mathcal{G}(t_0))$ induces the following unique extension up to isomorphisms

$$(1) \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X(-t_0) \otimes W^\vee \rightarrow 0$$

and the sheaf $\mathcal{E}$ in the middle satisfies the following:

(i) $h^1(\mathcal{E}(t)) = h^1(\mathcal{G}(t))$ for all $t \neq t_0$, and $h^1(\mathcal{E}(t_0)) = 0$;
(ii) \( h^i(\mathcal{E}(t)) = h^i(\mathcal{G}(t)) \) for all \( t \in \mathbb{Z} \) and all \( i \) with \( 2 \leq i \leq n - 1 \).

If \( \mathcal{G} \) is locally free, then \( \mathcal{E} \) is locally free.

The construction of aCM vector bundles in Proposition 2.5 and the one in Proposition 3.3 is an extension of the method in [7, Remark 4.3].

**Proposition 2.5.** Let \( X \subset \mathbb{P}^N \) be a projective Gorenstein scheme with pure dimension two and pure depth two such that \( h^1(\mathcal{O}_X(t)) = 0 \) for all \( t \in \mathbb{Z} \) and \( h^1(\mathcal{I}_{X,\mathbb{P}^N}) = 0 \). Assume \( X_{\text{reg}} \neq \emptyset \) and fix \( p \in X_{\text{reg}} \). Then there exists an aCM vector bundle \( \mathcal{E}_p \) of rank two on \( X \) fitting into the exact sequence

\[
0 \rightarrow \omega_X(1) \rightarrow \mathcal{E}_p \rightarrow \mathcal{I}_{p,X} \rightarrow 0.
\]

Moreover, if \( \deg(\omega_X) + \deg(X) \geq 0 \) and \( p, q \in X_{\text{reg}} \) with \( p \neq q \), then we have \( \mathcal{E}_p \nleq \mathcal{E}_q \).

**Proof.** Since \( X \) is Gorenstein, \( \omega_X(1) \) is a line bundle and we get

\[
\text{Ext}^1_X(\mathcal{I}_{p,X}, \omega_X(1)) \cong H^1(\mathcal{I}_{p,X}(-1))^\vee \cong k.
\]

So up to isomorphism there exists a unique sheaf \( \mathcal{E}_p \) fitting into an extension \( \mathcal{I}_{p,X}, \mathcal{E}_p, \omega_X(1) \) with a nonzero extension class. Since \( h^0(\mathcal{O}_X(-1)) = 0 \) and \( p \in X_{\text{reg}} \), the Cayley-Bacharach condition is satisfied for \( 2 \) and so \( \mathcal{E}_p \) is locally free; see [6]. Note that the restriction map

\[
H^0(\mathcal{O}_X(t)) \rightarrow H^0(\mathcal{O}_X(t)|_{\{p\}})
\]

is surjective for any \( t \geq 0 \). This implies that \( h^1(\mathcal{I}_{p,X}(t)) = 0 \) for any \( t \geq 0 \), because we have \( h^1(\mathcal{O}_X(t)) = 0 \). Then we see from [2] that \( h^1(\mathcal{E}_p(t)) = 0 \) for any \( t \geq 0 \). On the other hand, from \( \det(\mathcal{E}_p) \cong \omega_X(1) \), we get that \( h^1(\mathcal{E}_p(t)) = h^1(\mathcal{E}_p(\omega_X(-t)) = h^1(\mathcal{E}_p(-t - 1)) \neq 0 \) for \( t < 0 \) by Serre’s duality. Thus \( \mathcal{E}_p \) is aCM.

For the second assertion, assume \( \mathcal{E}_p \cong \mathcal{E}_q \). From the assumption \( \deg(\omega_X(t)) \geq 0 \), we get \( h^0(\mathcal{E}_p(-1)) \leq 1 \) with equality if and only if \( \omega_X \cong \mathcal{O}_X(-1) \). In particular, we have \( h^0(\mathcal{I}_{p,X} \otimes \omega_X(-1)) = 0 \). Then from the assumption \( h^1(\mathcal{O}_X) = 0 \) and \( [2] \), we get \( h^0(\mathcal{E}_p \otimes \omega_X(-1)) = 1 \) and that \( p \) is the only zero of a nonzero section of \( H^0(\mathcal{E}_p \otimes \omega_X(-1)) \). Thus we get \( p = q \). \( \square \)

**Theorem 2.6.** Let \( X \subset \mathbb{P}^N \) be a projective Gorenstein scheme with pure dimension two and pure depth two, satisfying that

- \( h^1(\mathcal{O}_X(t)) = 0 \) for all \( t \in \mathbb{Z} \) and \( h^1(\mathcal{I}_{X,\mathbb{P}^N}) = 0 \);
- \( X_{\text{reg}} \neq \emptyset \) and \( \deg(\omega_X) + \deg(X) \geq 0 \).

Then there exists a two-dimensional family of pairwise non-isomorphic aCM vector bundles of rank two on \( X \) whose very general member is indecomposable; here “very general” means outside countably many proper subvarieties.

**Proof.** By assumption \( X_{\text{reg}} \) is a two-dimensional quasi-projective smooth variety. By Proposition 2.5, there is a flat family of aCM vector bundles \( \{\mathcal{E}_p\}_{p \in X_{\text{reg}}} \) of rank two such that if \( p, q \in X_{\text{reg}} \) and \( p \neq q \), then \( \mathcal{E}_p \nleq \mathcal{E}_q \). Thus it is sufficient to prove that each \( \mathcal{E}_p \) is indecomposable. Assume that \( \mathcal{E}_p \) is decomposable. Since \( \mathcal{E}_p \) is a vector bundle of rank two, we get \( \mathcal{E}_p \cong \mathcal{A}_1 \oplus \mathcal{A}_2 \) with each \( \mathcal{A}_i \) a line bundle. Without loss of generality we assume \( h^0(\mathcal{A}_1 \otimes \omega_X(1)) \geq h^0(\mathcal{A}_2 \otimes \omega_X(1)) \). Since \( h^0(\mathcal{E}_p \otimes \omega_X(1)) = 1 \) from the proof of Proposition 2.5, we get \( h^0(\mathcal{A}_1 \otimes \omega_X(1)) = 1 \) and \( h^0(\mathcal{A}_2 \otimes \omega_X(1)) = 0 \). Thus a nonzero section \( \sigma \) of \( \mathcal{E}_p \otimes \omega_X(1) \) has either no zero or an effective Cartier divisor of \( X \) as its zero locus, contradicting the fact that \( \sigma \) vanishes only at \( p \), as shown in the proof of Proposition 2.5. \( \square \)

Throughout the article, as in Proposition 2.5, our construction of aCM sheaf of rank two on \( X \) is in terms of the following extension

\[
0 \rightarrow \omega_X \rightarrow \mathcal{E} \rightarrow \mathcal{I}_{Z,X}(a) \rightarrow 0
\]
with $Z$ a locally complete intersection of codimension two in $X$ and $a \in \mathbb{Z}$. Such extensions are parametrized by $\text{Ext}^1_X(\mathcal{I}_{Z,X}(a), \omega_X)$. In case when $X$ is a surface, the coboundary map associated to (3) is

$$
\delta_1 : H^1(\mathcal{I}_{Z,X}(a)) \to H^2(\omega_X) \cong \mathbb{k}
$$

and by Serre’s duality in \cite[Theorem 3.12]{Ballico} its dual is

$$
\mathbb{k} \cong \text{Hom}_X(\omega_X, \omega_X) \to \text{Ext}^1_X(\mathcal{I}_{Z,X}(a), \omega_X),
$$

which is obtained by applying the functor $\text{Hom}_X(-, \omega_X)$ to (3). Thus the coboundary map $\delta_1$ is surjective if and only if $h(3)$ is a non-trivial extension. Since we assume $h^1(\mathcal{O}_X) = h^1(\omega_X) = 0$, this implies that $h^1(\mathcal{E}) = h^1(\mathcal{I}_{Z,X}(a)) - 1$.

### 3. aCM vector bundle on surfaces in $\mathbb{P}^3$

We always assume that $X \subset \mathbb{P}^3$ is a surface of degree $m$, not necessarily smooth. In particular, its dualizing sheaf is $\omega_X \cong \mathcal{O}_X(m - 4)$ and we get $h^2(\mathcal{O}_X) = \binom{m-1}{3}$. We also have $h^0(\mathcal{O}_X) = 1$ and $h^1(\mathcal{O}_X) = 0$.

**Lemma 3.1.** Each line bundle $\mathcal{O}_X(t)$ with $t \in \mathbb{Z}$, is stable as an $\mathcal{O}_{\mathbb{P}^3}$-sheaf with pure depth 2.

**Proof.** It is enough to deal with the case $t = 0$. Assume the contrary and take a subsheaf $\mathcal{A} \subset \mathcal{O}_X$ such that $\mathcal{B} := \mathcal{O}_X/\mathcal{A}$ has depth 2 and normalized Hilbert polynomial at least the one of $\mathcal{O}_X$. Since $\mathcal{B}$ is a quotient of $\mathcal{O}_X$ with depth 2 and $X$ has no embedded component, we get $\mathcal{B} \cong \mathcal{O}_T$ for $T$ a union of some of the irreducible components of $X_{\text{red}}$ with at most the multiplicities appearing in $X$. This implies that $T \in |\mathcal{O}_{\mathbb{P}^3}(d)|$ for some integer $d$ with $1 \leq d < m$. Now the Hilbert polynomial of $\mathcal{O}_X$ is

$$
P_{\mathcal{O}_X}(t) = \binom{t + 3}{3} - \binom{t - m + 3}{3} = \left(\frac{m}{2}\right) t^2 + \left(2m - \frac{m^2}{2}\right) t + \left(\frac{m^3}{6} - m^2 + \frac{11m}{6}\right).
$$

Similarly, we get the Hilbert polynomial $P_{\mathcal{O}_T}(t)$ of $\mathcal{O}_T$ by replacing $m$ in $P_{\mathcal{O}_X}(t)$ by $d$. Then we see that $p_{\mathcal{O}_X}(t) < p_{\mathcal{O}_T}(t)$ for $t > 0$, a contradiction. \hfill $\square$

**Remark 3.2.** If either $\text{Pic}(X) \cong \mathbb{Z}(\mathcal{O}_X(1))$ or $X$ is integral, then every line bundle is stable. Note also that the proof of Lemma 3.1 shows that the ideal sheaf $\mathcal{I}_{Z,X}$ for any zero-dimensional subscheme $Z \subset X$, is also stable. If $X$ is integral, then any sheaf of rank 1 with positive depth is stable. Thus these sheaves are indecomposable.

**Proposition 3.3.** Let $X \subset \mathbb{P}^3$ be a surface of degree $m \geq 2$ with $X_{\text{reg}} \neq \emptyset$. Fix $p \in X_{\text{reg}}$, and let $\mathcal{E}_p$ be the unique non-trivial extension

$$
(4) \quad 0 \to \mathcal{O}_X(m - 3) \to \mathcal{E}_p \to \mathcal{I}_{p,X} \to 0.
$$

Then $\mathcal{E}_p$ is an aCM vector bundle of rank two on $X$ and $\mathcal{E} \not\cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$ for any $a, b \in \mathbb{Z}$. If one of the following holds, then $\mathcal{E}$ is indecomposable.

