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Stationary Process, 1963-64

1. Gaussian System

1. Definition. (a) A random variable $x(\omega)$ is called Gaussian, if its characteristic function is of the following form:

$$E(e^{izx(\omega)}) = \exp\{imz - \frac{v}{2} z^2\}, \quad -\infty < m < +\infty, \quad o \leq v < +\infty$$

In case (nondegenerate case): $v > o$,

$$P(x \in E) = \frac{1}{\sqrt{2\pi v}} \int_E \exp\left\{ - \frac{(\xi - m)^2}{2v} \right\} d\xi.$$

In case (degenerate case): $v = o$,

$$P(x = m) = 1.$$

(b) A System of random variables $x_\alpha(\omega)$, $\alpha \in A$, is called Gaussian if every linear expression $c_o + c_1 x_{\alpha_1} + \ldots + c_n x_{\alpha_n}$ is Gaussian.

2. Elementary Properties. (a) $(x_\alpha, \alpha \in A)$ is Gaussian, if they are independent and if each of them is Gaussian.

(b) A subsystem of a Gaussian system is Gaussian.

(c) Given a Gaussian system, the closed linear manifold (in $L^2(\Omega)$) generated by that system is also Gaussian.

(d) Consider a family of Gaussian systems

$$x^\lambda = (x_\alpha^\lambda, \ \alpha \in A_\lambda), \qquad \lambda \in \Lambda.$$

If they are independent, then the joint system

$$x = (x_\alpha^\lambda, \ \alpha \in A_\lambda, \ \lambda \in \Lambda)$$

is also Gaussian.

3. Existence Theorem. Let $x_\alpha(\omega)$, $\alpha \in A$, be a Gaussian system. Then

$$m_\alpha = E(x_\alpha(\omega))$$

$$v_{\alpha\beta} = E((z_\alpha - m_\alpha)(x_\beta - m_\beta))$$

are well defined and finite, and $v_{\alpha\beta}$ is symmetric and positive definite:

$$v_{\alpha\beta} = v_{\beta\alpha},$$

$$\sum_{ij} v_{\alpha_i\alpha_j} \xi_i \xi_j \geq 0$$

for every choice of n, $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in A^n$ and $(\xi_1, \xi_2, \ldots, \xi_n) \in R^n$.

The vector $(m_\alpha, \alpha \in A) \in R^A$ and the matrix $(v_{\alpha\beta}, \alpha, \beta \in A)$ (both are infinite dimensional if A is an infinite set) are called respectively the mean vector and the covariance matrix of the given Gaussian system.

Theorem. Given any vector $(m_\alpha, \alpha \in A)$ and any symmetric and positive-definite matrix $(v_{\alpha,\beta}, \alpha, \beta \in A)$, we can construct a Gaussian system with the mean vector (m_α) and the covariance matrix $(v_{\alpha\beta})$ on a suitable probability measure space. The crucial point of the proof is the following

Lemma. If $(v_{\alpha\beta}, \alpha, B \in A)$ is symmetric and positive definite, then there exists $(\xi_{\alpha\lambda}; \alpha \in A, \lambda \in A)$ such that

$$v_{\alpha\beta} = \sum_{\lambda\in\Lambda} \xi_{\alpha\lambda} \xi_{\beta\lambda}.$$

Note (due to C. Stein). Using Carleman's theorem in moment problems (see Shohat-Tamarkin: "The problem of moments" (1943)), we can generalize Theorem 2 (page 5) as follows.

Theorem 2*.

(A) $$\sum_n \left(\sum_{i=1}^{r} E(|x_{\alpha_i}|^n)\right)^{-1/n} = \infty \qquad \text{for every choice of } (\alpha_i)$$

implies $\mathcal{P}(x) = \mathcal{M}(x)$.

Remark 1. Since

$$E\left(\sum_{i=1}^{r} E(|x_{\alpha_i}|^n)/r \right)^{1/n}$$

is increasing in n, we have

$$\sum_n \left(\sum_i E(|x_{\alpha_i}|^n) \right)^{-1/n} = \infty$$

$$\Longleftrightarrow \sum_n \left(\sum_i E(|x_{\alpha_i}|^{2n})\right)^{-1/2n} = \infty$$

$$\Longleftrightarrow \sum_n \left(\sum_i E(|x_{\alpha_i}|^{3n})\right)^{-1/3n} = \infty$$

$\Longleftrightarrow$ etc.

Remark 3. Here are sufficient conditions for (A) which can be easily checked.

(B) (Stein's condition) There exists $0 < c(\alpha) < \infty$ and $a_n (a_n > 0$ and $\sum a_n = \infty)$ such that

$$E(|x_\alpha|^{2n})^{1/2n} \leq \frac{c(\alpha)}{a_n} \qquad \text{for every } \alpha \text{ and every } n$$

(C)

$$\lim_{c \downarrow 0} E(e^{c|x_\alpha|}) < \infty \qquad \text{for every } \alpha$$

(D) (The assumption in Theorem 2)

$$E\left(e^{c|x_\alpha|}\right) < \infty \qquad \text{for every } \alpha \text{ and } c > 0$$

In fact we have

$$(D) \Rightarrow (C) \Rightarrow (B) \Rightarrow (A) \Rightarrow \mathcal{P}(x) = \mathcal{M}(x)$$

Example. If each x_α is Gauss, Poisson or exponentially distributed, then (C) holds and so $\mathcal{P}(x) = \mathcal{M}(x)$.

Proof of Theorem 2^*. It is enough to show that $y \in \mathcal{M}(x) \ominus \mathcal{P}(x)$ implies $y = 0$. $d\mu \equiv y \cdot dP$ is a bounded signed measure with the absolute variation $d|\mu| = |y|\, dP$.

Consider $u = \sum_{i=1}^{r} \theta_i x_{\alpha_i}$ (θ_i real) and set $\nu = \mu u^{-1}$. Then $|\nu| = |y| u^{-1}$. Since $u^n \in \mathcal{P}(x)$,

$$\int_{R'} \xi^n \, d\nu(\xi) = \int_{\Omega} u^n \, d\mu = E(u^n \cdot y) = 0$$

On the other hand

$$\int_{R'} |\xi|^{2n} \, d|\nu| \, (\xi) = \int_{\Omega} |u|^{2n} \, d|\mu| = E(|u|^{2n} \, |y|)$$

$$\leq E(|u|^{4n})^{1/2} \, \|y\| \qquad \text{where } \|y\| = E(y^2)^{1/2}$$

$$\leq \theta^{2n} \, E(\sum_i |x_{\alpha_i}|^{4n})^{1/2} \, \|y\| \qquad \text{where } \theta = \sum |\theta_i|$$

$$\leq \theta^{2n} \, E(\sum_i |x_{\alpha_i}|^{2n}) \, \|y\|$$

and so

$$\sum [\int |\xi|^{2n} \, d|\nu|]^{-1/2n} \geq \theta^{-1} \, \|y\|^{-1/2n} \sum_n \left[\sum_i E(|x_{\alpha_i}|^{2n} \right]^{-1/2n}$$

This is divergent by our assumption (note Remark 1). Using Carleman's theorem we get $\nu = 0$ and so

$$(*) \quad \int_{\Omega} \exp\left(i \sum_{i=1}^{r} \theta_i x_{\alpha_i} \right) d\mu = \int_{\Omega} e^{iu} \, d\mu = \int_{R^1} e^{i\xi} \, d\nu(\xi) = 0$$

Denoting by π the mapping $\omega(\in \Omega) \to (x_{\alpha_1}(\omega), \ldots, x_{\alpha_r}(\omega)) \ (\in R^r)$, we get

$$\int_{R^r} \exp\left(i \sum_{i=1}^{r} \theta_i \xi_i \right) d\mu \pi^{-1} = 0$$

This is true for every (θ_i) and so we get $\mu\pi^{-1} = 0$.

Every $B \in \mathcal{B}(x_{\alpha_1} \cdots x_{\alpha_r})$ can be expressed as $B = \pi^{-1} E$ with some $E \in \mathcal{B}^r$. Therefore

$$\mu(B) = \mu\pi^{-1}(E) = 0$$

Given any $B \in \mathcal{B}(x_\alpha, \alpha \in A)$ and any $\varepsilon > 0$, there exists $B' \in \mathcal{B}(x_{\alpha_1}, \ldots, x_{\alpha_r})$ for some $\alpha_1 \cdots \alpha_r$ such that $|\mu| (B \sim B') < \varepsilon$ where $B \sim B' = B \cup B' - B \cap B'$. Therefore

$$|\mu(B)| \leq |\mu(B')| + |\mu(B) - \mu(B')| \leq |\mu| (B \sim B') < \varepsilon$$

Since ε is arbitrary, we get $\mu(B) = 0$ for $B \in \mathcal{B}(x_\alpha, \alpha \in A)$.

As y is measurable in $\mathcal{B}(x_\alpha, \alpha \in A)$, we get

$$\int_\Omega y \, d\mu = 0 \qquad \text{i.e.,} \qquad E(y^2) = 0$$

Remark. The direct proof of Theorem 2 is as follows. We get (*) by the term-by-term integration that is allowed by $E(\exp(\sum_i |\theta_i| \, |x_{\alpha_i}|)) < \infty$ which can be obtained from (D) by repeated use of the Schwarz inequality, and the rest of the proof is the same as above.

6. L^2 space over a Gaussian system.

(a) Let $x = (x_\alpha, \alpha \in A)$ be any system of real random variables of all moments finite defined on $\Omega(\mathcal{B}, P)$. Let us consider the closed linear subspaces of $L^2 \equiv L^2(\Omega, \mathcal{B}, P)$ generated by x in three different ways:

$\mathcal{L}(x)$ = the set of all linear combinations of $x_\alpha, \alpha \in A$ $\left(\sum_{i=1}^{n} c_i x_{\alpha_i}\right)$ and their L^2-limits,

$\mathcal{P}(x)$ = the set of all polynomials $(p(x_{\alpha_1} \cdots x_{\alpha_n}))$ and their L^2-limit,

$\mathcal{M}(x)$ = the set of all elements of L^2 measureable $\mathcal{B}(x) = \mathcal{B}(x_\alpha, \alpha \in A)$.

It is clear that

$$\mathcal{L}(x) \underset{\neq}{\subset} \mathcal{P}(x) \subset \mathcal{M}(x)$$

but we have

Theorem 2.

If

$$E\left(\exp\left[c \sum_{i=1}^{n} |x_{\alpha_i}|\right]\right) < +\infty \qquad \text{for any } c > 0,$$

any n and any $(\alpha_1, \dots, \alpha_n)$, then

$$\mathcal{P}(x) = \mathcal{M}(x).$$

Corollary. If $x = (x_\alpha, \alpha \in A)$ is a Gaussian system, then

$$\mathcal{P}(x) = \mathcal{M}(x)$$

(This fact contains the completeness of Hermite polynomials on $L^2(R^1, \mathcal{B}^1, e^{-x^2}dx)$ as a special case.)

(b). Let $x = (x_\alpha, \alpha \in A)$ be a Gaussian system with mean vector 0. Let $\hat{\mathcal{P}}_n(x)$ be the closed linear submanifold of $\mathcal{P}(x)$ spanned by all polynomials of degree $\leq n$. It is then clear that

$$\hat{\mathcal{P}}_0(x) \subset \hat{\mathcal{P}}_1(x) \subset \hat{\mathcal{P}}_2(x) \subset \cdots$$

$$\mathcal{P}(x) = \overline{\bigvee_n \hat{\mathcal{P}}_n(x)}$$ (= the smallest closed manifold containing all $\mathcal{P}_n(x)$).

Define $\mathcal{P}_n(x)$ by

$$\mathcal{P}_0(x) = \hat{\mathcal{P}}_0(x),$$

$$\mathcal{P}_n(x) = \hat{\mathcal{P}}_n(x) \ominus \hat{\mathcal{P}}_{n-1}(x)$$ (orthogonal complement) $(n = 1, 2, \dots)$.

Then we have

Theorem 3. $\mathcal{H}(x) = \mathcal{P}(x) = \Sigma \oplus \mathcal{P}_n(x)$ (direct sum)

It is clear that

$$\mathcal{P}_0(x) = R^1, \qquad \mathcal{P}_1(x) = \mathcal{L}(x)$$

(c). Cameron-Martin Expansion. Let $x = (x_\alpha, \alpha \in A)$ be a Gaussian system with mean vector 0.

Let $H_n(x)$ be the Hermite polynomial of degree n:

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}$$

$$- \cdots \begin{cases} + (-1)^{n/2} \dfrac{n!}{(\frac{n}{2})!} & \text{for even } n \\ + (-1)^{(n-1)/2} \dfrac{n!}{(\frac{n-1}{2})!}(2x) & \text{for odd } n \end{cases}$$

and modify this as

$$h_n(x) = \frac{H_n(x/\sqrt{2})}{\sqrt{2^n n!}}.$$

so that $h_0(x), h_1(x), \dots$ constitute a complete orthonormal system on $L^2(R^1, \mathcal{B}^1, N(dx))$, $N(dx)$ being the Gaussian distribution

$$\frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})\, dx.$$

Theorem 4. (Cameron-Martin) Let y_λ, $\lambda \in \Lambda$, be a complete orthonormal system in $\mathcal{L}(x)$ $(= \mathcal{P}_1(x))$. Then

$$h_{p_1}(y_{\lambda_1}) \cdots h_{p_\sigma}(y_{\lambda_\sigma}) \qquad \begin{array}{l} \lambda_1, \lambda_2, \ldots, \lambda_p \text{ different} \\ p_1 + p_2 + \cdots + p_\sigma = n \end{array}$$

constitute a complete orthonormal system in $\mathcal{P}_n(x)$. Therefore it follows from Theorem 4 that every $x \in \mathcal{M}(x)$ has an orthogonal expansion (Cameron-Martin expansion):

$$x = a_0 + \sum_{(\lambda)(p)} a\begin{pmatrix} \lambda_1 & \cdots & \lambda_\sigma \\ p_1 & \cdots & p_\sigma \end{pmatrix} h_{p_1}(y_{\lambda_1}) \cdots h_{p_\sigma}(y_{\lambda_\sigma})$$

$$= a_0 + \sum_n \sum_{(\lambda)(p)} a\begin{pmatrix} \lambda_1 & \cdots & \lambda_\sigma \\ p_1 & \cdots & p_\sigma \end{pmatrix} h_{p_1}(y_{\lambda_1}) \cdots h_{p_\sigma}(y_{\lambda_\sigma}) \qquad p_1 + \cdots + p_\sigma = n$$

(d) Predictor. Let $G = (x_\alpha, \alpha \in A)$ be a Gaussian system and H a subsystem of G. Take any $u \in \mathcal{M}(G)$.

The projection v_∞ of u onto $\mathcal{M}(G')$ minimizes $\|u - v\|$ in $v \in \mathcal{M}(G')$ and is called the predictor of degree ∞ of u over G', while the projection v_n of u onto $\hat{\mathcal{P}}_n(G')$ minimizes $\|u - v\|$ in $v \in \hat{\mathcal{P}}_n(G')$ and is called the predictor of degree n of u over G'.

It is clear that

$$\|u - v_0\| \geq \|u - v_1\| \geq \|u - v_2\| \geq \cdots \geq \|u - v_\infty\|,$$

but the Cameron-Martin expansion theorem implies.

Theorem 5. $u \in \mathcal{P}_n(G) \Longrightarrow v_n = v_\infty$; in particular

$$u = x_\alpha \Longrightarrow v_1 \quad (= \text{linear predictor}) = v_\infty,$$

which shows that the linear prediction is the same as the nonlinear prediction for Gaussian processes.

7. Multiple Wiener Integral (orthogonalized multiple integral with respect to a Gaussian random measure)

Let $M(E)$ be a Gaussian random measure on $T(\mathcal{B}_T, m)$ where m is atomless.

[Remark. m is called atomless if

(1) $0 < m(E) < \infty \Longrightarrow \exists F \subset E$ such that $0 < m(F) < m(E)$.

This condition is equivalent to each one of the following conditions.

