Iterated Loop Spaces and Oriented Matroids

Dai Tamaki
Depmartment of Mathematical Sciences,
Shinshu University

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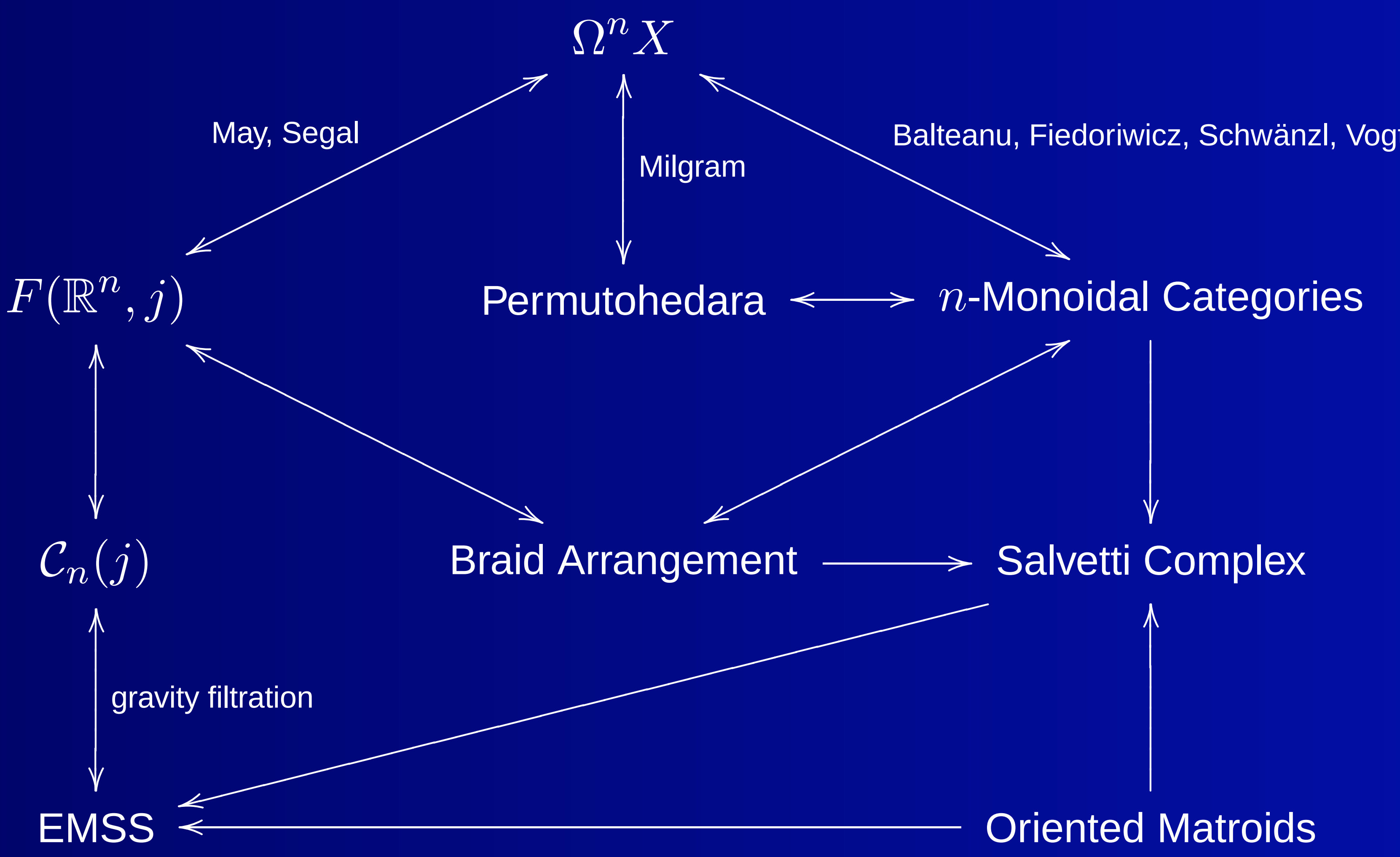
Introduction



Outline

- Iterated Loop Spaces and Little Cubes.
  - Iterated loop spaces, little cubes, and the braid arrangement.
  - The Eilenberg-Moore spectral sequence (EMSS).

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- Main Results.
  - Double loop spaces.
  - More highly iterated loop spaces and higher dimensional Salvetti complexes.

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  - Double loop spaces.
  - More highly iterated loop spaces and higher dimensional Salvetti complexes.
- More Arrangements?
  - The “center of mass” arrangement.

Iterated Loop Spaces and Little Cubes

Iterated Loop Spaces

- For a space X with a base point x_0, the n-fold loop space of X is

$$\Omega^n X = \{f : I^n \to X \mid \varphi(\partial I^n) = x_0\} = \Omega(\Omega^{n-1} X).$$

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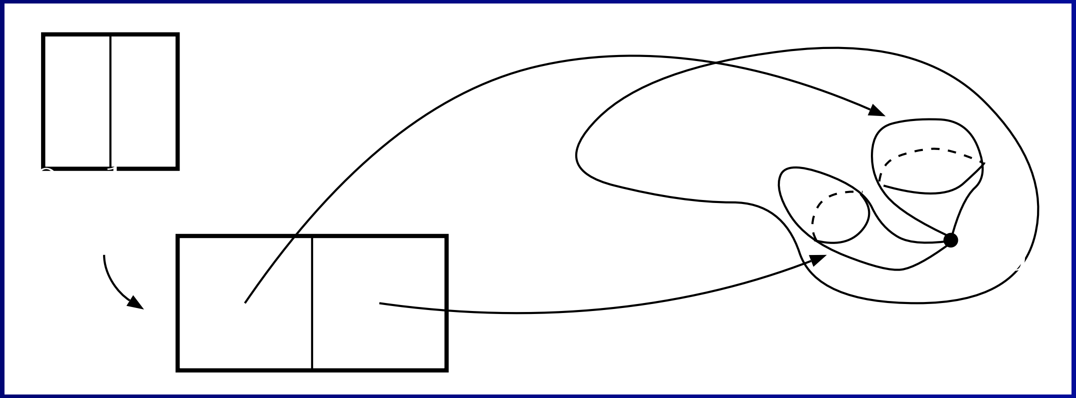
- It has n-kinds of multiplications

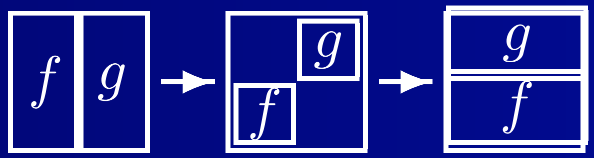
$$\mu_i : \Omega^n X \times \Omega^n X \longrightarrow \Omega^n X$$

for $1 \leq i \leq n$ corresponding to the coordinates of I^n.



