Apéry-Fermi pencil of $K3$-surfaces and 2-isogenies

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Abstract. Given a generic $K3$-surface $Y_k$ of the Apéry-Fermi pencil, we use the Kneser-Nishiyama technique to determine all its non isomorphic elliptic fibrations. These computations lead to determine those fibrations with 2-torsion sections $T$. We classify the fibrations such that the translation by $T$ gives a Shioda-Inose structure. The other fibrations correspond to a $K3$-surface identified by its transcendental lattice. The same problem is solved for a singular member $Y_2$ of the family showing the differences with the generic case. In conclusion we put our results in the context of relations between 2-isogenies and isometries on the singular surfaces of the family.

1. Introduction

The Apéry-Fermi pencil $F$ is realized with the affine equations

$$X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = k, \quad k \in \mathbb{C},$$

and taking $k = s + \frac{1}{2}$, is seen as the Fermi threefold $Z$ with compactification denoted $\overline{Z}$.

The projection $\pi_s : \overline{Z} \to \mathbb{P}^1(s)$ is called the Fermi fibration. In their paper [27], Peters and Stienstra proved that for $s \not\in \{0, \infty, \pm 1, 3 \pm 2\sqrt{2}, -3 \pm 2\sqrt{2}\}$ the fibers of the Fermi fibration are $K3$-surfaces with the Néron-Severi lattice of the generic fiber isometric to $M_6 = E_8[-1] \oplus U \oplus (-2 \times 6)$ and transcendental lattice isometric to $T = U \oplus (2 \times 6)$ ($U$ denotes the hyperbolic lattice and $E_8$ the unique positive definite even unimodular lattice of rank 8). Hence this family appears as a family of $M_6$-polarized $K3$-surfaces $Y_k$ with period $t \in H$. And we deduce from a result of Dolgachev [12] the following property. Let $E_t = \mathbb{C}/(Z + t\mathbb{Z})$ and $E_{t'} = \mathbb{C}/(Z + (-\frac{1}{6t})\mathbb{Z})$ be the corresponding pair of isogenous elliptic curves. Then there exists a canonical involution $\tau$ on $Y_k$ such that $Y_k/(\tau)$ is birationally isomorphic to the Kummer surface $E_t \times E_{t'}/(\pm 1)$.

This result is linked to the Shioda-Inose structure of $K3$-surfaces with Picard number 19 and 20 described first by Shioda and Inose [33] and extended by Morrison [21] (Corollary 6.4).

As observed by Elkies [14], the base of the pencil of $K3$-surfaces can be identified with the elliptic modular curve $X_0(6)/\langle w_2, w_3 \rangle$ where $w_2$ and $w_3$ denote the Atkin-Lehner involutions [1]. Indeed it can be derived from Peters and Stienstra [27].

Shioda considers the problem whether every Shioda-Inose structure can be extended to a sandwich, that is, given a $K3$-surface $S$, if there exists a unique Kummer surface
$K = Km(C_1 \times C_2)$ with two rational maps of degree 2, $S \to K$ and $K \to S$ where $C_1$ and $C_2$ are elliptic curves. In [31] Shioda proved a "Kummer sandwich theorem", for an elliptic $K$-surface $S$ (with a section) with two $II^*$-fibres.

In van Geemen-Sarti [16], Comparin-Garbagnati [10], Koike [18] and Schütt [28] (3.5, 4.4, 5.4), sandwich Shioda-Inose structures are constructed via elliptic fibrations with 2-torsion sections. Recently Bertin and Lecacheux [5] found all the elliptic fibrations of a singular member $Y_2$ of $\mathcal{F}$ (i.e. of Picard number 20) and observed that many of its elliptic fibrations are endowed with 2-torsion sections. Considering the minimal resolution of the quotient of $2$-torsion sections.

4.4, 5.4), sandwich Shioda-Inose structures are constructed via elliptic fibrations with

Theorem 1.1. Suppose $Y_k$ is a generic $K3$-surface of the family with Picard number 19.

Let $\pi : Y_k \to \mathbb{P}^1$ be an elliptic fibration with a torsion section of order 2 which defines an involution $i$ of $Y_k$ (van Geemen-Sarti involution) then the minimal resolution of the quotient $Y_k/i$ is either the Kummer surface $K_k$ associated to $Y_k$ given by its Shioda-Inose structure or a surface $S_k$ with transcendental lattice $T(S_k) = \langle -2 \rangle \oplus \langle 2 \rangle \oplus \langle 6 \rangle$ and Néron-Severi lattice $NS(S_k) = U \oplus E_8[-1] \oplus E_8[-1] \oplus \langle -2 \rangle \oplus \langle -6 \rangle$, which is not a Kummer surface. Thus, $\pi$ leads to an elliptic fibration either of $K_k$ or of $S_k$. Moreover there exist some genus 1 fibrations $\theta : K_k \to \mathbb{P}^1$ without section such that their Jacobian variety satisfies $J_\theta(K_k) = S_k$.

More precisely, among the elliptic fibrations of $Y_k$ (up to automorphisms) 12 of them have a two-torsion section. And only 7 of them possess a Morrison-Nikulin involution $i$ such that $Y_k/i = K_k$.

Remark 1.1. The fact that $S_k$ is not a Kummer surface follows from a result of Morrison [21].

The $K3$-surface $S_k$ is the Hessian $K3$-surface of a general cubic surface with 3 nodes studied by Dardanelli and van Geemen [13].

Theorem 1.2. In the Apéry-Fermi pencil, the $K3$-surface $Y_2$ is singular, meaning that its Picard number is 20. Moreover $Y_2$ has many more 2-torsion sections than the generic $K3$-surface $Y_k$: hence among its 20 van Geemen-Sarti involutions, 13 of them are Morrison-Nikulin involutions, 5 are symplectic automorphisms of order 2 (self-involutions) and the two remaining ones exchange two elliptic fibrations of $Y_2$.

The specializations to $Y_2$ of the 7 Morrison-Nikulin involutions of a generic member $Y_k$ are verified among the 13 Morrison-Nikulin involutions of $Y_2$. The specializations of the 5 remaining involutions between $Y_k$ and the $K3$-surface $S_k$ are among the 7 van Geemen-Sarti involutions of $Y_2$ which are not Morrison-Nikulin.

Remark 1.2. The fact that the specializations to $Y_2$ of the 7 Morrison-Nikulin involutions of $Y_k$ are Morrison-Nikulin involutions of $Y_2$ can be deduced from a general result
of Schütt [28].

This theorem provides an example of a Kummer surface $K_2$ defined by the product of two isogenous elliptic curves (actually the same elliptic curve of $j$-invariant equal to 8000), having many fibrations of genus one whose Jacobian surface is not a Kummer surface. A similar result but concerning a Kummer surface defined by two non-isogenous elliptic curves has been exhibited by Keum [17].

Throughout the paper we use the following result [36]. If $E$ denotes an elliptic fibration with a 2-torsion point $(0,0)$:

$$E : y^2 = x^3 + Ax^2 + Bx,$$

the quotient curve $E/\langle (0,0) \rangle$ has a Weierstrass equation of the form

$$E/\langle (0,0) \rangle : y^2 = x^3 - 2Ax^2 + (A^2 - 4B)x.$$

The paper is organized as follows.

In section 2 we recall the Kneser-Nishiyama method and use it to find all the 27 elliptic fibrations of a generic $K3$-surface of the family $F$. In section 3, using Elkies’s method of ”2-neighbors” [15], we exhibit an elliptic parameter giving a Weierstrass equation of the elliptic fibration. The results are summarized in Table 2. Thus we obtain all the Weierstrass equations of the 12 elliptic fibrations with 2-torsion sections. Their 2-isogenous elliptic fibrations are computed in section 5 with their Mordell-Weil groups and discriminants. Section 4 recalls generalities about Nikulin involutions and Shioda-Inose structure. Section 5 is devoted to the proof of Theorem 1.1 while section 6 is concerned with the proof of Theorem 1.2.

It is not easy to obtain a theorem similar to Theorem 1.2 for other singular $K3$-surfaces of the family, in particular to get all their fibrations. Nevertheless, in the last section 7, as a corollary of a result of Boissière, Sarti and Veniani [7], we shall explain why the existence of symplectic automorphisms of order two cannot be expected in all the singular $K3$-surfaces of the family. Precisely we prove this existence only on the singular $K3$-surfaces $Y_2$ and $Y_{10}$.

Computations were performed using partly the computer algebra system PARI [26] and mostly the computer algebra system MAPLE and the Maple Library “Elliptic Surface Calculator” written by Kuwata [20].

2. Elliptic fibrations of the family

We refer to [5], [29] for definitions concerning lattices, primitive embeddings, orthogonal complement of a sublattice into a lattice. We recall only what is essential for understanding this section and section 5.2.

2.1. Discriminant forms

Let $L$ be a non-degenerate lattice. The dual lattice $L^*$ of $L$ is defined by

$$L^* := \text{Hom}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q}/ b(x, y) \in \mathbb{Z} \text{ for all } y \in L\}$$
and the discriminant group $G_L$ by
\[ G_L := L^*/L. \]

This group is finite if and only if $L$ is non-degenerate. In the latter case, its order is equal to the absolute value of the lattice determinant $|\det(G(e))|$ for any basis $e$ of $L$.

A lattice $L$ is unimodular if $G_L$ is trivial.

Let $G_L$ be the discriminant group of a non-degenerate lattice $L$. The bilinear form on $L$ extends naturally to a $\mathbb{Q}$-valued symmetric bilinear form on $L^*$ and induces a symmetric bilinear form
\[ b_L : G_L \times G_L \to \mathbb{Q}/\mathbb{Z}. \]

If $L$ is even, then $b_L$ is the symmetric bilinear form associated to the quadratic form defined by
\[ q_L : G_L \to \mathbb{Q}/2\mathbb{Z}, \]
\[ q_L(x + L) \mapsto x^2 + 2\mathbb{Z}. \]

The latter means that $q_L(na) = n^2q_L(a)$ for all $n \in \mathbb{Z}$, $a \in G_L$ and $b_L(a,a') = \frac{1}{4}(q_L(a + a') - q_L(a) - q_L(a'))$, for all $a,a' \in G_L$, where $\frac{1}{4} : \mathbb{Q}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the natural isomorphism. The pair $(G_L, b_L)$ (resp. $(G_L, q_L)$) is called the discriminant bilinear (resp. quadratic) form of $L$.

The lattices $A_n = \langle a_1, a_2, \ldots, a_n \rangle$ ($n \geq 1$), $D_l = \langle d_1, d_2, \ldots, d_l \rangle$ ($l \geq 4$), $E_p = \langle e_1, e_2, \ldots, e_p \rangle$ ($p = 6, 7, 8$) defined by the following Dynkin diagrams are called the root lattices. All the vertices $a_j, d_k, e_l$ are roots and two vertices $a_j$ and $a'_j$ are joined by a line if and only if $b(a_j, a'_j) = 1$. We use Bourbaki’s definitions [8]. The discriminant groups of these root lattices are given below.

\[ A_n, \ G_A_n \]

Set
\[ [1]_{A_n} = \frac{1}{n+1} \sum_{j=1}^n (n - j + 1)a_j \]
then $A_n^* = (A_n^* [1]_{A_n})$ and
\[ G_{A_n} = A_n^*/A_n \simeq \mathbb{Z}/(n + 1)\mathbb{Z}. \]
\[ q_{A_n}([1]_{A_n}) = -\frac{n}{n+1}. \]

\[ D_l, \ G_D_l \]

Set
\[ [1]_{D_l} = \frac{1}{2} \left( \sum_{i=1}^{l-2} i d_i + \frac{1}{2} (l - 2) d_{l-1} + \frac{1}{2} l d_l \right) \]
\[ [2]_{D_l} = \sum_{i=1}^{l-2} i d_i + \frac{1}{2} (d_{l-1} + d_l) \]
\[ [3]_{D_l} = \frac{1}{2} \left( \sum_{i=1}^{l-2} i d_i + \frac{1}{2} (d_{l-1} + d_l) \right) \]
then $D^*_l = (D_l, [1]_{D_l}, [3]_{D_l})$.
\[ G_{D_l} = D^*_l/D_l \simeq \mathbb{Z}/4\mathbb{Z} \] if $l$ is odd,
\[ G_{D_l} = D^*_l/D_l \simeq \langle [1]_{D_l}, [2]_{D_l} \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \] if $l$ is even.
\[ q_{D_l}([1]_{D_l}) = -\frac{1}{4}, \ q_{D_l}([2]_{D_l}) = -1, \ b_{D_l}([1], [2]) = -\frac{1}{2}. \]
Let \( L \) be a Niemeier lattice (i.e. an unimodular lattice of rank 24). Denote \( L_{\text{root}} \) its root lattice. We often write \( L = Ni(L_{\text{root}}) \). Elements of \( L \) are defined by the glue code composed of glue vectors. Take for example \( L = Ni(A_{11}D_{7}E_{6}) \). Its glue code is generated by the glue vector \([1,1,1] \) where the first 1 means \([1]_{A_{11}} \), the second 1 means \([1]_{D_{7}} \) and the third 1 means \([1]_{E_{6}} \). In the glue code \( [1,\{0,1,2\}] \), the notation \((0,1,2) \) means any circular permutation of \((0,1,2) \). Niemeier lattices, their root lattices and glue codes used in the paper are given in Table 1 (glue codes are taken from Conway and Sloane [11]).

### 2.2. Kneser-Nishiyama technique

We use the Kneser-Nishiyama method to determine all the elliptic fibrations of \( Y_{k} \).

For further details we refer to [25], [29], [5], [3]. In [25], [5], [3] only singular K3 (i.e. of Picard number 20) are considered. In this paper we follow [29] we briefly recall.

Given an elliptic K3-surface \( S \), recall that \( H^{2}(S,\mathbb{Z}) \) with respect to the cup-product has the structure of an even lattice of rank 22 and the frame \( W(S) \) is the orthogonal complement in the Néron Severi lattice \( NS(S) \) of the lattice generated by the zero section and the general fiber. Nishiyama aims at embedding the frames \( W(S) \) of all elliptic fibrations into Niemeier lattices that are negative definite lattices of rank 24. For this purpose, Nishiyama determines an even negative definite lattice \( M \) such that

\[
q_{M} = -q_{NS(S)}, \quad \text{rank}(M) + \rho(S) = 26,
\]

<table>
<thead>
<tr>
<th>( L_{\text{root}} )</th>
<th>( L/L_{\text{root}} )</th>
<th>glue vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{8}^{1} )</td>
<td>( {0} )</td>
<td>([1,0])</td>
</tr>
<tr>
<td>( D_{16}E_{8} )</td>
<td>( \mathbb{Z}/2\mathbb{Z} )</td>
<td>([1,0,0])</td>
</tr>
<tr>
<td>( D_{10}E_{7}^{2} )</td>
<td>( \mathbb{Z}/2\mathbb{Z} ) (^{2})</td>
<td>([1,1,0],[1,0,1])</td>
</tr>
<tr>
<td>( A_{17}E_{7} )</td>
<td>( \mathbb{Z}/6\mathbb{Z} )</td>
<td>([3,1])</td>
</tr>
<tr>
<td>( D_{24} )</td>
<td>( \mathbb{Z}/2\mathbb{Z} )</td>
<td>([1])</td>
</tr>
<tr>
<td>( D_{12}^{4} )</td>
<td>( \mathbb{Z}/2\mathbb{Z} ) (^{4})</td>
<td>([1,2],[2,1])</td>
</tr>
<tr>
<td>( D_{6}^{4} )</td>
<td>( \mathbb{Z}/2\mathbb{Z} ) (^{4})</td>
<td>([1,2,2],[1,1,1],[2,2,1])</td>
</tr>
<tr>
<td>( A_{15}D_{9} )</td>
<td>( \mathbb{Z}/8\mathbb{Z} )</td>
<td>([2,1])</td>
</tr>
<tr>
<td>( E_{6}^{4} )</td>
<td>( \mathbb{Z}/3\mathbb{Z} ) (^{2})</td>
<td>([1,0,1,0])</td>
</tr>
<tr>
<td>( A_{11}D_{7}E_{6} )</td>
<td>( \mathbb{Z}/12\mathbb{Z} )</td>
<td>([1,1,1])</td>
</tr>
<tr>
<td>( D_{6}^{4} )</td>
<td>( \mathbb{Z}/2\mathbb{Z} ) (^{4})</td>
<td>(even permutations of ([0,1,2]))</td>
</tr>
<tr>
<td>( A_{1}^{4}D_{6} )</td>
<td>( \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} )</td>
<td>([2,4,0],[5,0,1],[0,5,3])</td>
</tr>
<tr>
<td>( A_{2}^{4}D_{7} )</td>
<td>( \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} )</td>
<td>([1,1,1,2],[1,7,2,1])</td>
</tr>
</tbody>
</table>

Table 1. Some Niemeier lattices and their glue codes [11].
\(\rho(S)\) being the Picard number of \(S\).

