# Bockstein maps and local cohomology

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1. Bockstein basics
2. Local cohomology
3. Stanley–Reisner theory
4. Main Theorem

# Bockstein Setup

| | |
|---|---|
| $R$ | commutative Noetherian ring |
| $M$ | $R$-module |
| $p$ | nzdv on $M$ |
| $\mathrm{F}\colon R\text{-mods} \longrightarrow R\text{-mods}$ | covariant additive $R$-linear $\delta$-functor. |

**Remark**

*More general setups are possible, including the case of singular cohomology with coefficients in the $R$-module $M$.*

*Needed is an endomorphism of a complex.*

# Bockstein definition

From

$$0 \to M \xrightarrow{p} M \to M/pM \to 0$$

we obtain

$$\mathrm{F}^k(M/pM) \xrightarrow{\delta_p^k} \mathrm{F}^{k+1}(M) \xrightarrow{p} \mathrm{F}^{k+1}(M) \xrightarrow{\pi_p^{k+1}} \mathrm{F}^{k+1}(M/pM).$$

The *Bockstein map* $\beta_p^k = \beta_p^k(\mathrm{F}, M)$ is

$$\pi_p^{k+1} \circ \delta_p^k \colon\ \mathrm{F}^k(M/pM) \to \mathrm{F}^{k+1}(M/pM).$$

A cohomology operation.

# Bockstein Examples

- $(R, p, \mathrm{F}) = (\mathbb{Z}, p, H^{\bullet}_{Sing})$.

  Classical case, used to study lens spaces.

  In 3D: $L(p, q) = \mathbb{S}^3 / \begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}$, $\omega^p = 1$. (Tietze 1908)

  Fact: $L(5, 1)$, $L(5, 2)$ have equal $\pi_1$ and $H_*$, but are not homotopy equivalent.

  Fact: $L(7, 1)$, $L(7, 2)$ homotopy equivalent but not homeomorphic.

- $(R, p, \mathrm{F}) = (R, f, H^{\bullet}(f; -))$.

  $H^0(f; R/pR) = (0 :_{R/pR} f)$, $\quad H^1(f; R/pR) = R/(p, f)R$,

$$\beta(r \bmod (p)) = \frac{fr}{p} \bmod (p, f).$$

- F=Local cohomology (next).

# Definition of local cohomology

## Setup

$R =$ commutative ring,
$(f_1, \ldots, f_t)\, R = I \subseteq R$ an ideal,
$M$ an $R$-module.

Consider $\check{C}_i^\bullet = \left(R \to R[f_i^{-1}]\right)$ and

$$\check{C}_f^\bullet = \bigotimes_1^t \check{C}_i^\bullet$$

## Definition

$$H_I^k(M) := H^k(M \otimes \check{C}_f^\bullet).$$

# Why care about local cohomology?

- If $\sqrt{I} = \sqrt{J}$ then $H^\bullet_I = H^\bullet_J$ ("geometric").
- $R = \mathbb{C}[x_1, \ldots, x_n]$, $V = \mathrm{Var}(I)$, $U = \mathbb{C}^n \smallsetminus V$.

$$E_2^{p,q} = H^p_{deRham}(H^q_I(R)) \Rightarrow H^{p+q+n-1}_{sing}(U).$$

- $(R, \mathfrak{m})$ local: $\left[H_{\mathfrak{m}}^{<\dim(R)}(R) = 0\right] \Leftrightarrow [R \text{ is CM}]$.
- Relations to hypergeometric systems, Riemann–Hilbert correspondence, tight closure, topology.

# Stanley–Reisner theory

## Setup

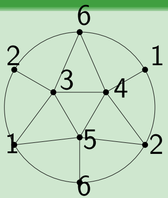
$R = \mathbb{C}[x_1, \ldots, x_n]$, $\Delta \subseteq \Delta_n = n$-simplex, $\mathfrak{m} = \mathbf{x}R$.

## Definition

$$I_\Delta = \{\mathbf{x}^\sigma \mid \sigma \notin \Delta\}R, \quad S_\Delta = R/I_\Delta$$

(squarefree monomial) Stanley–Reisner ideal, and ring.

## Example

$\Delta = \mathbb{RP}^2$: 

Here, $I_{\mathbb{RP}^2} = (x_1x_2x_5, \ldots, x_4x_5x_6)R$.