Iterated Loop Spaces

- $\mu_i \simeq \mu_j$ for any i, j



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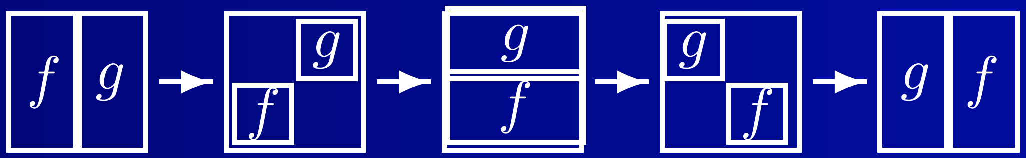

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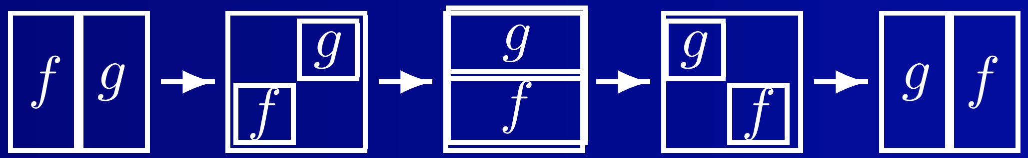

$\Longrightarrow$ The multiplication is homotopy commutative if $n \geq 2$.

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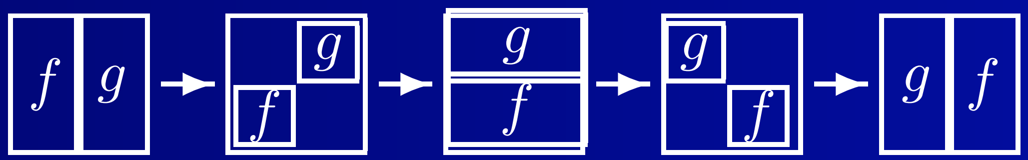

$\Longrightarrow$ The multiplication is homotopy commutative if $n \geq 2$.

- For $n \geq 1$, the space of little n-cubes is

$$\mathcal{C}_n(1) = \left\{ c : I^n \to I^n \;\middle|\; \begin{array}{l} c = \ell_1 \times \cdots \times \ell_n, \\ \ell_i : I \to I \text{ affine embedding} \end{array} \right\}.$$

- For $n \geq 1$ and $j \geq 1$, the configuration space of j little n-cubes is

$$\mathcal{C}_n(j) = \{(c_1, \cdots, c_j) \in \mathcal{C}_n(1)^j \mid c_i(\mathrm{Int} I^n) \cap c_k(\mathrm{Int} I^n) = \emptyset \text{ if } i \neq k\}$$

Little Cubes

- $\{\mathcal{C}_n(j)\}_j$ forms an operad $\mathcal{C}_n$.

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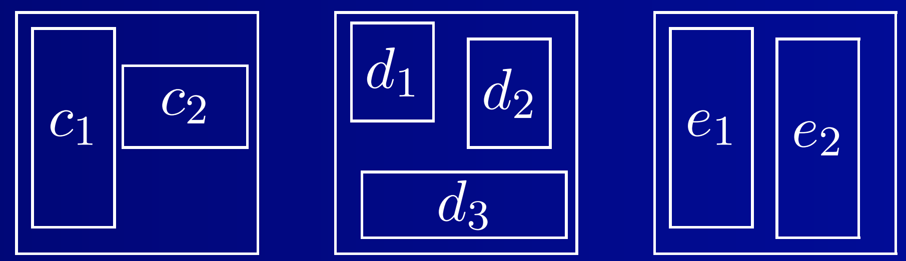
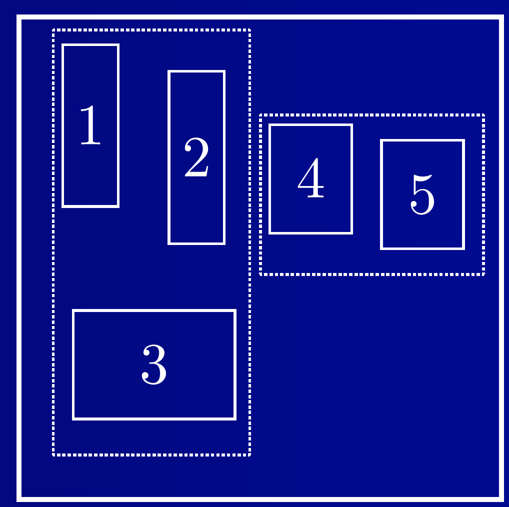


$$\in \mathcal{C}_2(2) \times \mathcal{C}_2(3) \times \mathcal{C}_2(2)$$

$\longmapsto$



$\in \mathcal{C}_2(5).$

Little Cubes and Iterated Loop Spaces

- (May's Recognition Principle. 1972) Y has a weak homotopy type of n-fold loop space if and only if Y admits an action of $\mathcal{C}_n$

$$\mathcal{C}_n(j) \times Y^j \longrightarrow Y.$$

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- (May's Approximation Theorem. 1972) When X is "well-pointed" and path-connected, we have a weak equivalence

$$\Omega^n \Sigma^n X \underset{w}{\simeq} C_n(X),$$

where $\Sigma^n X$ is the n-fold suspension of X and

$$C_n(X) = \left(\coprod_j \mathcal{C}_n(j) \times_{\Sigma_j} X^j \right) \Big/ _{\sim}.$$

Little Cubes and Iterated Loop Spaces

- In the construction of $C_n(X)$ we did not use the operad structure of $\mathcal{C}_n$.
  We only used

$$(c_1, \cdots, c_n) \in \mathcal{C}_n(j) \longmapsto (c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n) \in \mathcal{C}_n(j-1).$$

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- Define

$$F(\mathbb{R}^n, j) = \left\{ (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_j) \in (\mathbb{R}^n)^j \;\middle|\; \boldsymbol{x}_i \neq \boldsymbol{x}_k \text{ if } i \neq k \right\}.$$

  This is called the configuration space of j points in $\mathbb{R}^n$.

Configuration Spaces and Iterated Loop Spaces

- We have a Σ_j-equivariant homotopy equivalence

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- The collection $\{F(\mathbb{R}^n, j)\}_j$ is closed under the "removing points" operation.
- (Segal, 1973) We have a weak homotopy equivalence

$$\Omega^n \Sigma^n X \underset{w}{\simeq} \left(\coprod_j F(\mathbb{R}^n, j) \times_{\Sigma_j} X^j \right) \Big/ \sim$$

for a well-pointed path-connected space X.

Homology of Iterated Loop Spaces

Problem. Use $\mathcal{C}_n(j)$ or $F(\mathbb{R}^n, j)$ to study the homology of $\Omega^n\Sigma^n X$.

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- Several methods are known for studying the homology of iterated loop spaces.
  - The Serre spectral sequence.
  - Homology operations.
  - The Eilenberg-Moore spectral sequence.
  - . . .

The Serre Spectral Sequence

- We have a fibration

$$\Omega^n\Sigma^n X \longrightarrow P\Omega^{n-1}\Sigma^n X \longrightarrow \Omega^{n-1}\Sigma^n X.$$

where $PY = \{\ell : I \to Y \mid \ell(0) = y_0\} \simeq *$.

- And we obtain a spectral sequence

$$E^2_{s,t} \cong H_s(\Omega^{n-1}\Sigma^n X; h_t(\Omega^n\Sigma^n X)) \Longrightarrow h_{s+t}(P\Omega^{n-1}\Sigma^n X) \cong h_{s+t}(*)$$

for a homology theory $h_*(-)$.

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- When $h_*(-)$ is not an ordinary homology (e.g. K-theory), this method fails.