(2) $m(E) < \infty \implies \forall \varepsilon > 0 \ \exists \ E = E_1 \vee \cdots \vee E_n$ (disjoint)

such that

$$m(E_i) < \varepsilon, \qquad i = 1, 2, \ldots, n$$

(3) $m(E) < \infty \implies \forall_n \ \exists \ E = E_1 \vee \cdots \vee E_n$ (disjoint)

such that

$$m(E_i) = m(E)/n$$

(4) $m(E) < \infty \implies \forall \, 0 < \alpha < 1 \ \exists \ F \subset E$ such that $m(F) = \alpha \, m(E)$.]

Then

$$\mathcal{W}(M) = \mathcal{P}(M) = \sum_n \oplus \mathcal{P}_n(M), \quad \mathcal{P}_0(M) = R^1, \quad \mathcal{P}_1(M) = \mathcal{L}(M)$$

and the Cameron-Martin expansion theorem holds.

Let $L^2(T^n)$ be the L^2-space over the product space T^n associated with the direct product measure m^n and $S(T^n)$ be the set of all symmetric functions in $L^2(T^n)$. Then we can establish a one-to-one correspondence:

$$S(T^n) \ni f_n \to I_n(f_n) \in \mathcal{P}_n(M)$$

Case n = 0.

$$L^2(T^0) = S(T^0) = R^1$$

$$f_0 \in L^2(T^0) \to I_0(f_0) = f_0$$

Case n = 1.

$$L^2(T) = S(T)$$

Let $\mathcal{E}(T)$ be the class of all functions of the form $f_1 = \sum_{i=1}^{r} c_i e_{E_i}(t)$ with disjoint $E_1, E_2, \dots, E_r \in \mathcal{F}$. (Recall that $\mathcal{F}$ is the class of sets of finite m-measure.) Then $\mathcal{E}(T)$ is a dense linear subspace of $L^2(T)$.

Define $I_1(f_1) \equiv \int f_1(t)\, d M(t)$ for $f_1 \in \mathcal{L}^2(T)$ as $I_1(f_1) = \sum_i c_i M(E_i)$. $I_1(f_1)$ is well defined independently of the representation of f_1 and $I_1(f_1)$ is linear and isometric from $\mathcal{E}(T)$ into $L^2(\Omega)$. Therefore it can be extended to a linear isometric operator from $L^2(T) = \overline{\mathcal{E}(T)}$ into $L^2(\Omega)$.

It is easy to see $I_1(L^2(T)) = \mathcal{P}_1(M)$, so that I_1 gives an isomorphism: $L^2(T) \to \mathcal{P}_1(M)$.

Case n = 2.

$$L^2(T^2) \supsetneq S(T^2)$$

Let $\mathcal{E}(T^2)$ be the class of all functions of the form $f_2(t_1, t_2) = \sum_{i \neq j} c_{ij}\, e_{E_i}(t_1)\, e_{E_j}(t_2)$ with disjoint $E_1, E_2, \dots, E_r \in \mathcal{F}$. $\mathcal{E}(T^2)$ is a linear subspace of $L^2(T)$, and $\overline{\mathcal{E}(T^2)} = L^2(T^2)$ follows from our assumption that m is atomless.

Define $I_2(f_2) \equiv \iint f(t_1, t_2)\, d\,M(t_1) d\,M(t_2)$ for $f_2 \in \mathcal{E}(T)$ as

$I_2(f_2) = \sum_{i \neq j} c_{ij}\, M(E_i)\, M(E_j)$.

$I_2(f_2)$ is well defined independently of the representation of f_2 and $I_2(f_2)$ is linear.

Since $f_2(t_1, t_2) = g_2(t_1, t_2) \Longrightarrow I_2(f_2) = I_2(g_2)$ by the definition, we get

$$I_2(f_2) = I_2(\tilde{f}_2), \quad \tilde{f}_2(t_1, t_2) = \frac{f_2(t_1, t_2) + f_2(t_2, t_1)}{2} \in S(T^2)$$

It is easy to verify

$$\text{(1)} \qquad \|I_2(f_2)\|^2 \leq \|I_2(\tilde{f}_2)\|^2 = 2\|\tilde{f}_2\|^2 \leq 2\|f_2\|^2$$

Therefore $I_2(f_2)$ can be extended to a linear bounded operator from $L^2(T^2)$ into $L^2(\Omega)$, and (1) is true for this extension. In particular, if $f_2 \in S(T^2)$, then $f_2 = \tilde{f}_2$ and

$$\|I_2(f_2)\|^2 = 2\|f_2\|^2$$

It is clear that $I_2(L^2(T^2)) = I_2(S(T^2))$. We can prove that $I_2(L^2(T^2)) = \mathcal{P}_2(M)$. Therefore $2^{-1/2} I_2$ gives an isomorphism: $S(T^2) \to \mathcal{P}_2(M)$.

Case. general $n(\geq 3)$.

$$L^2(T^n) \underset{\neq}{\supset} S(T^n)$$

Starting with the definition of $I_n(f_n)$ for f_n in

$$\mathcal{E}(T^n) = \left\{ f_n : \begin{array}{l} f_n(t_1 \cdots t_n) = \sum_{i_\mu \neq i_\nu} c_{i_1 \cdots i_n} e_{E_{i_1}}(t_1) \cdots e_{E_{i_n}}(t_n) \\ \text{with a finite } \underline{\text{disjoint}} \text{ subsystem } \{E_i\} \text{ of } \mathcal{F} \end{array} \right\}$$

we can repeat the same arguments as for the case $n = 2$ to see that

$(n!)^{-1/2} I_n$ gives an isomorphism ; $S(T^n) \to P_n(M)$.

Theorem 6. Every $z \in \mathcal{M}(M)$ $(= \mathcal{P}(M))$ has an expansion:

$$z = \sum_n \int \cdots \int f_n(t_1 \cdots t_n)\, dM(t_1) \cdots dM(t_n)$$

with $f_n \in L^2(T^n)$; we can restrict f_n in $S(T^n)$ and the expansion is unique under this restriction.

Let us establish an interesting relation among complete orthonormal systems in $L^2(T^n)$, $S(T^n)$ and $\mathcal{P}_n(M)$.

Let $\phi_\lambda(t)$, $\lambda \in \Lambda$, be the complete orthonormal system of $L^2(T)$.

$$\mathcal{O}_n = \{\phi_{\lambda_1}(t_1) \cdots \phi_{\lambda_n}(t_n) \; ; \; (\lambda_1, \ldots , \lambda_n) \in \Lambda^n\}$$

constitute a complete orthonormal system in $L^2(T^n)$ and

$$\mathcal{O}'_n = \{(n!)^{-1/2} \sum_\pi \phi_{\lambda_1}(t_{\pi_1}) \cdots \phi_{\lambda_n}(t_{\pi_n})\}$$

(π runs over all permutations of $1, 2, \ldots , n$).

$$\equiv \left\{ \phi_{\begin{pmatrix} n_1 \cdots n_r \\ \sigma_1 \cdots \sigma_r \end{pmatrix}}(t_1 \cdots t_n) : \begin{array}{l} r = 1, 2, \ldots , n \\ \sigma_i \text{ different} \\ n_i > 0, \ldots , \sum n_i = n \end{array} \right\}$$

constitute a complete orthonormal system of $S(T^n)$.

On the other hand

$$y_\lambda = \int \phi_\lambda(t)\, dM(t) , \qquad \lambda \in \Lambda ,$$

constitute a complete orthonormal system in $\mathcal{P}_1$ and the Cameron-Martin expansion theorem shows that

$$\mathcal{H}_n = \left\{ h_{n_1}(y_{\sigma_1}) \cdots h_{n_r}(y_{\sigma_r}) : \begin{array}{l} r = 1, 2, 3, \ldots, n \\ \sigma_{\lambda_i} \text{ different} \\ \sum n_i = n, \quad n_i > 0 \end{array} \right\}$$

constitute a complete orthonormal system in $\mathcal{P}_n$.

<u>Theorem 7</u>. $(p!)^{-1/2} I_n$ carries $\mathcal{O}_n$ onto $\mathcal{H}_n$:

$$(n!)^{-1/2} \int \cdots \int \phi_{\lambda_1}(t_1) \cdots \phi_{\lambda_n}(t_n)\, dM(t_1) \cdots dM(t_n) = h_{n_1}(y_{\sigma_1}) \cdots h_{n_r}(y_{\sigma_r})$$

where $\sigma_1, \sigma_2, \ldots, \sigma_r$ are the different members among $\{\lambda_i\}$ and each n_i is the number of σ_i in $\{\lambda_i\}$.

The mapping $(n!)^{-1/2} I_n : \mathcal{O}_n \dashrightarrow \mathcal{H}_n$ is not one-to-one and if and only if there exists a permutation $\pi = (\pi_i)$ of $1, 2, \ldots, n$ such that $\mu_i = \lambda_{\pi_i}$,

$$\phi_{\lambda_1}(t_1) \cdots \phi_{\lambda_n}(t_n) \quad \text{and} \quad \phi_{\mu_1}(t_1) \cdots \phi_{\mu_n}(t_n)$$

goes over to the same element in $\mathcal{H}_n$ and therefore $(n!)^{-1/2} I_n$ is one-to-one: $\mathcal{O}_n' \dashrightarrow \mathcal{H}_n$.

Proof of $I_n(L^2(T^n)) = \mathcal{P}_n(M)$

1st step. $I_n(L^2(T^n)) \perp I_m(L^2(T^m))$ $(n < m)$

It is enough to prove that $I_n(\mathcal{E}(T^n)) \perp I_m(\mathcal{E}(T^m))$. Take any $y \in I_n(\mathcal{E}(T^n))$ and any $z \in I_m(\mathcal{E}(T^m))$ and express them as

$$y = \sum_{i_\mu \neq i_\nu} c_{i_1 \dots i_n} M(E_{i1}) \cdots M(E_{in})$$

$$z = \sum_{j_\mu \neq j_\nu} d_{j_1 \dots j_m} M(E_{j_1}) \cdots M(E_{j_m})$$

with a finite disjoint subsystem $\{E_i\}$ of $\mathcal{F}$. (Notice that we can take a common $\{E_i\}$ for y and z; if not, we can do that by using a common subdivision.) Since $n < m$, at least one of $j_1, j_2, \dots, j_m$ is different from any of $i_1 \dots i_n$. Noticing that $M(E_1), M(E_2), \dots$ are independent, we get $E(y \cdot z) = 0$.

2nd step.

Lemma 1. $f_1 \in L^2(T)$, $g_n \in L^2(T^n)$ $(n \geq 1)$

$\Longrightarrow$

$$f_1 \cap g_n(t_1 \dots t_{n+1}) \equiv f(t_1)\, g_n(t_2 \dots t_{n+1}) \in L^2(T^{n+1})$$

$$f_1 \cup g_n(t_1 \dots t_{r-1}) \equiv \sum_{k=1}^{n} \int f_1(t)\, g_n(t_1 \dots t_{k-1}\, t\, t_k \dots t_{n-1})\, dm(t) \in L^2(T^{n-1})$$

$$\|f_1 \cap g_n\| = \|f_1\|\, \|g_n\|,$$

$$\|f_1 \cup g_n\| \leq r \|f_1\|\, \|g_n\|$$

$$(*) \quad I_1(f_1) \cdot I_n(g_n) = I_{n+1}(f_1 \cap g_n) + I_{n-1}(f_1 \cup g_n)$$

Proof of Lemmaal.

Since both sides of (*) are bilinear in (f_1, g_r), it is enough to prove it for

$$f_1 = e_E(t), \qquad g_1 = e_{E_1}(t_1)\, e_{E_2}(t_2) \dots e_{E_n}(t_n) \qquad (E_1, \dots, E_n \text{ disjoint}),$$

Using subdivision, we can assume that either

(i) $\qquad E \cap (E_1 \cup \dots \cup E_n) = \emptyset$

or

(ii) $\qquad E = \text{some } E_i \quad (\text{say } E_1)$

In case (i)

$$f_1 \cap g_n = e_E(t_1)\, e_{E_1}(t_2) \dots e_{E_n}(t_n)$$

$$f_1 \cup g_n = 0$$

and (*) is clear. In case (ii)

$$f_1 \cap g_n = e_E(t_1)\, e_E(t_2)\, e_{E_2}(t_3) \dots e_{E_n}(t_{n+1})$$

$$f_1 \cup g_n = m(E)\, e_{E_2}(t_1) \dots e_{E_n}(t_{n-1})$$

Decomposing E as $E = \bigcup_{i=1}^{k} E_{k_i}$, $m(E_{k_i}) = m(E)/k$ as is possible by virtue of the atomless property of m, we have

$$f_1 \cap g_n = \sum_{i\neq j} e_{E_{ki}}(t_1)\, e_{E_{kj}}(t_2)\, e_{E_2}(t_3) \cdots e_{E_n}(t_{n+1})$$

$$+ \sum_i e_{E_{ki}}(t_1)\, e_{E_{k_i}}(t_2)\, e_{E_2}(t_3) \cdots e_{E_n}(t_{n+1})$$

$$= \varphi_k + \psi_k$$

$$\|\psi_k\|^2 = \sum m(E_{ki})^2\, m(E_2) \cdots m(E_n) = \frac{1}{k} m(E)\, m(E_2) \cdots m(E_n) \to 0 \qquad (k \to \infty)$$

Therefore

$$I_{n+1}(f_1 \cap g_n) = \lim_{k\to\infty} I_{n+1}(\varphi_k) = \lim_{k\to\infty} \sum_{i\neq j} M(E_{ki})\, M(E_{kj})\, M(E_2) \cdots M(E_n)$$

On the other hand

$$I_1(f_1)\, I_n(g_n) = M(E)^2\, M(E_2) \cdots M(E_n)$$

$$= \sum_{i\neq j} M(E_{ki})\, M(E_{kj})\, M(E_2) \cdots M(E_n) + \sum_i M(E_{ki})^2\, M(E_2) \cdots M(E_n)$$

$$= A_k + B_k$$

$$\lim_k A_k = I_{n+1}(f_1 \cap g_n) \qquad \text{(as proved above).}$$

$$\|B_k - I_{n-1}(f_1 \cup g_n)\|^2$$

$$= \|\sum_i M(E_{ki})^2\, M(E_2) \cdots M(E_n) - m(E)\, M(E_2) \cdots M(E_n)\|^2$$

$$= \|\sum_i M(E_{ki})^2 - m(E)\|^2\, \|M(E_2)\|^2 \cdots \|M(E_n)\|^2$$

(Notice that $M(E_{ki})$, $M(E_j)$, $i = 1, 2, \ldots, k$; $j = 2, 3, \ldots, n$ are independent.)

$$= \sum_i \| M(E_{ki})^2 - m(E_{ki}) \|^2 \, \| M(E_2) \|^2 \cdots \| M(E_n) \|^2 = \sum_i 3m(E_{ki}) \, m(E_2) \cdots m(E_n)$$

$$= \frac{3}{n} m(E) \, m(E_2) \cdots m(E_n) \longrightarrow 0$$

Thus (*) is proved.

As an immediate result of Lemma 1, we get

Lemma 2.

$$y \in I_n(L^2(T^n)), \qquad z \in I_1(L^2(T))$$

$$\Longrightarrow y \cdot z \in I_{n+1}(L^2(T^n)) \oplus I_{n-1}(L^2(T^{n-1}))$$

$$\Longrightarrow yz \in \sum_{r=0}^{n} \oplus \; I_r(L^2(T^r))$$

3rd step. By the result of the 1st step it is enough to prove that

$$\mathfrak{M}^n = \sum_{r=0}^{n} \oplus \; I_r(L^2(T^r)) \supset \hat{\mathcal{P}}_n(M) .$$

(Notice that the opposite inclusion is clear.) This is evident for $n = 1$. Assume that this is true for $n = 1, 2, \ldots, m$. In order to prove that this is true for $n = m+1$, it is enough to prove that any element of the form $M(E_1) \cdots M(E_k)$ (E_i not necessarily disjoint, $k \leq m+1$) belongs to $\mathfrak{M}^{m+1}$. If $k < m+1$, then the assumption of induction implies that $M(E_1) \cdots M(E_k) \in \mathfrak{M}^k \subset \mathfrak{M}^{m+1}$. Consider the case $k = m+1$. Then $M(E_1) \in \mathfrak{M}^1$ and $M(E_2) \cdots M(E_{m+1}) \in \mathfrak{M}^n$ and so

$$M(E_1) = y_0 + y_1$$

$$M(E_2) \cdots M(E_{m+1}) = z_0 + z_1 + \cdots + z_m$$

where

$$y_r, z_r \in I_r(L^2(T^r))$$

Thus we have

$$M(E_1)\, M(E_2) \cdots M(E_{m+1}) = \sum_{i,j} y_i z_j \in \mathcal{M}^{m+1}$$

because

$$y_0 \cdot z_j \in I_j(L^2(T^j)) \; ; \qquad y_1 \cdot z_0 \in I_1(L^2(T))$$

and

$$y_1 z_j \in \mathcal{M}^{j+1} \qquad \text{(by lemma 2)}.$$

Remark: Lemma 1, combined with the recursion formula for Hermite polynomials, proves Theorem 7 (page 15).