(i) $\text{Pic}(X) \cong \mathbb{Z}(\mathcal{O}_X(1))$,

(ii) $\mathcal{O}_X(t)$ for $t \in \mathbb{Z}$ are the only aCM line bundles on $X$, or

(iii) $m \geq 4$ and $X$ is integral.

**Proof.** By Proposition 2.5 it remains to deal with indecomposability of $\mathcal{E}_p$. First show that there are no integers $a, b$ such that $\mathcal{E}_p \cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$. Assume that such $a, b$ exist, say $a \geq b$. Since $h^0(\mathcal{E}_p(3 - m)) = 1$ and $h^0(\mathcal{E}_p(2 - m)) = 0$, we get $(a, b) = (m - 3, 0)$ and $m \geq 3$. Then we get $h^0(\mathcal{E}_p) = \binom{m}{3} + 1$, while (4) gives $h^0(\mathcal{E}_p) = \binom{m}{3}$.
Now assume that $E_p$ is decomposable. Since $E_p$ is locally free and it has rank 2, we have $E_p \cong A_1 \oplus A_2$ with each $A_i \in \text{Pic}(X)$. Since $E_p$ is aCM, each $A_i$ is aCM. In cases (i) and (ii) the assertion holds by above. Thus we assume the case (iii). By Lemma 3.1 and Remark 3.2, (4) is the Harder-Narasimhan filtration of $E_p$. Applying the functor $\text{Hom}_X(E_p, -)$ to (4), we get

$$0 \to \text{Hom}_X(E_p, \mathcal{O}_X(m - 3)) \to \text{Hom}_X(E_p, \mathcal{O}_p) \to \text{Hom}_X(E_p, \mathcal{I}_{p,X}) \to \text{Ext}_X^1(E_p, \mathcal{O}_X(m - 3)).$$

Note that $\text{hom}_X(E_p, \mathcal{O}_X(m - 3)) = h^2(E_p(-1)) = h^0(E_p) = \binom{m}{3}$ by Serre’s duality. By applying the functor $\text{Hom}_X(-, \mathcal{I}_{p,X})$ to (4), we get

$$\text{hom}_X(E_p, \mathcal{I}_{p,X}) = \text{hom}_X(\mathcal{I}_{p,X}, \mathcal{I}_{p,X}) = 1.$$

Thus we have

$$\binom{m}{3} \leq \text{hom}_X(E_p, \mathcal{O}_p) \leq 1 + \binom{m}{3}.$$

Since $h^0(\mathcal{O}_X) = 1$, we have $\text{hom}_X(A_i, A_i) = 1$ for each $i$. So we get

$$\text{hom}_X(E_p, \mathcal{O}_p) = 2 + \text{hom}_X(A_1, A_2) + \text{hom}_X(A_2, A_1).$$

Since $X$ is integral, each $A_i$ is stable and we get either $A_1 \cong A_2$ or $\text{hom}_X(A_i, A_{3-i}) = 0$ for each $i$. In the latter case we have $\text{hom}_X(E_p, \mathcal{O}_p) = 2 < \binom{m}{3}$, a contradiction. In the former case, we have $\text{hom}_X(E_p, \mathcal{O}_p) = 4$ and the only possibility is $m = 4$. But this is also impossible, since we would get $\text{hom}_X^{\geq 2} \cong \text{det}(E_p) \cong \mathcal{O}_X(1)$. \qed

**Proposition 3.4.** Let $X \subset \mathbb{P}^3$ be a surface of degree $m \geq 2$ and let $Z \subset X$ be a zero-dimensional subscheme of degree 3, which is not collinear. Assume that $Z$ is a locally complete intersection inside $X$, i.e. for each $p \in Z_{\text{red}}$ the ideal sheaf of $Z$ at $\mathcal{O}_{X,p}$ is generated by two elements of $\mathcal{O}_{X,p}$. Then there is a vector bundle $\mathcal{G}$ of rank two fitting into an exact sequence

$$0 \to \mathcal{O}_X(m - 4) \to \mathcal{G} \to \mathcal{I}_{Z,X} \to 0$$

with $h^1(\mathcal{G}(t)) = 0$ for all $t \neq 0$ and $h^1(\mathcal{G}) = 1$. There is also an exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{O}_X \to 0,$$

where $\mathcal{E}$ is an aCM vector bundle of rank three such that $\mathcal{E} \cong \mathcal{O}_X(a_1) \oplus \mathcal{O}_X(a_2) \oplus \mathcal{O}_X(a_3)$ for any $(a_1, a_2, a_3) \in \mathbb{Z}^{\geq 3}$. Moreover, if $\text{Pic}(X) \cong \mathbb{Z}(\mathcal{O}_X(1))$, then $\mathcal{E}$ is indecomposable.

**Proof.** Note that $\omega_X \cong \mathcal{O}_X(m - 4)$ and so we have $h^0(\mathcal{I}_{Z,X} \otimes \mathcal{O}_X(4 - m) \otimes \omega_X) = 0$ for all $p \in Z_{\text{red}}$. Since $Z$ is a locally complete intersection, the Cayley-Bacharach condition is satisfied and so there is a locally free $\mathcal{G}$ fitting into [5]; see [9]. From [5] we immediately get $h^1(\mathcal{G}(t)) = 0$ for all $t > 0$, because $Z$ is not collinear. Note that $\text{det}(\mathcal{G}) \cong \mathcal{O}_X(m - 4)$ and $\mathcal{G}$ is a vector bundle of rank two. This implies $\mathcal{G}^\vee \cong \mathcal{G}(4 - m)$. For $t < 0$, we have $h^1(\mathcal{G}(t)) = h^1(\mathcal{G}^\vee(-t) \otimes \omega_X) = h^1(\mathcal{G}(-t)) = 0$ by Serre’s duality. Now consider the coboundary map $\delta_1 : H^1(\mathcal{I}_{Z,X}) \to H^2(\mathcal{O}_X(m - 4)) \cong \mathbb{C}$ with $\ker(\delta_1) = H^1(\mathcal{G})$. The dual of $\delta_1$ is the map

$$\text{Hom}_X(\mathcal{O}_X(m - 4), \mathcal{O}_X(m - 4)) \to \text{Ext}_X^1(\mathcal{I}_{Z,X}, \mathcal{O}_X(m - 4))$$

sending the identity map to the element corresponding to $\mathcal{G}$. This implies that $\delta_1$ is surjective and $h^1(\mathcal{G}) = 1$.

Now we apply Lemma 2.3 to $\mathcal{G}$ to obtain an aCM vector bundle $\mathcal{E}$ of rank three fitting into [5]. Since $h^0(\mathcal{G}) = 1$ and $h^1(\mathcal{G}) = 0$, [5] and [6] give $h^0(\mathcal{E}) = h^0(\mathcal{G}) = \binom{m - 1}{3}$. Assume the existence of integers $a_1 \geq a_2 \geq a_3$ such that $\mathcal{E} \cong \oplus_{i=1}^3 \mathcal{O}_X(a_i)$. Since $\text{det}(\mathcal{E}) \cong \mathcal{O}_X(m - 4)$, we have $a_1 + a_2 + a_3 = m - 4$. If $2 \leq m \leq 3$, then we have $a_1 \geq 0$ from $a_1 + a_2 + a_3 = m - 4$. This implies that $h^0(\mathcal{O}_X(a_1)) > 0 = \binom{m - 1}{3} = h^0(\mathcal{E})$, a contradiction. If $m = 4$, then we have $h^0(\mathcal{E}) = 1$. Since $a_1 + a_2 + a_3 = 0$, we have $\sum_{i=1}^3 h^0(\mathcal{O}_X(a_i)) > 1$, a contradiction. Finally assume $m > 4$. From [5] and [6] we see that
\[ \mathcal{O}_X(m - 2) \] is the first non-trivial sheaf in the Harder-Narasimhan filtration of \( \mathcal{E} \). Thus \( a_1 = m - 4 \) and \( h^0(\mathcal{O}_X(a_1)) = \binom{m - 1}{3} \). Since \( a_2 + a_3 = 0 \), we have \( h^0(\mathcal{O}_X(a_2)) > 0 \) and so \( h^0(\mathcal{E}) > \binom{m - 1}{3} \), a contradiction. Hence we get \( \mathcal{E} \not\cong \oplus_{i=1}^3 \mathcal{O}_X(a_i) \) for any triple of integers \( (a_1, a_2, a_3) \).

It remains to show the last assertion. Assume \( \text{Pic}(X) \cong \mathbb{Z}(\mathcal{O}_X(1)) \) and that \( \mathcal{E} \) is decomposable; by the previous assertion we have \( \mathcal{E} \cong \mathcal{A}_1 \oplus \mathcal{A}_2 \) with \( \text{rank}(\mathcal{A}_i) = i \) for each \( i \) and \( \mathcal{A}_2 \) indecomposable. Set \( \mathcal{A}_1 \cong \mathcal{O}_X(a) \) for \( a \in \mathbb{Z} \). Since \( h^0(\mathcal{E}) = \binom{m - 1}{3} \), we have \( a < m - 4 \). From (5) and (6) we get the existence of a subsheaf \( \mathcal{F} \subset \mathcal{E} \) such that \( \mathcal{F} \cong \mathcal{O}_X(m - 4) \) and \( \mathcal{E}/\mathcal{F} \) is an extension \( \mathcal{H} \) of \( \mathcal{O}_X \) by \( \mathcal{I}_{Z,X} \). Note that \( \mathcal{H} \) is not locally free, because \( \mathcal{I}_{Z,X} \) has not depth 2. In particular, \( \mathcal{H} \) is not isomorphic to \( \mathcal{A}_2 \) and we get \( \mathcal{A}_1 \not\cong \mathcal{F} \); otherwise we would get that \( \mathcal{A}_2 \cong \mathcal{E}/\mathcal{A}_1 \cong \mathcal{H} \) is locally free. So we have \( a < m - 4 \). Now consider a restriction map

\[ u_{\{0\} \oplus \mathcal{A}_2} : \{0\} \oplus \mathcal{A}_2 \to \mathcal{O}_X. \]

If this restriction map is surjective, then its kernel is a line bundle, say \( \mathcal{O}_X(b) \). Since \( X \) is aCM, we get \( \mathcal{A}_2 \cong \mathcal{O}_X \oplus \mathcal{O}_X(b) \), a contradiction. Thus the restriction map is not surjective and so the other restriction map \( u_{\mathcal{A}_2 \oplus \{0\}} \) is not zero. In particular, we get \( a \leq 0 \). If \( a = 0 \), then we have \( \mathcal{A}_1 \cong \mathcal{O}_X \) and the map \( u_{\mathcal{A}_1 \oplus \{0\}} \) is an isomorphism. Thus (5) splits and we get \( h^1(\mathcal{E}) \geq h^1(\mathcal{G}) > 0 \), a contradiction. So we get \( a < 0 \). Since there is no nonzero map \( \mathcal{F} \to \mathcal{A}_1 \) by \( a < m - 4 \), \( \mathcal{F} \) is isomorphic to a subsheaf \( \mathcal{F}_1 \) of \( \mathcal{A}_2 \) and we get \( \mathcal{H} \cong \mathcal{O}_X(a) \oplus \mathcal{A}_2/\mathcal{F}_1 \). From \( a < 0 \) there is no nonzero map \( \mathcal{I}_{Z,X} \to \mathcal{O}_X(a) \). Since \( \mathcal{H} \) is an extension of \( \mathcal{O}_X \) by \( \mathcal{I}_{Z,X} \), we get that \( \mathcal{I}_{Z,X} \cong \mathcal{A}_2/\mathcal{F}_1 \) and so \( \mathcal{O}_X(a) \cong \mathcal{O}_X \), a contradiction.