The Eilenberg-Moore Spectral Sequence

- For a multiplicative homology theory $h_*(-)$ satisfying the Künneth isomorphism, we have a spectral sequence

$$E^2_{s,t} \cong \mathrm{Cotor}^{h_*(Y)}_{s,t}(h_*(*), h_*(*)) \Longrightarrow h_{s+t}(\Omega Y),$$

  which may or may not converge.
- When h_* is a nonconnective homology theory, such as K-theory, the spectral sequence behaves badly.

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  Does this converge?

My Thesis Problem

Problem: Filter $\mathcal{C}_n(j)$ or $F(\mathbb{R}^n, j)$ to study the convergence of the Eilenberg-Moore spectral sequence

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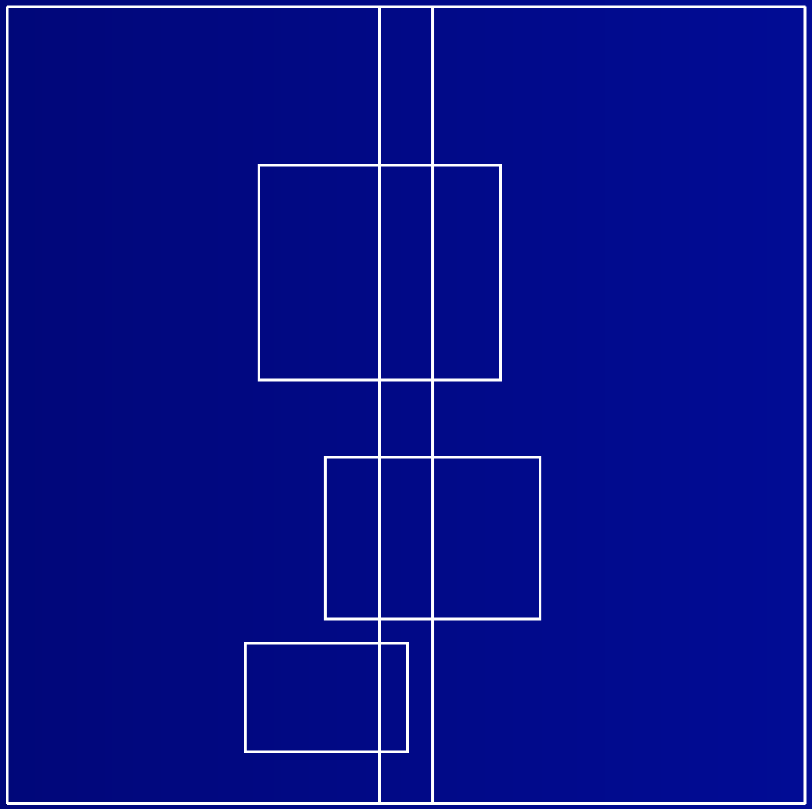
- Corresponding fi ltartion on $\mathcal{C}_n(j)$?

The Idea of Gravity Filtration

Idea: Filter $\mathcal{C}_n(j)$ by measuring the overlaps in the first coordinates.



The Gravity Filtration

- There exists a filtration on $\mathcal{C}_n(j)$

$$\emptyset = F_{-j-1}\mathcal{C}_n(j) \subset F_{-j}\mathcal{C}_n(j) \subset \cdots \subset F_{-2}\mathcal{C}_n(j) \subset F_{-1}\mathcal{C}_n(j) = F_0\mathcal{C}_n(j) = \mathcal{C}_n(j).$$

satisfying the following conditions:

— $\boldsymbol{c} = (c_1, \cdots, c_j) \in F_{-s}\mathcal{C}_n(j) \Longleftrightarrow$ there exists

$$\{1, \cdots, j\} = S_1 \amalg \cdots \amalg S_{s+k} \ (k \geq 0)$$

such that each collection of cubes $\{c_i \mid i \in S_\ell\}$ is "stable under gravity (and anti-gravity)" with respect to the first coordinate.

— In other words, we need to divide $\{c_1, \cdots, c_j\}$ into at least s groups each of which is "stable under gravity (and anti-gravity)" with respect to the first coordinate.

The Snaith Splitting

- The filtration on $\mathcal{C}_n(j)$ is not compatible with the base point relation in the definition of

$$C_n(X) = \left(\coprod_j \mathcal{C}_n(j) \times_{\Sigma_j} X^j \right) \Big/ \sim .$$

- (Snaith, 1974.) We have a stable homotopy equivalence

$$\Sigma^\infty C_n(X) \simeq \Sigma^\infty \left(\bigvee_j \mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j} \right)$$

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hence

$$\tilde{h}_*(C_n(X)) \cong \bigoplus_j \tilde{h}_*(\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}).$$

The Gravity Spectral Sequence

- The gravitiy filtration on each $\mathcal{C}_n(j)$ defines a filtration on the j-th Snaith summand $\mathcal{C}_n(j)_+ \wedge_\Sigma X^{\wedge j}$.
- We obtain a spectral sequence

$$E^2(j) \Longrightarrow \tilde{h}_*(\mathcal{C}_n(j)_+ \wedge_\Sigma X^{\wedge j})$$

for each j.

- By taking the direct sum, we obtain a spectral sequence

$$\begin{aligned} E^2 = \bigoplus_j E^2(j) \Longrightarrow & \bigoplus_j \tilde{h}_*(\mathcal{C}_n(j)_+ \wedge_\Sigma X^{\wedge j}) \\ \cong & \ \tilde{h}_*\left(\bigvee_j \mathcal{C}_n(j)_+ \wedge_\Sigma X^{\wedge j}\right) \\ \cong & \ \tilde{h}_*(C_n(X)) \\ \cong & \ \tilde{h}_*(\Omega^n\Sigma^n X). \end{aligned}$$

The Spectral Sequence

Theorem (T., 1994). *The spectral sequence has the following properties:*

- *Each summand converges strongly.*
- *When* $h_*(-)$ *satisfies the Künneth formula,*

$$E^2 \cong \mathrm{Cotor}^{h_*(\Omega^{n-1}\Sigma^n X)}(h_*(*), h_*(*)).$$

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Braid Arrangement?

The Braid Arrangement

- We have

$$\begin{aligned} \mathcal{C}_n(j) &\simeq F(\mathbb{R}^n, j) \\ &= \mathbb{R}^j \otimes \mathbb{R}^n - \bigcup_{i,i'} L_{i,i'} \otimes \mathbb{R}^n, \end{aligned}$$

where

$$L_{i,i'} = \{(x_1, \cdots, x_j) \in \mathbb{R}^j \mid x_i = x_{i'}\}.$$

- $\{L_{i,i'} \mid 1 \leq i < i' \leq j\}$: a central arrangement in $\mathbb{R}^j$.
- $\mathfrak{h}_j = \left\{(x_1, \cdots, x_j) \in \mathbb{R}^j \;\middle|\; x_1 + \cdots + x_j = 0\right\}$.
- $\mathcal{A}_{j-1} = \{L_{i,i'} \cap \mathfrak{h}_j \mid 1 \leq i < i' \leq j\}$: the braid arrangement.
- $\mathcal{A}_{j-1}$ is a real essential central arrangement in $\mathfrak{h}_j$.

Combinatorics

Question: What does the combinatorics of the braid arrangements $\{\mathcal{A}_{j-1}\}_{j\geq 1}$ tell us about the homology of $\Omega^n\Sigma^n X$?

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Question: Can we construct the "gravity spectral sequence" in terms of the combinatorics of the braid arrangements $\{\mathcal{A}_{j-1}\}_{j\geq 1}$?