2. COMPLEX GAUSSIAN SYSTEM

1. Definitions.

(a) A complex random variable $x(\omega)$ is called complex Gaussian if its probability distribution is an isotropic Gaussian distribution on the complex plane. Let x be complex Gaussian. Then $v = E(|x|^2) < \infty$ and

$$P(a < \mathcal{R}x < b, c < \mathcal{J}x < d) = \int_a^b \int_c^d \frac{1}{\pi v} \exp\left[-\frac{\xi^2 + \eta^2}{v}\right] d\xi\, d\eta \quad \text{in case } v > 0$$

$$P(x = 0) = 1 \qquad \text{in case } v = 0.$$

(b) A system of complex random variables $x_\alpha(\omega)$, $\alpha \in A$, is called complex Gaussian if every linear combination $\sum_{i=1}^{n} c_i x_{\alpha_i}$ with complex coefficients c_i is complex Gaussian.

(c) $(z_\alpha \equiv x_\alpha + iy_\alpha, \alpha \in A)$ is a complex Gaussian system, if and only if $(x_\alpha, y_\alpha, \alpha \in A)$ is a real Gaussian system with

$$E(x_\alpha) = E(y_\alpha) = 0$$

$$E(x_\alpha x_\beta) = E(y_\alpha y_\beta)$$

$$E(x_\alpha y_\beta) = -E(x_\beta y_\alpha) .$$

2. Elementary properties of complex Gaussian systems.

The properties (a), (b), (c), (d) of (real) Gaussian systems (page 1, § 2) hold for complex Gaussian systems.

3. Existence theorem.

Let x_α, $\alpha \in A$, be a complex Gaussian system. Then

$$E(x_\alpha) = 0$$

$$v_{\alpha\beta} = E(x_\alpha \bar{x}_\beta) \text{ exists and is finite.}$$

$(v_{\alpha\beta}, \alpha, \beta \in A)$ is called the covariance matrix and is positive definite in the sense that

$$\sum_{i,j=1}^{n} v_{\alpha_i\alpha_j} \xi_i \bar{\xi}_j \geq 0, \qquad v_{\alpha\beta} = \bar{v}_{\beta\alpha} .$$

for every n, every $(\xi_1 \ldots \xi_n) \in C^n$ (C = the space of complex numbers) and every $(\alpha_1, \ldots, \alpha_n) \in A^n$.

(The second condition follows from the first one.)

Theorem 1. Given any positive definite matrix $(v_{\alpha\beta}, \alpha, \beta \in A)$, we can construct a complex Gaussian system whose covariance matrix is $(v_{\alpha\beta})$.

To prove this we use

<u>Lemma</u>. If $(v_{\alpha\beta}, \alpha, \beta \in A)$ is positive definite, then there exists $(\xi_{\alpha\lambda}, \alpha \in \lambda, \lambda \in \Lambda)$ such that

$$v_{\alpha\beta} = \sum_{\lambda\in\Lambda} \xi_{\alpha\lambda} \bar{\xi}_{\beta\lambda}$$

4. <u>Independence in a complex Gaussian system</u>

Let G_1, G_2 be two subsystems of a Gaussian system G. If every element of G_1 is orthogoanl to every element of G_2 (in the Hilbert space $L^2(\Omega)$), then G_1 and G_2 are independent. This is true for any (even infinite) number of subsystems.

5. <u>Complex Gaussian random measures</u> can be defined in the same way as (real) Gaussian random measures (see page 4).

Let $M(E) = M_1(E) + iM_2(E)$ be a complex Gaussian random measure with

$$E(M(E) \cdot \overline{M(F)}) = m(E \cap F).$$

Then $M_i(E)$ is a (real) Gaussian random measure with

$$E(M_i(E) \cdot M_i(F)) = \frac{1}{2} m(E \cap F)$$

for each i and <u>these two random measures are independent</u>.

6. L^2 over a complex Gaussian system

(a) Given a complex Gaussian system $(x_\alpha, \alpha \in A)$, consider the following closed linear subspaces of $L^2 = L^2(\Omega, \mathcal{B}, P)$;

$\mathcal{L}(x)$ = the set of all linear combinations $\sum c_i x_{\alpha_i}$ and their L^2-limits.

$\mathcal{P}(x)$ = the set of all polynomials $p(x_{\alpha_1} \cdots x_{\alpha_n}, \bar{x}_{\beta_1} \cdots \bar{x}_{\beta_n})$ and their L^2-limits. (We allow some α_i to equal some β_j.)

$\hat{\mathcal{P}}_n(x)$ = the set of all polynomials of degree $\leq n$ and their L^2-limits.

$$\mathcal{P}_n(x) = \hat{\mathcal{P}}_n(x) \ominus \hat{\mathcal{P}}_{n-1}(x) = \sum_{p+q\leq n} \oplus \; P_{p,q}(x)$$

where

$\hat{\mathcal{P}}_{p,q}(x)$ = the set of all polynomials in $\mathcal{P}_n(x)$ of degree $\leq p$ in $(x_\alpha, \alpha \in A)$ and their L^2-limits

$$\mathcal{P}_{p,q}(x) = \hat{\mathcal{P}}_{p,q}(x) \ominus \hat{\mathcal{P}}_{p-1,q}(x) .$$

Theorem 2. $\mathcal{M}(x) = \mathcal{P}(x) = \sum_n \oplus \; \mathcal{P}_n(x) = \sum_{p,q} \oplus \; \mathcal{P}_{p,q}(x)$

(b) Cameron-Martin expansion for a complex Gaussian system.

Definition.

$$H_{p,q}(z,\bar{z}) = \sum_{n=0}^{p \wedge q} (-1)^n \frac{\underline{|p}\ \underline{|p}}{\underline{|n}\ \underline{|p-n}\ \underline{|q-n}} z^{p-n}\, \bar{z}^{\,q-n} . \quad (p \wedge q = \min(p,q))$$

Theorem 3.

(A) $$\exp(-t\bar{t} + t\bar{z} + \bar{t}z) = \sum_{p,q=0}^{\infty} \frac{1}{\underline{|p}\ \underline{|q}} H_{pq}(z,\bar{z})\, \bar{t}^p\, t^q$$

(B) $$H_{pq}(z,\bar{z}) = \exp(z\bar{z})\, (-1)^{p+q} \frac{\partial^{p+q}}{\partial \bar{z}^p\, \partial z^q} \exp(-z\bar{z}) \qquad (p,\ q \geq 0)$$

(C) $$\begin{cases} H_{p+1,q}(z,\bar{z}) - H_{p,q}(z,\bar{z})\, z + qH_{p,q-1}(z,\bar{z}) = 0 \\ H_{p,q+1}(z,\bar{z}) - H_{pq}(z,\bar{z})\, \bar{z} + pH_{p-1,q}(z,\bar{z}) = 0 \end{cases}$$

(D) $$\frac{\partial}{\partial z} H_{pq}(z,\bar{z}) = pH_{p-1,q}(z,\bar{z})$$

$$\frac{\partial}{\partial \bar{z}} H_{pq}(z,\bar{z}) = qH_{p,q-1}(z,\bar{z})$$

(E) $$\frac{\partial^2}{\partial z\,\partial \bar{z}} H_{pq}(z,\bar{z}) - \bar{z}\frac{\partial}{\partial \bar{z}} H_{pq}(z,\bar{z}) + qH_{pq}(z,\bar{z}) = 0$$

$$\frac{\partial^2}{\partial \bar{z}\,\partial z} H_{pq}(z,\bar{z}) - z\frac{\partial}{\partial z} H_{pq}(z,\bar{z}) + pH_{pq}(z,\bar{z}) = 0$$

(F) $H_{pq}(z,\bar{z})$, $p, q \geq 0$ form a complete orthonormal system in $L^2(C,N)$ where N is the measure

$$dN(z) = \frac{1}{\pi}\exp(-x^2 - y^2)\,dx\,dy\,, \qquad z = x + iy$$

on the space C of complex numbers.

(c) Let u_λ, $\lambda \in \Lambda$, be the complete orthonormal system in $\mathcal{P}_{1,0}(x)$. Then

$$H_{\begin{pmatrix} p_1 & \cdots & p_r \\ q_1 & \cdots & q_r \end{pmatrix}}(u,\bar{u}) = \prod_i H_{p_i q_i}(u,\bar{u})$$

$(p = \Sigma p_i,\ q = \Sigma q_i)$ form a complete orthonormal system in $\mathcal{P}_{p,q}(x)$.

7. Complex multiple Wiener integral.

Let $(T, \mathcal{B}_T, m)$ be a measure space and denote with $\mathfrak{F}$ the class of all sets $\in \mathcal{B}_T$ with finite m-measure. Let $M(E)$, $E \in \mathfrak{F}$, be a complex Gaussian measure with

$$E(M(E)\,\overline{M(F)}) = m(E \cap F) .$$

We shall define

$$I_{p,q}(f_{p,q}) = \int_{T^p \times T^q} \cdots \int f_{p,q}(t_1, \ldots, t_p; s_1, \ldots, s_q)\, dM(t_1) \cdots dM(t_p) \times \overline{dM(s_1)} \cdots \overline{dM(s_q)}$$

$$I_{pq}(L_2(T^p \times T^q)) = I_{pq}(S(T^p \times T^q)) = \mathcal{P}_{p,q}(M)$$

($S(T^p \times T^q)$ is the set of all functions $f(t_1, \ldots, t_p; s_1, \ldots, s_q) \in L^2(T^p \times T^q)$ that are symmetric in (t_i) as well as in (S_i).)

$$\frac{1}{\sqrt{p!q!}} I_{pq} : S(T^p \times T^q) \xrightarrow[\text{(isomorphism)}]{} P_{p,q}(M)$$

We can establish the facts similar to those proved in Section 7 (page 9).

Remark: $N(E) = \int_E \varphi(t)\, dM(t)$ $\qquad |\varphi(t)| = 1$

is also a complex Gaussian random measure with

$$E(N(E)\, N(F)) = m(E \cap F)$$

and

$$I_{pq}(f_{p,q};\, dN) = I_{pq}(g_{p,q};\, dM)$$

where

$$g_{p,q}(t_1, \dots, t_p; s_1, \dots s_q) =$$

$$= \varphi(t_1) \dots \varphi(t_p)\ \overline{\varphi(s_1)} \dots \overline{\varphi(s_q)}\, f(t_1, \dots, t_p; s_1, \dots, s_q)$$

This fact will be used later.

3. Ergodic Theorems for Strictly Stationary Processes

1. Definitions. (a) $x_t = x(t, \omega)$, $\omega \in \Omega(\mathcal{B}, P)$, is called a strictly stationary process (s.s.p.) if

$$P((x_{t_1}, \ldots, x_{t_n}) \in E) = P(x_{t_1+t}, \ldots, x_{t_n+t}) \in E)$$

for every choice of n, t_i, and E. We shall assume that $\mathcal{B}$ is the Borel algebra $\mathcal{B}(x)$ determined by x_t, $-\infty < t < \infty$.

(b). Given two measure spaces $\Omega(\mathcal{B}, P)$ and $\Omega'(\mathcal{B}', P')$, a set transformation (modulo null sets) $T: \mathcal{B} \to \mathcal{B}'$ is called an isomorphism if the following conditions are satisfied:

(i) $T(\bigcup_n B_n) = \bigcup_n TB_n$

(ii) $T(B^c) = (TB)^c$

(iii) $m(T(B)) = m(B)$

T is called an automorphism on $\Omega(\mathcal{B}, P)$ in case $\Omega'(\mathcal{B}', P') = \Omega(\mathcal{B}, P)$.

A family of automorphisms S_t, $-\infty < t < \infty$ on $\Omega(\mathcal{B}, P)$ is called an automorphism group on $\Omega(\mathcal{B}, P)$ if $S_{t+s} = S_t S_s$ and $S_0 = I$ (identity).

Let $S_t(S'_t)$ be an automorphism group on $\Omega(\mathcal{B}, P)$ $(\Omega'(\mathcal{B}', P'))$. They are called <u>equivalent</u> if there exists an isomorphism $T: \mathcal{B} \to \mathcal{B}'$ such that $S'_t = T S_t T^{-1}$ for every t.

(c). Given two Hilbert spaces H and H', a linear isometric transformation from H onto H' is called an isomorphism; it is called a <u>unitary operator</u> on H if $H = H'$.

A family of unitary operators U_t, $-\infty < t < \infty$ on H is called a unitary group, if $U_{t+s} = U_t U_s$ and $U_0 = I$.

Let $U_t(U'_t)$ be a unitary group on $H(H')$. These are called <u>equivalent</u> if there exists an isomorphism $V: H \to H'$ such that $U'_t = V U_t V^{-1}$.

(d) Let H denote $L^2(\Omega, \mathcal{B}, P)$. Then every automorphism S on $\Omega(\mathcal{B}, P)$ induces a unique unitary operator U on H determined by the condition

$$U e_B = e_{SB} \qquad \text{for every } B \in \mathcal{B}.$$

Therefore, every automorphism group S_t on $\Omega(\mathcal{B}, P)$ induces a unique unitary group U_t on H.

If $U_t(\leftarrow S_t)$ and $U'_t(\leftarrow S'_t)$ are equivalent, then S_t and S'_t are called <u>unitary equivalent</u> or <u>unitary isomorphic</u>.

If S_t and S'_t is equivalent, then it is unitary equivalent but the converse is not always true.

(c). Let $x_t(\omega)$ be a s.s.p. on $\Omega(\mathcal{B}, P)$ with $\mathcal{B} = \mathcal{B}(x)$. Then we can construct a unique isomorphism group S_t on $\Omega(\mathcal{B}, P)$ which satisfies

$$S_t\{\omega: (x_{t_1}, \ldots, x_{t_n}) \in E\} = \{\omega: (x_{t_1+t}, \ldots, x_{t_n+t}) \in E\}$$

for every choice of n, t_i, E and t.

S_t induces a unitary group U_t. Thus we have

$$(x_t) \to (S_t) \to (U_t) .$$

Now consider another s.s.p $x_t'(\omega')$, $\omega' \in \Omega'(\mathcal{B}', P')$. Then we have

$$(x_t') \to (S_t') \to (U_t').$$

If (S_t) and (S_t') are equivalent, then (x_t) and (x_t') are called <u>measure isomorphic</u>.

If (U_t) and (U_t') are equivalent, then (x_t) and (x_t') are called <u>unitary equivalent</u> or <u>unitary isomorphic</u>

equivalent $\Longrightarrow$ measure isomorphic

$\Longrightarrow$ unitary equivalent

The converse implication is not always true.

If there exists an isomorphism $T: \mathcal{B} \to \mathcal{B}'$ such that $T\{\omega : x_t \in E\} = \{\omega' : x_t' \in E\}$ for every t and every E then two processes (x_t) and (x_t') are called <u>equivalent</u>.

2. Ergodicity.

Definition 1. Let (S_t) be an automorphism group on $\Omega(\mathcal{B}, P)$. A set $B \in \mathcal{B}$ is called invariant if $S_t B = B$ for every t (it is clear that this means $S_t B = B$ modulo null sets for every t). If there exists no invariant set (modulo null set) besides $\emptyset$ and Ω, (S_t) is called ergodic or indecomposable.