**Remark 3.5.** In case \( m = 1 \), i.e. \( X = \mathbb{P}^2 \), we fail in obtaining an indecomposable aCM vector bundle of rank three, using the method in Proposition 3.4. Indeed, we get \( \mathcal{G} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{a,b} \) and the corresponding vector bundle of rank three is \( \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{a,b} \).

**Remark 3.6.** In case \( m = 2 \), i.e. \( X = \mathbb{P}^2 \) a smooth quadric surface, there exist exactly three aCM vector bundle up to twist: \( \mathcal{O}_2, \mathcal{O}_3(1,0) \) and \( \mathcal{O}_3(0,1) \). Thus we may set the bundle in Proposition 3.4 is \( \mathcal{E} \cong \mathcal{O}_3(a,b) \oplus \mathcal{O}_3(1,b_2) \oplus \mathcal{O}_3(c_1,c_2) \); since \( c_1(\mathcal{E}) \cong \mathcal{O}_3(-2,-2) \) and \( h^0(\mathcal{E}) = 0 \), there must be exactly one direct summand of the form \( \mathcal{O}_3(a,b) \). After a simple computation, we see that \( \mathcal{E} \cong \mathcal{O}_3(-1,-1) \oplus \mathcal{O}_3(-1,0) \oplus \mathcal{O}_3(0,-1) \).

**Corollary 3.7.** Let \( X \subset \mathbb{P}^3 \) be anion of multiple planes in which at least one plane occurs with multiplicity 1. Then there is an indecomposable aCM vector bundle of rank three on \( X \). If \( m > 4 \), we have a family of such aCM vector bundles of dimension 6.

**Proof.** Assume that \( X \) has one component \( H \) with multiplicity 1. In this case we take as \( Z \) a set of 3 general points in \( H \). Then the first assertion follows from Proposition 3.4. Note that the set of all such \( Z \) has dimension 6. Assume now that \( X \) has a component \( H \) with multiplicity 3. Fix a general point \( p \in H \) and take a general line \( L \subset \mathbb{P}^3 \) with \( p \in L \). Then set \( Z \) to be the connected component of the scheme \( X \cap L \) with \( p \) as its reduction. Then we may get the assertion from Proposition 3.4 and that \( \text{Pic}(X) \cong \mathbb{Z}(\mathcal{O}_X(1)) \) by [2, Lemma 2.5].

**Proposition 3.8.** Let \( X \subset \mathbb{P}^3 \) be a surface of degree \( m \geq 4 \) with an irreducible component \( Y \) appearing with multiplicity 2 in \( X \). Fix \( p \in Y_{\text{reg}} \) so that \( Y \) is the only irreducible component of \( X \) containing \( p \). For a general line \( L \subset \mathbb{P}^3 \) containing \( p \), let \( Z \subset X \) be the connected component of \( L \cap X \) with \( p \) as its reduction. We have \( \deg(Z) = 2 \) and there is an aCM vector bundle \( \mathcal{E}_Z \) of rank two fitting into an exact sequence

\[ 0 \to \mathcal{O}_X(m - 4) \to \mathcal{E}_Z \to \mathcal{I}_{Z,X} \to 0. \]

Moreover, there is an integral 4-dimensional variety \( \Delta \), a flat family of aCM vector bundles on \( X \) such that each isomorphism classes in \( \overline{\mathcal{M}_X} \) appears for a unique element in \( \Delta \) with the following properties.

(i) For any \( \mathcal{E}_Z \in \Delta \), there are no integers \( a, b \) with \( \mathcal{E}_Z \cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b) \).

(ii) A very general \( \mathcal{E}_Z \in \Delta \) is indecomposable.

(iii) If \( \text{Pic}(X) \cong \mathbb{Z}(\mathcal{O}_X(1)) \), then each \( \mathcal{E}_Z \in \Delta \) is indecomposable.
(iv) If $Z(\mathcal{O}_X(1))$ are the only aCM line bundles on $X$, then each $\mathcal{E}_Z \in \Delta$ is indecomposable.

**Proof.** Since no other component of $X$ but $Y$ contains $p$ and $p$ is a regular point of $X$, we have $\deg(Z) = 2$; it is enough to take as $L$ any line through $p$ not contained in the tangent plane $T_pY$ of $Y$.

Since $\omega_X \cong \mathcal{O}_X(m - 4)$, we have $h^0(\mathcal{O}_X(m - 4) \otimes \omega_X) = 1$ and $\mathcal{O}_X(m - 4) \otimes \omega_X$ is globally generated. Thus we have $h^0(T_pX \otimes \mathcal{O}_X(m - 4) \otimes \omega_X) = 0$. Since $Z$ is a locally complete intersection, the Cayley-Bacharach condition is satisfied for (7) and so there is a locally free $\mathcal{E}_Z$ fitting into (7); see [6].

Since $\mathcal{O}_X(1)$ is very ample and $\deg(Z) = 2$, we get $h^1(\mathcal{E}_Z(t)) = 0$ for all $t > 0$ by [4]. Note that $\det(\mathcal{E}_Z) \cong \mathcal{O}_X(m - 4)$ and $\mathcal{E}_Z$ is a vector bundle of rank two. This implies $\mathcal{E}_Z \cong \mathcal{E}_Z(4 - m)$. For $t < 0$, we have $h^1(\mathcal{E}_Z(t)) = h^1(\mathcal{E}_Z((m - t - 4)) = h^1(\mathcal{E}_Z(-t)) = 0$ by Serre’s duality. Now consider the coboundary map $\delta_1 : H^1(T_pX) \rightarrow H^2(\mathcal{O}_X(m - 4)) \cong \mathbb{k}$ with $\ker(\delta_1) = H^1(\mathcal{E}_Z)$. The dual of $\delta_1$ is the map

$$\text{Hom}_X(\mathcal{O}_X(m - 4), \mathcal{O}_X(m - 4)) \rightarrow \text{Ext}_X^1(\mathcal{I}_Z, \mathcal{O}_X(m - 4))$$

sending the identity map to the element corresponding to $\mathcal{E}_Z$. This implies that $\delta_1$ is non-zero and hence and $h^1(\mathcal{E}_Z) = 0$. Thus $\mathcal{E}_Z$ is aCM.

The set of all $p \in Y_{\text{reg}}$ such that $Y$ is the only irreducible component of $X$ containing $p$ is an irreducible 2-dimensional variety $\Delta'$. For each $p \in \mathbb{P}^3$ the set of all lines through $p$ is a $\mathbb{P}^2$. Define a variety $\Delta$ as follows:

$$\Delta := \{(p, L) \mid p \in \Delta' \text{ and } L \text{ a line in } \mathbb{P}^3 \text{ with } p \in L \text{ and } L \not\subseteq T_pY\}.$$ 

Since $m \geq 4$, we have $h^0(\mathcal{I}_{Z,X}(4 - m)) = 0$. Thus (7) gives $h^0(\mathcal{E}_Z(4 - m)) = 1$. Thus the isomorphism classes of $\mathcal{E}_Z$ uniquely determine $Z_i$, i.e. if $\mathcal{E}_Z \not\cong \mathcal{E}_{Z'}$, then we get $Z \not\cong Z'$. For two elements $(p_1, L_1), (p_2, L_2) \in \Delta$, let $Z_i$ be the subscheme of degree 2 determined by $(p_i, L_i)$ for each $i = 1, 2$. Since each $p_i$ is the reduction of $Z_i$ and $L_i$ is the line spanned by $Z_i$, the variety $\Delta$ uniquely parametrizes the isomorphism classes of the aCM vector bundles $\mathcal{E}_Z$.

Assume $\mathcal{E}_Z \cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$ for some integers $a, b$ with $a \geq b$. Since $\det(\mathcal{E}_Z) \cong \mathcal{O}_X(m - 4)$, we have $b = m - 4 - a$. But since $h^0(\mathcal{E}_Z(4 - m)) = 1$, the only possibility is that $a = 4 - m$ and $b < 0$, a contradiction. Thus we get (i). We may get (ii) as in the proof of Theorem 2.6. Now assume that $\mathcal{E}_Z$ is decomposable, say $\mathcal{E}_Z \cong \mathcal{A}_1 \oplus \mathcal{A}_2$ with each $\mathcal{A}_i$ a line bundle. Since $\mathcal{E}_Z$ is aCM, each $\mathcal{A}_i$ is also aCM. Thus (iii) and (iv) follow from (i). \hfill \square

**Remark 3.9.** In case $m = 2$, i.e. $X = 2H$ the double plane with a hyperplane $H \subset \mathbb{P}^3$, the vector bundle $\mathcal{E}_Z$ described in Proposition 3.8 is the vector bundle $\mathcal{O}_X(-1)^{\oplus 2}$.

**Theorem 3.10.** Let $X \subset \mathbb{P}^3$ be a surface of degree $m \geq 4$ with $X_{\text{reg}} \neq \emptyset$, i.e. $X$ has an irreducible component $Y$ appearing with multiplicity 1. We further assume that either Pic$(X) = \mathbb{Z}(\mathcal{O}_X(1))$ or $X$ is integral. For a fixed integer $s > 0$ and a set $S \subset X_{\text{reg}} \cap Y$ with $\sharp(S) = s$, a general sheaf $\mathcal{E}_S$ fitting into an exact sequence

$$0 \rightarrow \mathcal{O}_X(m - 3)^{\oplus s} \rightarrow \mathcal{E}_S \rightarrow \bigoplus_{p \in S} \mathcal{I}_{p,X} \rightarrow 0,$$

is a locally free, indecomposable and aCM sheaf of rank $2s$. Moreover, if $S' \subset X_{\text{reg}} \cap Y$ is another set with $\sharp(S') = s$ and $S' \neq S$, then we have $\mathcal{E}_{S'} \not\cong \mathcal{E}_S$.