The Salvetti Complex of the Braid Arrangement

Real Hyperplane Arrangements

- $\mathcal{V}$: a finite collection of nonzero vectors in a real inner product space V.
- $\mathcal{A}$: the associated real central hyperplane arrangement in V.
- $\mathcal{L}(\mathcal{A})$: the face poset.
- $\mathcal{L}^{(0)}(\mathcal{A})$: the set of chambers.
- (Gel′fand-Rybnikov, 1989)

$$\mathcal{L}(\mathcal{A}) \subset \mathrm{Map}(\mathcal{V}, S_1)$$

  as posets, where $S_1 = \{0 < +1, -1\}$.
- Matroid product

$$\circ : \mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{A}) \longrightarrow \mathcal{L}(\mathcal{A}).$$

The Face Lattice of the Braid Arrangement

- Define the set of ordered partition of $\{1, \cdots, j\}$ by

$$\Pi_j = \coprod_{s=1}^{j} \{\lambda : \{1, \cdots, j\} \to \{1, \cdots, s\} \mid \text{surjections}\}.$$

- For $\lambda \in \Pi_j$, define

$$F_\lambda = \left\{ (x_1, \cdots, x_j) \in \mathbb{R}^j \;\middle|\; \begin{array}{c} x_i < x_{i'} \text{ if } \lambda(i) < \lambda(i') \\ \text{and } x_i = x_{i'} \text{ if } \lambda(i) = \lambda(i') \end{array} \right\}$$

  Then

$$\mathcal{L}(\mathcal{A}_{j-1}) = \{F_\lambda \mid \lambda \in \Pi_j\}.$$

- Chambers correspond to bijections, hence

$$\mathcal{L}^{(0)}(\mathcal{A}_{j-1}) = \Sigma_j.$$

Complexification

- $S_2 = \{0, e_1, -e_1, e_2, -e_2\}$ with $0 < \pm e_1 < \pm e_2$.
- Define $i_1, i_2 : S_1 \hookrightarrow S_2$ by

$$\begin{aligned} i_1(0) = i_2(0) &= 0 \\ i_1(\pm 1) &= \pm e_1 \\ i_2(\pm 1) &= \pm e_2. \end{aligned}$$

- $$\mathrm{Map}(E, S_1) \underset{(i_2)_*}{\overset{(i_1)_*}{\rightrightarrows}} \mathrm{Map}(E, S_2)$$
- For $L \subset \mathrm{Map}(E, S_1)$,

$$L \otimes \mathbb{C} = \{(i_1)_*(F) \circ (i_2)_*(G) \mid F, G \in L\}.$$

Complements

Definition. For a real central arrangement $\mathcal{A}$ in V, define

$$\mathcal{L}^{(1)}(\mathcal{A}) = \{X \in \mathcal{L}(\mathcal{A}) \otimes \mathbb{C} \mid X(v) \neq 0 \text{ for all } v \in \mathcal{V}\}.$$

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Theorem (Salvetti (1987), Björner-Ziegler (1992))**.**

$$B\mathcal{L}^{(1)}(\mathcal{A}) \simeq V \otimes \mathbb{C} - \bigcup_{L \in \mathcal{A}} L \otimes \mathbb{C},$$

where

$$B : \textbf{Posets} \hookrightarrow \textbf{Small Categories} \xrightarrow{B} \textbf{Spaces}$$

is the classifying space functor (order complex).

The Salvetti Complex

Theorem (Salvetti, 1987). *The simplicial complex* $B\mathcal{L}^{(1)}(\mathcal{A})$ *has a structure of regular cell complex having cells* $D(F, C)$ *in one-to-one correspondence with pairs* (F, C) *of a chamber* C *and a face* $F \leq C$.

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- We denote this cell complex by

$$\mathrm{Sal}(\mathcal{A}) = \bigcup_{(F,C)} D(F,C).$$

- $\mathrm{Sal}(\mathcal{A})$ is called the Salvetti complex of $\mathcal{A}$.

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Double loop spaces and the Salvetti complex of the braid Arrangement?

Iterated Loop Spaces

- We have

$$\begin{aligned}\Omega^2\Sigma^2 X &\underset{w}{\simeq} C_2(X) \\ &= \left(\coprod_j \mathcal{C}_2(j) \times_{\Sigma_j} X^j\right)\Big/_{\sim} \\ &\simeq \left(\coprod_j F(\mathbb{C}, j) \times_{\Sigma_j} X^j\right)\Big/_{\sim}.\end{aligned}$$

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- The confi guration space $F(\mathbb{C}, j)$ is the complement of the complexifi cation of the braid arrangement $\mathcal{A}_{j-1}$.
- We have a Σ_j-equivariant homotopy equivalence

$$F(\mathbb{C}, j) \simeq_{\Sigma_j} \mathrm{Sal}(\mathcal{A}_{j-1}).$$

Iterated Loop Spaces

- The Snaith splitting

$$\Sigma^\infty C_2(X) \simeq \Sigma^\infty \bigvee_j \mathcal{C}_2(j)_+ \wedge_{\Sigma_j} X^{\wedge j}$$

implies

$$\Sigma^\infty C_2(X) \simeq \Sigma^\infty \bigvee_j \mathrm{Sal}(\mathcal{A}_{j-1})_+ \wedge_{\Sigma_j} X^{\wedge j}.$$

Iterated Loop Spaces

- The Snaith splitting

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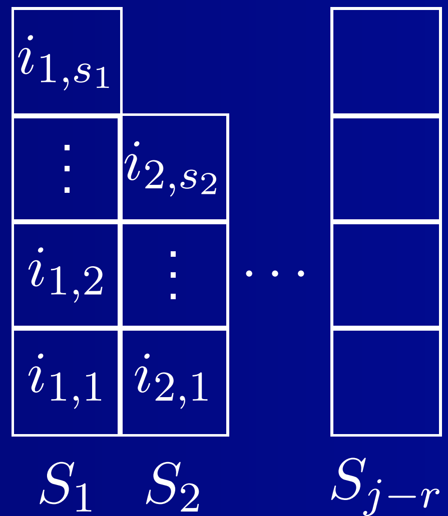
Question: Is there a filtration on $\mathrm{Sal}(\mathcal{A}_{j-1})$ corresponding to the gravity filtartion on $\mathcal{C}_2(j)$?

Answer: The skeletal filtration defined by Salvetti.