Defintion 2. Let $x_t(\omega)$, $\omega \in \Omega(\mathcal{B}, P)$ be a s.s.p. and (S_t) be an automorphism induced from (x_t). x_t is called ergodic if (S_t) is ergodic.

Let (S_t) be an automorphism group on $\Omega(\mathcal{B}, P)$ and (U_t) be a unitary group induced from (S_t).

Theorem 1. [S_t is ergodic] $\Longleftrightarrow$ [if $U_t f = f$ for every t, then $f = \text{const.}$]

Now we shall assume that (S_t) is weakly continuous, i.e., $P(S_t B \cap C)$ is continuous in t for ever choice of $B, C \in \mathcal{B}$. Then (U_t) is also weakly continuous, i.e., $(U_t f, g)$ is continuous in t for every (f, g) and therefore it has Stone's decomposition:

$$U_t = \int e^{i\lambda t}\, dE(\lambda) = e^{itA}, \qquad A = \int \lambda\, dE(\lambda)$$

Theorem 2 S_t is ergodic $\Leftrightarrow$ 0 is a simple eigenvalue.

We shall discuss a nice relation between the discrete spectra and the ergodicity in the next section.

Definition 3. (S_t) is called weakly mixing, if

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |P(S_t A \cap B) - P(A) P(B)|^2 dt = 0$$

for every pair $A, B \in \mathcal{B}$.

Corollary 1. (S_t) is weakly mixing, iff (U_t) $(\leftarrow (S_t))$ satisfies

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |(U_t f, g) - (f, 1) \overline{(g, 1)}|^2 dt = 0 .$$

for every pair $f, g \in H$.

Corollary 2. Weakly mixing $\Longrightarrow$ ergodic.

Theorem 3. (S_t) is weakly mixing $\Longleftrightarrow$ 0 is an eigenvalue with multiplicity 1 (eigenfunction $\equiv$ constant) and there is no other eigenvalue.

Definition 4. If

$$\lim_{t \to \infty} P(S_t A \cap B) = P(A) P(B)$$

for every pair $A, \mathcal{B} \in \mathcal{B}$, then (S_t) is called strongly mixing.

Corollary 3. Strongly mixing $\Longrightarrow$ weakly mixing $\Longrightarrow$ ergodic.

Supplementary Remarks for pages III.1-III.5.

[a] set homomorphism

Let $\Omega(\mathcal{B},P)$ and $\Omega'(\mathcal{B}',P')$ be given probability measure spaces and T be a transformation (modul null sets) from $\mathcal{B}$ into $\mathcal{B}'$.

Definition. T is called a set homomorphism: $\mathcal{B} \to \mathcal{B}'$ if the following three conditions are satisfied:

(1) T preserves finite or countable sum, i.e.,

$$T(\cup B_n) = \cup TB_n$$

(2) T preserves complement, i.e.,

$$T(B^c) = (TB)^c$$

(3) T preserves measure.

$$P'(TB) = P(B).$$

It follows from (1) and (2) that T preserves all usual set operations in measure theory.

Example. A measurable point transformation $\sigma : \Omega'(\mathcal{B}',P') \xrightarrow[\text{into}]{} \Omega(\mathcal{B},P)$ is called measure preserving (or equi-measure) if

$$p'(\sigma^{-1}B) = P(B) \qquad \text{for every } B \in \mathcal{B}$$

In this case $T = \sigma^{-1}$ is a set homomorphism from $\mathcal{B}$ into $\mathcal{B}'$. Notice that not every homomorphism can be induced from a point transformation.

[b] function homomorphism.

Set $\mathcal{S}(\Omega, \mathcal{B}, P)$ be the set of all complex-valued measurable functions on $\Omega(\mathcal{B}, P)$.

Given two probability measure spaces $\Omega(\mathcal{B}, P)$ and $\Omega(\mathcal{B}', P')$, consider

$$\mathcal{S} = \mathcal{S}(\Omega, \mathcal{B}, P) \quad \text{and} \quad \mathcal{S}' = \mathcal{S}(\Omega', \mathcal{B}', P').$$

Definition. Φ is called a function homomorphism: $\mathcal{S} \rightarrow \mathcal{S}'$ if the following three conditions are satisfied:

(1) Φ preserves functional relation:

Let F be any Borel measurable function : $C^n \to C^1$ (C^n = the complex n-dimensional space). Then

$$f(\omega) = F(f_1(\omega), \ldots, f_n(\omega)) \Longrightarrow \Phi f(\omega') = F[\Phi f_1(\omega'), \ldots, \Phi f_n(\omega')]$$

(2) Φ preserves limit:

$$f(\omega) \to f(\omega) \Longrightarrow \Phi f_n(\omega') \to \Phi f(\omega')$$

(3) Φ preserves integral:

$$f \geq 0 \Longrightarrow \int_{\Omega'} \Phi f \cdot dP' = \int_{\Omega} f \, dP$$

It follows from (1) that

$$\Phi \bar{f} = \overline{\Phi f}, \qquad \Phi \mathcal{R} f = \mathcal{R} \Phi f, \qquad \Phi \mathcal{J} f = \mathcal{J} \Phi f, \qquad \Phi |f| = |\Phi f|,$$

and therefore (3) implies that if f is integrable, then Φf is so and

$\int_{\Omega'} \Phi f dP' = \int_{\Omega} f \, dP.$

(c) <u>One-to-one correspondence between set homomorphisms and function homomorphisms.</u>

Using the same notations as above, we can see that the set homomorphisms $\mathcal{B} \dashrightarrow \mathcal{B}'$ correspond to the function homomorphisms $\mathcal{S} \dashrightarrow \mathcal{S}'$ in one-to-one as follows:

$T \dashrightarrow \Phi$

$$\Phi f = \lim_n \sum_{k,\ell} (2^{-n} \cdot k + i\, 2^{-n} \cdot \ell)\, e_{TB_{n,k,\ell}}$$

where

$$B_{n,k,\ell} = \{\omega : 2^{-n} \cdot k \leq \mathcal{R} f < 2^{-n}(k+1),\ 2^{-n} \cdot \ell \leq \mathcal{J} f < 2^{-n}(\ell+1)\}.$$

$$n = 1, 2, \ldots ;\ k, \ell = 0, \pm 1, \pm 2, \ldots .$$

$\Phi \longrightarrow T$

$$TB = \{\omega' : \Phi \cdot e_B(\omega') = 1\}$$

[d] L^2-homomophism

$$\Omega(\mathcal{B},P) \longrightarrow \mathcal{B} \longrightarrow \mathcal{S} \longrightarrow H = L^2(\Omega,\mathcal{B},P)$$

$$T \downarrow \qquad \Phi \downarrow \qquad V\downarrow$$

$$\Omega'(\mathcal{B}',P') \longrightarrow \mathcal{B}' \longrightarrow \mathcal{S}' \longrightarrow H' = L^2(\Omega,\mathcal{B},P)$$

$V = \Phi|H : H \underset{\text{into}}{\longrightarrow} H'$ homomorphism.

$$V(\alpha f + \beta g) = \alpha Vf + \beta Vg$$

$$(Vf, Vg) = (f,g)$$

Not every homomorphism from H into H' can be induced from a set of homomorphism (or equivalent a function homomorphism), but we have

Theorem. Let $V : H = L^2(\Omega, \mathcal{B}, P) \longrightarrow H' = L^2(\Omega, \mathcal{B}, P)$ be a homomorphism. If

$V(f\cdot g) = Vf \cdot Vg$ for every pair of bounded functions f, g,

then V can be induced from a set (function) homomorphism.

[e] A one-to-one onto homomorphism is called an isomorphism.

An isomorphism onto itself is called an automorphism. An automorphism on a Hilbert space is called a unitary operator.

[f] A one-parameter family of automorphisms S_t (Φ_t), $t \in R'$ is called an automorphism group if

$$S_{t+s} = S_t S_s \qquad (\phi_{t+s} = \phi_t \phi_s)$$

and

$$S_0 = I \text{ (identity)} \qquad (\phi_0 = I).$$

A one-parameter family of unitary operators U_t, $t \in R^1$, is called a unitary group, if

$$U_{t+s} = U_t U_s \qquad \text{and} \qquad U_0 = I.$$

Definition. (S_t) on $\mathcal{B}$ and (S'_t) on $\mathcal{B}'$ are called equivalent if there exists an isomorphism T such that

$$S'_t T = T S_t \qquad \text{for every } t.$$

We can define equivalence for (ϕ_t) and (U_t) similarly.

[g] ergodicity.

Theorem. $S_t \longleftrightarrow \phi_t \longleftrightarrow U_t$

$$S_t \text{ ergodic} \underset{\text{def}}{\Longleftrightarrow} (S_t B = B \Longrightarrow B = \emptyset \text{ or } \Omega)$$

$$\Longleftrightarrow (\phi_t f = f \Longrightarrow f = \text{const})$$

$$\Longleftrightarrow (U_t f = f \Longrightarrow f = \text{const})$$

[h] The automorphism group induced from a strictly stationary process.

Given a strictly stationary process x_t on $\Omega(\mathcal{B}, \mathcal{P})$ we can induce $S_t^{(x)}$, $\Phi_t^{(x)}$ and $U_t^{(x)}$ from x.

$$S_t = S_t^{(x)} : \mathcal{B}(x) \longrightarrow \mathcal{B}(x) \quad \text{by}$$

$$S_t\{\omega : (x_{t_1}, \ldots, x_{t_n}) \in E_n\} = \{\omega : (x_{t_1+t}, \ldots, x_{t_n+t}) \in E_n\}$$

$$\Phi_t = \Phi_t^{(x)} : \mathcal{S}(x) \longrightarrow \mathcal{S}(x) \quad (\mathcal{S}(x) = \mathcal{S}(\Omega, \mathcal{B}(x), P)) \quad \text{by}$$

$$\Phi_t F(x_{t_1}, \ldots, x_{t_n}) = F(x_{t_1+t}, \ldots, x_{t_n+t})$$

$$U_t = U_t^{(x)} : H(t) \longrightarrow H(x) \quad (H(x) = L^2(\Omega, \mathcal{B}, P)) \quad \text{by}$$

$$U_t = \Phi_t | H(x)$$

[i] Strictly stationary processes induced from an automorphism group.

Theorem. $\Omega(\mathcal{B}, P) \longrightarrow S_t (\longleftrightarrow \Phi_t)$: automorphism group. Take any fixed f in $\mathcal{S} = \mathcal{S}(\Omega, \mathcal{B}, \mathcal{P})$ and set

$$y_t(\omega) = \Phi_t f(\omega)$$

(i) $y_t(\omega)$ is strictly stationary.

(ii) The automorphism group $S_t^{(y)}$ on $S(y) \cong S(\Omega, \mathcal{B}(y), P)$ induced from y_t is the restriction S_t over $\mathcal{B}(y)$.

(iii) S_t is ergodic $\Longrightarrow$ y_t is ergodic.

Remark. In general $\mathcal{B}(y) \subset \mathcal{B}$, but if $\mathcal{B}$ is separable with respect to the metric $P(B_1,B_2) = P(B_1 \sim B_2)$, then we can take a function $f(\omega)$ for which $\mathcal{B} = \mathcal{B}(f)$, a fortiori $\mathcal{B} = \mathcal{B}(y)$.

3. The discrete spectra of ergodic automorphism group.

Let S_t be a weekly continuous ergodic automorphism group on $\Omega(\mathcal{B}, P)$ and U_t be the unitary group on $H \equiv L^2(\Omega, \mathcal{B}, P)$ induced from S_t. Let

$$U_t = \int e^{i\lambda t}\, dE(\lambda) = e^{itA}, \quad A = \int \lambda\, dE(\lambda)$$

be the Stone decomposition of U_t.

Let Λ be the set of all eigenvalues, i.e.

$$\Lambda = \{\lambda \,;\, E(\lambda+0) - E(\lambda-0) \neq 0\}$$

and $\mathfrak{M}_\lambda$ be the eigenspace for λ, i.e.

$$= (E(\lambda+0) - E(\lambda-0)) \cdot H .$$

Then we can prove the following facts due to J. v. Neumann.

(a) $f \in \mathfrak{M}_\lambda$ and $\|f\| = 1 \Longrightarrow |f| \equiv 1$

(b) Λ is a subgroup of the additive group of real numbers.

(c) Every $\lambda \in \Lambda$ is simple, i.e. $\dim \mathfrak{M}_\lambda = 1$; in particular $\mathfrak{M}_0 = \{f \equiv \text{constant}\}$.

(d) We can choose a representative φ_λ from each $\mathfrak{M}_\lambda$ such that $\varphi_\lambda / \varphi_\mu = \varphi_{\lambda-\mu}$ for every pair λ, μ and $|\varphi_\lambda| \equiv 1$ for every λ.

(e) $H_d \equiv \sum_\lambda \oplus \mathfrak{M}_\lambda = L^2(\Omega, \mathcal{B}_d, P|\mathcal{B}_d)$ where $\mathcal{B}_d = \mathcal{B}[H_d]$.

(f) It holds for φ_λ in (d) that

(i) each φ_λ is uniformly distributed on the unit circle in C,

(ii) if $\{\lambda_i\}$ are rationally dependent, then there is an algebraic relation among $\{\varphi_{\lambda_i}\}$, i.e. if $\sum_i k_i\lambda_i = 0$, then $\prod_i \varphi_{\lambda_i}^{k_i} = 1$,

(iii) if $\{\varphi_{\lambda_i}\}$ are rationally independent, then $\{\varphi_{\lambda_i}\}$ are independent random variables.

Proof of (d). Take any $f_\lambda \in \mathfrak{M}_\lambda$ with $\|f_\lambda\| = 1$ (i.e. $|f_\lambda| \equiv 1$ by (a)). It is enough to construct a function $\gamma(\lambda)$: $\Lambda \to C$ such that

(i) $|\gamma(\lambda)| \equiv 1$

and

(ii) $\prod_{j=1}^{n} (\gamma(\lambda_i) f_{\lambda_i})^{k_i} \equiv 1$ for every choice of n, k_i and $\lambda_i \in \Lambda$ with $\sum_i k_i \lambda_i = 0$;

once such $\gamma(\lambda)$ is constructed, $\varphi_\lambda \equiv \gamma(\lambda) f_\lambda$, $\lambda \in \Lambda$, are the representatives we wanted to get.

Consider the set Γ of all functions with $\mathfrak{D}(\gamma) \subset \Lambda$ that satisfy (i) and (ii) in $\mathfrak{D}(\gamma)$. Introduce a partial order $\prec$ in Γ by

$$\gamma \prec \gamma' \iff \mathfrak{D}(\gamma) \subset \mathfrak{D}(\gamma') \text{ and } \gamma = \gamma' | \mathfrak{D}(\gamma).$$

Then it is enough to see that Γ is inductive, i.e. that every linearly ordered subset of Γ has a supremum. Using Zorn's Lemma, we have a maximal element γ in Γ. If we can prove that $\mathfrak{D}(\gamma) = \Lambda$, then γ is what we wanted to construct. To do this, we shall prove that if $\mathfrak{D}(\gamma) \neq \Lambda$, then γ will have a proper extension $\in \Gamma$.

If $\mathfrak{D}(\gamma) \neq \Lambda$, take $\sigma \in \Lambda - \mathfrak{D}(\gamma)$ and set $M = \mathfrak{D}(\gamma)$ and $M' = M \cup \{\sigma\}$. We shall construct an extension $\gamma' (\in \Gamma)$ of γ with $\mathfrak{D}(\gamma') = M'$.