We have $\text{Ext}_X^1(\mathcal{I}_{p,X}, \mathcal{O}_X(m - 3)) = h^1(\mathcal{I}_{p,X}(-1)) = 1$ for each $p \in X_{\text{reg}}$ by Serre’s duality. So the extension $\mathcal{E}_S$ corresponds to an element in a finite dimensional vector space

$$\mathbb{E}(S) := \text{Ext}_X^1(\bigoplus_{p \in S} \mathcal{I}_{p,X}, \mathcal{O}_X(m - 3)^{\oplus s}) \cong \mathbb{k}^s.$$ 

If $s = 1$, say $S = \{p\}$, the dimension of $\mathbb{E}(S)$ is one. Thus there exists a unique non-trivial extension. Denote this non-trivial extension simply by $\mathcal{E}_p$.

In Theorem 3.10 a “general” choice of $\mathcal{E}_S$ means that there exists a non-empty Zariski open subset $U \subset \mathbb{E}(S)$ such that the middle term of any extension in $U$ is aCM, locally free and indecomposable.
4. Proof of Theorem 3.10

Set $E'(S)$ to be the set of all elements in $E(S)$ whose corresponding middle term is locally free and aCM.

**Lemma 4.1.** $E'(S)$ is a non-empty open subset of $E(S)$.

**Proof.** Let $\tilde{E}$ be the universal family over $E(S)$, i.e. let $\tilde{E}$ be the coherent sheaf over $X \times E(S)$ such that $E_a := \tilde{E}|_{X \times \{a\}}$ is the sheaf corresponding to $a \in E(S)$. Let $\pi_2 : X \times E(S) \to E(S)$ denote the projection onto the second factor, and set $\Gamma := \{(x, a) \in X \times E(S) \mid \tilde{E}$ is not locally free at $(x, a)\}$. Since local freeness is an open condition, $\Gamma$ is a closed subscheme of $X \times E(S)$. Since $\pi_2$ is proper, $\pi_2(\Gamma)$ is closed in $E(S)$ and hence $E(S) \setminus \pi_2(\Gamma)$ is open in $E(S)$. We have $E(S) \setminus \pi_2(\Gamma) = \{a \in E(S) \mid E_a$ is locally free$\}$.

On the other hand, we check that aCM is an open property for the set of all locally free $E \in E(S)$. Note that $h^1(\mathcal{I}_{S,X}(t)) = 0$ for any set $S \subset X$ with cardinality $s$ and all $t \geq s - 1$. In particular, we get $h^1(\mathcal{E}(t)) = 0$ for all $t \geq s - 1$ by \[5\]. Dualizing \[5\], or using the relative case of \[25\], Théorème 1 in page 268 with the fact that a locally free $E$ has depth 2, we get the existence of a negative integer $t_1$ such that $h^1(\mathcal{E}(t)) = 0$ for all $t < t_1$ and all locally free $E \in E(S)$. By the semicontinuity theorem for cohomology in \[15\], Theorem III.12.8 the set of all $E \in E(S)$ such that $h^1(\mathcal{E}(t)) = 0$ for all $t$ such that $t_1 \leq t \leq s - 2$ is an open subset $U$ of $E(S)$. A locally free $E \in E(S)$ is aCM if and only if $E \in U$.

Now we see that $E'(S)$ is an open subset of $E(S)$. Thus it remains to prove that $E'(S) \neq \emptyset$. Proposition 3.3 gives the case $s = 1$. For $s > 1$, we may find a direct sum of aCM vector bundles of rank two fitting into \[8\], i.e. take $\oplus_{p \in S} E_p$. This implies $E'(S) \neq \emptyset$.

**Remark 4.2.** In the set-up of \[8\] set $\mathcal{A} := e(\mathcal{O}_X(m - 3)^{\oplus s})$. By Lemma 3.1 and Remark 3.2 together with the assumption $m \geq 3$, we see that $\mathcal{A}$ is the first term of the Harder-Narasimhan filtration of $E_S$. Thus we get $f(\mathcal{A}) \subseteq \mathcal{A}$ for any $f \in End(E_S)$.

**Lemma 4.3.** If $E$ is the middle term of an extension $\varepsilon \in E'(S)$, then $E$ has no line bundle as a factor.

**Proof.** Assume that $E$ is a line bundle that is a factor of $E$, i.e. $E = \mathcal{L} \oplus \mathcal{G}$ for some aCM vector bundle $\mathcal{G}$ of rank $2s - 1$. Since $m \geq 3$, we have

$$h^0(\mathcal{L}(3 - m)) + h^0(\mathcal{G}(3 - m)) = h^0(\mathcal{E}(3 - m)) = s.$$ 

First assume $h^0(\mathcal{L}(3 - m)) = 0$ and $h^0(\mathcal{G}(3 - m)) = s$. Then we have $\mathcal{A} := e(\mathcal{O}_X(m - 3)^{\oplus s}) \subset \{0\} \oplus \mathcal{G}$ in \[8\] and so $\mathcal{L} \cong \mathcal{I}_{p,X}$ for some $p \in S$ by the uniqueness of the Harder-Narasimhan filtration \[8\], a contradiction. Thus we have $h^1(\mathcal{L}(3 - m)) > 0$ and so $h^0(\mathcal{G}(3 - m)) < s$. In particular, there is a nonzero map $u : \mathcal{O}_X(m - 3) \to \mathcal{L}$. Assume for the moment that $Pic(X) \cong \mathbb{Z}(\mathcal{O}_X(1))$ and write $\mathcal{L} \cong \mathcal{O}_X(a)$ for some $a \in \mathbb{Z}$. The map $u$ gives $a \geq m - 3$. Since $m \geq 3$, \[8\] is the Harder-Narasimhan filtration of $E$ and we get $a = m - 3$. Thus $\mathcal{G}$ fits into an exact sequence

$$0 \to \mathcal{O}_X(m - 3)^{\oplus (s - 1)} \to \mathcal{G} \to \oplus_{p \in S} \mathcal{I}_{p,X} \to 0.$$ 

Then we get $h^1(\mathcal{G}(-1)) \geq 1$ from $h^1(\mathcal{I}_{p,X}(-1)) = 1$ and $h^2(\mathcal{A}(m - 4)) = 1$. Thus $\mathcal{G}$ is not aCM, a contradiction. If $X$ is integral, then every line bundle is stable and so \[8\] is the Harder-Narasimhan filtration of $E$, we get either $\mathcal{L} \cong \mathcal{O}_X(m - 3)$; we get a contradiction as above, or $\mathcal{L}$ is a factor of $\oplus_{p \in S} \mathcal{I}_{p,X}$, which is not locally free, a contradiction.

Let $F(S)$ (resp. $F'(S)$) be the set of isomorphism classes of middle terms of extensions in $E(S)$ (resp. $E'(S)$). Let us denote by $E = E(\varepsilon)$ the middle term of the extension corresponding to $\varepsilon \in E'(S)$.

**Lemma 4.4.** For two non-empty finite sets $S_1, S_2 \subset X_{reg}$ with $\sharp(S_1) = s_1$, take $E_i \in F'(S_i)$ and call $\mathcal{A}_i$ the subsheaf of $E_i$ isomorphic to $\mathcal{O}_X(m - 3)^{\oplus s_i}$ for each $i = 1, 2$. If there exists a map $f : E_1 \to E_2$ with $f(\mathcal{E}_1) \subseteq \mathcal{A}_2$, then we have $S_1 \cap S_2 \neq \emptyset$.

**Proof.** Since $\text{Hom}_X(\mathcal{O}_X(m - 3), \mathcal{I}_{p,X}) = 0$ for all $p \in X$, we have $f(\mathcal{A}_1) \subseteq \mathcal{A}_2$. In particular, $f$ induces a nonzero map $\hat{f} : \oplus_{p \in S_1} \mathcal{I}_{p,X} \to \oplus_{q \in S_2} \mathcal{I}_{q,X}$. This implies that $S_1 \cap S_2 \neq \emptyset$. \[\square\]
Lemma 4.5. Assume that $E \in \mathbb{F}(S)$ is decomposable; $E \cong E_1 \oplus \cdots \oplus E_h$ with each $E_i$ indecomposable. Then there is a partition $S = \bigsqcup_{i=1}^h S_i$ with $E_i \in \mathbb{F}(S_i)$ for each $i$. If there is another decomposition $E \cong E'_1 \oplus \cdots \oplus E'_k$ with each $E'_i$ indecomposable, then we get $k = h$ and there is a permutation $\sigma : \{1, \ldots, h\} \to \{1, \ldots, k\}$ such that $E'_{\sigma(i)} \cong E_i$ for all $i$ and $E'_{\sigma(i)} \in \mathbb{F}(S(\sigma(i)))$.

Proof. We use induction on $s$. The case $s = 1$ is true, because each $E_p$ for $p \in X_{\text{reg}}$ is indecomposable by Proposition 3.3. Since $E$ is aCM by the definition of $\mathbb{F}(S)$, each $E_i$ is also aCM. We consider the subsheaf $A \cong O_X(m - 3)^{\oplus \ast} \subset E$ as in Remark 4.2 and set $G_i := A \cap E_i$. Since the Harder-Narasimhan filtration of $E$ is obtained from the ones of each factors, we have

$$A \cong \oplus_{i=1}^h G_i \quad \text{and} \quad \oplus_{p \in S} I_{p,X} \cong \oplus_{i=1}^h E_i / G_i.$$  

By Lemma 4.3 we have $G_i \subseteq E_i$ for all $i$. By Remark 3.2 we may write $S = \bigsqcup_{i=1}^h S_i$ with $E_i / G_i \cong \oplus_{p \in S_i} I_{p,X}$. Since $E_i / G_i \neq 0$, we have $S_i \neq \emptyset$ for all $i$. Thus the set $\{S_1, \ldots, S_h\}$ gives a partition of $S$.

To prove the first part of the lemma it is sufficient to prove that $\sharp(S_i) = \text{rank}(G_i)/2$ for all $i$. If this is not true, then there is $i \in \{1, \ldots, h\}$ with $\sharp(S_i) > \text{rank}(G_i)/2$, i.e. $\text{rank}(G_i) < \sharp(S_i)$. The exact sequence

$$0 \to G_i(-1) \to E_i(-1) \to \oplus_{p \in S_i} I_{p,X}(-1) \to 0$$

gives $h^1(E_i(-1)) \geq \sharp(S_i) - \text{rank}(G_i) > 0$. In particular, $E_i$ is not aCM, a contradiction.