By Nikulin \cite{nikulin1998}, \(M \oplus W(S)\) has a Niemeier lattice as an overlattice for each frame \(W(S)\) of an elliptic fibration on \(S\). Thus one has to determine the (inequivalent) primitive embeddings of \(M\) into Niemeier lattices \(L\). To achieve this, it is essential to consider the root lattices involved. In each case, the orthogonal complement of \(M\) into \(L\) gives the corresponding frame \(W(S)\).

Let us describe how to determine \(M\) in the case of the Apéry-Fermi pencil.

Let \(T(Y_k)\) be the transcendental lattice of \(Y_k\), that is the orthogonal complement of \(\text{NS}(Y_k)\) in \(H^2(Y_k, \mathbb{Z})\). The lattice \(T(Y_k)\) is an even lattice of rank \(r = 22 - \rho(Y_k) = 3\) and signature \((2,1)\). Let \(t := r - 2 = 1\). By Nikulin’s theorem (\cite{nikulin1998}, Theorem 1.12.4), \(T(Y_k)[-1]\) admits a primitive embedding into the following indefinite unimodular lattice:

\[
T(Y_k)[-1] \in U \oplus E_8[-1],
\]

where \(U\) denotes the hyperbolic lattice and \(E_8\) the unique positive definite even unimodular lattice of rank 8. Define \(M\) as the orthogonal complement of a primitive embedding of \(T(Y_k)[-1]\) in \(U \oplus E_8[-1]\). Since

\[
T(Y_k)[-1] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -12 & 0 \\ -1 & 0 & 0 \end{pmatrix},
\]

it suffices to get a primitive embedding of \((-12)\) into \(E_8[-1]\). From Nishiyama \cite{nishiyama1992} (p. 335) we find the following primitive embedding:

\[
v = (9e_2 + 6e_1 + 12e_3 + 18e_4 + 15e_5 + 12e_6 + 8e_7 + 4e_8) \in E_8[-1],
\]

giving \(v \frac{1}{2} E_8[-1] = A_2 \oplus D_5\). Now the primitive embedding of \(T(Y_k)[-1]\) in \(U \oplus E_8[-1]\) is defined by \(U \oplus v\); hence \(M = (U \oplus v)_{U \oplus E_8[-1]} = A_2 \oplus D_5\). By construction, this lattice \(M\) is negative definite of rank equal to \(8 - 1 = 7 = r + 4 = 26 - \rho(Y_k)\) and its discriminant form satisfies by Nikulin (\cite{nikulin1998} proposition 1.6.1),

\[
q_M = -q_{T(Y_k)[-1]} = q_{T(Y_k)} = -q_{\text{NS}(Y_k)}.
\]

Hence \(M\) takes exactly the shape required for Nishiyama’s technique.

All the elliptic fibrations come from all the primitive embeddings of \(M = A_2 \oplus D_5\) into all the Niemeier lattices \(L\). Since \(M\) is a root lattice, a primitive embedding of \(M\) into \(L\) is in fact a primitive embedding into \(L_{\text{root}}\). Whenever the primitive embedding is given by a primitive embedding of \(A_2\) and \(D_5\) in two different factors of \(L_{\text{root}}\), or for the primitive embedding of \(M\) into \(E_8\), we use Nishiyama’s results \cite{nishiyama1992} (4.1 and p.335). Otherwise we have to determine the primitive embeddings of \(M\) into \(D_l\) for \(l = 8, 9, 10, 12, 16, 24\). This is done in the following lemma.

\textbf{Lemma 2.1.} \textit{We obtain the following primitive embeddings.}

1. \(A_2 \oplus D_5 = (d_8, d_6, d_7, d_5, d_4, d_1, d_2) \in D_8\)

\[
(d_8, d_6, d_7, d_5, d_4, d_1, d_2)_{D_8} = (2d_1 + 4d_2 + 6d_3 + 6d_4 + 6d_5 + 6d_6 + 3d_7 + 3d_8) = (-12)
\]
2. $A_2 \oplus D_5 = (d_9, d_7, d_8, d_6, d_5, d_3, d_2) \hookrightarrow D_9$

$$(d_9, d_7, d_8, d_6, d_5, d_3, d_2)_{D_9} = \langle d_9 + 6d_7 + 2d_6 + 2d_5 + 2d_4 + d_3 - d_1, d_3 + 2d_2 + 3d_1 \rangle$$

with Gram matrix $\begin{pmatrix} -4 & 6 \\ 6 & -12 \end{pmatrix}$ of determinant 12.

3. $A_2 \oplus D_5 = (d_n, d_{n-2}, d_{n-1}, d_{n-3}, d_{n-4}, d_{n-7}, d_{n-6}) \hookrightarrow D_n, n \geq 10$

$$(d_n, d_{n-2}, d_{n-1}, d_{n-3}, d_{n-4}, d_{n-7}, d_{n-6})_{D_n} = \langle a, d_{n-6} + 2d_{n-7} + 3d_{n-8}, d_{n-9}, \ldots, d_1 \rangle$$

with $a = d_n + d_{n-1} + 2(d_{n-2} + \ldots + d_2) + d_1$.

$$(A_2 \oplus D_5)_{D_n}^{1/2} = D_{n-8}.$$ 

We have also the relation $2,[2]_{D_n} = a + d_1$, $a$ being the above root.

**Theorem 2.1.** There are 27 elliptic fibrations on the generic K3-surface of the Apéry-Fermi pencil (i.e. with Picard number 19). They are obtained from all the non isomorphic primitive embeddings of $A_2 \oplus D_5$ into the various Niemeier lattices. Among them, 4 fibrations have rank 0, precisely with the type of singular fibers and torsion:

- $A_{11}2A_22A_1$ 6 - torsion
- $E_6D_{11}$ 0 - torsion
- $E_7A_5D_5$ 2 - torsion
- $E_8E_8A_3$ 0 - torsion.

The list together with the rank and torsion is given in Table 2.

**Remark 2.1.** Fibrations of rank 0 are also already computed in [30].

**Proof.** The torsion groups can be computed as explained in [5] or [3]. Let us recall briefly the method.

Denote $\phi$ a primitive embedding of $M = A_2 \oplus D_5$ into a Niemeier lattice $L$. Define $W = (\phi(M))_L$ and $N = (\phi(M))_{L_{\text{root}}}$. We observe that $W_{\text{root}} = N_{\text{root}}$. Thus computing $N$ then $N_{\text{root}}$ gives the type of singular fibers. Recall also that the torsion part of the Mordell-Weil group is

$$\overline{W_{\text{root}}}/W_{\text{root}}(\subset W/N)$$

and can be computed in the following way [3]: let $l + L_{\text{root}}$ be a non trivial element of $L/L_{\text{root}}$. If there exist $k \neq 0$ and $u \in L_{\text{root}}$ such that $k(l + u) \in N_{\text{root}}$, then $l + u \in W$ and the class of $l$ is a torsion element.

We use also several facts.

1. If the rank of the Mordell-Weil group is 0, then the torsion group is equal to $W/N$. Hence fibrations #1($A_3E_6E_6$), #3($D_{14}E_6$), #7($D_5A_5E_7$), #20($A_{11}2A_12A_2$) have respective torsion groups (0), (0), $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$.

2. If there is a singular fiber of type $E_8$, then the torsion group is (0). Hence the fibrations #1, #2 and #6 have no torsion.
3. Using Lemma 2.2 below and the shape of glue vectors we prove that fibrations #11, #18, #21, #22, #25, #27 have no torsion.

4. Using Lemma 2.3 below and the shape of glue vectors we can determine the torsion for elliptic fibrations #5, #10, #13, #15 #23.

**Lemma 2.2.** Suppose $A_2$ primitively embedded in $A_n$, $A_2 = (a_1, a_2) \hookrightarrow A_n$. Then for all $k \neq 0$, $k[1]_{A_n} \notin [(A_2)_{A_n}]_{\text{root}}$.

**Proof.** It follows from the fact that $[1]_{A_n}$ is not orthogonal to $a_1$. \qed
LEMMA 2.3. Suppose \( A_2 \) primitively embedded in \( D_l \), \( A_2 = \langle d_l, d_{l-2} \rangle \hookrightarrow D_l \). Then \( 2.\|D_i \in \langle (A_2)_{\sqrt{2}} \rangle_{\text{root}} \) but there is no \( k \) satisfying \( k \|D_i \in \langle (A_2)_{\sqrt{2}} \rangle_{\text{root}}, i = 1, 3 \).

PROOF. It follows from Nishiyama \cite{25} \( (A_2)_{\sqrt{2}} = \langle y, x_4, d_{l-4}, \ldots, d_4 \rangle, \ l \geq 5 \), with \( y = d_l + 2d_{l-1} + 2d_{l-2} + d_{l-3}, x_4 = d_l + d_{l-1} + 2(d_{l-2} + d_{l-3} + \ldots + d_2) + d_1 \) and Gram matrix

\[
L_{l-3} = \begin{pmatrix}
-4 & 1 & 1 & 0 & \cdots & 0 \\
-1 & 1 & 0 & & & \\
1 & 0 & D_{l-3} & & \\
& \vdots & & & \\
0 & & & & &
\end{pmatrix}
\]

Moreover \( \langle (A_2)_{\sqrt{2}} \rangle_{\text{root}} = \langle x_4, d_{l-4}, \ldots, d_4 \rangle \). From there we compute easily the relation

\( 2.\|D_i = x_4 + d_{l-4} + 2(d_{l-5} + \ldots + d_1) \). The last assertion follows from the fact that \( [i]_{D_i} \) is not orthogonal to \( A_2 \).

We now give some examples showing the method in detail.

2.2.1. Fibration #17

It comes from a primitive embedding of \( A_2 \oplus D_5 \) into \( D_9 \) giving a primitive embedding of \( A_2 \oplus D_5 \) into \( N(A_{15}D_9) \) with glue code \((2, 1)\). Since by Lemma 2.1(2) \( N_{\text{root}} = A_{15} \), among the elements \( k.\langle 2, 1 \rangle \), only \( 4.\|D_5 \) satisfies \( 2.\|0 + u \rangle \in \langle k \rangle_{\text{root}} = A_{15} \) with \( u = 4.1 \). Hence the torsion group is \( \mathbb{Z}/2\mathbb{Z} \).

2.2.2. Fibration #19

It comes from a primitive embedding of \( A_2 = \langle e_1, e_3 \rangle \) into \( E_6^{(1)} \) and \( D_5 = \langle e_2, e_3, e_4, e_5, e_6 \rangle \) into \( E_6^{(2)} \) giving a primitive embedding of \( A_2 \oplus D_5 \) into \( N(E_6) \). In that case \( N(E_6)/E_6 \cong (\mathbb{Z}/3\mathbb{Z})^2 \) and the glue code is \( \langle 1, (0, 1, 2) \rangle \). Moreover \( D_5 \|E_6 = 3e_2 + 4e_1 + 5e_3 + 6e_4 + 4e_5 + 2e_6 = a \), \( (A_2)_{E_6} = \langle e_2, y \rangle \oplus \langle e_5, e_6 \rangle \) with \( y = 2e_2 + e_1 + 2e_3 + 3e_4 + 2e_5 + e_6 \). From the relation

\[
[1]_{E_6} = -\frac{1}{3}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6)
\]

we get

\[
-3[1]_{E_6} = a - 2e_1 - e_3 + e_5 + 2e_6 \in E_6
\]

and deduce that only \( [1, 0, 1, 2], [2, 0, 2, 1], [0, 0, 0, 0] \) contribute to the torsion thus the torsion group is \( \mathbb{Z}/3\mathbb{Z} \).

2.2.3. Fibration #10

The embeddings of \( A_2 = \langle d_{10}, d_8 \rangle \) into \( D_{10} \) and \( D_5 = \langle e_2, e_3, e_4, e_5, e_6 \rangle \) into \( E_7^{(1)} \) lead to a primitive embedding of \( A_2 \oplus D_5 \) into \( N(D_{10}E_7) \) satisfying \( N(D_{10}E_7)/D_{10}E_7 \cong (\mathbb{Z}/2\mathbb{Z})^2 \) with glue code \( \langle 1, 1, 0 \rangle, [3, 0, 1, 2] \). We deduce from Lemma 2.3 that no glue vector can contribute to the torsion which is therefore \( (0) \).
2.2.4. Fibration #18
The embeddings of \( A_2 = \langle a_1, a_2 \rangle \) into \( A_{15} \) and \( D_5 = \langle d_5, d_7, d_8, d_9, d_9 \rangle \) into \( D_9 \) lead to a primitive embedding of \( A_2 \oplus D_5 \) into \( N\iota(A_{15}D_9) \) satisfying \( N\iota(A_{15}D_9)/(A_{15}D_9) \simeq (\mathbb{Z}/8\mathbb{Z}) \) with glue code \( \langle [2, 1] \rangle \). We deduce from Lemma 2.2 that no glue vector can contribute to the torsion which is therefore (0).

2.2.5. Fibration #8
The primitive embeddings of \( A_2 = (e_1, e_3) \) into \( E_7^{(1)} \) and \( D_5 = (e_2, e_3, e_4, e_5, e_6) \) into \( E_7^{(2)} \) lead to a primitive embedding of \( A_2 \oplus D_5 \) into \( N\iota(D_{10}E_7^2) \) satisfying \( N\iota(D_{10}E_7^2)/(D_{10}E_7^2) \simeq (\mathbb{Z}/22\mathbb{Z})^2 \) with glue code \( \langle [1, 1, 0], [3, 0, 1] \rangle \). From Nishiyama [25] we get \( (A_2)^{\perp}_{E_7^{(1)}} \approx \langle e_2, e_4, e_7, e_6, e_5 \rangle \simeq A_5 \) with \( y = 2e_2 + e_1 + 2e_3 + 3e_4 + 2e_5 + e_6 \) and \( (D_5)^{\perp}_{E_7} = \langle (1), e_2 + e_3 + 2(e_4 + e_5 + e_6 + e_7) = (−2) \rangle \). Hence \( N = D_{10} \oplus A_5 \oplus (−4) \oplus A_1 \) and \( W_{\text{root}} = N_{\text{root}} = D_{10} \oplus A_5 \oplus A_4 \). Now

\[
-2\eta_7 = -2[1]_{E_7} = 2y - e_2 + e_5 + 2e_6 + 3e_7 \in ((A_2)^{\perp}_{E_7})_{\text{root}}
\]

and for all \( k \neq 0, k,[1]_{E_7} \notin (D_5)^{\perp}_{E_7} \). Hence only the generator \( [1, 1, 0] \) can contribute to the torsion group which is therefore \( \mathbb{Z}/2\mathbb{Z} \).

2.2.6. Fibration #24
The primitive embeddings \( A_2 = (d_6, d_4) \) into \( D_6^{(1)} \) and \( D_5 = (d_6, d_5, d_4, d_3, d_2) \) into \( D_6^{(2)} \) give a primitive embedding of \( A_2 \oplus D_5 \) into \( L = N\iota(D_6^4) \) with \( L/L_{\text{root}} \simeq (\mathbb{Z}/22\mathbb{Z})^4 \) and glue code \( \langle \text{even permutations of } [0, 1, 2, 3] \rangle \). From Nishiyama [25] we get \( (A_2)^{\perp}_{D_6} = \langle y = 2d_5 + d_6 + 2d_4 + d_3, x_4 = d_5 + d_6 + 2(d_4 + d_3) + d_2, d_2, d_1 \rangle, \ associated \( (A_2)^{\perp}_{D_6} \simeq A_3 \) and \( (D_5)^{\perp}_{D_6} = \langle x_6^2 = (d_5 + d_6 + 2(d_4 + d_3) + d_2, d_1) \rangle = (−4) \).