Cells of the Salvetti Complex of the Braid Arrangement

For an ordered partition $\lambda \in \Pi_j$ of rank r and $\sigma \in \Sigma_j$ which is a subdivision of λ, define a symbol $S(\lambda, \sigma)$ as follows:



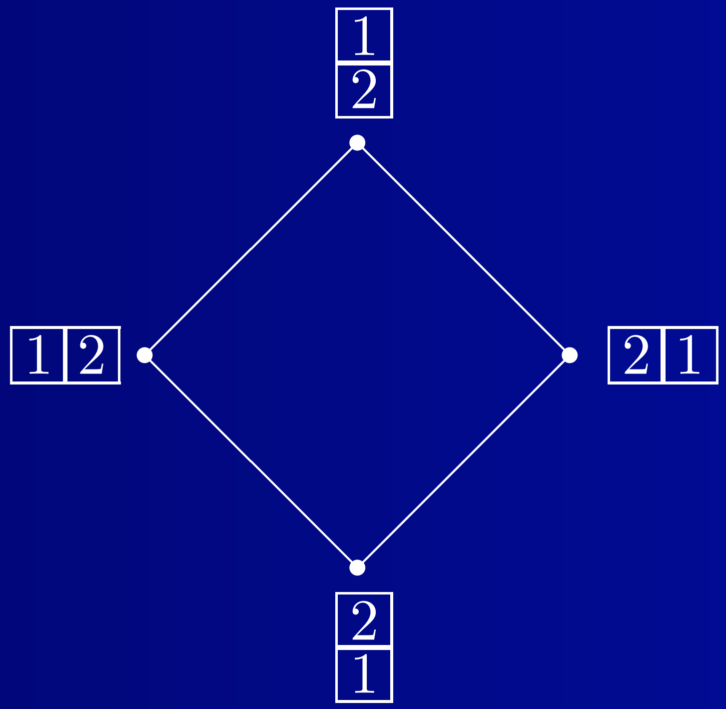
where

- for each $1 \leq i \leq j-r$, S_i is a vertically stacked squares of length $|\lambda^{-1}(i)|$, and
- labels in squares in S_i are given by $\lambda^{-1}(i)$ ordered by σ from the bottom to the top.

Example

$\mathcal{A}_1 = \{\{0\}\}$: an arrangement in $\mathfrak{h}_1 \cong \mathbb{R}$.

$\mathrm{Sal}(\mathcal{A}_1) =$



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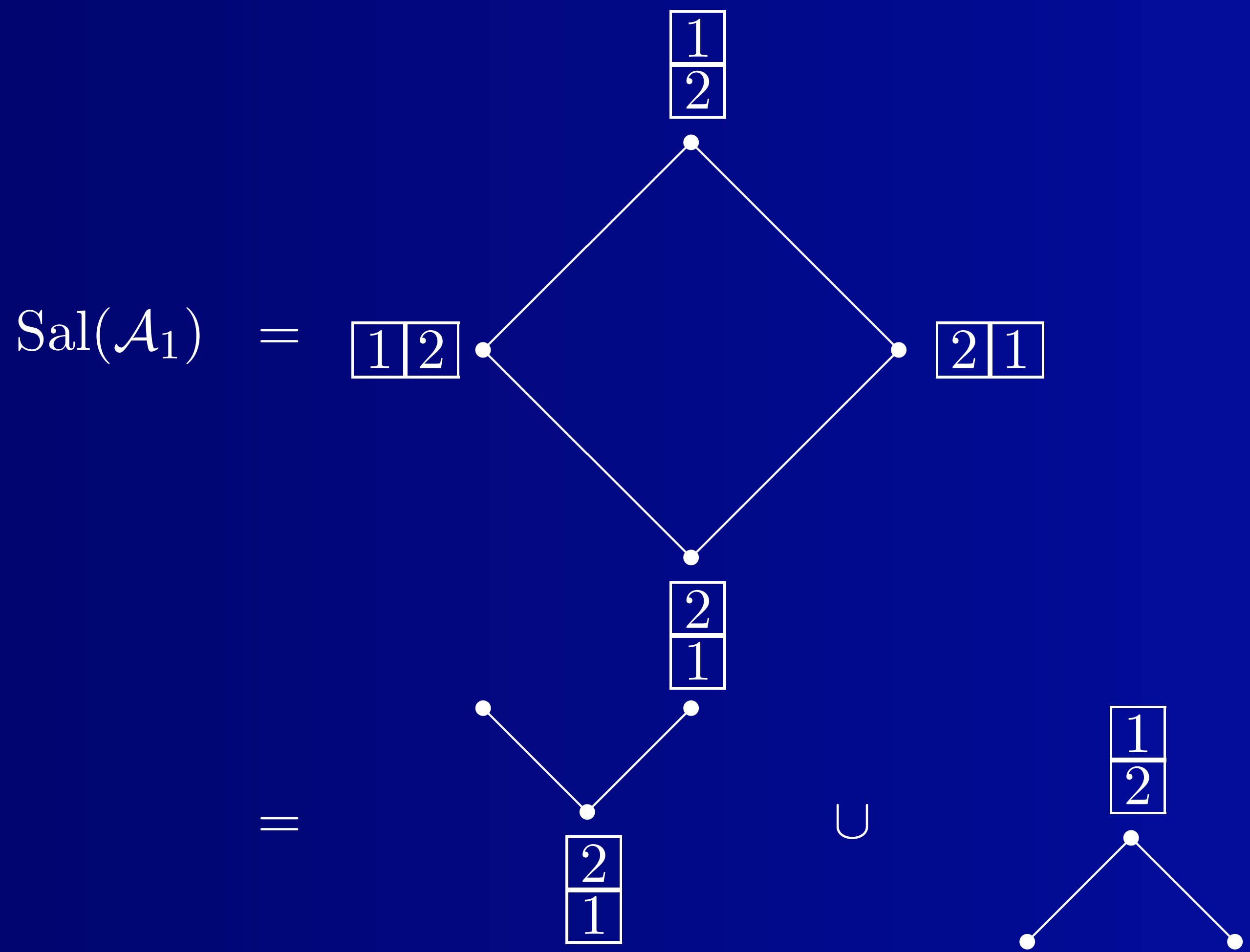

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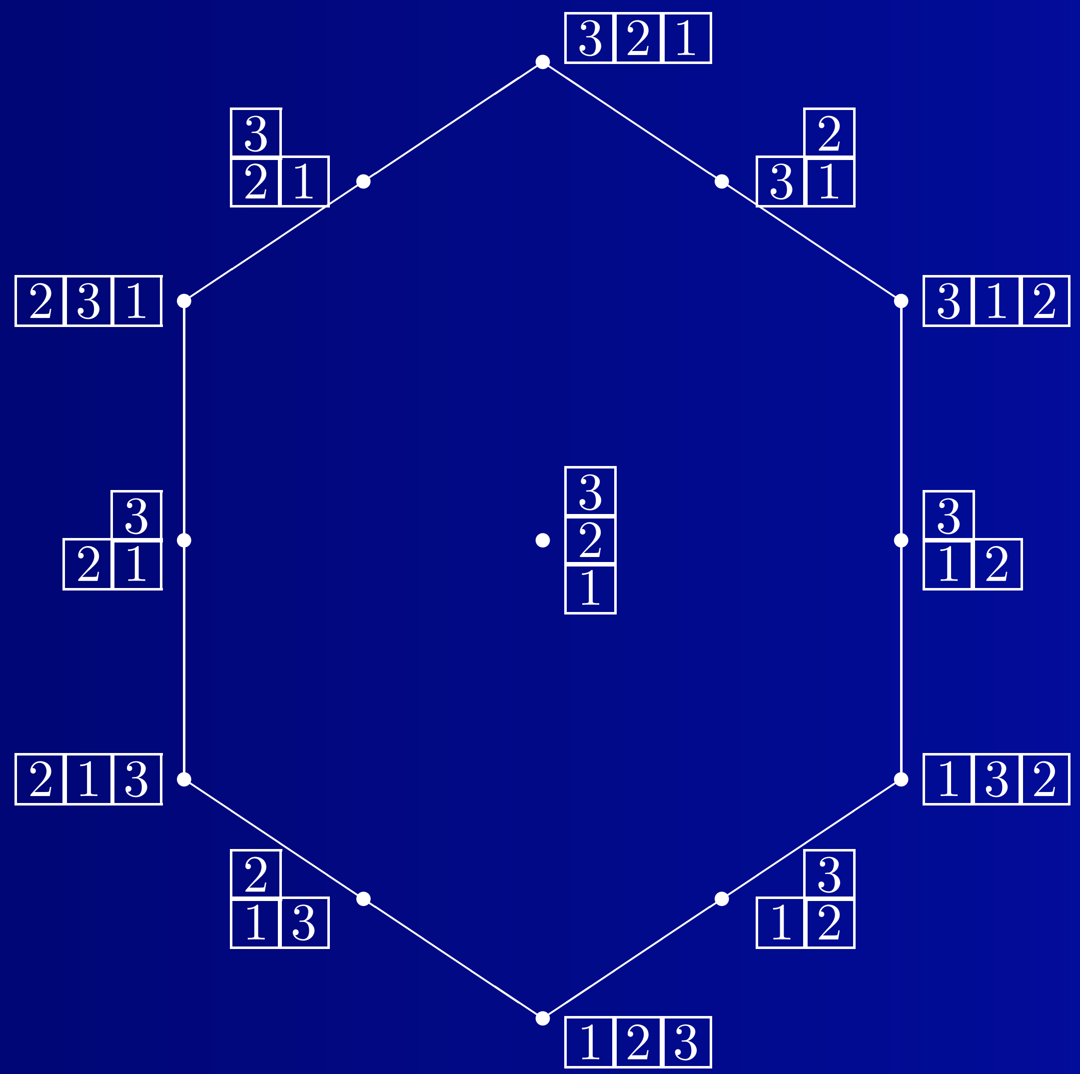