Now we check the last assertion of the lemma. Take two partitions

$$S = S_1 \sqcup \cdots \sqcup S_h = S'_1 \sqcup \cdots \sqcup S'_k$$

such that there is a decomposition

$$E \cong E_1 \oplus \cdots \oplus E_h \cong E'_1 \oplus \cdots \oplus E'_k$$

with $E_i \in \mathbb{F}(S_i)$ and $E'_i \in \mathbb{F}'(S'_i)$ indecomposable. By the Krull-Schmidt theorem in [1], we get $h = k$ and there is a permutation $\sigma : \{1, \ldots, h\} \to \{1, \ldots, h\}$ such that $B_{\sigma(i)} \cong E_i$ for all $i$. By renaming $\{E'_1, \ldots, E'_k\}$, we may assume that $E'_i \cong E_i$ for all $i$. This implies

$$\sharp(S_i) = \text{rank}(E_i)/2 = \text{rank}(E'_i)/2 = \sharp(S'_i).$$

Now fix an isomorphism $f_i : E_i \to E'_i$ for each $i$. Since [3] gives the Harder-Narasimhan filtrations of $E_i$ and $E'_i$, the map $f_i$ induces an isomorphism $\tilde{f}_i : \oplus_{p \in S_i} I_{p,X} \to \oplus_{p \in S'_i} I_{p,X}$. Since $p$ is the unique point of $X$ at which $I_{p,X}$ is not locally free, we get $S_i = S'_i$. For each $i$, let $A_i$ be the unique subsheaf of $E_i$ isomorphic to $O_X(m - 3)^{\oplus \ast}$. Then for any embedding $u : E_i \to E_i \oplus \cdots \oplus E_h$, the composition $v_j \circ \pi_j \circ u$

$$E_i \to E_i \oplus \cdots \oplus E_h \xrightarrow{\pi_i} E_j \to \oplus_{p \in S_i} I_{p,X}$$

is zero for any $j \neq i$ by Lemma 4.4, where $\pi_j : E \to E_j$ is the projection and $v_j : E_j \to \oplus_{p \in S_j} I_{p,X}$ is the surjection in [8] for $S_j$. Since $u$ is an embedding, we see that $v_j \circ \pi_j \circ u$ is surjective. Thus $G := \pi_i(u(E_i))$ is a subsheaf with $v_i(G) = \oplus_{p \in S_i} I_{p,X}$.

Lemma 4.6. With the setting as in Theorem 3.10, we have $\text{ext}_X^1(E_p, E_q) \geq 2$ for two points $p, q \in X_{\text{reg}}$, possibly $p = q$.

Proof. Set $F_o := E_o(3 - m)$ for $o \in \{p, q\}$. Since $\text{Ext}_X^1(E_p, E_q) \cong \text{Ext}_X^1(F_p, F_q)$, we have $\chi(E_p \otimes E_q^*) = \chi(F_p \otimes F_q^*)$. Since Euler's characteristic is constant in a flat family of vector bundles and $p, q \in X_{\text{reg}}$, it is sufficient to compute $\chi(F_p \otimes F_q^*)$ when $X$ is smooth. So from now on we assume that $X$ is smooth.

Since a smooth surface in $\mathbb{P}^3$ is connected, the same observation applied to a family of vector bundles on $X$ shows $\chi(F_p \otimes F_q^*) = \chi(F_p \otimes F_q^*)$. We have an exact sequence

$$(9) \quad 0 \to O_X \xrightarrow{w} F_p \xrightarrow{w} I_{p,X}(3 - m) \to 0$$
with $\det(F_p) \cong \mathcal{O}_X(3 - m)$ and $c_2(F_p) = 1$. Since $X \subset \mathbb{P}^3$ is a surface of degree $m$, we have $c_1(F_p)^2 = m(m - 3)^2$. By Riemann-Roch for $\mathcal{E}nd(F_p)$, we have

$$\chi(\mathcal{E}nd(F_p)) = c_1(F_p)^2 - 4c_2(F_p) + 4\chi(\mathcal{O}_X) = m(m - 3)^2 - 4 + 4\left(\frac{m - 1}{3}\right) + 4$$

$$= \frac{1}{6}(10m^3 - 60m^2 + 98m - 24).$$

In particular, we have $\chi \sim \frac{5}{4}m^3$ for $m \gg 0$. Note that by Serre’s duality we have $h^2(F_p \otimes F_p^\vee) = h^0(F_p \otimes F_p(m - 4)).$

**Claim 1:** We have $\text{hom}_X(F_p, F_p) = 1 + \binom{m}{3}^3$.

**Proof of Claim 1:** We have $\text{hom}_X(I_p, X(3 - m), \mathcal{O}_X) = h^0(\mathcal{O}_X(m - 3)) = \binom{m}{3}$ and any nonzero map $u : I_p, X(3 - m) \to \mathcal{O}_X$ induces an element $\tilde{u}$ in $\text{Hom}_X(F_p, F_p)$ with rank one as the following composition:

$$F_p \xrightarrow{w} I_p, X(3 - m) \to \mathcal{O}_X \xrightarrow{v} F_p.$$

This defines a one-to-one map $\text{hom}_X(I_p, X(3 - m), \mathcal{O}_X) \to \text{hom}_X(F_p, F_p)$, because for two $u, u' \in \text{hom}_X(I_p, X(3 - m), \mathcal{O}_X)$ we have $\text{Im}(\tilde{u}) \neq \text{Im}(\tilde{u}')$. The vector space $\text{Hom}_X(F_p, F_p)$ also contains the nonzero multiples of the identity map $F_p \to F_p$ and these maps have rank two. Thus we get $h^0(F_p \otimes F_p^\vee) \geq 1 + \binom{m}{3}^3$. On the other hand, for any $f \in \text{Hom}_X(F_p, F_p)$ we get $w \circ f \circ (v(\mathcal{O}_X)) \subseteq v(\mathcal{O}_X)$ from $h^0(I_p, X(3 - m)) = 0$. Thus $w \circ f \circ$ induces a map $f_1 : \mathcal{O}_X \to \mathcal{O}_X$, which is induced by the multiplication by $c \in k$. Hence $f - c \cdot \text{Id}_{F_p}$ is induced by a unique $g \in \text{Hom}_X(I_p, X(3 - m), F_p)$. Since $F_p$ is locally free and $X$ is smooth, we have $\text{Hom}_X(I_p, X(3 - m), F_p) = H^0(F_p(m - 3))$. By [8] we have $h^0(F_p(m - 3)) = \binom{m}{3}$ and so $\text{hom}_X(F_p, F_p) \leq 1 + \binom{m}{3}^3$.

**Claim 2:** We have $\text{hom}_X(F_p, F_p(m - 4)) \geq \binom{2m - 4}{3} + 2\binom{m - 1}{3} - \binom{m - 4}{3} - 1$.

**Proof of Claim 2:** For any $f \in \text{Hom}_X(F_p, F_p(m - 4))$, set $f_1 := f|_{\mathcal{O}_X}$. Since $h^0(\mathcal{O}_X(-1)) = 0$, we have $w \circ f_1 = 0$ and so $f_1(v(\mathcal{O}_X)) \subseteq v(\mathcal{O}_X(m - 4)))$. Take $f$ with $f_1 \equiv 0$. Such a map $f$ is uniquely determined by an element in $\text{Hom}_X(I_p, X(3 - m), F_p(m - 4))$ and the converse also holds. Since $F_p(m - 4)$ is locally free and $X$ is smooth at $p$, we have $\text{Hom}_X(I_p, X(3 - m), F_p(m - 4)) = \text{Hom}_X(\mathcal{O}_X(3 - m), F_p(m - 4)) = H^0(F_p(2m - 7))$. Since $h^1(\mathcal{O}_X(t)) = 0$ for any $t \in \mathbb{Z}$, [9] gives

$$h^0(F_p(2m - 7)) = h^0(\mathcal{O}_X(2m - 7)) + h^0(\mathcal{O}_X(m - 4)) - 1 = \binom{2m - 4}{3} - \binom{m - 4}{3} + \binom{m - 1}{3} - 1.$$

Note that a map $f$ obtained by a composition

$$F_p \xrightarrow{w} I_p, X(3 - m) \to \mathcal{O}_X(m - 4) \xrightarrow{v} F_p(m - 4)$$

has $f_1 \equiv 0$. Now for any linear subspace $W \subset \text{Hom}_X(F_p, F_p(m - 4))$ such that $f_1 \neq 0$ for any $f \in W \setminus \{0\}$, we would get

$$\dim \text{Hom}_X(F_p, F_p(m - 4)) \geq \binom{2m - 4}{3} - \binom{m - 4}{3} + \binom{m - 1}{3} - 1 + \dim W.$$

We may choose $W$ to consist of the compositions of the identity map $F_p \to F_p$ with the multiplication by an element of $H^0(\mathcal{O}_X(m - 4))$. Then we have $\dim W = \binom{m - 1}{3}$.

Combining Claims 1 and 2, we get

$$h^0(F_p \otimes F_p^\vee) + h^2(F_p \otimes F_p^\vee) \geq \binom{2m - 4}{3} + \binom{m - 4}{3} + 2\binom{m - 1}{3} - \binom{m - 4}{3}$$

$$= \frac{1}{6}(10m^3 - 60m^2 + 98m - 12).$$
Thus we have
\[ h^1(F_p \otimes F'_p) = h^0(F_p \otimes F'_p) + h^2(F_p \otimes F'_p) - \chi(\text{End}(F_p)) \geq 2 \]
and so we get the assertion.

**Proof of Theorem 3.10** By Remark 4.2, \( O \) is the Harder-Narasimhan filtration of \( \mathcal{E}_S \). Proposition 3.3 gives the case \( s = 1 \). For \( s > 1 \), we may find a direct sum of \( s \) vector bundles of rank 2 from the case \( s = 1 \), fitting into \([8]\). just take \( \oplus_{p \in \mathcal{E}_p} \). So a general extension in \( \mathcal{E}(S) \) has a locally free and aCM middle term, because being local free and aCM are both open conditions.

Note that \( h^0(\mathcal{E}_S(3 - m)) = s \) from \([8]\). In particular there is a unique subsheaf \( A \subset \mathcal{E}_S \) isomorphic to \( \mathcal{O}_X(m - 3) \oplus s \) and for each \( f \in \text{Hom}(\mathcal{O}_X(m - 3), \mathcal{E}_S) \) we have \( f(\mathcal{O}_X(m - 3)) \subseteq A \). Now by Lemma 3.1 and Remark 3.2 the extension \([8]\) is the Harder-Narasimhan filtration of \( \mathcal{E}_S \). By uniqueness of the Harder-Narasimhan filtration, we get \( \mathcal{E}_S \not\cong \mathcal{E}_{S'} \) for \( S \neq S' \).

Now it remains to show the indecomposability of \( \mathcal{E}_S \). By Lemma 4.3 there is no rank one factor of \( \mathcal{E}_S \).

**Claim 1:** For two distinct points \( p, q \) in \( X_{\text{reg}} \), we have
\[ \text{Hom}_X(I_{p,X}, I_{q,X}) = 0, \text{Hom}_X(\mathcal{E}_p, I_{q,X}) = 0 \text{ and } \text{Ext}^1_X(I_{p,X}, I_{q,X}) = 0. \]

**Proof of Claim 1:** By an extension theorem for locally free sheaves in [15] Exercise I.3.20, we have \( \text{Hom}_X(I_{p,X}, I_{q,X}) = \text{Hom}_X(\mathcal{O}_X, I_{q,X}) = 0 \). The second vanishing is obtained from the first vanishing and \( \text{Hom}_X(\mathcal{O}_X(m - 3), I_{q,X}) = 0 \). For the last vanishing, we apply the functor \( \text{Hom}_X(I_{p,X}, -) \) to the standard exact sequence for \( I_{q,X} \subset \mathcal{O}_X \) and obtain an exact sequence
\[ 0 \rightarrow \text{Hom}_X(I_{p,X}, \mathcal{O}_X) \rightarrow \text{Hom}_X(I_{p,X}, \mathcal{O}_q) \rightarrow \text{Ext}^1_X(I_{p,X}, I_{q,X}) \rightarrow \text{Ext}^1_X(I_{p,X}, \mathcal{O}_X) \]
by the first vanishing in the Claim. Here we have
\[ \text{Hom}_X(I_{p,X}, \mathcal{O}_X) \cong \text{Hom}_X(I_{p,X}, \mathcal{O}_q) \cong k \]
and \( \text{Ext}^1_X(I_{p,X}, \mathcal{O}_X) \cong H^1(I_{p,X}(m - 4)) \) by Serre’s duality. Then we get the assertion from the assumption that \( m \geq 4 \).