We deduce \( N_{\text{root}} = A_3 \oplus D_5 \oplus D_6 \). From the relations \( 2,[2]_{D_6} = x_4 + d_2 + d_1 \) we deduce that the glue vectors having 1, 2, 3 or 0 in the first position may belong to \( W \). From the relation \( 2,[2]_{D_6} = x_6^2 \) we deduce that only glue vectors with 2 or 0 in the second position may belong to \( W \). Finally only the glue vectors \( [0, 2, 3, 1], [1, 0, 3, 2], [1, 2, 0, 3], [2, 0, 1, 3], [2, 0, 2, 2], [3, 0, 2, 1], [3, 2, 1, 0], [0, 0, 0, 0] \) belong to \( W \). Since \( y \) and \( x_6^2 \) are not roots, only glue vectors with 0 or 2 in the first position and 0 in the second position may contribute to torsion that is \( [2, 0, 1, 3], [0, 0, 0, 0] \). Hence the torsion group is \( \mathbb{Z}/2\mathbb{Z} \).

2.2.7. Fibration #26
The primitive embeddings of \( A_2 = (d_5, d_3) \) into \( D_5^{(1)} \) and \( D_5 \) into \( D_5^{(2)} \) give a primitive embedding into \( L = N\iota(A_4^2D_5^5) \) with \( L/L_{\text{root}} \simeq \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \) and glue code \( \langle [1, 1, 2], [1, 7, 2, 1] \rangle \). From Nishiyama we get \( (A_2)^{\perp}_{D_5} = \langle y = 2d_4 + d_5 + 2d_3 + d_2, x_4 = d_5 + d_4 + 2d_3 + 2d_2 + d_1 \rangle \) and Gram matrix \( M_5^2 = \begin{pmatrix} -1 & -1 & 4 \\ 4 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \) of determinant 12. We also deduce \( N_{\text{root}} = E_7^{D_5}A_4^2, 2,[2]_{D_5} = x_4 + d_1 \in ((A_2)^{\perp}_{D_5})_{\text{root}} \).

Moreover neither \( k,[1]_{D_5} \) nor \( k,[3]_{D_5} \) belongs to \( (A_2)^{\perp}_{D_5} \). Thus only glue vectors with 2 or 0 in the third position can belong to \( W \) and eventually contribute to torsion, that is \( [1, 2, 2, 0], [4, 4, 0, 0], [6, 6, 2, 0], [2, 6, 0, 2], [6, 2, 0, 2], [4, 0, 2, 2], [0, 4, 2, 2], [0, 0, 0, 0] \). Since there is no \( u_4 \in D_5 \) satisfying \( 2(2 + u_4) = 0 \) or \( 4(2 + u_4) = 0 \), glue vectors with the last component equal to 2 cannot satisfy \( k(l + u) \in N_{\text{root}} \) with \( l \in L \) and \( u \in L_{\text{root}} = A_4^2D_5^5 \).

Hence only the glue vectors generated by \( \langle [2, 2, 2, 0] \rangle \) contribute to torsion and the torsion group is therefore \( \mathbb{Z}/4\mathbb{Z} \). □
3. Weierstrass Equations for all the elliptic fibrations of $Y_k$

The method can be found in [5], [15]. We follow also the same kind of computations used for $Y_2$ given in [5]. We give only explicit computations for 4 examples, #19, #2, #9, and #16. For #2 and #9 it was not obvious to find a rational point on the quartic curve. All the results are given in Table 3. For the 2 or 3-neighbor method [15] we give in the third column the starting fibration and in the forth the elliptic parameter. The terms in the elliptic parameter refer to the starting fibration.

3.1. Fibration #19

We take $u = \frac{X^3}{x^2}$ as a parameter of an elliptic fibration and with the birational transformation

$$ x = -u(1 + uZ)(u + Y), \quad y = u^2 ((u + Y)(uY - 1)Z + Y(Y + 2u + k) - 1) $$

we obtain a Weierstrass equation

$$ y^2 + ukyx + u^2 (u^2 + uk + 1) y = x^3, $$

where the point $(x = 0, y = 0)$ is a 3-torsion point and the point $(-u^2, -u^2)$ is of infinite order. The singular fibers are of type $IV^*(0, \infty), I_3 (u^2 + uk + 1)$ and $I_1 (27u^2 - k(k^2 - 27)u + 27)$. Moreover if $k = s + \frac{1}{s}$ the two singular fibers of type $I_3$ are above $u = -s$ and $\frac{1}{s}$.

3.2. Fibration #2

Using the 3-neighbor method from fibration #19 we construct a new fibration with a fiber of type $I^*$ and the parameter $m = \frac{ts}{(w + s)^2}$. Then we obtain a cubic $C_m$ in $w, u$, with $x = w (u + s)$

$$ C_m : (s + u) m^2 + u (s^2 w + u^2 s + w + u) m - w^3 s^2 = 0. $$

From some component of the fiber of type $I_3$ at $u = -s$ we obtain the rational point on $C_m : \omega_m = \left( u_1 = \frac{m^2 - 1}{s - m}, u_1 = \frac{m(s^2 - 1)}{s(s - m)} \right)$ which is not a flex point. The first stage is to obtain a quartic equation $Q_{wa} : y^2 = ax^4 + bx^3 + cx^2 + dx + e^2$. First we observe that $\omega_m$ is on the line $w = u + \frac{1}{s}$, so we replace $w$ by $K$ with $w = u + \frac{1}{s} + K$ and $u = u_1 + T$. The transformation $K = WT$ gives an equation of degree two in $T$, with constant term $FW + g$ where $f$ and $g$ belong to $Q(s, m)$. With the change variable $Wf + g = x$ we have an equation $M(x)T^2 + N(x)T + x = 0$. The discriminant of the quadratic equation in $T$ is $N(x)^2 - 4xM(x)$, a polynomial of degree 4 in $x$ and constant term a square. Easily we obtain the form $Q_{wa}$.

From the quartic form, setting $y = e + \frac{dx}{2e} + x^2X', x = \frac{8eX' - 4e^2 + d^2}{4x^2}$ we get

$$ Y'^2 + 4e (dX' - be) Y + 4e^2 \left( 8e^3 X' - 4e^2 + d^2 \right) (X'^2 - a) = 0. $$

Finally the following Weierstrass equation follows from standard transformations where we replace $m$ by $t$

$$ Y'^2 - X'^3 + \frac{1}{3}(s^2 + 1)(s^6 + 219s^4 - 21s^2 + 1)X $$
with a section $\Phi$ of height 12 corresponding to 
\[ (8e^3 X' - 4e^2 + d^2) = 0 \text{ and } Y' = 0. \]
The coordinates of $\Phi$, too long, are omitted but we can follow the previous computation to obtain it.

Writing the above form as
\[ y^2 = x^3 - 3\alpha x + \left( t + \frac{1}{t} \right) - 2\beta \]
we recover the values of the $j$ invariants of the two elliptic curves for the Shioda-Inose structure (see paragraph 4.5.1 and corollary 4.1 below).

3.3. Fibration #9

Let $g = \frac{XY}{Z^2}$. Eliminating $X$ and writing $Y = ZU$ we obtain an equation of bidegree 2 in $U$ and $Z$. If $k = s + \frac{1}{s}$ there is a rational point $U = -1$, $Z = -\frac{s}{g}$ on the previous curve. By standard transformations we get a Weierstrass equation
\[ y^2 = x^3 + \frac{1}{4}g^2 \left( s^4 + 14s^2 + 1 \right) x^2 + s^2 g^3 \left( g + s^2 \right) \left( g s^2 + 1 \right) x \]
and a rational point
\[ x = \frac{s^2 (g - 1)^2 \left( g + s^2 \right) \left( g s^2 + 1 \right)}{(s^2 - 1)^2}, \]
\[ y = \frac{1}{2} \frac{s^2 (g^2 - 1) \left( g + s^2 \right) \left( g s^2 + 1 \right) \left( 2g^2 s^2 + g \left( s^4 - 6s^2 + 1 \right) + 2s^2 \right)}{(s^2 - 1)^3}. \]
The singular fibers are of type $2III^* (\infty, 0)$, $2I_2 \left( -s^2, -\frac{1}{s^2} \right)$, $4I_1$.

3.4. Fibration #16

Using the fibration #9 we consider the parameter $t = \frac{s}{g(g + s^2)}$ and obtain a Weierstrass equation
\[ Y^2 = X^3 + \left( 4t \left( t^2 + s^2 \right) + t^2 \left( s^4 + 14s^2 + 1 \right) \right) X^2 + 16s^6 t^4 X. \]
The singular fibers are of type $I_4^* (\infty, 0)$, $4I_1$.

4. Nikulin involutions and Shioda-Inose structure

4.1. Background

Let $S$ be a $K3$-surface.

The lattice $H^2(S, \mathbb{Z})$ admits a Hodge decomposition of weight two
\[ H^2(S, \mathbb{C}) \simeq H^{2,0} \oplus H^{1,1} \oplus H^{0,2}. \]
Similarly, the transcendental lattice $T(S)$ has a Hodge decomposition of weight two
\[ T(S) \otimes \mathbb{C} \simeq T^{2,0} \oplus T^{1,1} \oplus T^{0,2}. \]
Table 3. Weierstrass equations of the elliptic fibrations of $Y_k$

| #1 | $y^2 + tkyx + t^2k((+1)y = x^3 - t^3((+1)^3)$ | $y_{(X+Z)}(Y+Z)$ | $\frac{1}{x^2}$ |
| #2 | $y^2 = x^3 - \frac{2}{5}t^4(s^2 + 1)(s^8 + 219s^4 - 21s^2 + 1)x$ | $\frac{2}{7}$ | |
| #3 | $y^2 = x^3 + \frac{1}{7}t(4t^4s^4 + (s^2 - 10s^4 + 1)t + 12)x^3$ | $\frac{2}{7}$ | |
| #4 | $y^2 = x^3 + \frac{1}{7}t\left(t^3 + \frac{2}{9}(s^2 + 1)(s^8 + 219s^4 - 21s^2 + 1)t\right)(s^2 + 14s^4 + 1)(s^8 + 44s^4 + 1) x^3 + 16a^{10}x$ | $\frac{2}{7}$ | |
| #5 | $y^2 - k((+1)y = x^3 + (t^3 - 3)x^2 + 3x - 1$ | $\frac{1}{x^2}$ | |
| #6 | $y^2 = x^3 + \left(\frac{2}{7}t^2(s^4 + 14s^4 + 1) + t^3s^2\right)x^3$ | $\frac{3}{7}$ | |
| #7 | $y^2 = x^3 - \frac{1}{7}t\left(t((s^2 - 10s^4 + 1) + 8s^4\right) x^3 - t^2s^2(t - s^2)x$ | $\frac{3}{7}$ | |
| #8 | $y^2 - k((+1)y = x = (x^3 - t^3)$ | $\frac{X+Z}{Y^2}$ | |
| #9 | $y^2 = x^3 + \frac{1}{7}t\left(t((s^2 + 14s^4 + 1) x^3 + t^2s^2((t + s^2)(t - s^2)$ | $\frac{X+Z}{Y^2}$ | |
| #10 | $y^2 + t((s^2 + 1)(t + s^2)x = (t + s^2)\left(x^3 + t^3s^4\right)$ | $\frac{X+Z}{Y^2}$ | |
| #11 | $y^2 + t((s - 1 - s^2)y = x^2((x + s((s^2 - 1) - s^2(1)))$ | $\frac{y + z}{x^2}$ | |
| #12 | $y^2 = x^3 + \left(t(s^2 + \frac{1}{4}(s^2 + 14s^4 + 1) + (s^4 + 1))x^3$ | $\frac{2}{7}$ | |
| #13 | $y^2 = x^3 + \frac{1}{7}t\left(4t^4s^4 + (s^2 - 10s^4 + 1)t + 4s^4\right)x^3$ | $\frac{2}{7}$ | |
Table 4. Weierstrass equations of the elliptic fibrations of $Y_6$

<table>
<thead>
<tr>
<th>No</th>
<th>Weierstrass Equation</th>
<th>From</th>
<th>Param.</th>
</tr>
</thead>
<tbody>
<tr>
<td>#14</td>
<td>$y^2 = x^3 + \left( t \left( t^2 + t \right) + \frac{1}{4} t^2 \left( t^2 + 4 t + 1 \right) \right) x^2 + s^4 t x^4$</td>
<td>$I^\infty_6 \left( \infty, 0 \right), I_0 \left( 0 \right), 4 f_1 \left( - \frac{1}{4}, -4 \right) \left( \frac{x^2 - 1}{x^2 - 1} \right)^2$</td>
<td></td>
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<tr>
<td></td>
<td>$x_p = \frac{x^2 \left( \frac{x^2 - 1}{x^2 - 1} \right)^2}{\left( x^2 - 1 \right)^2}$</td>
<td>#9</td>
<td>$(t+x^2) \left( t^2 + 1 \right)$</td>
</tr>
<tr>
<td>#15</td>
<td>$(y - t x) \left( y - s^2 t x \right) = x \left( x - t x^2 \right) \left( x - t s^2 (t + 1)^2 \right)$</td>
<td>$I^\infty_6 \left( \infty, 0 \right), I_0 \left( 0 \right), I_1 \left( 1 \right), 3 f_1 \left( 1, -4 \right) \left( \frac{x^2 - 1}{x^2 - 1} \right)^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p = \frac{x s t}{t \left( x^2 + x \right)}$</td>
<td>#16</td>
<td>$(X Y + 1) Z$</td>
</tr>
<tr>
<td>#16</td>
<td>$y^2 = x^3 + \left( 4 t^2 + t \right) x^2 + 10 s^4 t^2 x + 10 s^3 t^4 x^2$</td>
<td>$I^\infty_6 \left( \infty, 0 \right), 4 f_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p = \frac{-4 t^2 \left( t^2 + 1 \right)^2}{\left( t^2 + 1 \right)^2 \left( x^2 + 1 \right)}$</td>
<td>#9</td>
<td>$(t+x^2)$</td>
</tr>
<tr>
<td>#17</td>
<td>$y^2 - \frac{1}{4} \left( x^4 + 14 s^2 + 1 - s^2 t^2 \right) y z = x \left( x - 4 s^2 \right) \left( x - 4 s^2 \right)$</td>
<td>$I_{16} \left( \infty, 0 \right), 8 f_1 \left( \pm 4 \frac{x^2 - 1}{x^2 - 1}, \ldots \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p_1 = 4 x^2; \quad x_p_2 = \frac{4 t \left( t^2 + 1 \right)^2}{\left( t^2 + 1 \right)^2 \left( x^2 + 1 \right)}$</td>
<td>#16</td>
<td>$\frac{y}{t x^2 + x^2}$</td>
</tr>
<tr>
<td>#18</td>
<td>$y^2 + (\tilde{t} - t^2 + \left( s^2 - 1 \right) t - 2 s^2 t) \left( x^2 + s^2 t^2 y \right) y^2 + s^4 t y^2 = x^2 \left( x - s^4 t \right)$</td>
<td>$I_{13} \left( \infty, 0 \right), I_0 \left( 0 \right), 5 f_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p = 0$</td>
<td>#15</td>
<td>$\frac{y + t x}{\left( x^2 + t^2 \right)}$</td>
</tr>
<tr>
<td>#19</td>
<td>$y^2 - y x \left( t^2 - k t + 1 \right) = x \left( x - 1 \right) \left( x - t^2 - k \right)$</td>
<td>$I_{12} \left( \infty, 0 \right), 2 f_1 \left( s, \frac{1}{2}, 1 \right), 2 f_1 \left( 0, k \right), 2 f_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$r = 0$</td>
<td>#20</td>
<td>$X + Y + Z$</td>
</tr>
<tr>
<td>#21</td>
<td>$y^2 = x^3 + \frac{1}{4} t^2 \left( t^2 + 2 \left( s^2 - 1 \right) t + \left( s^4 - 10 s^2 + 11 \right) \right) x^2$</td>
<td>$I_3 \left( \infty, 0 \right), 4 I \left( 0 \right), 2 f_1 \left( 1, -s^2 \right), 3 f_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p = \frac{s^2 t^2}{s^2 t^2 \left( x^2 + 4 \right)}$</td>
<td>#16</td>
<td>$\frac{y - x^2}{\left( t^2 - 1 \right)^2 \left( x^2 + 4 \right)}$</td>
</tr>
<tr>
<td>#22</td>
<td>$y^2 + \left( t \left( 1 - s^2 \right) + s^2 t^2 \right) y z + s^5 t^2 z = x \left( x - s^4 t \right) \left( x - t^2 s \left( 1 - 1 \right) \right)$</td>
<td>$I^\infty_6 \left( \infty, 0 \right), I_0 \left( 0 \right), 6 f_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p_1 = 1; \quad x_p_2 = s^2 t$</td>
<td>#23</td>
<td>$\frac{Z \left( X Y + Z + Z + s \right)}{1 + Y Z}$</td>
</tr>
<tr>
<td>#23</td>
<td>$y^2 + \left( 2 t^2 - k t + 1 \right) y z = x \left( x - t^2 \right) \left( x - t^2 \right)$</td>
<td>$I^\infty_6 \left( \infty, 0 \right), I_3 \left( 0 \right), 6 f_1 \left( \frac{1}{4 \gamma + 2 \gamma}, \ldots \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p_1 = t^2; \quad x_p_2 = \frac{t k - 1}{t k - 1}$</td>
<td>#24</td>
<td>$\frac{1}{X Y Y}$</td>
</tr>
<tr>
<td>#24</td>
<td>$y^2 + \left( 2 s^2 + 1 \right) t y z = x \left( x - t^2 s^2 \right) \left( x - s^2 \left( t + 1 \right)^2 \right)$</td>
<td>$2 I_2 \left( \infty, 0 \right), I_2 \left( 1, -1 \right), 4 f_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p_1 = t + 1; \quad x_p_2 = t^2 s^2$</td>
<td>#25</td>
<td>$\frac{Z}{X Y + s Z}$</td>
</tr>
<tr>
<td>#25</td>
<td>$y^2 + \left( s + t \right) \left( t s + 1 \right) y z + t^2 s^2 \left( t \left( s^2 - 1 \right) + s \right) y = x \left( x - s \right) \left( x - t^2 s \left( 1 - s \right) \right)$</td>
<td>$I_2 \left( \infty, 0 \right), I_0 \left( 0 \right), 4 f_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p_1 = t s; \quad x_p_2 = -t^2 s^2$</td>
<td>#26</td>
<td>$\frac{Y - Y + Z}{X Y + Z}$</td>
</tr>
<tr>
<td>#26</td>
<td>$y^2 + \left( s - 1 \right) \left( t - s \right) y z = x \left( x - t^2 s^2 \right)^2$</td>
<td>$2 f_1 \left( \infty, 0 \right), I_2 \left( s, \frac{1}{2} \right), 4 f_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p = t s$</td>
<td>#27</td>
<td>$Z$</td>
</tr>
<tr>
<td>#27</td>
<td>$y^2 - \left( t \left( s^2 - 1 \right) + s^2 \right) y z + s^4 t^2 \left( t + 1 \right) y = x \left( x + t^2 s^2 \left( 1 + 1 \right) \right)$</td>
<td>$I^\infty_6 \left( \infty, 0 \right), I_0 \left( 0 \right), I_5 \left( - 1 \right), 4 f_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_p = 0$</td>
<td>#28</td>
<td>$\frac{Z}{X + Y}$</td>
</tr>
</tbody>
</table>
An isomorphism between two lattices that preserves their bilinear form and their Hodge decomposition is called a Hodge isometry.