Consider the set K of the integers k such that

$$k\sigma + k_1\lambda_1 + \cdots + k_n\lambda_n = 0$$

for some n, $k_i \in Z$ and $\lambda_i \in M$. (Z = the set of all integers).
Then K is a subgroup of the additive group Z and so
either

$$K = \{0\}$$

or

$$K = \{\ell h \; ; \; \ell \in Z\} .$$

In the first case we can get an extension γ' of γ by defining $\gamma'(\sigma) = 1$ and $\gamma'(\lambda) = \gamma(\lambda)$ for $\lambda \in \Lambda$. In fact if $k\sigma + k_1\lambda_1 + \cdots + k_n\lambda_n = 0$, then $k \in K$ *i.e.* $k = 0$, so that $k_1\lambda_1 + \cdots + k_n\lambda_n = 0$. Since $\lambda_i \in M$, we have

$$(\gamma(\lambda_1)f_{\lambda_1})^{k_1} \cdots (\gamma(\lambda_n)f_{\lambda_n})^{k_n} \equiv 1$$

i.e.

$$(\gamma'(\sigma)f_\sigma)^{k}(\gamma'(\lambda_1)f_{\lambda_1})^{k_1} \cdots (\gamma'(\lambda_n)f_{\lambda_n})^{k_n} \equiv 1.$$

In the second case we have, by $h \in K$,

(*) $$h\sigma + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3 + \cdots + h_m\sigma_m = 0$$

for some $h_i \in Z$ and $\sigma_i \in M$. Then we have

$$U_t(f_\sigma^{h} f_{\sigma_1}^{h_1} \cdots f_{\sigma_m}^{h_m}) = (U_t f_\sigma)^{h}(U_t f_{\sigma_1})^{h_1} \cdots (U_t f_{\sigma_m})^{h_m}$$

$$= e^{i(h\sigma + h_1\sigma_1 + \cdots + h_m\sigma_m)t} f_\sigma^{h} f_{\sigma_1}^{h_1} \cdots f_{\sigma_m}^{h_m}$$

$$= f_\sigma^{h} f_{\sigma_1}^{h_1} \cdots f_{\sigma_m}^{h_m} \qquad (\text{by } (*))$$

and so

$$f_\sigma^h f_{\sigma_1}^{h_1} \cdots f_{\sigma_m}^{h_m} \equiv \beta \qquad \text{(constant)}.$$

It is clear that $|\beta| = 1$.

Take a complex number z_0 which satisfies

$$z_0^h \, \gamma(\sigma_1)^{h_1} \cdots \gamma(\sigma_m)^{h_m} = \beta^{-1}$$

and define γ' by $\gamma'(\sigma) = z_0$ and $\gamma'(\lambda) = \gamma(\lambda)$ for $\lambda \in M$.

To complete the proof, it is enough to verify (i) and (ii) for γ'. (i) is easy to see. To verify (ii), suppose

(**) $\qquad k\sigma + k_1\lambda_1 + \cdots + k_n\lambda_n = 0$

$$(k, k_1, \ldots, k_n \in Z, \ \lambda_i \in M).$$

Then $k \in K$ and so $k = \ell h$ with some $\ell \in Z$. It follows from (*) and (**) that

$$k_1\lambda_1 + \cdots + k_n\lambda_n - \ell h_1\sigma_1 - \cdots - \ell h_m\sigma_m = 0.$$

Since $\lambda_i, \sigma_j \in M$, this implies

(***) $\qquad (\gamma(\lambda_1)f_{\lambda_1})^{k_1} \cdots (\gamma(\lambda_n) f_{\lambda_n})^{k_n} (\gamma(\sigma_1)f_{\sigma_1})^{-\ell h_1} \cdots (\gamma(\sigma_m)f_{\sigma_m})^{-\ell h_m}$

$$= 1$$

and so

$$(\gamma'(\sigma)f_\sigma)^k (\gamma'(\lambda_1)f_{\lambda_1})^{k_1} \cdots (\gamma'(\lambda_n)f_{\lambda_n})^{k_n}$$

$$= (z_0 f_\sigma)^k (\gamma(\lambda_1)f_{\lambda_1})^{k_1} \cdots (\gamma(\lambda_n)f_{\lambda_n})^{k_n}$$

$$= (z_0 f_\sigma)^{\ell h} (\gamma(\sigma_1)f_{\sigma_1})^{\ell h_1} \cdots (\gamma(\sigma_m)f_{\sigma_m})^{\ell h_m} \qquad \text{(by (***))}$$

$$= (z_0^h \gamma(\sigma_1)^{h_1} \cdots \gamma(\sigma_m)^{h_m} f_\sigma^h f_{\sigma_1}^{h_1} \cdots f_{\sigma_m}^{h_m})^\ell$$

$$= (\beta^{-1} \cdot \beta)^\ell = 1.$$

4. Ergodic Automorphism Groups with ~~No-Continuous-Spectrum~~ pure point spectra

We shall use the same notations as in the preceeding section. If $H_d = H$, then we say that our automorphism group (or the corresponding unitar group) has pure point spectra.

Theorem 1. (J.v. Neumann). If S_t has pure point spectra, then

(i) Λ is a subgroup of the additive group of real numbers.

(ii) $\dim \mathfrak{M}_\lambda = 1$.

(iii) $\exists \varphi_\lambda \in \mathfrak{M}_\lambda$ such that $|\varphi_\lambda| \equiv 1$ and $\varphi_\lambda / \varphi_\mu = \varphi_{\lambda-\mu}$

(iv) $H = \sum_\lambda \oplus \mathfrak{M}_\lambda$.

Theorem 2. (J.v. Neumann). For any additive group $\Lambda \subset R^1$, we can construct one and only one (up to equivalence) weekly continuous ergodic automorphism group with pure point spectra and the set of eigenvalues $= \Lambda$.

Proof. Let Γ be the multiplicative group of complex numbers with absolute value 1, and G be the direct product group Γ^Λ which is abelian and compact by Tihonov's theorem. Consider a homomorphic mapping $h: R^1 \xrightarrow[\text{(into)}]{} G$

$$h(s) = (e^{i\lambda s}, \lambda \in \Lambda).$$

Set $A = h(R^1)$ and $\Omega = \bar{A}$. Then Ω is also a compact abelian group. Denote its Haar measure with P. We normalize P as $P(\Omega) = 1$. Define $\mathcal{B}$ as the least Borel algebra which makes all continuous function measurable. S_t is defined to be the set transformation induced by the point transformation $(\omega_\lambda, \lambda \in \Lambda) \to (e^{-i\lambda t}\omega_\lambda, \lambda \in \Lambda)$. S_t induces a unitary opeator U_t. Set $\varphi_\lambda(\omega) = \omega_\lambda$ for $\omega = (\omega_\lambda, \lambda \in \Lambda)$. Then φ_λ is the unique (up to constant factor) eigenvector for $\lambda \in \Lambda$ and φ_λ, $\lambda \in \Lambda$, form a complete orthonormal system in $H = L^2(\Omega, \mathcal{B}, P)$.

Let S_t on $\Omega(\mathcal{B},P)$ and S'_t on $\Omega'(\mathcal{B}',P')$ be two weakly continuous automorphism groups.

Suppose

$$S_t \to U_t \quad \text{on } H \equiv L^2(\Omega,\mathcal{B},P) \to \Lambda \Longrightarrow \{\varphi_\lambda\}$$

$$S'_t \to U'_t \quad \text{on } H' \equiv L^2(\Omega,\mathcal{B},P) \to \Lambda' \to \{\varphi'_\lambda\}$$

Suppose that $\Lambda' = \Lambda$. Then $\mathcal{B} = \mathcal{B}[\varphi_\lambda, \ \lambda \in \Lambda]$, $\mathcal{B}' = \mathcal{B}[\varphi'_\lambda, \ \lambda \in \Lambda]$ and $\{\varphi_\lambda, \lambda \in \Lambda\}$ $(\{\varphi_\lambda, \lambda \in \Lambda\})$ is a complete orthonormal system (c.o.n.s.) on $H(H')$.

Let V be an isomorphism H onto H' with $V\varphi_\lambda = \varphi'_\lambda$ for every $\lambda \in \Lambda$. Then

$$(*) \qquad V(f \cdot g) = (Vf)(Vg) \qquad \text{for all bounded } f, g \in H.$$

It follows from $\varphi_\lambda\varphi_\mu = \varphi_{\lambda+\mu}$ and $\varphi'_\lambda\varphi'_\mu = \varphi'_{\lambda+\mu}$ that (*) holds for f, g, polynomials of φ_λ, $\lambda \in \Lambda$. To verify it for general bounded $f, g \in H$, it is enough to observe that $\mathcal{B} = \mathcal{B}[\varphi_\lambda, \lambda \in \Lambda]$, $|\varphi_\lambda| \equiv 1$, $\bar{\varphi}_\lambda = \varphi_{-\lambda}$ and $\varphi_\lambda\varphi_\mu = \varphi_{\lambda+\mu}$.

It follows from (*) that V is induced from an isomorphism T from H onto H'.

Since $VU_t\varphi_\lambda = Ve^{i\lambda t}\varphi_\lambda = e^{i\lambda t}V\varphi_\lambda = U'_tV\varphi_\lambda$ and $\{\varphi_\lambda\}$ is a c.o.n.s., we get $VU_t = U'_tV$, so that $TS_t = S'_tT$, i.e., S_t and S'_t are equivalent.

Example 1.

$\Omega = R/Z$

$\mathcal{B}$ = the Borel algebra generated by open subsets of Ω

P = Lebesgue measure

$H = L^2(\Omega, \mathcal{B}, P)$,

α: given number $\neq o$

$S_t \omega = \omega - \alpha t$

$S_t B = (S_t \omega : \omega \in B)$

$U_t f(\omega) = f(\omega + \alpha t)$

$\psi_k(\omega) \equiv e^{i2\pi k \omega}$, $k \in Z$: c.o.n.s. in H,

$U_t \psi_k = e^{i2\pi k \alpha t} \psi_k$.

Since $\alpha \neq o$, $k\alpha$ are all different and so U_t has only discrete spectra $2\pi k \alpha$, $k \in Z$, each having multiplicity 1 and no continuous spectrum. S_t is clearly ergodic and

$$\Lambda = \{2k\pi\alpha, \quad k \in Z\}; \quad q_{2\pi k \alpha} = \psi_k .$$

Example 2.

$\Omega = R^2/Z^2$

$\mathcal{B}$ = the Borel algebra generated by open subsets of Ω

P = Lebesgue measure

$H = L^2(\Omega, \mathcal{B}, P)$

α, β: given rationally independent real numbers

$$S_t(\omega_1, \omega_2) = (\omega_1 - \alpha_t, \ \omega_2 - \beta t)$$

$$S_t B = (S_t \omega : \omega \in B)$$

$$U_t f(\omega_1, \omega_2) = f(\omega_1 + \alpha t, \ \omega_2 + \beta t)$$

$$\psi_{kh} = \exp(i 2\pi (k\omega_1 + h\omega_2)), \quad k, h \in Z: \ c.o.n.s \ \text{in } H$$

$$U_i \psi_{kh} = \exp(i 2\pi (k\alpha + h\beta)) \varphi_{kh}$$

Since α and β are rationally independent, $k\alpha + h\beta$, $k, h \in Z$, are all different and so U_t has only discrete spectra $k\alpha + h\beta$, $k, h \in Z$, each having multiplicity 1 and no continuous spectrum. S_t is therefore ergodic and

$$\Lambda = (2k\pi A, \ k \in Z)$$

$$\varphi_{2\pi(k\alpha + h\beta)} = \psi_{k,h} .$$

<u>NOTE</u> Using Pontriagin's duality theorem we can give a group-theoretical proof to Theorem 2.

Let Ω be the character group of the (discrete) group Λ. Then Ω is a compact abelian group. Set $\varphi_\lambda(\omega)=(\lambda,\omega)$, $\lambda\in\Lambda$, $\omega\in\Omega$. Let $\mathcal{B}$ be the least Bórel algebra of subsets of Ω that makes all continuous functions measurable and P the normalized Haar measure on Ω. Then $f(\in H \equiv L^2(\Omega,\mathcal{B},P))$ can be expressed as a generalized Fourier series: $f(\omega)=\sum_\lambda f_\lambda\varphi_\lambda(\omega)$, $f_\lambda=(f,\varphi_\lambda)$. Therefore $\varphi_\lambda(\omega)$, $\lambda\in\Lambda$, form a complete orthonormal system. Since $e^{i\lambda t}$ is a character of Λ, we have $e^{i\lambda t}=\varphi_\lambda(\omega_t)$ with a unique $\omega_t\in\Omega$. $\omega_{t+s}=\omega_t\omega_s$ follows from $e^{i\lambda(t+s)}=e^{i\lambda t}e^{i\lambda s}$. Now let S_t denote the set transformation induced by a point transformation $S_t\omega\equiv\omega_t\omega$. Then S_t induces a unitary group U_t:

$$\begin{aligned}U_t f(\omega) &= f(\omega_t\omega) = \sum_\lambda f_\lambda\varphi_\lambda(\omega_t\omega)\\ &=\sum_\lambda f_\lambda\varphi_\lambda(\omega_t)\varphi_\lambda(\omega)\\ &=\sum_\lambda f_\lambda e^{i\lambda t}\varphi_\lambda(\omega),\end{aligned}$$

which proves the existence part of the Theorem.

If there exist an ergodic automorphism group with only point spectra Λ, then such group should be equivalent to that constructed above, as is clear from Theorem 1.

5. Hinčin decomposition and Kolmogorov-Cramer decomposition.

Let x_t be a weakly stationary process continuous in L^2-norm. Then

(1) $E(x_t) = a$ (constant)

(2) $E((x_t-a)(\overline{x_s-a})) = r(t-s)$.

Using Bochner's decomposition of positive definite functions and Stone's decomposition of unitary groups, we get

Hincin decomposition

(3) $r(t) = \int_{R^1} e^{i\lambda t}\, m(d\lambda)$

with

(4) $m \geq 0, \quad m(R^1) < \infty$.

and

Kolmogorov-Cramer decomposition

(5) $x_t = a + \int_{R^1} e^{i\lambda t}\, M(d\lambda)$

with

(6) $(M(\Lambda_1), M(\Lambda_2)) = m(\Lambda_1 \cap \Lambda_2)$

(7) $(M(\Lambda), 1) = 0$,

and

(8) $\text{c.l.m.}\left[x_t - a,\ t \in R^1\right] = \text{c.l.m.}\left[M(\Lambda),\ \Lambda \in \mathcal{B}^1\right]$.

Remark. It holds for $M(\Lambda)$ that if $\Lambda_1, \Lambda_2, \ldots$ are disjoint, then $M(\Lambda_1), M(\Lambda_2), \ldots$ are orthogonal and

$$M(\bigcup_n \Lambda_n) = \sum_n M(\Lambda_n) \qquad \text{(convergence in } L^2\text{-norm)}.$$

M is therefore called an orthogonal random measure.

6. Ergodic strictly stationary processes with purely discontinuous Hincin measure.

Suppose that x_t is

(1) strictly stationary

(2) second order moment finite

(3) continuous in L^2-norm.

Then we ca apply the decomposition theorems(section 5) to this process.

Suppose in addition that

(4) the Hinčin measure is purely discontinuous.

Then

$$r(t) = \sum_n c_n^2 e^{i\lambda_n t} \qquad c_n > 0$$

$$x_t = a + \sum_n e^{i n t} c_n \psi_n(\omega)$$

$$(\psi_n, \psi_m) = \delta_{nm}, \quad (\psi_n, 1) = 0.$$

$$\text{c.l.m.}[x_t + a, \ t \in R^1] = \text{c.l.m.}[\psi_n, \ n = 1, 2, \ldots]$$

Theorem 1. Such a process x_t is ergodic iff it is equivalent to

$$y(t) = a + \sum_n e^{i\lambda_n t} a_n \varphi_{\lambda_n}(\omega), \quad a_n^2 = c_n,$$

where Ω is the character group of the (discrete) group generated by $\{\lambda_n\}$, $\mathcal{B}$ is the least Borel algebra that makes all continuous functions measurable, P is the normalized Haar measure on $\Omega(\mathcal{B})$, and $\varphi_\lambda(\omega) = (\lambda, \omega)$.

Theorem 2. Two erdodic processes x_t and x_t' satisfying (1), (2), (3) and (4) are measure isomorphic (i.e. induce equivalent automorphic groups) iff λ_n (x_t) and λ_n' (x_t') generate the same additive group.