(a) First assume \( s = 2 \) and take two distinct points \( p, q \) in \( X_{\text{reg}} \).

**Claim 2:** If there exists a sheaf \( \mathcal{G} \not\cong \mathcal{E}_p \oplus \mathcal{E}_q \) fitting into the exact sequence
\[ 0 \rightarrow \mathcal{E}_p \rightarrow \mathcal{G} \rightarrow \mathcal{E}_q \rightarrow 0, \tag{10} \]
then the case \( s = 2 \) is true.

**Proof of Claim 2:** Such a sheaf \( \mathcal{G} \) would be locally free and aCM with rank 4. Since \( h^1(\mathcal{O}_X) = 0 \) and \([8]\) gives the Harder-Narasimhan filtrations of \( \mathcal{E}_p \) and \( \mathcal{E}_q \) by Lemmas 3.1 and Remark 3.2 \( \mathcal{G} \) has a subsheaf \( \mathcal{F} \cong \mathcal{O}_X(m - 3) \oplus s \) such that \( \mathcal{G}/\mathcal{F} \) is an extension of \( I_{q,X}(1) \) by \( I_{p,X}(1) \). Claim 1 gives \( \mathcal{G}/\mathcal{F} \cong I_{p,X} \oplus I_{q,X} \) and so we get \( \mathcal{G} \cong \mathcal{E}_S \) with \( S = \{p, q\} \).

**Claim 3:** If \( \mathcal{G} \cong \mathcal{E}_p \oplus \mathcal{E}_q \) for all \( \mathcal{G} \) in \([10]\), then we have \( \text{Ext}^1_X(\mathcal{E}_p, \mathcal{E}_p) = 0 \).

**Proof of Claim 3:** Let \( \mathcal{G} \cong \mathcal{E}_p \oplus \mathcal{E}_q \) fitting into \([10]\) correspond to \( \varepsilon \in \text{Ext}^1_X(\mathcal{E}_p, \mathcal{E}_p) \). Then it is sufficient to prove that \( \varepsilon = 0 \), or \( \ker(v) \cong \mathcal{E}_p \oplus \{0\} \). But since \( \ker(v) \cong \mathcal{E}_p \), it is sufficient to prove that either \( \mathcal{E}_p \oplus \{0\} \cong \ker(v) \) or \( \mathcal{E}_p \oplus \{0\} \cong \ker(v) \). Assume \( v(\mathcal{E}_p \oplus \{0\}) \neq 0 \). Since \( \text{Hom}_X(\mathcal{E}_p, I_{q,X}) = 0 \) by Claim 1, we have \( v(\mathcal{E}_p \oplus \{0\}) \subseteq \mathcal{O}_X(m - 3) \). This implies that the restriction of the surjection \( \mathcal{E}_q \rightarrow I_{q,X} \) to \( v(\{0\} \oplus \mathcal{E}_q) \) is surjective. Since \( h^0(\mathcal{O}_X) = 1 \) and \( \text{Hom}_X(\mathcal{E}_p, \mathcal{O}(m - 3), I_{q,X}) = 0 \), we get either \( v(\{0\} \oplus \mathcal{O}_X(m - 3)) = 0 \) or \( v \) induces an isomorphism \( \{0\} \oplus \mathcal{O}_X(m - 3) \rightarrow \mathcal{O}_X(m - 3) \). Assume for the moment \( v(\{0\} \oplus \mathcal{O}_X(m - 3)) = 0 \). Since \( v(\mathcal{E}_p \oplus \{0\}) \) maps to \( 0 \) in \( I_{q,X} \), we get that \( v(\{0\} \oplus \mathcal{E}_q) \) is a subsheaf of \( \mathcal{E}_q \) which maps isomorphically onto \( I_{q,X} \). So we get \( \mathcal{E}_q \cong \mathcal{O}_X(m - 3) \oplus I_{q,X} \), a contradiction. Now assume \( v(\{0\} \oplus \mathcal{O}_X(m - 3) = 0 \). Since \( v(\{0\} \oplus \mathcal{E}_q) \) maps surjectively onto \( I_{q,X} \), the surjection \( v \) induces an isomorphism \( \{0\} \oplus \mathcal{E}_q \rightarrow \mathcal{E}_q \). Hence we get \( \mathcal{E}_q \oplus \{0\} \subseteq \ker(v) \).

□
Since \( \text{Ext}_X^1(\mathcal{E}_q, \mathcal{E}_p) \neq 0 \) by Lemma 4.6, Claim 3 concludes the proof of the case \( s = 2 \).

(b) Assume \( s > 2 \) and that Theorem 3.10 holds for smaller numbers. On \( \mathcal{E}(S) \) there is a universal family of extensions, i.e. a coherent sheaf \( \mathcal{E}_\mathcal{V} \) over \( \mathcal{E}(S) \times X \) such that for each \( \varepsilon \in \mathcal{E}(S) \) the sheaf \( \mathcal{V}_\varepsilon \) is the middle term \( \mathcal{E} \) of the extension corresponding to \( \varepsilon \). We call \( \mathcal{V} \) the restriction of \( \mathcal{V} \) to \( \mathcal{E}(S) \times X \); we thus consider the family of acM vector bundles induced from the extensions in \( \mathcal{E}(S) \).

Call \( \pi_1 : \mathcal{E}(S) \times X \to \mathcal{E}(S) \) the projection onto the first factor, and set \( A_S := \pi_1 \mathcal{Hom}(\mathcal{V}', \mathcal{V}) \). Since \( \pi_1 \) is a proper morphism, \( A_S \) is a coherent sheaf on \( \mathcal{E}(S) \). This sheaf has \( \mathcal{E}(S) \) as its support, because every vector bundle has the identity map. Since \( \mathcal{E}(S) \) is an integral variety, there is a non-empty open subset \( \mathcal{E}(S)_0 \subseteq \mathcal{E}(S) \) such that \( (A_S)_{|\mathcal{E}(S)_0} \) is locally free. Set \( \mathcal{V}_0 := (\mathcal{V})_{|\mathcal{E}(S)_0 \times X} \). Note that for each \( \varepsilon \in \mathcal{E}(S)_0 \) the fiber of \( A_S \) at \( \varepsilon \) is the vector space \( \text{End}(\mathcal{E}(\varepsilon)) \).

Define \( \Gamma(S) \) as a subset of the total space of \( A_S \) as follows:

\[
\Gamma(S) := \{ (\varepsilon, \varphi) \mid \varepsilon \in \mathcal{E}(S)_0 \text{ and } \varphi \in \text{End}(\mathcal{E}(\varepsilon)) \text{ with } \varphi^2 = \varphi \}.
\]

Note that \( \varphi \) is a projection of \( \mathcal{E}(\varepsilon) \) onto a factor of \( \mathcal{E}(\varepsilon) \), with the exception when \( \varphi = \text{Id}_{\mathcal{E}(\varepsilon)} \) or \( \varphi \equiv 0 \); if \( \mathcal{E}(\varepsilon) \) is indecomposable, only \( (\varepsilon, \text{Id}_{\mathcal{E}(\varepsilon)}(\varepsilon)) \) and \( (\varepsilon, 0) \) are contained in \( \Gamma(S) \). Indeed, for any vector bundle \( \mathcal{G} \), there exists a one-to-one correspondence:

\[
\{ \varphi \in \text{End}(\mathcal{G}) \mid \varphi^2 = \varphi \} \leftrightarrow \{ \text{factors of } \mathcal{G} \}
\]

via \( \varphi \mapsto \text{Im}(\varphi) = \ker(\text{Id}_{\mathcal{G}} - \varphi) \), with \( \mathcal{G} \) being associated to \( \text{Id}_{\mathcal{G}} \) and 0 associated to the zero map. Thus \( \mathcal{G} \) is decomposable if and only if \( \text{End}(\mathcal{G}) \) has a non-trivial idempotent. Note that \( \Gamma(S) \) is closed in the total space of the vector bundle \( \pi_1 \mathcal{Hom}(\mathcal{V}_0, \mathcal{V}_0) \) over \( \mathcal{E}(S)_0 \). By Lemma 4.5 for each \( \mathcal{E}(\varepsilon) \) there is a unique partition of \( S \) associated to any decomposition of \( \mathcal{E}(\varepsilon) \) with only finitely many indecomposable factors by the Krull-Schmidt theorem in [1]. By Lemma 4.5 for each \( \mathcal{E} \in \mathcal{F}(S) \) each isomorphism class of factors of \( \mathcal{E} \) corresponds to a unique subset of \( S \); \( \mathcal{E} \) and 0 correspond to \( S \) and \( \varnothing \), respectively. For each \( (\varepsilon, \varphi) \in \Gamma(S) \), let \( S(\varphi) \) be the subset of \( S \) associated to \( \text{Im}(\varphi) \) by Lemma 4.5.

Set

\[
\Gamma_0(S) := \{ (\varepsilon, \varphi) \in \Gamma(S) \mid \varphi \neq 0 \text{ and } \varphi \neq \text{Id}_{\mathcal{E}(\varepsilon)} \}.
\]

The goal is to show that \( \Gamma_0(S) \) is not dominant over \( \mathcal{F}(S) \) for a general \( S \).

Note that up to now we did not use that \( S \) is contained in the same connected component \( Y \cap X_{\text{reg}} \) of \( X_{\text{reg}} \). In particular the case \( s = 2 \) holds even if \( X \) has more than one irreducible components with multiplicity one and the two points of \( S \) belong to different connected components of \( X_{\text{reg}} \).