An automorphism of a K3-surface $S$ is called symplectic if it acts on $H^{2,0}(S)$ trivially. Such automorphisms were studied by Nikulin in [21] who proved that a symplectic involution $i$ (Nikulin involution) has eight fixed points and that the minimal resolution $Y \to S/i$ of the eight nodes is again a K3-surface.

We have then the rational quotient map $p : S \to Y$ of degree 2. The transcendental lattices $T(S)$ and $T(Y)$ are related by the chain of inclusions

$$2T(Y) \subseteq p^*T(S) = T(S)(2) \subseteq T(Y),$$

which preserves the quadratic forms and the Hodge structures.

In this paper, K3-surfaces are given as elliptic surfaces. If we have a 2-torsion section $\tau$, we consider the symplectic involution $i$ (van Geemen-Sarti involution) given by the fiberwise translation by $\tau$. In this situation, the rational quotient map $S \to Y$ is just an isogeny of degree 2 between elliptic curves over $\mathbb{C}(t)$, and we have a rational map $Y \to S$ of degree 2 as the dual isogeny.

### 4.2. Fibrations of some Kummer surfaces

Let $E_i$ be an elliptic curve with invariant $j_i$, defined by a Weierstrass equation in the Legendre form

$$E_i : y^2 = x(x - 1)(x - l).$$

Then $l$ satisfies the equation $j_i = 256 \frac{(1 - t^2)^3}{(1 - t^2)^3}$. For a fixed $j$ the six values of $l$ are given by $l = \frac{1}{2}, 1 - l, \frac{1}{l - 1}, l, \frac{1}{l - 1}, l - 1$.

Consider the Kummer surface $K$ given by $E_{i_1} \times E_{i_2}/\pm 1$ and choose as equation for $K$

$$x_1(x_1 - 1)(x_1 - l_1)t^2 = x_2(x_2 - 1)(x_2 - l_2).$$

Following [19] we can construct different elliptic fibrations. In the general case we can consider the three elliptic fibrations $F_i$ of $K$ defined by the elliptic parameters $m_i$, with corresponding types of singular fibers

- $F_6 : m_6 = \frac{x_1}{x_2}, \quad \frac{x_2}{x_1} = \frac{x_1 - l_1}{x_1 - l_2}$, $\quad 2I_2^*, 4I_2$
- $F_8 : m_8 = \frac{x_1}{x_2}, \quad \frac{x_2}{x_1} = \frac{x_1 - l_1}{x_1 - l_2}$, $\quad 3I_2^*, 3I_2$, $\quad I_1$
- $F_5 : m_5 = \frac{x_1}{x_2}, \quad \frac{x_2}{x_1} = \frac{x_1 - l_1}{x_1 - l_2}$, $\quad I_0^*, 6I_2$

In the special case when $E_1 = E_2$ and $j_1 = j_2 = 8000$ we obtain the following fibrations

- $F_6 : l_1 = l_2 = 3 + 2\sqrt{2}, \quad m_6 = \frac{x_1}{x_2}$, $\quad 2I_2^*, 4I_2$
- $F_8 : l_1 = 3 + 2\sqrt{2}, l_2 = \frac{1}{11}, \quad m_8 = \frac{(x_1 - l_2)(x_1 - x_2)}{l_2(x_2 - l_1)(x_1 - l_1)}$, $\quad 3I_2^*, I_2^*, I_4, I_2, I_1$
- $G_8 : l_1 = 3 + 2\sqrt{2}, l_2 = l_1, \quad m_8 = \frac{(x_1 - x_2)(x_1 - l_1) + (l_1 - 1)x_2}{(l_2x_1 - x_2)(x_1 - l_1 + (l_1 - 1)x_2)}$, $\quad 3I_2^*$
- $F_5 : l_1 = l_2 = 3 + 2\sqrt{2}, \quad m_5 = \frac{x_1}{x_2}, \quad \frac{x_2}{x_1} = \frac{x_1 - l_1}{x_1 - l_2}$, $\quad I_0^*, 4I_2$. 

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4.3. Nikulin involutions and Kummer surfaces

Proposition 4.1. Consider a family $S_{a,b}$ of K3-surfaces with an elliptic fibration, a two-torsion section defining an involution $i$ and two singular fibers of type $I_4^\ast$,

$$S_{a,b}: Y^2 = X^3 + \left( \frac{1}{t} + a \right) X^2 + b^2 X.$$

Then the K3-surface $S_{a,b}/i$ is the Kummer surface $(E_1 \times E_2)/\langle \pm \text{Id} \rangle$ where the $j_i$ invariants of the elliptic curves $E_i$, $i=1,2$ are given by the formulae

$$j_1j_2 = \frac{4096 \left( a^2 - 3 + 12b^2 \right)^3}{b^2},$$

$$(j_2 - 1728)(j_1 - 1728) = \frac{1024a^2(2a^2 - 9 - 72b^2)^2}{b^2}.$$

Proof. Recall that if $E_i$, $i=1,2$ are two elliptic curves in the Legendre form

$$E_i: y^2 = x(x-1)(x-l_i),$$

the Kummer surface $K$ is defined by the following equation

$$K: (E_1 \times E_2)/\langle \pm \text{Id} \rangle$$

is defined by the following equation

$$x_1(x_1-1)(x_1-l_1)t^2 = x_2(x_2-1)(x_2-l_2).$$

The Kummer surface $K$ admits an elliptic fibration with parameter $u = m_6 = \frac{w_1}{w_2}$ and Weierstrass equation $H_u$

$$H_u: Y^2 = X(X-u(u-1)(ul_2-l_1))(X-u(u-l_1)(l_2u-1)).$$

The 2-isogenous curve $S_{a,b}/\langle(0,0)\rangle$ has the following Weierstrass equation

$$Y^2 = X(X-t(t^2+(a-2b)t+1))(X-t(t^2+(a+2b)t+1))$$

with two singular fibers of type $I_2^\ast$ above 0 and $\infty$.

We easily prove that $S_{a,b}/\langle(0,0)\rangle$ and $H_u$ are isomorphic on the field $\mathbb{Q}(\sqrt{w_2})$ where

$$l_1 = w_1'w_2 = \frac{w_2}{w_1}, \quad l_2 = \frac{1}{w_1'w_2'} = w_1w_2 \text{ and } t = w_1u,$$

$w_1, w_1'$ and $w_2, w_2'$ being respectively the roots of polynomials $t^2+(a-2b)t+1$ and $t^2+(a+2b)t+1$.

Recall that the modular invariant $j_i$ of the elliptic curve $E_i$ is linked to $l_i$ by the relation

$$j_i = \frac{256 \left( 1 - l_i + l_i^2 \right)^3}{l_i^2(1-l_i)^2}.$$
By elimination of \( w_1 \) and \( w_2 \), it follows the relations between \( j_1 \) and \( j_2 \)
\[
\begin{align*}
\frac{j_1 j_2}{b^2} &= 4096 \frac{(a^2 - 3 + 12b^2)^3}{b^2} \\
(j_2 - 1728)(j_1 - 1728) &= \frac{1024a^2(2a^2 - 9 - 72b^2)^2}{b^2}.
\end{align*}
\]

\( \square \)

In the Fermi family, the \( K^3 \)-surface \( Y_k \) has the fibration #16 with two singular fibers \( I^*_4 \), a 2-torsion point and Weierstrass equation
\[
y^2 = x^3 + x^2 t \left( 4(t^2 + s^2) + t \left( s^4 + 14s^2 + 1 \right) \right) + 16t^4s^6x.
\]

Taking
\[
y = y't^3 \left( 2\sqrt{s} \right)^3, \quad x = x't^2 \left( 2\sqrt{s} \right)^2 \quad \text{and} \quad t = t's,
\]
we obtain the following Weierstrass equation
\[
y'^2 = x'^3 + \left( t' + \frac{1}{b} + \frac{1}{4} \frac{s^4 + 14s^2 + 1}{s} \right) + s^4x'.
\]

By the previous proposition with \( a = \frac{1}{4} s^4 + 14s^2 + 1 \) and \( b = s^2 \), we derive the corollary below.

**Corollary 4.1.** The surface obtained with the 2-isogeny of kernel \((0,0)\) from fibration #16, is the Kummer surface associated to the product of two elliptic curves of \( j \)-invariants \( j_1, j_2 \) satisfying
\[
\begin{align*}
\frac{j_1 j_2}{s^{10}} &= \left( s^2 + 1 \right)^3 \left( s^6 + 219s^4 - 21s^2 + 1 \right) \frac{s^{10}}{s^{10}} \\
(j_1 - 12^3)(j_2 - 12^3) &= \frac{(s^4 + 14s^2 + 1)(s^8 - 548s^6 + 198s^4 - 44s^2 + 1)}{s^{10}}.
\end{align*}
\]

**Remark 4.1.** If \( s = 1 \) we find \( j_1 = j_2 = 8000 \).

**Remark 4.2.** If \( b = 1 \) we obtain the family of surfaces studied by Narumiya and Shiga, [22]. Moreover if \( a = \frac{9}{4} \) (resp. 4) we find the two modular surfaces associated to the modular groups \( \Gamma_1(7) \) (resp. \( \Gamma_1(8) \)). In these two cases we get \( j_1 = j_2 = -3375 \) (resp. \( j_1 = j_2 = 8000 \)).

**Remark 4.3.** With the same method we can consider a family of \( K^3 \)-surfaces with Weierstrass equations
\[
E_v : Y^2 + XY - (v + \frac{1}{v} - k)Y = X^3 - (v + \frac{1}{v} - k)X^2,
\]
singular fibers of type \( 2I^*_4, 2I_2, 2I_4 \) and the point \( P_v = (0,0) \) of order 4. The elliptic curve \( E'_v = E_v/\langle 2P_v \rangle \) has singular fibers of type \( 2I^*_2, 4I_2 \). An analog computation gives
$$E_c = (E_1 \times E_2) / (\pm \text{Id})$$ and
$$j_1j_2 = (256k^2 - 16k - 767)^3 \quad (j_1 - 12^3)(j_2 - 12^3) = (32k - 1)^2(128k^2 - 8k - 577)^2.$$  

4.4. Shioda-Inose structure

**Definition 4.1.** A $K3$-surface $S$ has a Shioda-Inose structure if there is a symplectic involution $i$ on $S$ with rational quotient map $S \to Y$ such that $Y$ is a Kummer surface and $i^*$ induces a Hodge isometry $T(S)(2) \simeq T(Y)$.

Such an involution $i$ is called a Morrison-Nikulin involution.

An equivalent criterion is that $S$ admits a (Nikulin) involution interchanging two orthogonal copies of $E_8[-1]$ in $NS(S)$.

Or even more abstractly: $2E_8[-1] \hookrightarrow NS(S)$ [21] (Theorem 6.3).

Applying this criterion to fibrations #17 and #8 and the van Geemen-Sarti involution we get the following result.

**Proposition 4.2.** The translation by the two-torsion point of the fibrations #17 and #8 endows $Y_k$ with a Shioda-Inose structure.

**Proof.** Fibration #17 has a fiber of type $I_{16}$ at $t = \infty$. The idea [10] is to use the components $\Theta_{-2}, \Theta_{-1}, \Theta_0, \Theta_1, \Theta_2, \Theta_3, \Theta_4$ of $I_{16}$ and the zero section to generate a lattice of type $E_8[-1]$. The two-torsion section intersects $\Theta_8$ and the translation by the two-torsion point on the fiber $I_{16}$ transforms $\Theta_8$ in $\Theta_{n+8}$. The translation maps the lattice $E_8[-1]$ on another disjoint $E_8[-1]$ lattice and defines a Shioda-Inose structure.

For fibration #8, the fiber above $t = 0$ is of type $I_6$ and the section of order 2 specializes to the singular point $(0,0)$. Then after a blow up, it will not meet the 0-component. If we denote $\Theta_{0,i}, 0 \leq i \leq 5$, the six components, then the zero section meets $\Theta_{0,0}$ and the 2-torsion section meets $\Theta_{0,3}$, the translation by the 2-torsion section induces the permutation $\Theta_{0,i} \mapsto \Theta_{0,i+3}$.

The fiber above $t = \infty$ is of type $I_6^*$. The simple components are denoted $\Theta_{\infty,0}, \Theta_{\infty,1}$ and $\Theta_{\infty,2}, \Theta_{\infty,3}$; the double components are denoted $C_i$ with $0 \leq i \leq 6$ and $\Theta_{\infty,0}C_0 = \Theta_{\infty,1}C_0 = 1; \Theta_{\infty,2}C_6 = \Theta_{\infty,3}C_6 = 1$. Then the 2-torsion section intersects $\Theta_{\infty,2}$ or $\Theta_{\infty,3}$ and the translation by the 2-torsion section induces the transposition $C_i \leftrightarrow C_{6-i}$.