The Eilenberg-Moore Spectral Sequence

- The picture in the previous slide looks like an element in $\mathcal{C}_2(j)$.

The Eilenberg-Moore Spectral Sequence

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Theorem (T., math/0602085). *1. By mapping $S(\lambda, \sigma)$ to the corresponding "picture" in $\mathcal{C}_2(j)$, we obtain a filtration preserving Σ_j-equivariant map*

$$\mathrm{Sal}(\mathcal{A}_{j-1}) \hookrightarrow \mathcal{C}_2(j).$$

2. These maps induce an isomorphism of the associated spectral sequences.

The Eilenberg-Moore Spectral Sequence

Corollary. *The skeletal filtration on* $\{\mathrm{Sal}(\mathcal{A}_{j-1})\}_{j\geq 1}$ *induces the Eilenberg-Moore spectral sequence*

$$E^2 \cong \mathrm{Cotor}^{h_*(\Omega\Sigma^2 X)}(h_*, h_*) \Longrightarrow h_*(\Omega^2\Sigma^2 X).$$

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$$E^2 \cong \mathrm{Cotor}^{h_*(\Omega\Sigma^2 X)}(h_*, h_*) \Longrightarrow h_*(\Omega^2\Sigma^2 X).$$

Corollary. *The* E^1*-term can be described as*

$$E^1_{-s,*} \cong \bigoplus_j C_{j-s}(\mathrm{Sal}(\mathcal{A}_{j-1})) \otimes_{\Sigma_j} \tilde{h}_*(\Sigma X)^{\otimes j}$$
$$\cong \bigoplus_j h_*(*)\langle S(\lambda, (1|\cdots|j)) \mid \lambda \in O_{j,j-s}\rangle \otimes \tilde{h}_*(\Sigma X)^{\otimes j}$$

where

$$O_{j,j-s} = \{\lambda \in \Pi_j \mid \lambda : \{1, \cdots .j\} \to \{1, \cdots, s\} \text{order preserving}\}.$$

Salvetti Cells

Consider the cell

$$\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \in C_1(\mathrm{Sal}(\mathcal{A}_1)).$$

We have

$$\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \otimes x_{2n-1}^2 \in C_1(\mathrm{Sal}(\mathcal{A}_1)) \otimes_{\Sigma_2} \widetilde{H}_*(S^{2n-1};\mathbb{F}_2)^{\otimes 2} \in E^1_{-1,*}.$$

The d^1-differential is

$$\begin{aligned} d^1(\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \otimes x_{2n-1}^2) &= \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \otimes x_{2n-1}^2 + \begin{array}{|c|c|}\hline 2&1\\\hline\end{array} \otimes x_{2n-1}^2 \\ &= \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \otimes x_{2n-1}^2 + \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \otimes x_{2n-1}^2 \\ &= 2\begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \otimes x_{2n-1}^2 \\ &= 0. \end{aligned}$$

Classical Computation with EMSS

- Consider

$$E^2 = \mathrm{Cotor}^{H_*(\Omega S^{2n+1};\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) \Longrightarrow H_*(\Omega^2 S^{2n+1};\mathbb{F}_2).$$

- $H_*(\Omega S^{2n+1};\mathbb{F}_2) \cong \mathbb{F}_2[x_{2n}]$ as primitively generated Hopf algebras.
- $\mathrm{Cotor}^{\mathbb{F}_2[x_{2n}]}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathrm{Ext}^{(\mathbb{F}_2[x_{2n}])^*}(\mathbb{F}_2, F_2)$.
- As algebras

$$(\mathbb{F}_2[x_{2n}])^* \cong \bigotimes_{a\geq 0} \mathbb{F}_2[(x_{2n}^{2^a})^*]/(((x_{2n}^{2^a})^*)^2).$$

- $\mathrm{Ext}^{\mathbb{F}_2[y]/(y^2)}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathbb{F}_2[\tau(y)]$. ($\deg \tau(y) = \deg y - 1$.)
- $E^2 \cong \mathbb{F}_2\left[\tau((x_{2n}^{2^a})^*) \mid a \geq 0\right] \cong E^\infty \cong H_*(\Omega^2 S^{2n+1};\mathbb{F}_2)$ as algebras.

Salvetti Cells

- We denote

$$Q_1^a(x_{2n-1}) = \tau((x_{2n}^{2^a})^*).$$

- $Q_1^a(x_{2n-1}) \in E^2_{-1,*}$.
-

$$Q_1(x_{2n-1}) \longleftrightarrow \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \otimes x_{2n-1}^2.$$

-

$$Q_1^2(x_{2n-1}) \longleftrightarrow \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \otimes x_{2n-1}^4.$$

Higher Dimensions

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Theorem (Björner-Ziegler (1992), De Concini-Salvetti (2000))**.**

$$B\mathcal{L}^{(n-1)}(\mathcal{A}) \simeq V \otimes \mathbb{R}^n - \bigcup_{L \in \mathcal{A}} L \otimes \mathbb{R}^n.$$

Cellular Structure

- For real central arrangement $\mathcal{A}$, $\mathcal{L}^{(n-1)}(\mathcal{A})$ is a fi nite poset.
- $B\mathcal{L}^{(n-1)}(\mathcal{A})$ is a fi nite simplicial complex.
- There are too many simplices in $B\mathcal{L}^{(n-1)}(\mathcal{A})$.

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Proposition (De Concini-Salvetti, 2000). *There exists a structure of regular cell complex on $B\mathcal{L}^{(n-1)}(\mathcal{A})$ whose cells are labelled by sequences of faces $(C, F_1, \cdots, F_{n-1})$ where C is a chamber, F_1 is a face of C, F_2 is a face of F_1, and so on.*

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- $B\mathcal{L}^{(n-1)}(\mathcal{A})$ equipped with the above cell structure is denoted by $\mathrm{Sal}^{(n-1)}(\mathcal{A})$.