Now we use a monodromy argument, which requires that \( S \) is contained in a connected component of \( T := X_{\text{reg}} \cap Y \) and that \( S \) is general in \( Y \). Set \( S = \{ p_1, \ldots, p_s \} \) and fix an ordering of the points in \( S \), along which we get an ordering of the indecomposable factors of the sheaf \( \oplus_{p \in S} \mathcal{I}_{p,X} \). Together with the usual ordering on the factors of \( \mathcal{O}_X(m - 3)^{\oplus s} \), we may see any \( \varepsilon \in \mathcal{E}(S) \) as an \((s \times s)\)-square matrix, say \( \varepsilon = (\varepsilon_{ij}) \) with \( 1 \leq i, j \leq s \), where \( \varepsilon_{ij} \) is an element of the 1-dimensional vector space \( \text{Ext}_X^1(\mathcal{I}_{p,X}, \mathcal{O}_X(m - 3)) \). In particular, if \( \varepsilon \in \mathcal{E}(S) \) is general, then the associated \((s \times s)\)-square matrix is also general in the space of all such \((s \times s)\)-square matrices. Note that for a fixed integer \( j \), each \( \varepsilon_{ij} \) with \( i = 1, \ldots, s \), is an element of the same 1-dimensional vector space. We write \( \mathcal{O}_X(m - 3)^{\oplus s} = C^* \otimes \mathcal{O}_X(m - 3) \). In Claim 4 below, we assume that \( S \) is general in \( T \), so that we may use the inductive assumption for all proper subsets of \( S \).

Claim 4: \( \mathcal{E} = \mathcal{E}(\varepsilon) \) has two indecomposable factors, one of them being \( \text{Im}(\varphi) \) and the other one being \( \ker(\varphi) \).

Proof of Claim 4: Since \( \varphi^2 = \varphi \), we have \( \mathcal{E} \cong \mathcal{F}_1 \oplus \mathcal{F}_2 \) with \( \mathcal{F}_1 := \text{Im}(\varphi) \) and \( \mathcal{F}_2 = \ker(\varphi) \). By the definition of \( A \), we get an exact sequence

\[
0 \to \mathcal{O}_X(m - 3)^{\oplus k} \to \mathcal{F}_1 \to \oplus_{p \in A} \mathcal{I}_{p,X} \to 0,
\]

with \( k := \sharp(A) \). Since neither \( \varphi \equiv 0 \) nor \( \varphi = \text{Id}_{\mathcal{E}} \), we have \( 0 < k < s \). Then by Lemma 4.5 we get an exact sequence

\[
0 \to \mathcal{O}_X(m - 3)^{\oplus (s-k)} \to \mathcal{F}_2 \to \oplus_{p \in S \setminus A} \mathcal{I}_{p,X} \to 0.
\]
Now we need to prove that each $F_i$ is indecomposable. By the inductive assumption it is sufficient to prove that $F_1$ and $F_2$ are the middle terms of general extensions and , respectively. Since gives the Harder-Narasimhan filtration of each $F_i$, there are linear subspaces $V_1, V_2 \subset \mathbb{C}^s$ such that \[ \dim V_1 = k, \dim V_2 = s - k \] and \[ v(\mathbb{C}^s \otimes \mathcal{O}_X(m - 3)) \cap F_i = V_i \otimes \mathcal{O}_X(m - 3) \] for each $i$. From $E \cong F_1 \oplus F_2$ we see that $C^s = V_1 \oplus V_2$. Now we reorder the points in $S$ so that all points of $A$ are smaller than any points of $S \setminus A$. Then $\varepsilon$ can be understood as an $(s \times s)$-square matrix in a block form:

\[
\varepsilon = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}
\]

Here the $(k \times k)$-matrix $B_{11}$ in the upper left corner, is associated to the extension and similarly the $((s - k) \times (s - k))$-matrix $B_{22}$ in the lower right corner, is associated to the extension . The matrix of $\varepsilon$ also has a $(k \times (s - k))$-submatrix $B_{12}$ and an $((s - k) \times k)$-submatrix $B_{21}$. By assumption the $(s \times s)$-square matrix corresponding to $\varepsilon$ is general, and this implies that each block matrix $B_{ij}$ is also general in the space of the corresponding sized matrices. In particular, $B_{11}$ and $B_{22}$ are general and this implies that each $F_i$ is general. The inductive assumption gives that each $F_i$ is indecomposable. □

Assume that a general $E = E(\varepsilon)$ has two indecomposable factors, i.e. the set $\Gamma_0(S)$ is dominant over $\mathbb{F}(S)$. Let $\Gamma'(S)$ be an irreducible component of $\Gamma_0(S)$ dominant over $\mathbb{F}(S)$ and set $A := S(\varphi)$, where $(\varepsilon, \varphi)$ is any element of $\Gamma'(S)$. Now assume that $(\varepsilon, \varphi)$ is general in $\Gamma'(S)$ and set $E := E(\varepsilon)$. Note that the subset $A \subset S$ is invariant as $(\varepsilon, \varphi)$ varies in $\Gamma_0(S)$, due to the irreducibility of $\Gamma_0(S)$. Let $\text{Sym}^s(T)_0$ denote the set of all subsets of $T \cap X_{\text{reg}}$ with cardinality $s$. Below we find a contradiction under the assumptions that $E$ is decomposable and that $S$ is general in $\text{Sym}^s(T)_0$. On $\text{Sym}^s(T)_0 \times X$, we have a family $\mathcal{E}_T$ of relative Ext$^1$-group, whose fibre over $S \in \text{Sym}^s(T)_0$ is $E(S)$. Denoting its universal family by $\mathcal{V}_T$, choose a non-empty subset $\mathcal{V}_T \subset \mathcal{V}_S$ corresponding to the locally free aCM extensions. For $\pi_2 : \mathcal{E}_T \times X \to X$ the second projection, set $\mathcal{A}_T := \pi_2_* \text{Hom}(\mathcal{V}_T, \mathcal{V}_T)$. Note that an element of $\mathcal{A}_T$ represents a triple $(S, \varepsilon, \varphi)$ with $(S, \varepsilon) \in \Delta_0$ and $\varphi : E(\varepsilon) \to E(\varepsilon)$ an endomorphism. Since $E_T$ is an integral variety, there is a non-empty open subset $\Delta_0 \subset \mathcal{E}_T$ such that $\mathcal{A}_T|_{\Delta_0}$ is locally free. Then by restricting $\Delta_0$ and $E(S)_0$, we may assume that $\mathcal{A}_T$ is an algebraic subset whose fibre over $S \in \text{Sym}^s(T)_0$ is $\Gamma_0(S)$, with a projection map $\psi : \mathcal{A}_T \to \text{Sym}^s(T)_0$. If $\psi$ is not dominant, then it would imply that there exists a 2s-dimensional family of pairwise not isomorphic indecomposable aCM vector bundles of rank $2s$ on $X$. Thus we may assume that $\psi$ is dominant. We fix a general $S \in \text{Sym}^s(T)$ and fix an irreducible component $\Gamma'(S)$ of $\Gamma_0(S)$ to which we apply the previous construction with the partition $A \sqcup (S \setminus A)$ of $S$ attached to $\Gamma'(S)$. Let $\mathcal{A}_T$ be any irreducible component of $\mathcal{A}_T$ containing $\Gamma'(S)$ such that $\psi^{\mathcal{A}_T}$ is dominant.

Let $\mathcal{U}$ denote a non-empty Zariski open subset of $\text{Sym}^s(T)$ containing $S$ with $A = S(\varphi)$ such that for every $S' \in U$ a general $\mathcal{E}_{S'} \in \mathcal{E}(S')$ has exactly two indecomposable factors, one associated to a subset $F$ of $S'$ with $|F| = |A| = k$ and the other one associated to $S' \setminus F$. Now we fix $p \in A$ and $q \in S \setminus A$. Since $Y_{\text{reg}}$ is a connected manifold and $p, q \in Y_{\text{reg}}$, there exists a non-empty Zariski open subset $U \subset \mathcal{A}$ with a map $\varphi : U \to Y_{\text{reg}}$ such that $\varphi(t_0) = p$ and $\varphi(t_1) = q$ for some $t_0, t_1 \in U$, and $\varphi(U)$ passes no other points of $S$. Similarly we may consider a map $\varphi' : U \to Y_{\text{reg}}$ with $\varphi'(t_1) = p$ and $\varphi'(t_0) = q$ such that $\varphi(t) \neq \varphi'(t)$ for any $t \in U$. For each $t \in U$, set

\[ A_t := (A \setminus \{p\}) \cup \{\varphi(t)\} \quad \text{and} \quad S_t := (S \setminus \{p, q\}) \cup \{\varphi(t), \varphi'(t)\}, \]

e.g. $(A_t, S_t) = (A_t, S_t) = (A, S)$. Restricting $U$ to an open neighborhood of $\{t_0, t_1\}$, we may assume that $S_t \in \mathcal{V}$ for all $t \in U$. Then for each $t \in U$ we have a partition $S_t = A_t \sqcup (S_t \setminus A_t)$ such that a general $\mathcal{E}_{S_t} \in \Gamma'(S_t)$ has exactly two indecomposable factors, one associated to $A_t$ and the other associated to $S_t \setminus A_t$, due to the choice of $\mathcal{A}_T$.

We start from $t = t_0$ and vary $t$ in $U$ to arrive at $t = t_1$, where we have $S_{t_1} = S = A_q \sqcup (S \setminus A_q)$ with $A_q = (A \setminus \{p\}) \cup \{q\}$. Since $s > 2$, we have $\{A, S \setminus A\} \neq \{A_q, S \setminus A_q\}$, contradicting the assumption that $E_S$ has exactly two indecomposable factors. □
Proof of Theorem 1.1: The family $\Sigma$ of all $S \subset X_{\text{reg}}$ with $\sharp(S) = s$ clearly has dimension $2s$. By Theorem 3.10 if $S$ and $S'$ are two distinct sets in $\Sigma$, then we get $\mathcal{E}_S \not\cong \mathcal{E}_{S'}$. Now there is a universal family on any Ext$^1$-group of families of sheaves with $\Sigma \times X$ as its base; see [19] Proposition 3.1. Thus, we get a family of aCM locally free and indecomposable vector bundles with as a parameter space a rank $s^2$ vector bundle over $\Sigma$; the fibre of this vector bundle over $S \in \Sigma$ is $\mathcal{E}(S)$, corresponding to $S$. Choose a non-empty open subset $V$ of $\Sigma$ on which this vector bundle is trivial. Then a non-zero section of this bundle over $V$ parametrizes pairwise non-isomorphic, aCM and indecomposable vector bundles. □

Remark 4.7. We start with an observation by H. Matsumura and P. Monsky. Let $Y \subset \mathbb{P}^{n+1}$ with $n \geq 2$ be a smooth hypersurface of degree $d \geq 3$. Then the set of all $f \in \text{Aut}(\mathbb{P}^{n+1})$ such that $f(Y) = Y$ is finite by [21] Theorem 1. For any projective scheme $X$, A. Grothendieck proved that the set $\text{Aut}(X)$ of all automorphisms of $X$ is locally algebraic, i.e. it is a countable disjoint union of algebraic schemes; see [22] Theorem 5.23 and Exercise at page 133. The connected component $\text{Aut}^0(X)$ of $\text{Aut}(X)$ containing the identity map is thus a finite-dimensional algebraic group, but it may have infinitely many (countable) connected components and even modulo the connected component of the identity it may not be finitely presented. However, for a smooth surface $X \subset \mathbb{P}^3$ with $m := \text{deg}(X) > 4$, the situation is simpler for the following reason, as explained in [21] in general; see also [24] Theorem 1.8. Every automorphism $f$ of $X$ preserves $\omega_X \cong \mathcal{O}_X(m - 4)$ and hence induces a linear isomorphism $H^0(\mathcal{O}_X(m - 4)) \to H^0(\mathcal{O}_X(m - 4))$. In particular, it also induces an automorphism of $H^0(\mathcal{O}_X(1))$ and so a projective linear transformation of $X$, because we have

$$H^0(\mathcal{O}_X(m - 4)) \cong H^0(\mathcal{O}_{\mathbb{P}^3}(m - 4)) \cong \text{Sym}^{m-4} H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \cong \text{Sym}^{m-4} H^0(\mathcal{O}_X(1)).$$

Thus [21] Theorem 1] gives that $\text{Aut}(X)$ is finite. For $m = 4$ the situation is different. There are smooth quartic surfaces $X \subset \mathbb{P}^3$ with discrete automorphism group and with an automorphism of infinite order; refer to [23] part (2) of Theorem 1]. See [9] [23] and references therein for many other very interesting automorphism groups of K3 surfaces. Obviously since each automorphism of $X$ preserves the singular locus we know that $\text{Aut}(X)$ is small for singular surfaces $X$.