The class of the components $C_0, C_1, C_2, \Theta_{\infty,0}, \Theta_{\infty,1}$, the zero section, $\Theta_{0,0}$ and $\Theta_{0,1}$ define a copy of $E_8[-1]$. The Nikulin involution defined by the two-torsion section maps this $E_8[-1]$ to another copy of $E_8[-1]$ orthogonal to the first one; so the Nikulin involution is a Morrison-Nikulin involution. □

4.5. Base change and van Geemen-Sarti involutions

If a $K3$-surface $S$ has an elliptic fibration with two fibers of type $II^*$, this fibration can be realized by a Weierstrass equation of type

$$E_h : y^2 = x^3 - 3ax + (h + 1/h - 2\beta).$$

Moreover, Shioda [31] deduces the “Kummer sandwiching”, $K \to S \to K$, identifying the Kummer $K = E_4 \times E_2 / \pm 1$ with the help of the $j$-invariants of the two elliptic curves.
Apéry-Fermi pencil of $K^3$-surfaces and 2-isogenies

$E_1, E_2$ and giving the following elliptic fibration of $K$

$$y^2 = x^3 - 3ax + (t^2 + 1/t^2 - 2\beta).$$

This can be viewed as a base change of the fibration $E_h$ of $S$.

**4.5.1. Alternate elliptic fibration**

We shall now use an alternate elliptic fibration of $S$ ([29] example 13.6) to show that this construction is indeed a 2-isogeny between two elliptic fibrations of $S$ and $K$. In the next picture we consider a divisor $D$ of type $I_{12}^*$ composed of the zero section $0$ and the components of the $II^*$ fibers enclosed in dashed lines. The far double components of the $II^*$ fibres can be chosen as sections of the new fibration. Take $\omega$ as the zero section and $\mu$ for the other one. More precisely with the new parameter $u := x$ ($x$ from $E_h$) and the variables $Y = yh$ and $X = h$, we obtain the Weierstrass equation

$$Y^2 = X^3 + (u^3 - 3au - 2\beta)X^2 + X.$$ 

The section $\mu$ defined by $X = h = 0$ and $Y = 0$ is a two-torsion section.

In this equation, if we substitute $X(= h)$ by $t^2$, we obtain an equation in $W, t$ with $Y = Wt^2$, which is the equation for the 2-isogenous elliptic curve. Indeed the birational transformation

$$y = 4Y + 4U^3 + 2UA, \quad x = 2 \frac{Y + U^3}{U}$$

with inverse

$$U = 1/2 \frac{y}{x + A}, \quad Y = 1/8 \frac{(-y^2 + 2x^3 + 4xA^2 + 2xA^2)y}{(x + A)^3}$$

transforms the curve $Y^2 = U^6 + AU^4 + BU^2$ in the Weierstrass form

$$y^2 = (x + A)(x^2 - 4B).$$

This is an equation for the 2-isogenous curve of the curve $Y^2 = X^3 + AX^2 + BX$. On the curve $Y^2 = U^6 + AU^4 + BU^2$, the involution $U \mapsto -U$ means adding the two-torsion
point \((x = -A, y = 0)\).

Using this above process with \(A = (u^3 - 3\alpha u - 2\beta)\), the 2-isogenous curve \(E_u\) has a Weierstrass equation

\[
E_u : Y^2 = (X + (u^3 - 3\alpha u - 2\beta)) \left( X^2 - 4 \right)
\]

with singular fibers of type \(I_5^6, 6I_2^l\).

The coefficients \(\alpha\) and \(\beta\) can be computed using the \(j\)-invariants

\[
\alpha^3 = J_1J_2; \quad \beta^2 = (1 - J_1)(1 - J_2); \quad j_1 = 1728J_1.
\]

If the elliptic curve is put in the Legendre form \(y^2 = x'(x' - 1)(x' - \ell)\) then \(j = 256(1 + t^2)^3\), so

\[
\alpha^3 = \frac{16}{27} \frac{(1 - l_1 + l_1^2)(1 - l_2 + l_2^2)}{l_1^2(l_1 - 1)l_2^2(l_2 - 1)^2},
\]

\[
\beta = \frac{1}{27} \frac{(2l_1 - 1)(l_2 - 2)(2l_2 - 1)(l_2 - 1)(l_1 + 1)(l_2 + 1)}{l_1l_2(l_1 - 1)(l_2 - 1)}.
\]

On the Kummer surface \(E_1 \times E_2/ \pm 1\) of equation

\[
X_1 (X_1 - 1) (X_1 - l_1) Z^2 = X_2 (X_2 - 1) (X_2 - l_2)
\]

we consider an elliptic fibration (case \(J_5\) of [19]) with the parameter

\[
z = \frac{(l_2X_1 - X_2)(X_1 - l_1 + X_2(l_1 - 1))}{X_2(X_1 - 1)}
\]

(in fact \(z = \frac{-l_1(l_2 - 1)}{m_{5,-1}}\) cf. 4.2) and obtain the Weierstrass equation

\[
Y^2 = (X - 2l_1l_2(l_1 - 1)(l_2 - 1))(X + 2l_1l_2(l_1 - 1)(l_2 - 1))
\]

\[
(\alpha z^3 + 4(-2l_1l_2 + l_1 + l_2 + 1)z^2)
\]

\[
+4(l_1l_2 - l_1 - l_2)z + 2l_1l_2(l_1 - 1)(l_2 - 1).
\]

Substituting \(z = w - \frac{1}{3} (-2l_1l_2 + l_1 + l_2 + 1)\) it follows

\[
Y^2 = (X - 2l_1l_2(l_1 - 1)(l_2 - 1))(X + 2l_1l_2(l_1 - 1)(l_2 - 1))
\]

\[
\left( X + 4w^3 - \frac{4}{3}(l_1^2 - l_2)(l_1^2 + 1)w 
\]

\[
+ \frac{2}{27}(l_2 - 2)(2l_2 - 1)(l_1 - 2)(2l_1 - 1)(l_2 + 1)(l_1 + 1) \right).
\]

Up to an automorphism of this Weierstrass form we recover the equation of \(E_u\).

The previous results can be used to show the following proposition

**Proposition 4.3.** The translation by the two-torsion point of the elliptic fibration \#4 gives to \(Y_\ell\) a Shioda-Inose structure.
5. Proof of Theorem 1.1

Notation 5.1. If we consider an elliptic fibration \( \phi \) of a K3-surface \( S \) with a two-torsion point \( T \), we will write \( \phi(T) \) for the elliptic fibration of \( S/i \) if \( i \) denotes the involution given by the fiberwise translation by \( T \).

From the Shioda-Tate formula (cf. e.g. [32], Corollary 1.7) we have the relation

\[
12 = \frac{\vert \Delta \vert \prod m_v(1)}{|\text{Tor}|^2}
\]

where \( \Delta \) is the determinant of the height-matrix of a set of generators of the Mordell-Weil group, \( m_v(1) \) the number of simple components of a singular fiber and \( |\text{Tor}| \) the order of the torsion group of the Mordell-Weil group. From a set of infinite sections this formula allows us to determine generators of the Mordell-Weil group except for fibration #4. Using the 2-isogeny we determine also the Mordell-Weil group of \( \#n(T) \). The discriminant is either \( 12 \times 2 \) or \( 12 \times 8 \).

Proposition 5.1. The translation by the two-torsion point of the fibration \( \#16 \) gives to \( Y_k \) a Shioda-Inose structure.

Proof. From the Proposition 4.1, the translation by the two-torsion point of \( \#16 \) gives to the quotient a Kummer structure. The fibration \( \#16 \) is of rank one, its Mordell-Weil group is generated by the point \( P \) of \( x \)-coordinate \( x_P \) in Table 4, and the two-torsion point. By computation we can see that the Mordell-Weil group of the 2-isogenous curve on \( C(t) \) is generated by the image of \( P \) and torsion sections. So we can compute the discriminant of the Néron-Severi group which is \( 12 \times 8 \). The second condition, \( T(Y_k)(2) \cong T(K_k) \), is then verified. \( \square \)

Remark 5.1. The K3-surface of Picard number 20 given by the elliptic fibration

\[
Y^2 = X^3 - \left( t + \frac{1}{t} - \frac{3}{2} \right) X^2 + \frac{1}{16} X
\]

or

\[
y^2 = x^3 - \frac{1}{2} t \left( 2 t^2 + 2 - 3 t \right) x^2 + \frac{1}{16} t^4 x
\]

has rank 1. The Mordell-Weil group is generated by \((0,0)\) and \( P = (x = \frac{1}{t}, y = \frac{(t-1)^2}{8}) \). The determinant of the Néron-Severi group is equal to 12. By computation we find that the image of \( P \) by the 2-isogeny is equal to \( 2Q \) with \( Q = (t(t-1)(t^2-t+1), -t^3(t-1)(t^2-t+1)) \) of height \( \frac{1}{2} \). The determinant of the Néron Severi group of the 2-isogenous curve is then 12 not \( 12 \times 2^2 \). So the involution induced by the two-torsion point is not a Nikulin-Morrison involution. Moreover the 2-isogenous elliptic curve is a fibration of the Kummer surface \( E \times E/\pm 1 \) where \( j(E) = 0 \).

For fibrations \( \#n(T) \) with discriminant of the transcendental lattice \( 12 \times 8 \) we prove the Shioda-Inose structure in the following way: from corollary 4.1 this is true for \( \#16(T) \), from Proposition 4.3 this is true for \( \#4(T) \) and from Proposition 4.2 this is true for
\#17(T), \#8(T). The other fibrations \#n(T) can be obtained by 2- or 3-neighbor method from \#16(T), \#8(T) or \#17(T). The results are given in the Table 5. In the second column are written the Weierstrass equations for the \#n elliptic fibration and its 2-isogenous fibration, singular fibers and the \(x\)-coordinates of generators of the Mordell lattice of \#n(T). In the third column we give the starting fibration for the 2- or 3-neighbor method and in the last column the parameter used from the starting fibration.

5.1. The \(K3\)-surface \(S_k\)

For the remaining fibrations, (discriminant \(12 \times 2\), using also the 2- or 3-neighbor method, they are proved to lie on the same surface \(S_k\). Except for the case \#7 the results are collected in the Table 6 with the same format. The case \#7 needs an intermediate method, they are proved to lie on the same surface \(S\) and in the last column the parameter used from the starting fibration.

Starting with the fibration \#7(T) and using the parameter \(m_7 = \frac{w}{x(1-x^2)}\) it follows the Weierstrass equation

\[
Y^2 + 2\left(m_7^2s^2 - 2\right)YX - 16m_7^2s^4Y = (X - 8m_7^2s^2)(X + 8m_7^2s^2)\]

with singular fibers \(I_9(\infty), IV^*(0), 8I_1\).

Then the parameter \(m_{15} = \frac{y}{s}x + s(2s^2)\) leads to the fibration \#15(T).

For the last part of Theorem 1.1 we give properties of \(S_k\). First we prove that \(S_k\) is the Jacobian variety of some genus 1 fibrations of \(K_k\).

Starting with the fibration \#26(T) and Weierstrass equation

\[
y^2 = x \left( x + 4t^2s^2 \right) \left( x + \frac{1}{4} (t - s)^2 (ts - 1)^2 \right),
\]

the new parameter \(m := \frac{y}{t(x + \frac{1}{4} (t - s)^2 (ts - 1)^2)}\) defines an elliptic fibration of \#26(T) with Weierstrass equation

\[
E_m : Y^2 = m \left( s + 1 \right) YX = X \left( X - s^2m^2 \right) \left( X + \frac{1}{4} (2m - s)^2 (2m + s)^2 \right)
\]

and singular fibers are of type \(4I_4 \left( 0, \pm \frac{1}{2}s, \infty \right), 8I_1\).

Then setting as new parameter \(n = \frac{w}{m}\), it follows a genus one curve in \(m\) and \(Y\). Its equation, of degree 2 in \(Y\), can be transformed into

\[
w^2 = -16n(-n + s^2)m^4 + n(s^4 (8 + n) - 10ns^2 + n(1 + 4n))m^2 - ns^4(-n + s^2).
\]

Let us recall the formula giving the Jacobian of a genus one curve defined by the equation \(y^2 = ax^4 + bx^3 + cx^2 + dx + e\). If \(c_4 = 2^4(12ae - 3bd + c^2)\) and \(c_6 = 2^5(72ace - 27ad^2 - 27b^2e + 9bed - 2e^3)\), then the equation of the Jacobian curve is

\[
\tilde{y}^2 = \tilde{x}^3 - 27c_4\tilde{x} - 54c_6.
\]
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<th>Weierstrass Equation</th>
<th>From</th>
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</thead>
<tbody>
<tr>
<td>#8</td>
<td>( y^2 - k (t-1) y x = x (x-1) (x-t^2) )</td>
<td>see Prop 9</td>
<td>L_2^0(\infty,0), I_2(0), J_2(1), 4I_1, 2I_2(\infty,0), 4I_1</td>
</tr>
<tr>
<td>#16</td>
<td>( y^2 = x^3 + \frac{1}{4} (4t^4 - t^2k^2 + 2k^2 + 4 - k^2) x^2 + \frac{1}{16} (t-1)^2 (4t^2 + t (4-k^2) + (k-2)^2) x ) ( L_2^0(\infty,0), I_2(0), J_2(1), 4I_1 )</td>
<td>( x_Q = - \frac{1}{2} (t-1) (4t^2 + t (4-k^2) + (k-2)^2) )</td>
<td>( 2I_2(\infty,0), 4I_1 )</td>
</tr>
<tr>
<td>#17</td>
<td>( y^2 - \frac{1}{2} (s^4 + 14sx + 1 - s^2t^2) y x = x (x-4s^2) (x-4s^2) ) ( L_2^0(\infty,0), J_2(0), 6I_1, \frac{1}{2} (s^2 \pm 4s - 1, \ldots) )</td>
<td></td>
<td>( I_2(\infty,0), 6I_2, \frac{1}{2} (s \pm 4s - 1, \ldots) )</td>
</tr>
<tr>
<td>#23</td>
<td>( y^2 = x (x+\frac{1}{4} ((tk-2) -1) (4t^2 - (k-2) t +1)) ) ( L_2^0(\infty,0), J_2(0), 6I_1, 6I_2, \frac{1}{2} (t \pm 2, \ldots) )</td>
<td></td>
<td>( x_Q = 1, \frac{1}{2} ((tk-2) t +1) ((tk-2) t +1) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>#24</td>
<td>( y^2 + (s^2 + 1) ty x = x (x-t^2) (x-4s^2) ) ( L_2^0(\infty,0), I_1(0), 6I_1, \frac{1}{2} (t \pm 2, \ldots) )</td>
<td></td>
<td>( x_Q = 1, \frac{1}{2} ((tk-2) t +1) ((tk-2) t +1) )</td>
</tr>
<tr>
<td>#26</td>
<td>( y^2 = x (x+\frac{1}{4} ((tk-2) -1) (4t^2 - (k-2) t +1)) ) ( L_2^0(\infty,0), I_2(0), 6I_1, 6I_2, \frac{1}{2} (t \pm 2, \ldots) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Fibrations with discriminant 12 \times 8 (Fibrations of the Kummer \( K_k \))
<table>
<thead>
<tr>
<th>No</th>
<th>Weierstrass Equation</th>
<th>From</th>
<th>Param.</th>
</tr>
</thead>
<tbody>
<tr>
<td>#7</td>
<td>( y^2 = x^3 + \frac{1}{4} (t (s^4 - 10s^3 + 1) + 8s^4) x^2 - t^2 s^3 (t - s)^2 x )</td>
<td>( 2II^* (\infty), I_2^* (0), I_3 (s^2) ).</td>
<td>2I2</td>
</tr>
<tr>
<td></td>
<td>( y^2 = x^3 + \frac{1}{4} (t (s^8 - 10s^7 + 1) + 8s^8) x^2 + \frac{1}{16} t^3 (64s^8 - 20s^7 - 90s^6 - 20s^5 + 1) t + 16s^4 (s^2 + 1)^2 x )</td>
<td>( 2II^* (\infty), I_2^* (0), I_3 (s^2) ).</td>
<td>2I2</td>
</tr>
<tr>
<td>#9</td>
<td>( y^2 = x^3 + \frac{1}{4} (s^4 + 14s^3 + 1) t^3 x^2 + t^4 s^3 (s^2 + 1) t (s^4 + 1) x )</td>
<td>( 2II^* (\infty, 0), 2I_3 (-s^2, -\frac{1}{16}), 2I_2 ).</td>
<td>2</td>
</tr>
<tr>
<td>#14</td>
<td>( y^2 = x (x - \frac{1}{4} (s^2 - 1)^2 - \frac{1}{4} (s^4 + 14s^3 + 1) t x - ts^2) )</td>
<td>( I_6^* (\infty), I_4^* (0), 4I_3 (\frac{-1}{4}, \frac{-4s^2}{(s^2 - 1)^2}, \ldots) ).</td>
<td>2</td>
</tr>
<tr>
<td>#15</td>
<td>( (y - tx) (y - st x) = x (x - ts^2) (x - ts^2 (t + 1)^2) )</td>
<td>( I_4^* (\infty), I_4^* (0), I_4 (-1), 3I_3 (\frac{-1}{4}, \frac{(s^2 - 1)^2}{(s^2 - 4s + 1)} x, \ldots) ).</td>
<td>20</td>
</tr>
<tr>
<td>#20</td>
<td>( y^2 - (t^4 s - (s^4 + 1) t + 3s) y x - s^2 (t - s) (ts - 1) y = x^3 )</td>
<td>( I_{12} (\infty), 2I_3 (s, \frac{1}{2}), 2I_2 (0, \frac{s - 1}{2}), 2I_1 ).</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 6. Fibrations with discriminant 12 × 2 (Fibrations of \( S_k \))

In our case we obtain

\[
y^2 = x \left( x + n^3 s^2 - \frac{1}{4} n^2 (s^2 - 1)^2 \right) \left( x + n^3 s^2 - \frac{1}{4} (s^2 - 4s - 1) (s^2 + 4s - 1) n^2 + 4ns^2 \right)
\]

which is precisely the fibration #15(T).