Spectral Sequence

Skeletal filtration on $\mathrm{Sal}^{(n-1)}(\mathcal{A}_{j-1})$

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$\rightsquigarrow$ Stable filtration on $\Omega^n\Sigma^n X$.

Theorem. *For any homology theory $h_*(-)$, we obtain a strongly convergent spectral sequence*

$$E^1 \cong \bigoplus_j C_*\left(\mathrm{Sal}^{(n-1)}(\mathcal{A}_{j-1})\right) \otimes_{\Sigma_j} \tilde{h}_*\left(X^{\wedge j}\right) \Longrightarrow h_*\left(\Omega^n\Sigma^n X\right).$$

Spectral Sequence

- When $h_*(-)$ is multiplicative and satisfi es the strong form of the Künneth formula, the E^1-term is a functor of $\tilde{h}_*(X)$,

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The oriented matroid of $\mathcal{A}_{j-1} \Longrightarrow$ Homology of $\Omega^n\Sigma^n X$

Alternative Constructions

- Analogous cell complexes have been discovered independently.
  - Milgram's model for $\Omega^n \Sigma^n X$ by using the permutohedra.
  - Free iterated monoidal categories by Balteanu, Fiedorowicz, Schwänzl, and Vogt.
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  - $\cdots$
- Relations?

Milgram's Model

- Milgram constructed a cell complex $J_n(j)$ by gluing permutohedra

$$P_n = \mathrm{Conv}(\{(\sigma(1), \cdots, \sigma(n)) \mid \sigma \in \Sigma_n\}) \subset \mathbb{R}^n.$$

- He also proved

$$\Omega^n \Sigma^n X \simeq \left(\coprod_j J_n(j) \times_{\Sigma_j} X^j \right) \Big/ \sim .$$

Iterated Monoidal Category

- Balteanu, Fiedorowicz, Schwänzl, and Vogt introduced then notion of n-fold monoidal category in their study of iterated loop spaces.
- For any set S (or a small category), they defi ned a free n-fold monoidal category $\mathcal{F}_n(S)$ generated by S.

$$(i_1 \Box_2 i_2) \Box_1 (i_3 \Box_2 i_4 \Box_2 i_5)$$

- Let $\mathcal{M}_n(j)$ be the full subcategory of $\mathcal{F}_n(\{1, \cdots, j\})$ consisting of objects in which each i appears exactly once.
- Let $\mathcal{J}_n(j)$ be the full subcategory of $\mathcal{M}_n(j)$ consisting of objects in which $\Box_1$ appears in the outer most level, $\Box_2$ appears in the next level, and so on.

Relatioins

- $\mathcal{J}_n(j)$ is a poset and is isomorphic to $\mathcal{L}^{(n-1)}(\mathcal{A}_{j-1})$ as posets. Thus
$$B\mathcal{J}_n(j) = \mathrm{Sal}^{(n-1)}(\mathcal{A}_{j-1}).$$
- When $n = 2$, we have a homeomorphism
$$J_2(j) \cong B\mathcal{J}_2(j).$$
- When $n > 2$, $J_n(j)$ and $B\mathcal{J}_n(j)$ are different but there exists a cellular surjection
$$J_n(j) \longrightarrow B\mathcal{J}_n(j)$$
which is a homotopy equivalence.

Other Arrangements?

The "Center-Of-Mass" Arrangement $\mathcal{C}^t_{k-1}$

- $I, J \subset \{1, \cdots, k\}$.
- $L_{I,J} = \left\{(x_1, \cdots, x_k) \in \mathbb{R}^k \;\middle|\; |J| \sum_{i\in I} x_i = |I| \sum_{j \in J} x_j\right\}$.
- $\mathcal{C}^t_{k-1} = \{L_{I,J} \mid I, J \subset \{1, \cdots, k\}, |I| = |J| = t, I \neq J\}$.
- $\mathcal{C}^t_{k-1} = \{L_{I,J} \mid I, J \subset \{1, \cdots, k\}, |I| = |J| \leq t, I \cap J = \emptyset\}$
- We have the following inclusions

$$\mathcal{A}_{k-1} = \mathcal{C}^1_{k-1} \subset \mathcal{C}^2_{k-1} \subset \cdots \subset \mathcal{C}^{[\frac{k}{2}]}_{k-1} \supset \cdots \supset \mathcal{C}^{k-1}_{k-1} = \mathcal{A}_{k-1}.$$

- $M_t(\mathbb{C}, k) = \mathbb{C}^k - \bigcup_{L_{I,J} \in \mathcal{C}^t_{k-1}} L_{I,J} \otimes \mathbb{C}$.

Cohen-Kamiyama Conjecture

Conjecture (F. Cohen-Kamiyama, math/0611732). *Let p be an odd prime. The inclusion of arrangements*

$$i^p_k : \mathcal{A}_{k-1} \hookrightarrow \mathcal{C}^p_{k-1}$$

induces an isomorphism

$$i^p_k : H_*(M_p(\mathbb{C}, k)/\Sigma_k; \mathbb{F}_p(\pm 1)) \xrightarrow{\cong} H_*(F(\mathbb{C}, k)/\Sigma_k; \mathbb{F}_p(\pm 1)).$$

- If the conjecture were to be true, $P^{2np+1}(p) = S^{2np} \cup_p e^{2np+1}$ retracts off from $\Sigma^2\Omega^2 S^{2n+1}_{(p)}$ for an odd prime p.
- This is the main conjecture in Gray's program (Trans. A.M.S. 1993) of unstable homotopy theory of Moore spectra à la Cohen-Moore-Neisendorfer.

Experimental Calculations: 2-primary case

- Cohen and Kamiyama proved that

$$i_4^2 : H_*(M_2(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2) \longrightarrow H_*(F(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2)$$

is not an isomorphism, by using homology and cohomology operations.

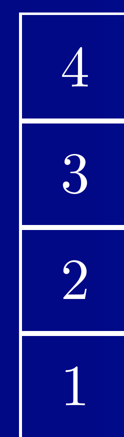
- We can directly compute both homology groups by using the Salvetti complex.

Experimental Calculations: 2-primary case

Proposition.

$$H_3(F(\mathbb{C},4)/\Sigma_4;\mathbb{F}_2) \cong \mathbb{F}_2.$$

The generator corresponding to $Q_1^2(x) \in H_7(\Omega^2 S^3;\mathbb{F}_2)$ *is denoted by*



Experimental Calculations: 2-primary case

Proposition. $C_3(M_2(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ *generated by*

$$\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}^{\,+}, \quad \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}^{\,-}.$$

Proposition. $H_3(M_2(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2) \cong \mathbb{F}_2$ *generated by*

$$\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}^{\,+} + \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}^{\,-}$$

Experimental Calculations: 2-primary case

The inclusion map $M_2(\mathbb{C}, 4) \hookrightarrow F(\mathbb{C}, 4)$ induces

$$\begin{array}{|c|}\hline 4\\ \hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array}^{+} + \begin{array}{|c|}\hline 4\\ \hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array}^{-} \longmapsto \begin{array}{|c|}\hline 4\\ \hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array} + \begin{array}{|c|}\hline 4\\ \hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array} = 0.$$

Thus

$$i_4^2 : H_3(M_2(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2) \longrightarrow H_3(F(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2)$$

is not an isomorphism.