6. Ergodic strictly stationary processes with almost periodic covariance function.

Theorem 1. Given a positive-definite almost periodic function

$$r(t) = \sum_n c_n^2 e^{i\lambda_n t} \qquad (c_n > 0,\ \sum_n c_n^2 < \infty),$$

the process of the form:

$$x_t = a + \sum_n a_n e^{i\lambda_n t} \varphi_{\lambda_n}(\omega)$$

($a, a_n \in C$ and $|a_n| = c_n$ and $\varphi_\lambda(\omega)$ is the eigenvector for λ for the automorphism group S_t with only point spectra Λ (=the additive group generated by $\{\lambda_n\}$). See Neumann's theorem for S_t)

is an strictly stationary process with the covariance function $r(t)$ and vice versa.

Theorem 2. Two ergodic strictly stationary processes x_t and x'_t with almost periodic covariance functions $r(t)$ and $r'(t)$ are measure isomorphic iff the sets of spectra for $r(t)$ and $r'(t)$ generate the same additive group.

We shall sketch the proof of Theorem 1.

If x_t is of the form described above, then

$$\begin{aligned} x_t &= a + \sum_n a_n e^{i\lambda_n t} \varphi_{\lambda_n}(\omega) \\ &= a + \sum_n a_n U_t \varphi_{\lambda_n}(\omega) \qquad (S_t \to U_t) \\ &= U_t(a + \sum_n a_n \varphi_{\lambda_n}(\omega)) \\ &= U_t x_0 = \Phi_t x_0 \qquad (S_t \to \Phi_t \to U_t). \end{aligned}$$

As S_t is ergodic, so is x_t by page III 5.6 [1] Theorem (iii).

If x_t is an ergodic strictly statationary process with the covariance function $r(t)$ mentioned above, then the Kolmogorov-Cramer decomposition of the process is

$$x_t = a + \sum_n c_n e^{i\lambda_n t} \xi_n(\omega). \qquad (\xi_n, 1) = 0,\ (\xi_n, \xi_m) = \delta_{nm}$$

Since $x_{t+s} = U_t x_s$, we have $U_t \xi_n = e^{i\lambda_n t} \xi_n$

Since our process x_t is ergodic, $|\xi_n| \equiv 1$. Therefore we can use Segal's Theorem to get

$$L^2(\Omega, \mathcal{B}(x), P) = L^2(\Omega, \mathcal{B}(\xi), P)$$
$$= \text{c.l.m.}\left[\xi_1^{k_1} \cdots \xi_n^{k_n}\ \bar{\xi}_1^{\ell_1} \cdots \bar{\xi}_m^{\ell_m},\ k, \ell, m, n\right]$$

$\xi_1^{k_1} \cdots \xi_n^{k_n} \cdot \bar{\xi}_1^{\ell_1} \cdots \bar{\xi}_n^{\ell_n}$ is an eigenvector for $\lambda = \sum k_i \lambda_i - \sum \ell_j \lambda_j$.

Thus $S_t (\to U_t)$ has only point spectra Λ, and $\varphi_{\lambda_n} = \varepsilon_n \xi_n \{|\varepsilon_n| = 1\}$.

example 1. If $r(t)$ is periodic with period 2π, then

$$x_t = a + \sum_n a_n e^{int} e^{in\omega} \qquad \omega \in [0, 2\pi]$$

example 2. If $r(t)$ is of the form

$$r(t) = \sum_{m,n} a_{mn} e^{i(m\lambda + n\mu)t}, \quad \lambda, \mu \text{ rationally independent}$$

then

$$x_t = a + \sum_{m,n} a_{mn} e^{i(m\lambda + n\mu)t} e^{i(m\omega_1 + n\omega_2)}, \qquad (\omega_1, \omega_2) \in [0, 2\pi]^2$$

7. Ergodic properties of Gaussian stationary processes.

First we shall consider the complex Gaussian stationary processes. As far as we are concerned with complex Gaussian processes, the covariance function

$$(1) \qquad r(t,s) = E(x_t \cdot \bar{x}_s)$$

determines the processes x_t up to equivalence in law and the weak stationarity:

$$(2) \qquad r(t,s) = r(t-s,\ 0) \qquad (= r(t-s))$$

implies the strict stationarity. Assuming that our process x_t is continuous in the L^2-norm, we shall have the decompositions:

$$(3) \qquad r(t) = \int e^{i\lambda t}\, m(d\lambda)$$

$$(4) \qquad x_t = \int e^{i\lambda t}\, M(d\lambda)\ .$$

Since $\text{c.l.m}[x_t,\ t \in R^1] = \text{c.l.m.}[M(\Lambda),\ \Lambda \in \mathcal{B}^1]$ we can see that

$$(5) \qquad H = L^2(\Omega, \mathcal{B}(x), P) = L^2(\Omega, \mathcal{B}(M), P)$$

and that

(6) M is a complex Gaussian random measure with

$$(M(\Lambda_1),\ M(\Lambda_2)) = m(\Lambda_1 \cap \Lambda_2).$$

Theorem 1. If m has a jump, then x_t is not ergodic.

Proof: Let λ_0 be a discontinuity point of m and set

$$M_0 = M(\{\lambda_0\}) , \qquad m_0 = m(\{\lambda_0\}) .$$

Then

$$\|M_0\|^2 = m_0 > 0$$

and

$$U_t M_0 = e^{i\lambda t} M_0 \qquad (\text{by } U_t x_s = x_{t+s}) .$$

Therefore $U_t |M_0| = |M_0|$. If x_t were ergodic, we would have

$$|M_0| = \text{constant} = \sqrt{m_0} ,$$

in contradiction with the fact that M_0 is complex Gaussian.

Theorem 2 (G. Maruyama).

If m has no jump, then x_t is weakly mixing, a fortiori ergodic.

Proof: It is enough to prove that if λ is an eigenvalue for U_t, then $\lambda = 0$ and λ is simple. Let f be an eigenvector for λ with $\|f\| = 1$ and consider its multiple Wiener integral expansion (which exists because of the continuity of m):

$$f = f_0 + \sum_{p+q\geq 1} I_{pq}(f_{pq})$$

where $f_0 = \text{const} = (f,1)$ and $f_{pq} = f_{pq}(\lambda_1 \dots \lambda_p, \mu_1 \dots \mu_q)$ is symmetric in (λ_j) as well as in (μ_k). Then

$$U_t f = f_o + \sum_{p+q\geq 1} I_{pq}[f_{pq} \exp(i(\Sigma \lambda_j - \Sigma \mu_k)t)]$$

because of $U_t\, M(d\lambda) = e^{i\lambda t}\, M(d\lambda)$, and

$$e^{i\lambda t} f = e^{i\lambda t} f_o + \sum_{p+q\geq 1} I_{pq}[f_{pq}\, e^{i\lambda t}].$$

Since f is an eigenvector for λ, we have, for every t,

$$0 = \|U_t f - e^{i\lambda t} f\|^2 = |1-e^{i\lambda t}|^2\, |f_o|^2 + \sum_{p+q\geq 1} p!q! \int_{R^p}\int_{R^q} |f_{pq}|^2\, |e^{i(\Sigma\lambda_j - \Sigma\mu_k)t} - e^{i\lambda t}|^2 \times m^p(d\lambda)\, m^q(d\mu) ,$$

and so each term of the right side should vanish for every t.

Since m is continuous,

$$|\exp[i(\Sigma\lambda_j - \Sigma\mu_k)t] - e^{i\lambda t}|^2 \neq 0$$

except on a $(\lambda_1 \dots \lambda_p, \mu_1 \dots \mu_q)$-set of $m^p \times m^q$-measure 0. Therefore we have

$$f_{pq} = 0 \qquad \text{a.e.}(m^p \times m^q) \qquad \text{for } p+q \geq 1,$$

and so

$$f \equiv f_o \equiv \text{constant}$$

which shows that λ is simple. If $\lambda \neq 0$, then $|1 - e^{i\lambda t}| \neq 0$ for some t and so $|f| \equiv |f_0| = 0$, in contradiction with $\|f\| = 1$.

<u>Theorem 3</u>. (J. L. Doob). If $r(t) \to 0$ as $t \to \infty$, then x_t is strongly mixing.

<u>Remark</u>. If the Hinčin measure is absolutely continuous, then $r(t) \to 0$ by the Riemann-Lebesgue theorem, but not <u>vice versa</u>.

<u>Lemma</u>. Suppose that m is a measure on R^1 with $m(R^1) < \infty$ and that $f \in L^1(R^n, m^n)$. If $\int_{R^1} e^{i\lambda t}\, m(d\lambda) \to 0$ as $t \to \infty$, then

$$\int_{R^n} \exp(i \sum_{j=1}^{n} \lambda_j t_j)\, f(\lambda_1 \ldots \lambda_n)\, m^n(d\lambda) \longrightarrow 0 \quad \text{as} \quad \sum_{j=1}^{n} |t_j| \longrightarrow \infty$$

<u>Proof of Theorem 3</u>. Expanding f, g, we have

$$U_t f = (f,1) + \sum_{p+q\geq 1} I_{pq}[f_{pq} \exp(i(\Sigma\lambda_j - \Sigma\mu_k)t)]$$

$$g = (g,1) + \sum_{p+q\geq 1} I_{pq}[g_{pq}]$$

and so

$$(U_t f, g) = (f,1)(g,1) + \sum_{p+q\geq 1} p!q! \int f_{pq}\, \bar{g}_{pq} \exp(i(\Sigma\lambda_j - \Sigma\mu_k)t)\, m^p(d\lambda)\, m^q(d\mu)$$

$$\left| \sum_{p+q>n} {}'' \right| \leq \sum_{p+q>n} p!\cdot q! \, \|f_{pq}\| \, \|g_{pq}\|$$

$$\leq \left[\sum_{p+q>n} p!q! \, \|f_{pq}\|^2 \right]^{1/2} \left[\sum_{p+q>n} p!q! \, \|g_{pq}\|^2 \right]^{1/2} \longrightarrow 0$$

(uniformly in t).

Each term in Σ converges to 0 as $t \to \infty$ by the lemma. Therefore

$$(U_t f, g) \longrightarrow (f,1)\ (g,1) \qquad \text{as} \qquad t \to \infty.$$

Now consider an L^2-continuous real Gaussian process x_t with

$$a = E(x_t)$$

$$r(t-s) = E((x_t - a)(x_s - a))$$

$$r(t) = \int e^{i\lambda t}\, m(d\lambda)$$

$$x_t = a + \int e^{i\lambda t}\, M(d\lambda)$$

It is easy to see that

(a) m is symmetric, i.e., $m(-\Lambda) = m(\Lambda)$

(b) M is conjugate-symmetric, i.e., $M(-\Lambda) = \overline{M(\Lambda)}$.

(c) $\mathcal{R}\, M(\Lambda)$, $\Lambda \in \mathcal{B}[0, +\infty)$ is a real Gaussian random measure with

$$E[\mathcal{R}\, M(\Lambda_1) \cdot \mathcal{R}\, M(\Lambda_2)] = \frac{1}{2}\, m(\Lambda_1 \cap \Lambda_2)$$

and $\mathcal{J}\, M(\Lambda)$, $\Lambda \in \mathcal{B}[0, +\infty)$ is also a real Gaussian random measure with

$$E[\mathcal{J}\, M(\Lambda_1) \cdot \mathcal{J}\, M(\Lambda_2)] = \frac{1}{2}\, m(\Lambda_1 \cap \Lambda_2)$$

and these two random measures are independent.

(d) $M(\Lambda)$, $\Lambda \in \mathcal{B}[0, +\infty)$, is a complex Gaussian random measure.

Using (b) and (d), we can easily verify

Theorem 4. Theorems 1, 2 and 3 hold for real Gaussian stationary processes.

Combining Maruyama's theorem with the results in the last section, we can get

Theorem 5. Given any positive definite function $r(t)$, we can construct an ergodic strictly stationary process with the covariance function $r(t)$. In addition if $r(t)$ is real, we can construct an ergodic strictly stationary real-valued process with the covariance function $r(t)$.

8. Spectral measure and multiplicity.

Let H be a separable Hilbert space and U_t be a unitary group on H with the Stone decomposition:

(1) $$U_t = \int e^{it\lambda} dE(\lambda).$$

Let us consider for $\varphi \in H$

(2) $$H(\varphi) \equiv \text{c.l.m.}\left[U_t\varphi,\ t \in R^1\right] = \text{c.l.m.}\left[E(\Lambda)\varphi,\ \Lambda \in \mathcal{B}^1\right]$$

and

(3) $$\mu_\varphi(\Lambda) = (E(\Lambda)\varphi, \varphi), \quad \Lambda \in \mathcal{B}^1.$$

It is easy to see that

(4) $$H(\varphi) = \left\{ \int f(\lambda) dE(\lambda)\varphi,\ f \in L^2(R^1, \mathcal{B}^1, \mu_\varphi) \right\}$$

and (5) $\psi \equiv \int f(\lambda) dE(\lambda)\varphi \longrightarrow f$

gives an isomorphism: $H(\varphi) \longrightarrow L^2(R^1, \mathcal{B}^1, \mu_\varphi)$

for which

(6) $$\psi \to f \quad \Rightarrow \quad U_t\psi \to e^{it\lambda} f(\lambda).$$

Theorem 1. There exists $\varphi \in H$ such that

(7) $$\mu_\psi \prec \mu_\varphi \qquad \text{for every } \psi \in H. \quad [\prec = \text{absolutely continuous}]$$

μ_φ is defined to be the spectral measure of U_t which is clearly determined uniquely up to equivalence.

Theorem 2. The spectral measure of $U_t \big| E(\Lambda)H$ is equal to the restriction of that of U_t over Λ.

Definition 1. U_t is called simple if $H = H(\varphi)$ for some $\varphi \in H$. In this case φ is called generator.

Theorem 3. If U_t is simple with generator φ, then μ_φ is the spectral measure of U_t.

Definition 2. If H is decomposed as a direct sum of invariant subspaces H_n ($U_t H_n \subset H_n$) which are orthogonal to each other:

$$H = \sum_{0 \le n < m} \oplus H_n \qquad (m = 1, 2, \ldots, \text{or } \infty)$$

such that $U_t \mid H_n$, $0 \le n < m$ are all simple and have the same spectral measure as U_t, then U_t is said to have constant multiplicity m.

Theorem 4. Let μ be the spectral measure of U_t. Then R^1 is decomposed as a disjoint sum:

$$R^1 = \bigcup_{1 \le n \le \infty} \Lambda_n \quad (\text{modulo } \mu\text{-null sets}) \quad \left[\begin{array}{l}\text{some } \Lambda_n\text{'s} \\ \text{may not appear}\end{array}\right]$$

such that each $U_t \mid E(\Lambda_n)\cdot H$, has constant multiplicity n. Such decomposition is unique modulo μ-null sets.

The function defined by

$$m(\lambda) = n \qquad \text{for } \lambda \in \Lambda_n, \ n = 1, 2, \ldots, \infty$$

is called the multiplicity (function) of U_t.

Theorem 5. (Hellinger-Hahn). The spectral measure and the multiplicity function $m(\lambda)$ determine U_t up to equivalence (= unitary isomorphism). In fact U_t is equivalent to the following unitary group U'_t:

$$\mu_k(\Lambda) = \mu(\Lambda \cap \{\lambda : m(\lambda) \ge k\}) \qquad k = 1, 2, \ldots,$$

$$H' = \sum_k \oplus L^2(R^1, \mathcal{B}^1, \mu_k), \quad (\text{this may end with finite term})$$

$$H' \ni f = (f_1(\lambda), f_2(\lambda), \ldots, f_\infty(\lambda))$$

$$\longrightarrow U'_t f = (e^{it\lambda} f_1(\lambda), e^{it\lambda} f_2(\lambda), \ldots, e^{it\lambda} f_\infty(\lambda));$$

Remark. Since $m(\lambda)$ is determined up to μ-measure 0, we cannot consider the value of $m(\lambda)$ at a particular value of λ. However, if λ_0 is an eigenvalue, then μ has a jump at λ_0, so that $m(\lambda_0)$ has a definite meaning which coincides with the multiplicity of λ_0 in usual sense.

Definition. The spectral measure and the multiplicity function of an automorphism group S_t are defined to be those of the unitary group induced by S_t.