**Remark 5.2.** Using the new parameter \( p_1 = \frac{Y}{m^2 (X + \frac{1}{4} (2m-s)^2) (2m+s)^2} \), another result can be derived from \( E_m \), leading to

\[
E_{p_1} : Y^2 - 2s (2p_1 - 1) (2p_1 + 1) Y X = \]

Apéry-Fermi pencil of $K^3$-surfaces and 2-isogenies

Let $X = (X + 6s^2p_2^2)(X + (2sp_1 + 1)(2sp_1 - 1)(s + 2p_1)(s - 2p_1))$, with singular fibers $2I^*_1, 4I_2, 4I_1$. From $E_{p_1}$ and the new parameter $k = \frac{X}{p_1^2}$ we obtain a genus one fibration whose Jacobian is $\#14(T)$.

Starting from the fibration $\#26(T)$ (previous equation), the parameter $q = \frac{s}{t^2}$ leads to a genus one fibration whose Jacobian is a fibration of $S_k$ leading to $\#15(T)$.

5.2. Transcendental and Néron-Severi lattices of the surface $S_k$

We shall indentify the $K^3$-surface $S_k$ by its transcendental and Néron-Severi lattices. As a corollary, using again the Kneser-Nishiyama technique this allows to recover the yet known elliptic fibrations of $S_k$ but also all of them. We shall see that all the fibrations can be obtained from the primitive embeddings of $M = A_1 \oplus A_1 \oplus A_5$ into the various Niemeier lattices. Since $M$ is composed of 3 root lattices of type $A_n$, the $K^3$-surface $S_k$ will possess probably more twice elliptic fibrations than $Y_k$.

Lemma 5.1. The $K^3$-surface $S_k$ has the transcendental lattice

$$T(S_k) = \langle -2 \rangle \oplus \langle 2 \rangle \oplus \langle 6 \rangle$$

and Néron-Severi lattice

$$NS(S_k) = U \oplus E_8[-1] \oplus E_7[-1] \oplus \langle -2 \rangle \oplus \langle -6 \rangle.$$ 

Proof. We consider the elliptic fibration $\#7(T)$ with Weierstrass equation given in Table 6 and draw the graph of the singular fibers, the zero and two-torsion sections of the elliptic fibration

$$Y^2 = X^3 + \left( \left( -1/2 s^4 + 5 s^2 - 1/2 \right) t^2 - 4 s^4 t \right) X^2$$

$$+ \left( 4 s^2 t^5 + \left( 1/16 s^8 - 45/8 s^4 - 5/4 s^2 - 5/4 s^6 + 1/16 \right) t^4 + s^4 \left( s^2 + 1 \right)^2 t^3 \right) X$$

with singular fibers $III^*(\infty), I_2^*(0), I_3(s^2), 2I_2(t_1, t_2)$.

With the parameter $m = \frac{s}{t}$ we obtain another fibration with singular fibers $II^*(\infty)$ (shown by $\oplus$), $I_2^*(0)$ (shown by $\odot$), $I_3(\frac{1}{4}s^2(s^2 - 1)^6)$ (part of it shown by $\boxdot$), $I_2(4s^4)$
We deduce the discriminant form, since $b_{\Sigma_k}$ has no torsion, rank 0, Weierstrass equation
\[
y^2 = x^3 + 2m((-s^4 + 10s^2 - 1)m + 2s^4(s^2 + 1)^2)x^2 \\
+ (m - 4s^4)m^3((s^8 - 20s^6 - 90s^4 - 20s^2 + 1)x + 256m^5s^2(m - 4s^4)^2
\]
and Néron-Severi group
\[
NS(\Sigma_k) = U \oplus E_8 \oplus D_6 \oplus A_2 \oplus A_1.
\]
By Morrison ([21], Corollary 2.10 ii), the Néron-Severi group of an algebraic $K3$-surface $X$ with $12 \leq \rho(X) \leq 20$ is uniquely determined by its signature and discriminant form. Thus we compute $q_{NS(\Sigma_k)}$ with the help of the fibration $\Sigma_k$. From
\[
D^*_6/D_6 = ([1]_{D_6}, [3]_{D_6}) \text{ and } q_{D_6}([1]_{D_6}) = q_{D_6}([3]_{D_6}) = -\frac{3}{2},
\]
we deduce the discriminant form, since $b_{D_6}([1]_{D_6}, [3]_{D_6}) = 0$,
\[
(G_{NS(\Sigma_k)}, q_{NS(\Sigma_k)}) = \mathbb{Z}/2\mathbb{Z}(-\frac{3}{2}) \oplus \mathbb{Z}/2\mathbb{Z}(-\frac{3}{2}) \oplus \mathbb{Z}/3\mathbb{Z}(-\frac{2}{3}) \oplus \mathbb{Z}/2\mathbb{Z}(-\frac{1}{2}) \mod 2\mathbb{Z}
\]
\[
= \mathbb{Z}/2\mathbb{Z}(\frac{1}{2}) \oplus \mathbb{Z}/6\mathbb{Z}(\frac{1}{6}) \oplus \mathbb{Z}/2\mathbb{Z}(\frac{1}{2}).
\]
From Morrison ([21] Theorem 2.8 and Corollary 2.10) there is a unique primitive embedding of $NS(S_k)$ into the $K3$-lattice $\Lambda = E_8[-1]^2 \oplus U^3$, whose orthogonal complement is by definition the transcendental lattice $T(S_k)$. Now from Nikulin([24] Proposition 1.6.1), it follows
\[
G_{NS}(S_k) \simeq (G_{NS(\Sigma_k)})^\perp = G_{T(S_k)}, \quad q_{T(S_k)} = -q_{NS(\Sigma_k)}.
\]
In other words the discriminant form of the transcendental lattice is
\[
(G_{T(S_k)}, q_{T(S_k)}) = \mathbb{Z}/2\mathbb{Z}(\frac{1}{2}) \oplus \mathbb{Z}/6\mathbb{Z}(\frac{1}{6}) \oplus \mathbb{Z}/2\mathbb{Z}(\frac{1}{2}).
\]
From this last relation we prove that $T(S_k) = (-2) \oplus (6) \oplus (2)$. Denoting $T'$ the lattice $T' = (-2) \oplus (6) \oplus (2)$, we observe that $T'$ and $T(S_k)$ have the same signature and discriminant form. Since $|\det(T')| = 24$ is small, there is only one equivalence class of forms in a genus, meaning that such a transcendental lattice is, up to isomorphism, uniquely determined by its signature and discriminant form ([11] p. 395).

Now computing a primitive embedding of $T(S_k)$ into $\Lambda$, since by Morrison ([21] Corollary 2.10 i) this embedding is unique, its orthogonal complement provides $NS(S_k)$. Take the primitive embedding $(-2) \hookrightarrow E_8$, $(2) = (u_1 + u_2) \hookrightarrow U$, $(6) = (u_1 + 3u_2) \hookrightarrow U$, $(u_1, u_2)$ denoting a basis of $U$. Hence we deduce
\[
NS(S_k) = U \oplus E_8[-1] \oplus E_7[-1] \oplus (-2) \oplus (-6).
\]
Corollary 5.1. All the elliptic fibrations of $S_k$ are obtained from the primitive embeddings of $A_1 \oplus A_1 \oplus A_5$ in the various Niemeier lattices.

Proof. In that purpose, embed $T(S_k)[-1]$ into $U \oplus E_8[-1]$ in the following way: $(-2) \oplus (-6)$ primitively embedded in $E_8[-1]$ as in Nishiyama (25, p. 334) and $(2) = \langle u_1 + u_2 \rangle \hookrightarrow U$. We obtain $M = (T(S_k)[-1])_{U \oplus E_8[-1]} = A_1 \oplus A_1 \oplus A_5$. Now all the elliptic fibrations of $S_k$ are obtained from the primitive embeddings of $M$ into the various Niemeier lattices, as explained in section 2.

□

Using their Weierstrass equations and a 2-neighbor method [15], it was proved in the previous subsection that all the fibrations #7($T$), #9($T$), #14($T$), #15($T$), #20($T$) are on the same $K3$-surface. Using the Kneser-Nishiyama method we can identify each of these elliptic fibrations with a primitive embedding into a certain Niemeier lattice.

This identification will be performed comparing singular fibers and Mordell-Weil lattices.

5.2.1. Take the primitive embedding into $Ni(D_{10}E_7^2)$, given by $A_5 = \langle e_2, e_4, e_5, e_6, e_7 \rangle \hookrightarrow E_7$ and $A_7^2 = \langle d_{10}, d_7 \rangle \hookrightarrow D_{10}$.

Since $(A_9)^{E_7}_E = A_2$ and $(A_7^2)^{D_{10}}_{D_{10}} = A_1 \oplus A_1 \oplus D_6$, it follows $N = N_{\text{root}} = 2A_1A_2D_6E_7$, det $N = 24 \times 4$, thus the rank is 0 and the torsion group $\mathbb{Z}/2\mathbb{Z}$. Hence this fibration can be identified with the elliptic fibration #7($T$).

5.2.2. The primitive embedding is into $Ni(D_{10}E_7^2)$, given by $A_5 \oplus A_7^2 = \langle d_{10}, d_8, d_7, d_6, d_5, d_{10} + d_9 + 2(d_8 + d_7 + d_6 + d_5 + d_4) + d_3, d_3 \rangle \hookrightarrow D_{10}$.

We get $A_5 \oplus A_7^2 = (-6) \oplus \langle x \rangle \oplus \langle d_1 \rangle = (-6) \oplus A_1 \oplus A_1$

with

$x = d_9 + d_{10} + 2(d_8 + d_7 + d_6 + d_5 + d_4 + d_3 + d_2) + d_1$

and

$(-6) = 3d_9 + 2d_{10} + 4d_8 + 3d_7 + 2d_6 + d_5$.

Thus $N_{\text{root}} = A_1A_1E_7^2$ and the rank of the fibration is 1. Since $2[2]_{D_{10}} = x + d_1$ and there is no other relation with $[1]_{D_{10}}$ or $[3]_{D_{10}}$, among the glue vectors $\langle [1, 1, 0] \rangle, \langle [3, 0, 1] \rangle$ generating $Ni(D_{10}E_7^2)$, only $\langle [2, 1, 1] \rangle$ contributes to torsion.

Hence the torsion group is $\mathbb{Z}/2\mathbb{Z}$. Moreover the 2-torsion section is $2F + 0 + \langle [2], [1], [1] \rangle$.
with height $4 - (1/2 + 1/2 + 3/2 + 3/2) = 0$. The infinite section is

$$3F + 0 + [(-6), 0, 0]$$

with height 6. Hence this fibration can be identified with the fibration #9(T).

### 5.2.3.

Take the primitive embedding into $N_i(D_8^2)$, given by $A_5 = \langle d_8, d_6, d_5, d_4, d_3 \rangle \hookrightarrow D_8^{(1)}$ and $A_7^2 = \langle d_8, d_4 \rangle \hookrightarrow D_8^{(2)}$. We compute $(A_5)_{D_8}^1 = (-6) \oplus (x_1 = (-2)) \oplus (d_1)$ with $x_1 = d_7 + d_8 + 2d_6 + d_5 + d_4 + d_3 + d_2 + d_1$

$$(A_7^2)_{D_8} = (d_7) \oplus (x_1 = d_7 + d_8 + 2(d_6 + d_5 + d_4 + d_3 + d_2) + d_1)$$

$$(A_7^2)_{D_8} = (d_7) \oplus (x_1 = d_7 + d_8 + 2(d_6 + d_5 + d_4 + d_3 + d_2) + d_1)$$

We deduce $N_{\text{root}} = 4A_1D_4D_8$ (hence the fibration has rank 1) and the relations

1. $2[2]_{D_8} = x_1 + d_1$
2. $2([2]_{D_8} - (d_1 + d_2)) = x_3 + 2d_3 + 2d_4 + d_5$
3. $2[3]_{D_8} = x_1 + 2x_3 + d_3 + 2d_4 + d_5 + d_7$
4. $2([1]_{D_8} - (d_6 + d_8)) = x_1 + x_3 + d_3 + 2d_4 + 2d_5 + d_7$
5. $2([1]_{D_8} - (d_6 + d_7 + d_8)) = 2x_3 + 3d_5 + 4d_4 + 3d_3 + 2d_2 + d_1 - d_7$.

Thus, among the glue vectors $[1, 2, 2], [1, 1, 1], [2, 2, 1]$ generating the Niemeier lattice, only vectors $[0, 3, 3], [2, 1, 2]$ contribute to torsion and the torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

From relations (1) to (5) we deduce the various contributions and heights of the following sections (see Table 7).

Hence this fibration can be identified with the fibration #14(T).

### 5.2.4.

The primitive embedding is into $N_i(D_8^2)$ and given by

$$A_5 = \langle d_8, d_6, d_5, d_4, d_3 \rangle \hookrightarrow D_8^{(1)}$$

$$A_7^2 = \langle d_8 \rangle \hookrightarrow D_8^{(2)}$$

$$A_1 = \langle d_8 \rangle \hookrightarrow D_8^{(3)}.$$

As previously $(A_5)_{D_8}^1 = (-6) \oplus (x_1) \oplus (d_1)$; we get also $\langle d_8 \rangle_{D_8} = \langle d_7 \rangle \oplus (x_4 = d_7 + d_5 + 2d_6 + d_5, d_4, d_3, d_2, d_1) = A_1 \oplus D_6$. Hence $N_{\text{root}} = 4A_12D_6$, and the rank is 1. Moreover it follows the relations

6. $2[2]_{D_8} = x_1 + d_1$
7. $2[2]_{D_8} = x_3 + d_5 + 2d_4 + 2d_3 + 2d_2 + 2d_1$
We deduce that among the glue vectors generating $N_i(D_0^2)$, only $\langle 0, 3, 3 \rangle, [2, 1, 2 \rangle$ contribute to torsion. So the torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. From relations (6) to (9) we deduce the various contributions and heights of the following sections (see Table 8).

Hence this fibration can be identified with the fibration #15(T).

5.2.5.

Take the primitive embedding onto $N_i(A_4^2D_4)$ given by $A_5 \hookrightarrow A_5$, $A_1 \oplus A_1 = \langle d_4, d_1 \rangle \hookrightarrow D_4$. Since $\langle d_4, d_1 \rangle \mathfrak{D}_A = A_4^2$, we get $N = N_{\text{root}} = 3A_2^2A_1$; thus the rank of the fibration is 0 and since $\det(N) = 24 \times 6^2$, the torsion group is $\mathbb{Z}/6\mathbb{Z}$.