Experimental Calculations: odd primary case

- $M_p(\mathbb{C}, k) \hookrightarrow F(\mathbb{C}, k)$?
- We have the following inclusions

$$\mathcal{A}_{k-1} = \mathcal{C}^1_{k-1} \subset \mathcal{C}^2_{k-1} \subset \cdots \subset \mathcal{C}^{[\frac{k}{2}]}_{k-1} \supset \cdots \supset \mathcal{C}^{k-1}_{k-1} = \mathcal{A}_{k-1}.$$

- If $p < \frac{k}{2}$,

$$\mathcal{A}_{k-1} = \mathcal{C}^1_{k-1} \subset \mathcal{C}^2_{k-1} \subset \cdots \subset \mathcal{C}^p_{k-1} \subset \cdots \subset \mathcal{C}^{[\frac{k}{2}]}_{k-1}.$$

- If $k > p > \frac{k}{2}$, $\mathcal{C}^p_{k-1} = \mathcal{C}^{k-p}_{k-1}$

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- If $k > p > \frac{k}{2}$, $\mathcal{C}^p_{k-1} = \mathcal{C}^{k-p}_{k-1}$ and we have

$$\mathcal{A}_{k-1} = \mathcal{C}^1_{k-1} \subset \mathcal{C}^2_{k-1} \subset \cdots \subset \mathcal{C}^{k-p}_{k-1} \subset \cdots \subset \mathcal{C}^{[\frac{k}{2}]}_{k-1}.$$

Experimental Calculations: $p = 3$ and $k = 4$

- If $p = 3$ and $k = 4$,

$$\mathcal{A}_3 = \mathcal{C}^1_3 \subset \mathcal{C}^2_3 \supset \mathcal{C}^3_3 (= \mathcal{A}_3)$$

- The map

$$\begin{aligned} H_*(M_2(\mathbb{C},4)/\Sigma_4;\mathbb{F}_3(\pm 1)) &\cong H_*(\mathrm{Sal}(\mathcal{C}^2_3)/\Sigma_4;\mathbb{F}_3(\pm 1)) \\ &\longrightarrow H_*(\mathrm{Sal}(\mathcal{C}^1_3)/\Sigma_4;\mathbb{F}_3(\pm 1)) \\ &\cong H_*(F(\mathbb{C},4)/\Sigma_4;\mathbb{F}_3(\pm 1)) \end{aligned}$$

is not an isomorphism.
- $\mathcal{C}^2_3$ doesn't behave very well.
- But we don't need $\mathcal{C}^2_3$ in order to go from $\mathcal{C}^3_3$ to $\mathcal{C}^1_3$.

Experimental Calculations: $p = 3$ and $k = 4$

Theorem.

$$H_i(\mathrm{Sal}(\mathcal{C}_3^2)/\Sigma_4; \mathbb{F}_3(\pm 3)) \cong \begin{cases} 0, & i = 0 \\ \mathbb{F}_3, & i = 1 \\ \mathbb{F}_3 \oplus \mathbb{F}_3, & i = 2 \\ \mathbb{F}_3, & i = 3. \end{cases}$$

Note that

$$H_i(\mathrm{Sal}(\mathcal{C}_3^1)/\Sigma_4; \mathbb{F}_3(\pm 3)) \cong \begin{cases} 0, & i = 0 \\ \mathbb{F}_3, & i = 1 \; (\beta Q_1(x)x) \\ \mathbb{F}_3, & i = 2 \; (Q_1(x)x) \\ 0, & i = 3. \end{cases}$$

Experimental Calculations: $p \geq 5$ and $k = 4$

Theorem. *If $p \geq 5$, the map*

$$\begin{aligned} H_*(M_2(\mathbb{C},4)/\Sigma_4;\mathbb{F}_p(\pm 1)) &\cong H_*(\mathrm{Sal}(\mathcal{C}^2_3)/\Sigma_4;\mathbb{F}_p(\pm 1)) \\ &\longrightarrow H_*(\mathrm{Sal}(\mathcal{C}^1_3)/\Sigma_4;\mathbb{F}_p(\pm 1)) \\ &\cong H_*(F(\mathbb{C},4)/\Sigma_4;\mathbb{F}_p(\pm 1)) \end{aligned}$$

is an isomorphism.

Experimental Calculations: odd primary case

Conjecture. *If $p < \frac{k}{2}$, the inclusions*

$$\mathcal{A}_{k-1} = \mathcal{C}^1_{k-1} \subset \mathcal{C}^2_{k-1} \subset \cdots \subset \mathcal{C}^p_{k-1}$$

induce isomorphisms of homology groups

$$\begin{aligned}
H_*(M_p(\mathbb{C},k)/\Sigma_k;\mathbb{F}_p(\pm)) &\cong H_*(\mathrm{Sal}(\mathcal{C}^p_{k-1})/\Sigma_k;\mathbb{F}_p(\pm 1)) \\
&\xrightarrow{\cong} H_*(\mathrm{Sal}(\mathcal{C}^{p-1}_{k-1})/\Sigma_k;\mathbb{F}_p(\pm 1)) \\
&\xrightarrow{\cong} \cdots \\
&\xrightarrow{\cong} H_*(\mathrm{Sal}(\mathcal{C}^1_{k-1})/\Sigma_k;\mathbb{F}_p(\pm 1)) \\
&\xrightarrow{\cong} H_*(\mathrm{Sal}(\mathcal{A}_{k-1})/\Sigma_k;\mathbb{F}_p(\pm 1)) \\
&\cong H_*(F(\mathbb{C},k);\mathbb{F}_p(\pm 1)).
\end{aligned}$$

Experimental Calculations: odd primary case

Conjecture. *If $\frac{k}{2} < p < k$, the inclusions*

$$\mathcal{A}_{k-1} = \mathcal{C}^1_{k-1} \subset \mathcal{C}^2_{k-1} \subset \cdots \subset \mathcal{C}^{k-p}_{k-1}$$

induce isomorphisms of homology groups

$$\begin{aligned} H_*(M_p(\mathbb{C},k)/\Sigma_k;\mathbb{F}_p(\pm)) &\xrightarrow{\cong} H_*(\mathrm{Sal}(\mathcal{C}^p_{k-1})/\Sigma_k;\mathbb{F}_p(\pm 1)) \\ &= H_*(\mathrm{Sal}(\mathcal{C}^{k-p}_{k-1})/\Sigma_k;\mathbb{F}_p(\pm 1)) \\ &\xrightarrow{\cong} H_*(\mathrm{Sal}(\mathcal{C}^{k-p-1}_{k-1})/\Sigma_k;\mathbb{F}_p(\pm 1)) \\ &\xrightarrow{\cong} \cdots \\ &\xrightarrow{\cong} H_*(\mathrm{Sal}(\mathcal{C}^{1}_{k-1})/\Sigma_k;\mathbb{F}_p(\pm 1)) \\ &\xrightarrow{\cong} H_*(\mathrm{Sal}(\mathcal{A}_{k-1})/\Sigma_k;\mathbb{F}_p(\pm 1)) \\ &\cong H_*(F(\mathbb{C},k);\mathbb{F}_p(\pm 1)). \end{aligned}$$