Theorem. Let μ and $m(\lambda)$ be the spectral measure and the multiplicity function of S_t. If $\mu \sim \delta_0$ + Lebesgue measure and if $m(\lambda) = \infty$ for $\lambda \neq 0$, $m(0) = 1$, then S_t is strongly mixing ($\longrightarrow$ weakly mixing $\longrightarrow$ ergodic).

9. Kolmogorov automorphism group.

(a) Definition. An automorphism group S_t is called Kolmogorov automorphism group if there exists a Borel subalgebra $\mathcal{B}_0$ of $\mathcal{B}$ satisfying

1. $$S_t\mathcal{B}_0 = \bigwedge_{s>t} S_s\mathcal{B}_0 ,$$
2. $$\bigwedge_t S_t\mathcal{B}_0 = \{\emptyset , \Omega\}$$ (= trivial Borel Algebra),
3. $$\bigvee_t S_t\mathcal{B}_0 = \mathcal{B} .$$

Theorem (A. Kolmogorov). The μ and $m(\lambda)$ of any Kolmogorov automorphism group are

$$\mu \sim \delta_0 + \text{Lebesgue measure}$$

$$m(\lambda) = \infty \quad \text{for } \lambda \neq 0, \quad m(0) = 1.$$

Corollary. Any Kolmogorov automorphism group is strongly mixing ($\longrightarrow$ weakly mixing $\longrightarrow$ ergodic)

(b) To prove Kolmogorov theorem we shall use a part of the following

Theorem (Kolmogorov-Wiener). Let x_t be a weakly stationary process with mean 0 and the covariance function $r(t) = \int e^{i\lambda t}\, dF(\lambda)$. Let L_t be the c.l.m. spanned by x_s, $s \leq t$. Then the following are equivalent

1. $\bigcap_t L_t = 0$ (trivial subspace)
2. dF is absolutely continous and the density $\varphi(\lambda) \neq 0$ (a.e.) and $$\int \frac{\log \varphi(\lambda)}{1+\lambda^2}\, d\lambda > -\infty .$$

6. Let μ be the spectral measure of U_t, Then

(a) $U_t\varphi = \varphi$ only for $\varphi = 0$ <==> μ has no jump at 0.

(b) $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |(U_t\varphi, \psi)|^2 dt = 0$ for every φ, ψ

<==> μ is continuous

(c) $\lim_{t\to\infty} (U_t\varphi, \psi) = 0$ for every φ, ψ

<==> the Fourier transform of μ vanishes at ∞.

Definition 3: If $\mu \sim$ Lebesgue measure and if $m(t) \equiv$ constant, then U_t is called multiply Lebesgue. In the special case in which this constant is ∞, U_t is called σ-Lebesgue.

Theorem 7: The following three conditions are equivalent:

(i) $\{U_t\}$ is multiply Lebesgue,

(ii) $\{U_t\}$ is isomorphic with the following unitary group U_t' on

$H' = \Sigma \oplus L^2(R^1, \mathcal{B}^1, ds)$ (at most countable direct sum)

defined by $U_t' : (f_1(s), f_2(s), \ldots) \to (f_1(s-t), f_2(s-t), \ldots)$,

(iii) there exists a subspace H_0 of H with the following properties:

(α) $U_tH_0 \subset U_sH_0$ for $t < s$

(β) $\bigwedge_t U_tH_0 = 0$ (= trivial subspace)

(γ) $\bigwedge_t U_tH_0 = H$:

To prove this theorem we shall use a part of the following

Theorem (Kolmogorov-Wiener). Let L be the Hilbert space $L^2(R^1, \mathcal{B}^1 dF)$ with $0 < F(R^1) < \infty$, and L_t the closed linear subspace of L spanned by $f_s(\lambda) \equiv e^{i\lambda s}$, $s \leq t$. If $\bigwedge_t L_t = 0$, then $dF(\lambda) \ll d\lambda$, $F'(\lambda) > 0$ (a.e. $(d\lambda)$) and

$$\int \frac{\log F'(\lambda)}{1+\lambda^2} d\lambda > -\infty, \text{ and vice versa},$$

Proof of Theorem 6.

(a) $U_t\ \varphi = \varphi$ for $\varphi \neq U$

$$\Longrightarrow \mu_\varphi\{0\} = \|E\{0\}\ \varphi\|^2 = \|\varphi\|^2 > 0$$

$$\Longrightarrow \mu\{0\} > 0 \qquad (\because\ \mu_\varphi \prec \mu)$$

$$\mu\{0\} > 0$$

$$\Longrightarrow \|E\{0\}\ \varphi_0\|^2 = \mu\{0\} > 0 \qquad (\text{Take } \varphi_0 \text{ with } \mu_{\varphi_0} = \mu)$$

$$\Longrightarrow U_t\ \varphi = \varphi \quad \text{for} \quad \varphi = E\{0\}\ \varphi_0 \neq 0$$

(b) Noticing

$$(U_t\ \varphi,\ \psi) = \frac{1}{4}\ (U_t(\varphi + \psi),\ (\varphi + \psi)) + \frac{1}{4}\ (U_t(\varphi + \psi),\ (\varphi - \psi))$$
$$+ \frac{i}{4}\ (U_t(\varphi + i\psi),\ (\varphi + i\psi)) + \frac{i}{4}\ (U_t(\varphi - i\psi),\ (\varphi - i\psi))$$

we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |(U_t\ \varphi,\ \psi)|^2\ dt = 0 \qquad \text{for } \forall\varphi,\ \psi$$

$$\Longleftrightarrow \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |(U_t\ \varphi,\ \varphi)|^2\ dt = 0 \qquad \text{for } \forall\varphi$$

But

$$(U_t \varphi, \varphi) = \int e^{i\lambda t} \mu_\varphi(d\lambda)$$

and so

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |(U_t \varphi, \varphi)|^2 dt = \sum_{\mu_\varphi\{\lambda\} > 0} \mu_\varphi(\lambda)^2 .$$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |(U_t \varphi, \varphi)|^2 dt = 0 \qquad \text{for every } \varphi$$

<==> μ_φ is continuous for every φ

<==> μ is continuous

(c) $\lim_{t \to \infty} (U_t \varphi, \psi) = 0$ for every (φ, ψ)

<==> $\lim_{t \to \infty} (U_t \varphi, \varphi) = 0$ for every φ

<==> $\lim_{t \to \infty} \int e^{i\lambda t} d\mu_\varphi(\lambda) = 0$

<==> $\lim_{t \to \infty} \int e^{i\lambda t} d\mu(\lambda) = 0$ (Use Lemma in page III.21)

Proof of Theorem 7. (i) → (ii)

It is enough to prove that

$$U_t: H = L^2(R^1, \mathcal{B}^1, \rho(\lambda)\, d\lambda) \ni f(\lambda) \to e^{i\lambda t} f(\lambda)$$

is isomorphic with

$$U'_t: H' = L^2(R^1, \mathcal{B}^1, ds) \ni g(s) \to g(s - t).$$

But U_t is isomorphic with

$$U''_t: H'' = L^2(R^1, \mathcal{B}^1, d\lambda) \ni h(\lambda) \to e^{i\lambda t}\, h(\lambda)$$

by

$$V: H'' \ni f(\lambda) \to h(\lambda) = \sqrt{\rho(\lambda)} \cdot f(\lambda) \in H'.$$

U''_t is isomorphic with U'_t by

$$V': H' \ni g(s) \to h(\lambda) = (\mathcal{F} g)(\lambda) = \frac{1}{\sqrt{2\pi}} \operatorname*{l.i.m.}_{T \to \infty} \int_{-T}^{T} e^{i\lambda s}\, g(s)\, ds \in H''$$

which completes the proof.

(ii)→(iii)

Let U_t be isomorphic with U'_t by $V: H \to H'$, and set

$$H'_0 = \{f = (f_1(s), f_2(s), \dots) \in H', \forall n\ f_n(s) = 0 \text{ for } s \geq 0\}$$

Then H'_0 satisfies (α) (β) (γ). Now set $H_0 = V^{-1} H'_0$. Then H_0 is what we wanted.

(iii) → (i)

To prove this we shall use the following Lemma which is essentially due to Olf Hanner: Deterministic and non-deterministic stationary random processes, Arkiv för Matematik 1 (1952), 161-177, in particular p. 166-169.

Lemma. Except in the trivial case $H = 0$, under the conditions (α), (β), and (γ), there exists an interval function $z(t, s)$ with

(1) $z(t, s) \in H_{ts}$ where $H_t = U_t H_0$, $H_{ts} = H_s \ominus H_t$

(2) $z(t, x) + z(s, u) = z(t, u)$ for $t < s < u$

(3) $U_h z(t, s) = z(t+h, s+h)$

(4) $\|z(t, s)\|^2 = s-t$

Proof of Lemma. Given $x \in H$ and u, v $(u < v)$, let us consider

$$z(t, s) \equiv z(t, s: u, v, x) = \int_{-\infty}^{\infty} P_{ts} U_\theta P_{uv} x \, d\theta, \qquad t \leq s$$

where $P_{\alpha\beta}$ = projection onto $H_{\alpha\beta}$. Since

$$P_{ts} U_\theta P_{uv} x \neq 0 \qquad \text{only where } t-v < \theta < s-u,$$

$z(t, s)$ is well defined. It is easy to verify (1) and (2). To verify (3), notice

$$(*) \qquad U_h P_{ts} = P_{t+h, s+h} U_h$$

to see

$$\begin{aligned} U_h z(t, s) &= \int_{-\infty}^{\infty} U_h P_{ts} U_\theta P_{uv} x d\theta \\ &= \int_{-\infty}^{\infty} P_{t+h, s+h} U_h U_\theta P_{uv} x d\theta \\ &= \int_{-\infty}^{\infty} P_{t+h, s+h} U_{h+\theta} P_{uv} x d\theta \\ &= \int_{-\infty}^{\infty} P_{t+h, s+h} U_\theta P_{uv} x d\theta = z(t+h, s+h) \end{aligned}$$

Now we shall observe that

$$(**) \qquad \lim_{t\uparrow u, s\downarrow u} P_{ts} = 0 \qquad \text{(strong limit)}$$

if otherwise,

$$\tilde{H}_u = (\lim_{t\uparrow u, s\downarrow u} P_{ts}) \cdot H \neq 0$$

and so $\tilde{H}_v = U_{v-u} \tilde{H}_u \neq 0$ for every v, which combined with $\tilde{H}_u \perp \tilde{H}_v$ $(u \neq v)$, would imply that H is not separable, contrary to the assumption.

By (**) it is clear that $z(t, s)$ is continuous in (t, s). Then $\|z(t, s)\|^2$ is also continuous in (t, s). If $t < s < u$, then $z(t,s) \perp z(s,u)$ and so

$$\|z(t, u)\|^2 = \|z(t, s)\|^2 + \|z(s, u)\|^2.$$

It follows from (3)

$$\|z(t+h, s+h)\|^2 = \|z(t, s)\|^2 .$$

Thus we have

$$\|z(t, s)\|^2 = c(s-t), \qquad s < t ,$$

where $c = c(x, u, v)$ is a non-negative constant independent of (s, t).

If we can take x, u, v so that $c > 0$, it is enough to consider $z(t, s)/\sqrt{c}$ instead of $z(t, s)$ to complete the proof of Lemma.

It is clear that $H_{01} \neq 0$; if otherwise, we could have

$$H = \sum \oplus H_{n,n+1} = \sum_n \oplus U_n H_{01} = 0$$

Take an arbitrary $x_0 (\neq 0) \in H_{01}$. Then

$$\frac{1}{2\gamma} \int_{-\gamma}^{\gamma} P_{01} U_\Theta x_0 \, d\Theta \to P_{01} U_0 x_0 = P_{01} x_0 = x_0 \neq 0$$

as $\gamma \to 0$. Take $\gamma > 0$ with

$$A \equiv \int_{-\gamma}^{\gamma} P_{01} U_\Theta x_0 \, d\Theta \neq 0$$

A can be expressed as

$$A = \sum_{i=1}^{n} \int_{-\gamma}^{\gamma} P_{\frac{i-1}{n}\frac{i}{n}} U_\Theta x_0 \, d\Theta$$

$$= \sum_{i=1}^{n} \int_{-\gamma}^{\gamma} P_{\frac{i-1}{n}\frac{i}{n}} U_\Theta \left[P_{-\infty, \frac{i-1}{n}-\gamma} x_0 + P_{\frac{i-1}{n}-\gamma, \frac{i}{n}+\gamma} x_0 + P_{\frac{i}{n}+\gamma, \infty} x_0 \right] d\Theta$$

$$= \sum_{i=1}^{n} \int_{-\gamma}^{\gamma} P_{\frac{i-1}{n}\frac{i}{n}} U_\Theta P_{\frac{i-1}{n}-\gamma, \frac{i}{n}+\gamma} x_0 \, d\Theta$$

(Use (*) and $P_{\alpha\beta} P_{\epsilon\delta} = 0$ if $(\alpha,\beta] \cap (\epsilon, \delta] = \emptyset$)

On the other hand, consider

$$B_n = \sum_{i=1}^{n} \int_{-\infty}^{\infty} P_{\frac{i-1}{n}\frac{i}{n}} U_\theta P_{\frac{i}{n} - \gamma, \frac{i}{n} + \gamma} x_0 \, d\theta$$

$$= \sum_{i=1}^{n} \int_{-\frac{1}{n} - \gamma}^{\frac{1}{n} + \gamma} P_{\frac{i-1}{n}\frac{i}{n}} U_\theta P_{\frac{i-1}{n} - \gamma, \frac{i}{n} + \gamma} x_0 \, d\theta \qquad \text{(use (*))}$$

Then

$$\|A - B_n\| \le \sum_{i=1}^{n} \int_{\gamma}^{\gamma + \frac{1}{n}} \|" \quad "\| \, d\theta + \sum_{i=1}^{n} \int_{-\gamma - \frac{1}{n}}^{-\gamma} \|" \quad "\| \, d\theta$$

$$\left\| P_{\frac{i-1}{n}\frac{i}{n}} U_\theta P_{\frac{i}{n} - \gamma, \frac{i}{n} + \gamma} x_0 \right\|$$

$$= \left\| U_\theta P_{\frac{i-1}{n} - \theta, \frac{i}{n} - \theta} P_{\frac{i}{n} - \gamma, \frac{i}{n} + \gamma} x_0 \right\|$$

$$= \|P_{s_i, t_i} x_0\| \qquad s_i = \max\left(\frac{i-1}{n} - \theta, \frac{i}{n} - \gamma\right)$$
$$t_i = \min\left(\frac{i}{n} - \theta, \frac{i}{n} + \gamma\right)$$

$$\le \sup\{\|P_{s,t} x_0\|, \ 0 \le t-s < \frac{1}{n}, \ -1 \le t, \ s \le 1\} = \sigma_n$$

$\|A - B_n\| = 2\,\sigma_n \to 0$ as $n \to \infty$ (recall the continuity of $P_{st}\, x_0$ in (s,t)).

Since $A \neq 0$, $B_n \neq 0$ for some n. Therefore,

$$\int_{-\infty}^{\infty} P_{\frac{i-1}{n}\frac{i}{n}}\, U_\Theta\, P_{\frac{i}{n}-\gamma,\, \frac{i}{n}+\gamma}\, x_0\, d\Theta \neq 0$$

for some n and i. Now set

$$u = \frac{i}{n} - \gamma, \qquad v = \frac{i}{n} + \gamma \qquad \text{and} \qquad x = P_{\frac{i}{n}-\gamma,\, \frac{i}{n}+\gamma}\, x_0 .$$

Then $z(\frac{i-1}{n}, \frac{i}{n}) \neq 0$, i.e., $c|\frac{i}{n} - \frac{i-1}{n}| \neq 0$, i.e., $c \neq 0$.

Now let us return to the proof of (iii) $\to$ (i). Let us call a $\{U_t\}$-invariant subspace H' of H simple Lebesgue if U_t restricted on H' has simple Lebesgue spectrum.