This fibration can be identified with the fibration #20(T).

Remark 5.3. From fibration #20(T) the surface $S_k$ appears to be a double cover of the rational elliptic modular surface associated to the modular group $\Gamma_0(6)$ given in Beauville’s paper [2]

$$(x + y)(y + z)(z + x)(t - s)(ts - 1) = 8xyz.$$

6. Proof of Theorem 1.2

We recall first on Table 8 the results obtained by Bertin and Lecacheux in [3].

Comparing to the fibrations of the family, you remark more elliptic fibrations with 2-torsion sections on $Y_2$. Some of them are specializations for $s = 1$ of the generic ones. They are denoted for example #17(18 – m) which means the following: it is the fibration (18 – m) in the last Table of [2] ($m$ denotes the elliptic parameter of the fibration numbered 18) and the specialization for $k = 2$ of the fibration #17 of the generic case. Those generic elliptic fibrations with Morrison-Nikulin involutions possess specializations to $Y_2$ with Morrison-Nikulin involutions, by a Schütt’s Lemma [28], namely #4 (16 – a), #8 (9 – r), #16 (14 – t), #17 (18 – m), #23 (2 – k), #24 (5 – d) a), #26 (1 – s). Others (#15 (17 – q), #24 (ψ), #10 (10 – c), #15 (6 – p) c), #24 (5 – d) b, c)) are specific to $K_2$ and cannot be deduced from elliptic fibrations of the generic Kummer. To identify them, we have to use the distinguished property of $Y_2$, that is $Y_2$ is a singular $K3$-surface with Picard number 20.

Hence $Y_2$ inherits of a Shioda-Inose structure, that is the quotient of $Y_2$ by an involution is isomorphic to a Kummer surface $K_2$ realized from the product of CM elliptic curves [33, 34] provided in the following way.

<table>
<thead>
<tr>
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<td>1/2</td>
<td>1+1/2</td>
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<tr>
<td>$[2,2,1]$</td>
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<tr>
<td>$[3,3,3]$</td>
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<td>0</td>
<td>1</td>
<td>1+1/2</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 8. Contributions and heights of the sections of 5.2.4

\[2([1]_{D_0} - (d_5 + d_6 + d_7 + d_8)) = 2x_3 + d_5 + 4d_4 + 3d_3 + 2d_2 + d_1 - d_7\]

\[2([3]_{D_0} = 3x_3 + d_7 + 2d_5 + 4d_4 + 3d_3 + 2d_2 + d_1 \in A_1 \oplus D_6.\]
\[ L_{\text{root}} \quad L/L_{\text{root}} \quad \text{Fibers} \quad \text{R. Tor.} \]

| #1(11 – f) | \( A_1 \subset E_8 \) | \( D_5 \subset E_8 \) | 0 \( (0) \) |
| #2(13 – h) | \( A_1 \oplus D_6 \subset E_8 \) | \( A_1 E_8 E_8 \) | 1 \( (0) \) |

\[ E_6 D_{16} \quad 2 \times 2Z \]

| #3(30 – a) | \( A_1 \subset E_8 \) | \( D_6 \subset D_{16} \) | \( E_7 D_{11} \) | 0 \( (0) \) |
| #4(16 – o) | \( A_1 \oplus D_3 \subset E_8 \) | \( A_1 D_{16} \) | 1 \( Z/2Z \) |
| #5(17 – q) | \( D_5 \subset E_8 \) | \( A_1 \subset D_{16} \) | \( A_4 A_1 D_{14} \) | 0 \( Z/2Z \) |
| #6(25 – d) | \( A_1 \oplus D_7 \subset D_{16} \) | \( E_8 A_1 D_{14} \) | 0 \( (0) \) |

\[ E_4 D_{10} \quad 2 \times 2Z \]

| #7(29 – b) | \( A_1 \subset E_7 \) | \( D_5 \subset D_{10} \) | \( E_6 D_6 D_5 \) | 0 \( Z/2Z \) |
| #8(17 – r) | \( A_1 \subset E_7 \) | \( D_6 \subset D_{7} \) | \( D_6 A_1 D_{10} \) | 1 \( Z/2Z \) |
| #9(19 – v) | \( A_1 \oplus D_6 \subset E_7 \) | \( E_7 D_{10} \) | 1 \( Z/2Z \) |
| #10(12 – g) | \( A_1 \oplus D_5 \subset D_{10} \) | \( E_7 E_7 A_4 \) | 0 \( Z/2Z \) |

\[ E_7 A_{17} \quad 2 \times 2Z \]

| #11(19 – n) | \( D_5 \subset E_7 \) | \( A_1 \subset A_{17} \) | \( A_1 A_{15} \) | 2 \( (0) \) |

\[ D_{24} \quad 2 \times 2Z \]

| #12(23 – i) | \( A_1 \oplus D_5 \subset D_{24} \) | \( A_1 D_{17} \) | 0 \( (0) \) |

\[ D_{12} \quad 2 \times 2Z \]

| #13(26 – p) | \( A_1 \subset D_{12} \) | \( D_6 \subset D_{12} \) | \( A_1 D_{16} D_{7} \) | 0 \( Z/2Z \) |
| #14(22 – q) | \( A_1 \oplus D_5 \subset D_{12} \) | \( A_1 D_6 D_{12} \) | 0 \( Z/2Z \) |

\[ D_{24} \quad 2 \times 2Z \]

| #15(6 – p) | \( A_1 \subset D_6 \) | \( D_6 \subset D_{8} \) | \( A_1 D_6 A_3 D_8 \) | 0 \( Z/2Z \) |

\[ D_{24} \quad 2 \times 2Z \]

| #16(14 – l) | \( A_1 \oplus D_6 \subset D_{8} \) | \( A_1 D_6 D_8 \) | 0 \( Z/2Z \) |

\[ D_{24} \quad 2 \times 2Z \]

| #17(18 – n) | \( A_1 \oplus D_6 \subset D_{9} \) | \( A_1 A_1 A_1 A_{15} \) | 0 \( Z/2Z \) |
| #18(28 – o) | \( D_5 \subset D_9 \) | \( A_1 \subset A_{15} \) | \( D_4 A_{13} \) | 1 \( (0) \) |

\[ E_8 \quad 2 \times 2Z \]

| #19(8 – b) | \( A_1 \subset E_6 \) | \( D_5 \subset E_6 \) | \( A_5 E_6 E_6 \) | 1 \( Z/2Z \) |

\[ A_{11} E_6 D_7 \quad 2 \times 2Z \]

| #20(17 – m) | \( A_1 \subset E_6 \) | \( D_5 \subset D_7 \) | \( A_6 A_1 A_1 A_{11} \) | 0 \( Z/2Z \) |
| #21(27 – n) | \( A_1 \subset A_{11} \) | \( D_5 \subset D_7 \) | \( A_6 A_1 A_{11} E_6 \) | 1 \( (0) \) |
| #22(15 – l) | \( A_1 \oplus D_5 \subset D_7 \) | \( A_1 E_6 A_1 \) | 0 \( Z/2Z \) |

| #23(2 – k) | \( D_5 \subset E_8 \) | \( A_1 \subset D_7 \) | \( A_1 A_3 A_1 D_7 \) | 0 \( Z/2Z \) |

\[ D_{24} \quad 2 \times 2 \times 2 \times 2 \]

| #24(5 – a) | \( A_1 \subset D_8 \) | \( D_5 \subset D_8 \) | \( A_1 D_1 D_4 D_8 \) | 1 \( Z/2Z \) |

\[ D_{24} A_{1} \quad 2 \times 2 \times 2 \times 2 \]

| #25(3 – v) | \( D_5 \subset D_6 \) | \( A_1 \subset A_9 \) | \( A_7 A_9 \) | 2 \( (0) \) |

\[ D_{24} A_{2} \quad 2 \times 2 \times 2 \times 2 \]

| #26(1 – s) | \( D_5 \subset D_5 \) | \( A_1 \subset D_7 \) | \( A_1 A_4 A_7 A_7 \) | 0 \( Z/2Z \) |
| #27(4 – a) | \( D_5 \subset D_5 \) | \( A_1 \subset A_7 \) | \( D_7 A_2 A_7 \) | 1 \( (0) \) |

Table 9. The elliptic fibrations of \( Y_2 \)

Since the transcendental lattice of \( Y_2 \) is \( T(Y_2) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \), we get \( b^2 - 4ac = -8 \), \( \tau_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \), \( \tau_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2} \), hence \( \tau_1 = \tau_2 = i\sqrt{2} \).

We deduce \( K_2 = E \times E / \pm 1 \) with \( E = \mathbb{C}/(\mathbb{Z} + \tau_1 \mathbb{Z}) \) and \( j(E) = j(i\sqrt{2}) = 8000 \). The fact that the two CM elliptic curves are equal and satisfy \( j(E) = 8000 \) can be obtained also by specialization from the Shioda-Inose structure of the family (see Remark 4.1).
The elliptic curve $E$ can be also put in the Legendre form:

$$E \ y^2 = x(x - 1)(x - l),$$

$l$ satisfying the equation $j = 8000 = \frac{256(1-\ell^2)^3}{\ell(\ell-1)^2}$. Thus $l = 3 \pm 2\sqrt{2}$ or $l = -2 \pm 2\sqrt{2}$ or $l = \frac{1+\sqrt{7}}{2}$.

**Proposition 6.1.** The elliptic fibrations $(24 - \psi) (T)$ and $\#10(10 - e) (T)$ are elliptic fibrations of $K_2$.

**Proof.** It follows from the $4 \cdot 2$ fibration $F_8$ that the fibration $\#10(10 - e) (T)$ with Weierstrass equation

$$Y^2 = X^3 - 2U^2(U - 1)X^2 + U^3(U + 1)^2(U - 4)X,$$

singular fibers $III^*(0), I_2^*(\infty), I_4^*(-1), I_2^*(4), I_4^*(-1/2)$, and $\mathbb{Z}/2\mathbb{Z}$-torsion is an elliptic fibration of $K_2$. Similarly from the $4 \cdot 2$ fibration $G_8$, we deduce that the elliptic fibration $(24 - \psi) (T)$ with Weierstrass equation

$$Y^2 = X^3 + 2(t + 5\ell^2)X^2 + t^2(4t + 1)(t^2 + 6t + 1)X,$$

singular fibers $III^*(\infty), I_3^*(0), 3I_2^*(-1/4, t^2 + 6t + 1)$ and $\mathbb{Z}/2\mathbb{Z}$-torsion is an elliptic fibration of $K_2$. □

To achieve the proof of Theorem 1.2 we need also the following lemma.

**Lemma 6.1.** The Kummer surface $K_2$ has exactly 4 extremal elliptic fibrations given by Shimada-Zhang [35] with the type of their singular fibers and their torsion group

1. $E_7 A_7 A_3 A_1, \mathbb{Z}/2\mathbb{Z}$,
2. $D_9 A_7 A_1 A_1, \mathbb{Z}/2\mathbb{Z}$,
3. $D_6 D_5 A_7, \mathbb{Z}/2\mathbb{Z}$,
4. $A_7 A_3 A_3 A_3 A_1 A_1, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

From Lemma 6.1 (2) we obtain the fibration $\#5(17 - q) (T)$ and from Lemma 6.1 (3) the fibration $\#15(6 - p) (T)$.

We notice also that fibrations $\#17(18 - m) (T)$ and $\#26(1 - s) (T)$ obtained by specialization are also fibration (4) of Lemma 6.1 and fibration $\#23(2 - k) (T)$, by a 2-neighbor process of parameter $m = \frac{X}{x^2(x^2 + 4)}$ gives fibration (3) of Lemma 6.1.

Finally, by a 2-neighbor process of parameter $m = \frac{X}{x^2(x^2 + 4)}$, fibrations $\#24(5 - d) (T)$ $b$ and $c$ give fibration $\#16(14 - t) (T)$, hence are elliptic fibrations of $K_2$.

**Corollary 6.1.** As a byproduct of the proof we get Weierstrass equations for extremal fibrations of Lemma 6.1 (2), (3), (4).

All these previous results are listed in Table 10. Other Van-Geemen Sarti involutions are in the Table 11, where we can see easily on the equations the self-involutions and isogenies which exchange two elliptic fibrations of $Y_2$. 
<table>
<thead>
<tr>
<th>No</th>
<th>Weierstrass Equation</th>
<th>From or to</th>
</tr>
</thead>
<tbody>
<tr>
<td>#4(16 - o)</td>
<td>( y^2 = x^3 + (\omega^5 - 5\omega + 2)x^2 + x )</td>
<td>Spec.#4(T)</td>
</tr>
<tr>
<td>#5(17 - q)</td>
<td>( y^2 = x^3 + (q^2 + q + 2 - 2x^2 + (1 - 2q)x )</td>
<td>Lemma 6.1 (2)</td>
</tr>
<tr>
<td>#8(9 - r)</td>
<td>( y^2 = x^3 - r(x^2 - r + 2x^2 + r^2x )</td>
<td>Spec.#8(T)</td>
</tr>
<tr>
<td>(24 - ψ)</td>
<td>( y^2 = x^3 - (\psi + 5\omega + 2)x^2 + \psi x )</td>
<td>Prop. 6.1</td>
</tr>
<tr>
<td>#10(10 - e)</td>
<td>( y^2 = x^3 + 2(p + 2)x^2 + X^2 + (p - 1)^2v^2x^2 + (r + 1)X^2 )</td>
<td>Prop. 6.1</td>
</tr>
<tr>
<td>#15(6 - p)</td>
<td>( y^2 = x(x-p)(x-p+p+1)X^2 )</td>
<td>Lemma 6.1 (3)</td>
</tr>
<tr>
<td>( T = (p, 0) )</td>
<td>( y^2 = x^3 + p(p^2 + 2p - 1)x^2 + p^2(p + 2)x )</td>
<td>Spec.#16(T)</td>
</tr>
<tr>
<td>#16(14 - t)</td>
<td>( y^2 = x^3 + t^2 + 4t + 1)x^2 + ct )</td>
<td>Lemma 6.1 (4)</td>
</tr>
<tr>
<td>#17(18 - m)</td>
<td>( y^2 = x^3 + \frac{1}{2}(m^2 - 4)x^2 + 1 )</td>
<td>Spec.#17(T)</td>
</tr>
<tr>
<td>#23(2 - k)</td>
<td>( y^2 = x^3 + \frac{1}{2}(k^2 - 1)x^2 + k^2x )</td>
<td>X ( \in \mathbb{Q}^{(e)} ) to Lemma 6.1 (3)</td>
</tr>
<tr>
<td>( #24(5 - d) )</td>
<td>( y^2 = x(x + d^2)(x - (d^2 + d)) )</td>
<td>a) Spec.#24(T)</td>
</tr>
<tr>
<td>( a) T = (0, 0) )</td>
<td>( y^2 = x^3 + 2d + 1)x^2 + d^2 + d^2x )</td>
<td>b) ( m = \frac{x}{d^2 + d} ) to #16(14 - t) (T)</td>
</tr>
<tr>
<td>( b) T = (d + 2, 0) )</td>
<td>( y^2 = x^3 + 2d + 1)x^2 + d^2(d + 1)x )</td>
<td>c) similar to b</td>
</tr>
<tr>
<td>( c) T = (d^2 + d^2, 0) )</td>
<td>( y^2 = x^3 + 2d + 1)x^2 + d^2(d + 1)x )</td>
<td></td>
</tr>
<tr>
<td>#26(1 - s)</td>
<td>( y^2 = x^3 + x^2\left(\frac{1}{2}(s - 1)^3 - 2s^2\right) + s^4x )</td>
<td>Spec.#26(T)</td>
</tr>
<tr>
<td>( #24(5 - d) )</td>
<td>( y^2 = x(x + d^2)(x - (d^2 + d)) )</td>
<td>a) Spec.#24(T)</td>
</tr>
<tr>
<td>( a) T = (0, 0) )</td>
<td>( y^2 = x^3 + 2d + 1)x^2 + d^2 + d^2x )</td>
<td>b) ( m = \frac{x}{d^2 + d} ) to #16(14 - t) (T)</td>
</tr>
<tr>
<td>( b) T = (d + 2, 0) )</td>
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<td>c) similar to b</td>
</tr>
<tr>
<td>( c) T = (d^2 + d^2, 0) )</td>
<td>( y^2 = x^3 + 2d + 1)x^2 + d^2(d + 1)x )</td>
<td></td>
</tr>
<tr>
<td>#26(1 - s)</td>
<td>( y^2 = x^3 + x^2\left(\frac{1}{2}(s - 1)^3 - 2s^2\right) + s^4x )</td>
<td>Spec.#26(T)</td>
</tr>
</tbody>
</table>