Hence over any uncountable algebraically closed field, there is an integer $t_0$ such that for every even integer $r$, $X$ has a family of dimension at least $r - t_0$, consisting of indecomposable aCM vector bundles of rank $r$ on $X$ with each isomorphism class of vector bundles appearing at most countably many times in this family. If $m > 4$ we may drop the assumption that the base field is uncountable and find a family such that each isomorphism class only appears finitely many times in the family.

5. Non-locally free aCM sheaf

In this section, we let $X \subset \mathbb{P}^N$ be a closed subscheme with pure dimension $n$ at least two. Assume that $X$ is aCM with respect to $\mathcal{O}_X(1)$, i.e. $h^i(\mathcal{I}_{X,\mathbb{P}^N}(t)) = 0$ for all $t \in \mathbb{Z}$ and all $1 \leq i \leq n$, and that each local ring $\mathcal{O}_{X,x}$ with $x \in X$ has positive depth. The exact sequence

$$0 \to \mathcal{I}_{X,\mathbb{P}^N}(t) \to \mathcal{O}_{\mathbb{P}^N}(t) \to \mathcal{O}_X(t) \to 0$$

shows that $h^i(\mathcal{I}_{X,\mathbb{P}^N}(t)) = h^{i-1}(\mathcal{O}_X(t))$ for all $i \geq 2$. Hence we may restate our assumption as $h^1(\mathcal{I}_{X,\mathbb{P}^N}(t)) = 0$ and $h^i(\mathcal{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$ and $i = 1, \ldots, n - 1$. By [23] Théorème 1 in page 268 the condition that $h^1(\mathcal{O}_X(-x)) = 0$ for $x > 0$ and $i = 1, \ldots, n - 1$, plus having positive depth at each $x \in X$, is equivalent to all $\mathcal{O}_{X,x}$ having depth $n$. Since $h^1(\mathcal{I}_{X,\mathbb{P}^N}) = 0$, we have $h^0(\mathcal{O}_X) = 1$ and in particular $X$ is connected. Since $h^1(\mathcal{I}_{X,\mathbb{P}^N}(1)) = 0$, $X$ is linearly normal in the linear subspace of $\mathbb{P}^N$ spanned by $X$. Since $n \geq 2$ we have $h^1(\mathcal{O}_X) = 0$ an so Pic($X$) is a finitely generated abelian group.

Lemma 5.1. Assume $X$ aCM. Let $C \subset X$ be a reduced aCM subvariety of pure dimension $n - 1$. Then its ideal sheaf $\mathcal{I}_{C,X}$ is an aCM $\mathcal{O}_X$-sheaf such that

- it is locally free outside $C$ and
- for any closed subscheme $Y \subseteq X$, it is not an $\mathcal{O}_Y$-sheaf.
Proof. Since $C$ is aCM as a closed subscheme of $\mathbb{P}^N$ and $C$ has pure dimension $n - 1$, we have $h^1(I_{C,B}\mathcal{N}(t)) = 0$ for all $t \in \mathbb{Z}$. Thus the restriction map $\rho_t : H^0(\mathcal{O}_{\mathbb{P}^N}(t)) \to H^0(\mathcal{O}_C(t))$ is surjective for any $t \in \mathbb{Z}$. Since $\rho_t$ factors through the restriction map $\eta_t : H^0(\mathcal{O}_X(t)) \to H^0(\mathcal{O}_C(t))$, $\eta_t$ is surjective. Since $\eta_t$ is surjective and $h^1(\mathcal{O}_X(t)) = 0$, we have $h^1(I_{C,X}(t)) = 0$. This implies that $I_{C,X}$ is aCM. From $I_{C,X} \subset \mathcal{O}_X(-C)$, we see that $I_{C,X}$ is locally free and of rank 1 outside $C$. Since $C$ is not an irreducible component of $X_{\text{red}}$ and $I_{C,X}$ is locally free of positive rank outside $C$, there is no closed subscheme $Y \subset X$ with $I_{C,X}$ an $\mathcal{O}_Y$-sheaf.

Lemma 5.2. Assume that $X \subset \mathbb{P}^N$ is an aCM close subscheme with an aCM irreducible component $Y$ of $X_{\text{red}}$. For a fixed integer $e > 0$ and any integral divisor $C \in |\mathcal{O}_Y(e)|$, define

$$\Sigma_C := \{ p \in Y \mid I_{C,X} \text{ is not locally free at } p \}.$$

(i) If $X$ is not reduced at a general point of $X$, then we have $\Sigma_C = C$, i.e. for all $p \in C$ the sheaf $I_{C,X}$ is not locally free at $p$. For any two integral curves $C_1, C_2 \in |\mathcal{O}_Y(e)|$, we have $I_{C_1,X} \cong I_{C_2,X}$ if and only if $C_1 = C_2$.

(ii) Assume that $X$ is reduced at a general point of $Y$ and that $X$ is not integral. Let $F$ be the intersection of $Y$ with the other irreducible components of $X$. Then we have $F \neq \emptyset$ and $F$ has pure dimension $n - 1$. Moreover, we have $\Sigma_C = (F \cap C)_{\text{red}}$ and $F \cap C \neq \emptyset$.

(iii) For any two integral divisors $C_1, C_2 \in |\mathcal{O}_Y(e)|$ such that $I_{C_1,X} \cong I_{C_2,X}$, we have $\Sigma_{C_1} = \Sigma_{C_2}$; in case (i) we have the converse.

Proof. By Lemma 5.1 the sheaf $I_{C,X}$ is aCM and locally free with rank 1 at all $p \in X \setminus C$. Fix $p \in C$ and assume that $I_{C,X}$ is locally free at $p$. Then there is $w \in (I_{C,X,p})_p$ such that $w$ is not a zero-divisor of $\mathcal{O}_{X,p}$ and $(I_{C,X})_p \cong w\mathcal{O}_{X,p}$ as a module over the local ring $\mathcal{O}_{X,p}$. We get that in a neighborhood of $p$ the divisor $C$ is a Cartier divisor of $X$. Let $I \subset \mathcal{O}_{X,p}$ be the ideal of $Y$ and $J \subset \mathcal{O}_{X,p}$ the ideal of $C$. We have $I \subset J$. First assume that $X$ is not reduced at a general point of $X$. Since the support of the nilradical $\eta \subset \mathcal{O}_X$ of the structural sheaf $\mathcal{O}_Y$ is a closed subset of $X_{\text{red}}$, $X$ is not reduced at any point of $Y$ and in particular it is not reduced at $p$. Thus there is a nonzero $h \in I$ such that $h^m = 0$ for some $m > 0$. Since $I \subset J$, we have $h \in J$ and so $h$ is divided by $w$. Thus we get $w^m = 0$ and so $w$ is a zero-divisor, a contradiction.

Now assume that $X$ is reduced at a general point of $Y$. Obviously $I_{C,X}$ is locally free outside the support of $C$. Since $X$ is aCM, it is connected and so $F \neq \emptyset$. More precisely, since all local rings $\mathcal{O}_{X,x}$ have depth $n$, $X_{\text{red}}$ is locally connected in dimension $\leq n - 1$ and so $F$ has pure dimension $n - 1$. Since $C \in |\mathcal{O}_Y(e)|$, $C$ is a Cartier divisor of $Y$. Thus $C$ is a Cartier divisor of $X$ at all points of $C \setminus (C \cap F)$. Since $e > 0$, $C$ is an ample divisor of $Y$. In particular, we get $F \cap C \neq \emptyset$. Fix $p \in F \cap C$. Any local equation $w$ of $C$ at $p$ vanishes on each irreducible component of $X_{\text{red}}$ containing $p$, because $w$ is assumed to be a non-zero divisor of $\mathcal{O}_{X,p}$. There is at least one another irreducible component of $X_{\text{red}}$ containing $p$, because $p \in F$.

Part (iii) is obvious.

As a corollary of Lemma 5.2 we get the following result, which shows that $X$ is of wild representation type in a very strong form.

Proposition 5.3. Let $X \subset \mathbb{P}^N$ be a non-integral closed aCM subscheme with pure dimension at least two such that there exists an aCM irreducible component $Y$ of $X_{\text{red}}$. For a fixed integer $w > 0$, there is an integral quasi-projective variety $\Delta$ and a flat family $\{ F_a \}_{a \in \Delta}$ of aCM sheaves of rank one on $X$ with each $F_a$ locally free outside a one-codimensional subscheme $C_a$ and for each $a \in \Delta$ the set of all $b \in \Delta$ such that $F_a \cong F_b$ is contained in an algebraic subscheme $\Delta_a \subset \Delta$ with dim $\Delta - \dim \Delta_a \geq w$.

Proof. First assume that $Y$ has the multiplicity at least two. Fix a positive integer $e$ such that $\dim |\mathcal{O}_Y(e)| \geq w$ and take as $\Delta$ the family of all integral $C \in |\mathcal{O}_Y(e)|$. Then we may apply (i) of Proposition 5.2. In this case we may find $\Delta$ with the additional condition that for all $a, b \in \Delta$ we have $F_a \cong F_b$ if and only if $a = b$.
Now assume that the multiplicity of $Y$ in $X$ is one. Write $F \subset Y$ as in (ii) of Lemma 5.2. Fix an integer $e > 0$ such that $h^0(\mathcal{O}_X(e)) - h^0(\mathcal{O}_X(e)(-F)) > w$ and let $\Delta$ be the set of all integral divisors $C \in |\mathcal{O}_X(e)|$ not contained in $F$ and such that the scheme $F \cap C$ is reduced. Since $F$ has pure dimension $n - 1$ and $C$ is an ample divisor, the set $(F \cap C)_{\text{red}}$ has pure codimension 2. Fix any finite set $B \subset F$. For $e \gg 0$ we may find $C \in |\mathcal{O}_Y(e)|$ containing no irreducible component of $F$ and with $B \subset C$. Take $|B| \geq w$. Then we may apply (ii) of Lemma 5.2.

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