By Zorn's Lemma there exists a maximal system of mutually orthogonal simple Lebesgue subspaces $\{H', H'', \ldots\}$. To complete the proof it is enough to prove $H^* \equiv H \ominus (H' \oplus H'' \oplus \cdots) = 0$. Suppose $H^* \neq 0$. H^* is $\{U_t\}$-invariant. Set $H_0^* = P_{H^*} H_0$. Then

$$P_{H^*} U_t = U_t P_{H^*}$$

$$\therefore \qquad \wedge U_t H_0^* = 0$$

$$\vee U_t H_0^* = H^* \neq \emptyset$$

We can apply the Lemma to H^* to get $z(t, s)$ satisfying the conditions in the Lemma. Then

$$\tilde{H} = \{\int f(t)\, dz(t) : f \in L^2(R', \mathcal{B}', dt)\}$$

is a Lebesgue simple subspace of H; in fact

$$U_h \int f(t)\, dz(t) = \int f(t-h)\, dz(t)$$

$$\|\int f(t)\, dz(t)\|^2 = \int |f(t)|^2\, dt,$$

and so U_t over $\tilde{H}$ is isomorphic with

$$\tilde{U}_h : f \in L^2(R', \mathcal{B}', dt) \to (\tilde{U}_h f)(s) = f(s-h)$$

It is also clear that $\tilde{H}$ is orthogonal to H', H'', ... Thus $(\tilde{H}, H', H'', \ldots)$ is also a system of mutually orthogonal simple Lebesgue subspaces in contradiction with the maximal property of $(H', H'', \ldots)$.

9. The Spectral Measure and Multiplicity for Automorphism Groups.

Let S_t be an automorphism group and U_t the unitary group (acting on $H = L^2(\Omega, \mathcal{B}, P)$) induced by S_t. In this case 0 is always an eigenvalue and any function ≡ constant is always an eigenvector for 0. It is therefore enough to observe how U_t acts on $H' \equiv \{1\}^{\perp}$.

Definition 1. The spectral measure and the multiplicity function of an automorphism group S_t are defined to be those of the unitary group U_t acting on H'. If $U_t|H'$ is σ-Lebesgue, then S_t is called σ-Lebesgue.

Theorem 1. Let μ be the spectral measure of S_t. Then

(1) S_t is ergodic iff μ has no jump at 0,

(2) S_t is weakly mixing iff μ is continuous,

(3) S_t is strongly mixing iff the Fourier transform of μ vanishes at $+\infty$,

Defintion 2. $\{S_t\}$ is called Kolmogorov automorphism group (or K-flow) if there exists a Borel subalgebra $\mathcal{B}_0$ of $\mathcal{B}$ with

(α) $S_t \mathcal{B}_0 \subset S_s \mathcal{B}_0$ for $t < s$

(β) $\bigcap_t S_t \mathcal{B}_0$ = trivial algebra, i.e., $\{\emptyset, \Omega\}$ (modulo null sets)

(γ) $\bigcup_t S_t \mathcal{B}_0 = \mathcal{B}$

Theorem 2. (Kolmogrov-Sinai)

Any Kolmogorov automorphism group is σ-Lebesgue.

IV. STRICTLY STATIONARY SEQUENCES

In Chapters II and III we have discussed the ergodic properties of strictly stationary processes. Similar theorems hold for strictly stationary sequences.

1. Definitions.

(a) strictly stationary sequence

$$x_n \equiv x(n,\omega), \quad n \in Z = \{\ldots -2, -1, 0, 1, 2, \ldots\}$$

(b) If S is an automorphism on $\Omega(\mathcal{B},P)$, then S^n, $n \in Z$, form an automorphism group with discrete parameter.

(c) If U is a unitary operator on H, then U^n, $n \in Z$ form a unitary group with discrete parameter.

(d) S automorphism ⟶ unitary operator U

$$Ue_B = e_{SB}$$

(e) x_n, $n \in Z$, strictly stationary sequence

⟶ S automorphism (shift)

⟶ U unitary operator on $L^2(\Omega, \mathcal{B}, \mathcal{P})$

(f) equivalent, measure isomorphic, unitary equivalent

2. Ergodic Properties.

S automorphism ---> U unitary operator

(a) S is called ergodic if $S \cdot B = B \Longrightarrow B = \emptyset$ or Ω

S is ergodic $\rightleftarrows$ $Uf = f \Longrightarrow f = \text{const.}$

(b) S is called weakly mixing if

$$\lim_{N \to \infty} \frac{1}{2N} \sum_{-N}^{N} |P(S^n A \cap B) - P(A)\, P(B)|^2 \, dt = 0$$

for every (A,B).

S is weakly mixing $\rightleftarrows$ $\displaystyle\lim_{N \to \infty} \frac{1}{2N} \sum_{-N}^{N} |(U^n f, g) - (f,1)\, \overline{(g,1)}|^2 \, dt \to 0$

(c) strongly mixing $P(S^n A \cap B) \longrightarrow P(A)\, P(B)$ for every (A,B).

(d) Spectral decomposition of U:

$$U = \int_{\Gamma} e^{i2\pi\lambda} \, dE(\lambda) \qquad \Gamma = R^1 | Z \, .$$

(e) Theorem of Hellinger-Hahn

Let U be a unitary operator on a separable Hilbert space H.

Then there exists a decompositon of H: $H = \sum_n \oplus H_n$ with the following properties

1) $U|H_n$ is isomorphic with V_n on $L^2(\Gamma, \mathcal{B}_\Gamma, d\mu_n)$: $(V_n f_n)(\lambda) = e^{i2\pi\lambda} f_n(\lambda)$.

2) $\mu_1 \succ \mu_2 \succ \cdots$.

$(\mu_1, \mu_2, \ldots)$ is uniquely determined by U up to equivalence and is called the Hellinger-Hahn measures of U.

(f) U is called multiple Lebesgue (or is said to have multiple Lebesgue spectrum) if all Hellinger-Hahn measures are equivalent to the uniform measure on Γ. If the multiplicity is ∞, then it is called σ-Lebesgue.

U is multiple Lebesgue with multiplicity m

$\rightleftarrows$ U is the direct sum of m copies of the unitary operator

$$L^2(\Gamma, \mathcal{B}, d\lambda) \ni f(\lambda) \to e^{i2\pi\lambda} f(\lambda)$$

$\rightleftarrows$ U is the direct sum of m copies of the shift operator on

$$\ell^2(Z) = \{(a_n) : \sum_{n \in Z} |a_n|^2 < \infty\}$$

3. Kolmogorov-Sanai's theorem for the discrete parameter case.

Lemma. Let U be a unitary operator on a separable Hilbert space H. If there exists a subspace H_0 of H with the properties

1. $UH_0 \supset H_0$
2. $\bigvee_n U^n H_0 = H$
3. $\bigwedge_n U^n H_0 = 0$

then U is multiple Lebesgue with multiplicity $m = \dim[UH_0 - H_0]$, provided $H \neq 0$.

Definition: S is called a Kolmogorov automorphism if there exists a Borel subalgebra $\mathcal{B}_0$ of $\mathcal{B}$ with the properties

1. $S\mathcal{B}_0 \supset \mathcal{B}_0$
2. $\bigvee_n S^n \mathcal{B}_0 = \mathcal{B}$
3. $\bigwedge_n S^n \mathcal{B}_0 = \mathcal{N}$ (trivial algebra)

Theorem (Kolmogorov-Sinai). The unitary operator U associated with any Kolmogorov automorphism is σ-Lebesgue on $H = L^2(\Omega, \mathcal{B}, P) \ominus \{1\}$, provided $\mathcal{B} \neq \mathcal{N}$.

Proof: By the above lemma it is enough to show that

$$\dim[UH_0 \ominus H_0] = \infty \qquad \text{for} \quad H_0 = L^2(\Omega, \mathcal{B}_0, P) \ominus \{1\}$$

namely

$$\dim[L^2(\Omega, U\mathcal{B}_0, P) \ominus L^2(\Omega, \mathcal{B}_0, P)] = \infty.$$

Lemma 1. $B \in S^n \mathcal{B}_0$, $S^m B = B$ for some $m \neq 0$ and $n \Longrightarrow B \in \mathcal{X}$.

Proof: It follows from the assumption that

$$B \in \bigwedge_k S^{mk+n} \mathcal{B}_0 = \bigwedge_k S^k \mathcal{B}_0 = \mathcal{X}.$$

Lemma 2. For any given $B \in S\mathcal{B}_0 - \mathcal{B}_0$ there exist $B_1 \in S\mathcal{B}_0 - \mathcal{B}_0$ with $\mathcal{B}_1 \subsetneq \mathcal{B}$.

Proof: $P(B) = P(S^{-1}\mathcal{B}) = P(S^{-2}\mathcal{B}) = \cdots > 0$

$$P(\bigcup_{i \geq 0} S^{-i}\mathcal{B}) \leq P(\Omega) = 1$$

$\exists$ $m > 0$ and n

$$P(S^{-n}B \wedge S^{-n-m}B) > 0$$

$$P(B \wedge S^{-m} B) > 0$$

$B = B \wedge S^{-m}B \Longrightarrow B \subset S^{-m}B \Longrightarrow B = S^{-m}B$

(by Lemma 1) $\Longrightarrow B \in \mathcal{X} \subset \mathcal{B}_0$ (contradiction)

$\therefore$ $P(B - S^{-m}\mathcal{B}) > 0$

Either $B' \equiv B \wedge S^{-m}B \notin \mathcal{B}_0$ or $B'' = B - S^{-m}B \notin \mathcal{B}_0$ because $B' \vee B'' = B \notin \mathcal{B}_0$. Therefore $B_1 = B'$ or B'' will satisfy our condition.

Lemma 3. For any given $B \in S\mathcal{B}_0 - \mathcal{B}_0$, there exist C_1 and $C_2 \in C\mathcal{B}_0 - \mathcal{B}_0$ such that $C_1 \wedge C_2 = \emptyset$ and $C_1 \vee C_2 \subset B$.

Proof: Consider the class $\underline{C}$ of all well-ordered (with respect to the inclusion $\supset$) subsystems of $(S\mathcal{B}_0 - \mathcal{B}_0) \wedge 2^B$. The single system (B) belongs to $\underline{C}$. Since $C \supsetneqq C'$ implies $P(C) > P(C')$, every system $\mathcal{C} \in \underline{C}$ is countable. Using Zorn's lemma we can find a maximal system $\mathcal{M}$ in $\underline{C}$. If $\mathcal{M}$ has the last element (= the smallest set) M, then we can apply Lemma 2 to get $M' \in S\mathcal{B}_0 - \mathcal{B}_0$ such that $M' \subsetneqq M$. Then $\mathcal{M} \vee (\mu') \in \mathcal{C}$ in contradiction with the maximal property of $\mathcal{M}$. Thus $\mathcal{M}$ has no last element. Therefore the intersection M'' of all sets $\in \mathcal{M}$ is smaller than every set $\in \mathcal{M}$. If $M'' \notin \mathcal{B}_0$, i.e., $\in S\mathcal{B}_0 - \mathcal{B}_0$, then $\mathcal{M} \vee (M'') \in \mathcal{C}$ which is again a contradiction. Thus $M'' \in \mathcal{B}_0$. It is clear that the first element in $\mathcal{M}$ is B because of the maximal property of $\mathcal{M}$. Thus we have

$$B = \bigvee_{C \in \mathcal{M}} (C - C^*) \vee M'' \qquad \text{(countable disjoint sum)}$$

where C^* is the element next to C in $\mathcal{M}$. Since $M'' \in \mathcal{B}_0$ and $B \notin \mathcal{B}_0$, there exists at least one $C - C^* \notin \mathcal{B}_0$. Then $C_1 = C^*$ and $C_2 = C - C^*$ are the sets which satisfy our condition.

Since there exists at least one set $B \in S\mathcal{B}_0 - \mathcal{B}_0$ (if otherwise, $S\mathcal{B}_0 = \mathcal{B}_0$ and so $\mathcal{B} = \bigvee S^n \mathcal{B}_0 = \mathcal{B}_0 = \bigwedge S^n \mathcal{B}_0 = \mathcal{N}$ in contradiction with $\mathcal{B} \neq \mathcal{N}$), we can prove the following lemma by applying Lemma 3 recursively.

Lemma 4. There exists an infinite sequence (F_n) such that

$$F_i \in S\mathcal{B}_0 - \mathcal{B}_0, \qquad i = 1, 2, \ldots$$

and

$$F_i \wedge F_j = \emptyset \qquad (i \neq j).$$

Now we shall come back to the proof of our theorem. It is enough to construct a linearly independent system $\{g_1, g_2, \ldots, g_n\} \in L^2(\Omega, S\mathcal{B}_0, P) \ominus L^2(\Omega, \mathcal{B}_0, P)$ for any given n. Take $F_1, F_2, \ldots$ by Lemma 4. Rearrange $F_1, F_2, \ldots, F_{n(n+1)}$ as

$$\begin{matrix} F_{11} & F_{12} & \cdots & F_{1\,n+1} \\ F_{21} & F_{22} & \cdots & F_{2\,n+1} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{n\,n+1} \end{matrix}$$

Determine a_{kj}, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n+1$ such that

$$a_{kj} > 0,$$

$$\frac{a_{ij}}{a_{k\ell}} \neq \frac{a'_{i'j'}}{a_{k'\ell'}} \quad \text{if} \quad \begin{cases} (i\ j\ k\ \ell) \neq (i'\ j'\ k'\ \ell') \\ \text{and} \\ (ij\ i'j') \neq (k\ \ell\ k'\ell') \end{cases}$$

Write e_{ij} for the indicator function of F_{ij} and set

$$f_i = \sum_{j=1}^{n+1} a_{ij}\ e_{ij}$$

$$g_i = f_i - E(f_i | \mathcal{B}_o) \equiv f_i - P_{L^2(\Omega,\ \mathcal{B}_o, P)}\ f_i\ .$$

Then $g_i \in L^2(\Omega,\ S\mathcal{B}_o, P) \ominus L^2(\Omega, \mathcal{B}_o, P) = UH_o \ominus H_o$

In order to see that $g_1, g_2, \cdots, g_n$ are linearly independent, it is enough to prove that $\sum \lambda_i g_i \neq 0$, i.e.,

$$f \equiv \sum_i \lambda_i f_i \equiv \sum_{i=1}^{n} \sum_{j=1}^{n+1} \lambda_i\ a_{ij}\ e_{ij} \notin L^2(\Omega,\ \mathcal{B}_o, P)$$

unless $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Suppose $\lambda_i \neq 0$ for some i, say $i = 1$. Then <u>either</u> there exist $i_1, j_1, i_2, j_2, \cdots, i_{n+1}, j_{n+1}$ such that

$$\begin{aligned}
\lambda_1 a_{11} &= \lambda_{i_1} a_{i_1 j_1} & (i_1, j_1) &\neq (1, 1) \\
\lambda_1 a_{12} &= \lambda_{i_2} a_{i_2 j_2} & (i_2, j_2) &\neq (1, 2) \\
&\cdots\cdots\cdots\cdots\cdots\cdots && \\
\lambda_1 a_{1\, n+1} &= \lambda_{i_{n+1}} a_{i_{n+1} j_{n+1}} & (i_{n+1}, j_{n+1}) &\neq (1, n+1)
\end{aligned}$$

<u>or</u> there exists j_o such that $\lambda_1 a_{1j_o} \neq \lambda_i a_{ij}$ for every $(i, j) \neq (1, j_o)$.

In the first case all $\lambda_{i_\nu} \neq 0$ by virtue of $\lambda_1 \neq 0$ and $a_{ij} > 0$. Since $i_1, i_2, \ldots , i_{n+1}$ are among $1, 2, \ldots , n$, there exists at least one pair p, q $(p \neq q)$ with $i_p = i_q$. Then

$$\frac{a_{1p}}{a_{1q}} = \frac{a_{i_p j_p}}{a_{i_q j_q}},$$

which is impossible because of $p \neq q$ and $(1, p) \neq (i_p, j_p)$.

In the second case we have

$$(\omega : f = \lambda_1 a_{1j_o}) = F_{ij} \notin \mathcal{B}_o,$$

i.e., $$f \notin L^2(\Omega, \mathcal{B}_o, P).$$