Table 10. Morrison-Nikulin involutions of \( Y_2 \) (fibrations of \( K_2 \))
<table>
<thead>
<tr>
<th>No</th>
<th>Weierstrass Equation</th>
</tr>
</thead>
</table>
| #7(29 – β) | \[ y^2 = x^3 + 23^2(β - 1)x^2 + 23^3(β - 1)^2x \]  
            | \[ I_0(∞), I_1(0), I_2(1) \]  
            | \[ Y^2 = X^3 - 43^2(β - 1)X^2 + 43^3(β - 1)^2X \]  
            | \[ I_0(∞), I_1(0), I_2(1) \] |
| #9(12 - g) | \[ y^2 = x^3 + 4g^2x^2 + g(g + 1)x \]  
            | \[ 2I_0(∞), I_1(0), I_2(1) \]  
            | \[ Y^2 = X^3 - 8g^2X^2 - 4g(g + 1)x^2 \]  
            | \[ 2I_0(∞), I_1(0), I_2(1) \] |
| #13(26 – τ) | \[ y^2 = x^3 + x^2(τ^2 - 2τ - 2) + \tau^2(2τ + 1)x \]  
            | \[ I_0(∞), I_1(0), I_2(-1/2), I_3(1/4) \]  
            | \[ Y^2 = X^3 - 2X^2(τ^2 - 2τ - 2) + \tau^2τ(2τ + 1)x \]  
            | \[ I_0(0), I_1(∞), I_2(4), I_3(-1/2) \] |
| #14(22 – u) | \[ y^2 = x^3 + u(2u^2 + 4u + 2)x^2 + u^2x \]  
            | \[ I_0(∞), I_1(0), I_2(-2), I_3(1) \]  
            | \[ Y^2 = (X - u(2u^2 + 4u + 2))X(X - u) \]  
            | \[ I_0(0), I_1(∞), I_2(-1/2), I_3(-2) \] |
| #15(6 – p) | \[ \text{a): } Y^2 = X[4 + 6p - 4p^2]X[4 - 6p + 4p^2] \]  
            | \[ I_0(0), I_1(∞), I_2(-1), I_3(-2) \]  
            | \[ \text{b): } Y^2 = X^2 - 2p(2p^2 + 4p + 1)X^2 + p^2X^2 \]  
            | \[ I_0(0), I_1(∞), I_2(-1), I_3(-2) \] |
| #20(7 – w) | \[ y^2 = x^3 - (2 - w^2 - \frac{4}{3}w^2)x^2 - (w^2 - 1)x \]  
            | \[ I_{12}(∞), I_6(0), I_{16}(±1), I_{22}(±2\sqrt{2}) \]  
            | \[ Y^2 = X^3 - 2(2 - w^2 - \frac{4}{3}w^2)x^2 + \frac{16}{3}w^2(w^2 + 8)X^2 \]  
            | \[ I_{12}(0), I_{6}(∞), I_{22}(±2\sqrt{2}), I_{22}(±1) \] |

Table 11. Self and exchanging involutions of $y_2$

7. 2-isogenies and isometries

Theorem 1.2 where the 2-isogenous K3-surfaces of $y_2$ are either its Kummer surface $K_2$ or $Y_2$ itself, cannot be generalized in extenso to all the other singular K3-surfaces of the Apéry-Fermi family. The first reason is the difficulty of exhibiting all the elliptic fibrations of singular K3-surfaces $Y_k$. For example, considering the K3-surface $Y_{10}$, these elliptic fibrations are given by all the primitive embeddings into Niemeier lattices of the lattice $A_1 \oplus A_2 \oplus N$ where $N$ is not a root lattice. The fact that $A_1$ (resp. $A_2$) embeds primitively into all the $A_n$, (resp. $A_n$, $n \geq 2$) leads to considerably many more elliptic fibrations on $Y_{10}$ than on $Y_2$. The fact that $N$ is not a root lattice requires new accurate techniques we shall explain in a forthcoming paper. Another reason is the relation with a Theorem of Boissière, Sarti and Veniani [7], telling when $p$-isogenies (p prime) between complex projective K3-surfaces $X$ and $Y$ define isometries between their rational transcendental lattices $T(X)_Q$ and $T(Y)_Q$ (these lattices are isometric if there exists $M \in \text{GL}(n, \mathbb{Q})$ satisfying $T(X)_Q = M^*T(Y)_Q.M$). Let us recall the part of their Theorem related to 2-isogenies.

**Theorem 7.1.** [7] Let $γ : X \to Y$ be a 2-isogeny between complex projective K3-surfaces $X$ and $Y$. Then $\text{rk}(T(Y)_Q) = \text{rk}(T(Y)_Q) = r$.

1. If $r$ is odd, there is no isometry between $T(Y)_Q$ and $T(Y)_Q$. 
2. If \( r \) is even, there exists an isometry between \( T(Y)_{\mathbb{Q}} \) and \( T(X)_{\mathbb{Q}} \) if and only if \( T(Y)_{\mathbb{Q}} \) is isometric to \( T(Y)_{\mathbb{Q}}(2) \). This property is equivalent to the following: for every prime number \( q \) congruent to 3 or 5 modulo 8, the \( q \)-adic valuation \( \nu_q(\det T(Y)) \) is even.

As a corollary we deduce the following result.

**Theorem 7.2.** Among the singular K3-surfaces of the Apéry-Fermi family defined for \( k \) rational integer, only \( Y_2 \) and \( Y_{10} \) possess symplectic automorphisms of order 2 that are “self 2-isogenies”.

**Proof.** The singular K3-surfaces of the Apéry-Fermi family defined for \( k \) rational integer are

\[
Y_0, \ Y_2, \ Y_3, \ Y_6, \ Y_{10}, \ Y_{18}, \ Y_{102}, \ Y_{198}.
\]

This list has been computed numerically by Boyd [9]. Using the notation [35], that is writing the transcendental lattice \( T(Y) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) as \( T(Y) = [a \ b \ c] \) we get:

\[
T(Y_0) = [4 \ 2 \ 4] \quad T(Y_2) = [2 \ 0 \ 4] \quad T(Y_6) = [2 \ 0 \ 12].
\]

They are obtained by specialization of fibration #20 for \( k = 0 \), 2 and 6. For \( k = 0 \) the elliptic fibration has rank 0 and singular fibers of type \( I_{12}, I_4, 2I_3, 2I_1 \). For \( k = 2 \), the transcendental lattice is already known. For \( k = 6 \), the elliptic fibration has rank 0 and type of singular fibers \( I_{12}, 2I_3, 3I_2 \). Now using Shimada-Zhang table [35], we derive the previous announced transcendental lattices.

The transcendental lattices \( T(Y_3) \) and \( T(Y_{18}) \) were computed in the paper [4]. With the method used there, we can compute the transcendental lattices of \( Y_{10}, Y_{102} \) and \( Y_{198} \). We obtain:

\[
T(Y_3) = [2 \ 1 \ 8] \quad T(Y_{10}) = [6 \ 0 \ 12] \quad T(Y_{18}) = [10 \ 0 \ 12] \\
T(Y_{102}) = [12 \ 0 \ 26] \quad T(Y_{198}) = [12 \ 0 \ 34].
\]

Applying Bessière, Sarti and Veniani’s Theorem, we conclude that only \( Y_2 \) and \( Y_{10} \) may have self isogenies. By Theorem 1.2, \( Y_2 \) has self isogenies. We shall prove that \( Y_{10} \) satisfies the same property.

Consider the following elliptic fibration of rank 0 of \( Y_{10} \) (one of the elliptic fibrations of \( Y_{10} \) obtained in a forthcoming paper):

\[
y^2 = x^3 + x^2(9(t + 5)(t + 3) + (t + 9)^2) - xt^3(t + 5)^2
\]

with singular fibers \( III^*(\infty), I_6(0), I_4(-5), I_3(-9), I_2(-4) \) and 2-torsion. Its 2-isogenous curve has a Weierstrass equation

\[
Y^2 = X^3 + X^2(-20u^2 - 180u - 432) + 4X(u + 4)^2(u + 9)^3
\]

with singular fibers \( III^*(\infty), I_6(-9), I_4(-4), I_3(0), I_2(-5), \) rank 0 and 2-torsion. Hence this 2-isogeny defines an automorphism of order 2 of \( Y_{10} \) given by \( t + u = 9, x = -\frac{X}{2}, \)
\[ y = \frac{Y}{2\sqrt{2}}. \]

Moreover we observe that
\[
T(Y_2) = [2 \ 0 \ 4], \quad T(Y_2)_\mathbb{Q} = [2 \ 0 \ 1],
\]
\[
T(K_2) = [4 \ 0 \ 8], \quad T(K_2)_\mathbb{Q} = [2 \ 0 \ 1],
\]
Similarly
\[
T(Y_{10})_\mathbb{Q} = [6 \ 0 \ 3], \quad T(K_{10})_\mathbb{Q} = [3 \ 0 \ 6].
\]
Hence we suspect some relations between the rational transcendental lattices of \( K_i \) and of \( S_i \) for singular \( Y_i \). We give some examples of such relations in the following proposition.

**Proposition 7.1.** Even if the 2-isogenies from \( Y_0, Y_6 \) are not isometries, the following rational transcendental lattices satisfy the relations

1. \( T(K_0)_\mathbb{Q} = T(S_0)_\mathbb{Q} \),
2. \( T(K_6)_\mathbb{Q} = T(S_6)_\mathbb{Q} \).

Moreover the K3-surfaces \( S_3 \) and \( K_3 \) are the same surface.

**Proof.** 1. For \( k = 0 \) we get two elliptic fibrations of rank 0, namely #20 and #8. The fibration #8(\( T \)) gives a rank 0 elliptic fibration of \( K_0 \) with Weierstrass equation
\[
y^2 = x^3 + 2x^2(t^3 + 1) + x(t - 1)^2(t^2 + t + 1)^2,
\]
with singular fibers \( D_7, 3A_3, A_2 \), 4-torsion and \( T(K_0) = [8 \ 4 \ 8] \), using Shimada-Zhang’s list \([35]\). On the other end the fibration #20(\( T \)) gives a rank 0 elliptic fibration of \( S_0 \):
\[
y^2 = x(x - \frac{1}{4}(t - 3i)(t + i)^3)(x - \frac{1}{4}(t + 3i)(t - i)^3) \quad (i^2 = -1)
\]
with singular fibers \( 3A_5 \ (\infty, \pm i), \ 3A_1 \ (0, \pm 3i) \), torsion group \( \mathbb{Z}/2 \times \mathbb{Z}/6 \), 3-torsion points being \((\frac{1}{4}(t^2 + 1)^2, \pm \frac{1}{4}(t^2 + 1)^2) \). Hence, by Shimada-Zhang’s list \([35]\), \( T(S_0) = [2 \ 0 \ 6] \). Now we can easily deduce the relation
\[
\begin{pmatrix} 1/2 & 0 \\ 1/2 & -1 \end{pmatrix} \begin{pmatrix} 8 & \ 4 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}.
\]
2. For \( k = 6 \) the elliptic fibration #20 has rank 0 and #20(\( T \)) gives a rank 0 elliptic fibration of \( S_0 \):
\[
y^2 = x^3 + x^2(\frac{t^4}{4} + 6t^3 - 21t^2 + 18t + \frac{3}{2}) + x^3 \frac{(t - 3)^2}{16}(t^2 - 6t + 1)^3,
\]
with singular fibers \( 2I_0(t^2 - 6t + 1), I_6(\infty), I_4(3), 2I_1(0, 6) \), and torsion group \( \mathbb{Z}/6\mathbb{Z} \).
Using Shimada-Zhang’s list [35], we find $T(S_6) = [4 0 6]$. Since

$$T(K_6) = \begin{pmatrix} 4 & 0 \\ 0 & 24 \end{pmatrix} \sim \begin{pmatrix} 10 \\ 0 \end{pmatrix} \mod 6$$

and

$$T(S_6) = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 10 \\ 0 \end{pmatrix} \mod 6$$

we get straightforward

$$T(K_6) \equiv T(S_6) \mod 6.$$ 

Finally we prove that $S_3$ and $K_3$ are the same surface. Consider the elliptic fibration #20 of $Y_3$ with Weierstrass equation

$$y^2 = x^3 + \frac{1}{4}(t^4 - 6t^3 + 15t^2 - 18t - 3)x^2 - t(t - 3)x,$$

singular fibers $I_{12}(\infty), 2I_3(t^2 - 3t + 1), 2I_2(0, 3), 2I_1(t^2 - 3t + 9)$, rank 1 and 6-torsion. The infinite section $P_3 = (t, -\frac{1}{2}t(t^2 - 3t + 3))$, of height $\frac{5}{2}$ generates the free part of the Mordell-Weil group, since $\det(T(Y_3)) = 15$ by the previous theorem and by the Shioda-Tate formula

$$\det(T(Y_3)) = \frac{5}{4} \cdot 12 \cdot 3^2 \cdot 2^2 \cdot 6^2 = 15.$$

Its 2-isogenous curve has Weierstrass equation

$$y^2 = x^3 + \left(-\frac{1}{2}t^4 + 3t^3 - \frac{15}{2}t^2 + 9t + \frac{3}{2}\right)x^2 + \frac{1}{16}(t^2 - 3t + 9)(t^2 - 3t + 1)x,$$

singular fibers $3I_6(\infty), t^2 - 3t + 1, 2I_2(t^2 - 3t + 9), 2I_1(3, 0)$, rank 1 and 6-torsion. The section $Q_3$ image by the 2-isogeny of the infinite section $P_3$ is an infinite section of height $\frac{5}{2}$. Since neither $Q_3$ nor $Q_3 + (0, 0)$ are 2-divisible, the section $Q_3$ generates the free part of the Mordell-Weil group. Hence by the Shioda-Tate formula, it follows

$$\det(T(S_3)) = \frac{5}{2} \cdot 6^3 \cdot 2^2 \cdot 6^2 = 60 = \det(T(K_3)).$$

We can show that $K_3$ and $S_3$ are the same surface. To prove this property we show that a genus one fibration is indeed an elliptic fibration. We start with the fibration of $K_3$ obtained from #26($T$) and parameter $m = \frac{w}{t(x + \frac{1}{2}(t-s)^2(t-s-1))}.$ If $k = 3$ and $s = s_3 := \frac{3+\sqrt{5}}{2}$ we get $E_m$. Then changing $X = s_3^2x$ and $Y = s_3^2y$ it follows

$$y^2 - 3mxy = x(x - m^2) \left(x - \frac{1}{8} \left(112 - 48\sqrt{5}\right)m^4 - 16m^2 + 7 + 3\sqrt{5}\right).$$

The next fibration is obtained with the parameter $n = \frac{x}{m^2}$. Now if $w = \frac{x}{m^2}$ it gives the
following quartic in $w$ and $m$

$$w^2 - 3mnw + 2\left(3\sqrt{5} - 7\right)n(n-1)m^4 - n(n-1)(n-2)m^2 - \frac{1}{9}\left(3\sqrt{5} + 7\right)n(n-1).$$

Notice the point $(w = -\frac{1}{4}(7 + 3\sqrt{5})n(n+1), m = \frac{1}{4}(2 + \sqrt{5})(2n - 1 + \sqrt{5}))$ on this quartic, so it is an elliptic fibration of $K_3$ which is $\#15(T)$. □

**Remark 7.1.** The Kummer surface $K_0$ is nothing else than the Schur quartic [7] (section 6.3) with equation

$$x^4 - xy^3 = z^4 - zt^3.$$

**References**


[27] C. Peters, J. Stienstra, A pencil of K3-surfaces related to Apéry’s recurrence for \( \zeta(3) \) and Fermi