# Stanley–Reisner, local cohomology, Bockstein

**Theorem (Hochster, simplified)**

*For* $0 \neq p \in \operatorname{Spec} \mathbb{Z}$*,*

$$H^i_{\mathfrak{m}}(S_\Delta / pS_\Delta)_0 = \widetilde{H}^{i-1}(\Delta, \mathbb{Z}/p\mathbb{Z}).$$

(Really, Hochster describes all graded pieces. . . )

**Theorem (SW)**

*The Bockstein construction respects the decomposition of Hochster's theorem.*
*Moreover, the following are equivalent:*

- *Bocksteins are zero on* $H^\bullet_{\mathfrak{m}}(S_\Delta / pS_\Delta)$*;*
- *Bocksteins are zero on* $H^\bullet_{I_\Delta}(R/pR)$*;*
- *Bocksteins are zero on certain subcomplexes of* $\Delta$*.*

# Bockstein and Stanley–Reisner

## Example

A nonzero Bockstein in topology:

$$H^1_{sing}(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z}) = H^2_{sing}(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$$

Resolution of the Stanley–Reisner ideal:

$$0 \to R \to R^7 \to R^{15} \to R^{10} \to R \to R/I_\Delta \to 0.$$

Last map: $(x_1, \ldots, x_6, 2)$.
Diagram chase: the 2 is responsible for nonzero

$$\beta_2^2 \colon H^3_{I_\Delta}(R/2R) \to H^4_{I_\Delta}(R/2R)$$

via $(0,0,0,0,0,0,1) \in \mathrm{Ext}^3_{R/2R}(R/(2, I_\Delta), R/2R) \to$
$H^3_{I_\Delta}(R/2R) \to H^4_{I_\Delta}(R/2R) \leftarrow Ext^3_{R/2R}(R/(2, I_\Delta), R/2R) \ni (1)$.

# Finiteness of local cohomology

**Setup**

$R$ Noeth., $M$ fin. gen., $I$ ideal.

- Grothendieck: $\mathfrak{m}$ maximal $\Rightarrow H^\bullet_{\mathfrak{m}}(M)$ Artinian
- Huneke: $\operatorname{Ass}(H^i_I(R))$ finite ?
- Huneke–Sharp, Lyubeznik: $R$ regular containing field then yes.
- Singh: In general, no (next slide).
- Lyubeznik: $R$ mixed char., regular, local, unramified, then yes.
- Lyubeznik: $R$ regular then $H^i_I(R)$ finite Ass ?
  (open for $\mathbb{Z}[x_1, \ldots, x_n]$ !!!)

# Example (Singh)

**Setup**

$R = \mathbb{Z}[x,y,z,u,v,w]/(xu+yv+zw)$, $I = (x,y,z)$, $0 < p \in \operatorname{Spec}\mathbb{Z}$.

- $\eta_p = [(u/yz, -v/xz, w/xy)] \in H^2_I(R/pR)$.
- $\beta^2_p(\mathrm{Frob}(\eta_p)) = \left[\frac{(ux)^p+(vy)^p+(wz)^p}{p(xyz)^p}\right] \neq 0$.
- Thus, $\delta\colon H^2_I(R/pR) \to H^3_I(R)$ sends $\mathrm{Frob}(\eta_p)$ to $\delta(\mathrm{Frob}(\eta_p)) \neq 0$.
- $p$ kills $\mathrm{Frob}(\eta_p)$, hence also $\delta(\mathrm{Frob}(\eta_p))$.
- So $H^3_I(R)$ can't have finite Ass.

# Main Theorem

## Theorem

*$R = \mathbb{Z}[x_1, \ldots, x_n]$, $I = (f_1, \ldots, f_t)R$, $p$ nzdv on $H^{k+1}(f; R)$. Then*

$$\beta_p^k \colon H_I^k(R/pR) \to H_I^{k+1}(R/pR)$$

*is zero.*

## Remark

$$\left| \bigcup_{k \in \mathbb{N}} \mathrm{Ass}(H^{k+1}(f; R)) \right| < \infty.$$

*Counterexamples to Lyubeznik can't be found with Singh's method!*

# Idea of proof

- $H^k_I(R) = \varinjlim_{i \in \mathbb{N}} H^k(f^i; R)$
- Let $p$ nzdv on $H^{k+1}_I(R)$.
- $\psi\colon R \to R$, $\psi_p(x_i) = x_i^p$ is flat.
- $\Psi\colon R\text{-mods} \to R\text{-mods}$, $\Psi(M) = M \otimes_R {}^{\psi}R$ exact.
- $\Psi^e \left( H^{k+1}(f; R) \xrightarrow{p} H^{k+1}(f; R) \right)$ injective.
- Bockstein to $\Psi^e(f)$ is zero.
- Mod $p$, $\Psi(f) = f^p$, prove comparison for Bocksteins, done.

Thank you!