Mathematical Society of Japan (MSJ) Seasonal Institute (SI) 2009 on Arrangements of Hyperplanes

Graphical Arrangements and Braid Arrangements

-- Graph orientations and linear extensions --

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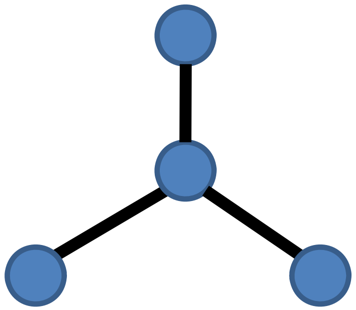
Research Institute for Mathematical Sciences (RIMS)

Kyoto University, Japan



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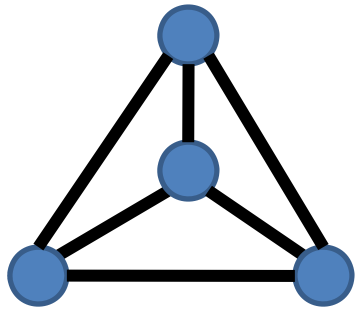

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Graphical arrangements

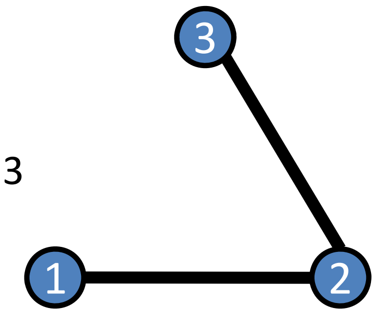
Def. $\Gamma = (V(\Gamma), E(\Gamma))$: a graph , $V(\Gamma) = [n] = \{1,2,\ldots,n\}$.

The graphical arrangement associated with Γ is:

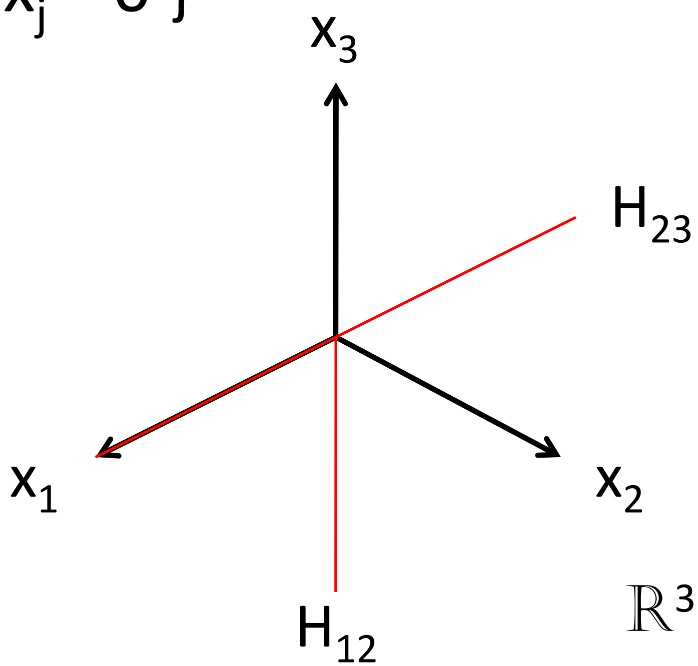
$\mathcal{A}(\Gamma) := \{ H_{ij} \mid ij \in E(\Gamma) \}$ (in $\mathbb{R}^n$)

where $H_{ij} := \{ x \in \mathbb{R}^n \mid x_i - x_j = 0 \}$

$\Gamma = P_3$



$\mathcal{A}(\Gamma)$



Braid arrangements

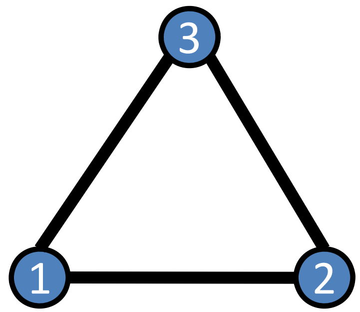
Def. The braid arrangement in $\mathbb{R}^n$ is:

$B_n := A(K_n) = \{ H_{ij} \mid 1 \leqq i < j \leqq n \}$

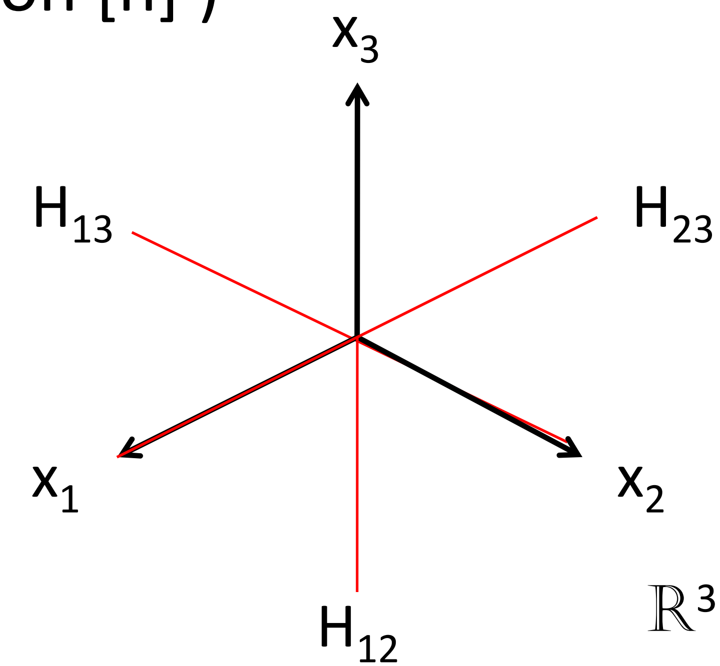
where $H_{ij} := \{ x \in \mathbb{R}^n \mid x_i - x_j = 0 \}$

(K_n : the complete graph on [n])

K_3



B_3



Γ-cones and chambers

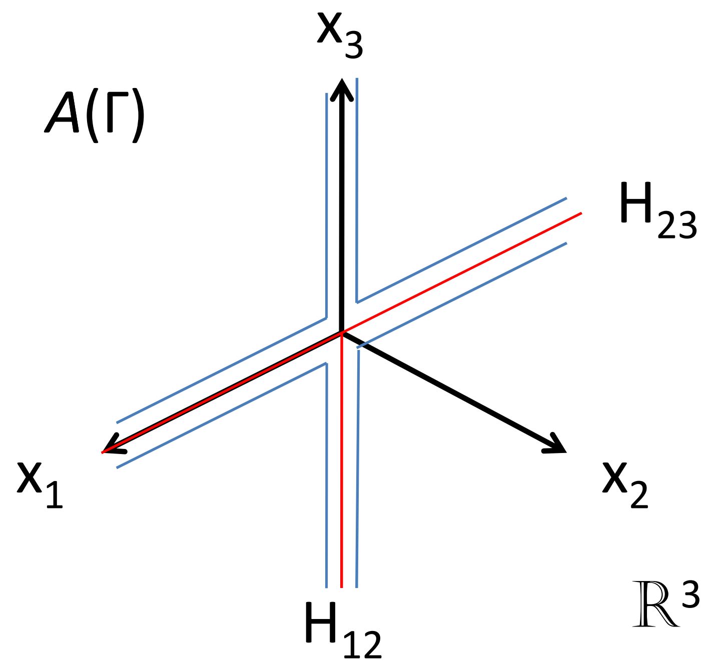
Def.

a Γ-cone

= a connected component of

$\mathbb{R}^n - \bigcup_{H \in A(\Gamma)} H$

(This terminology is due to K. Saito.)



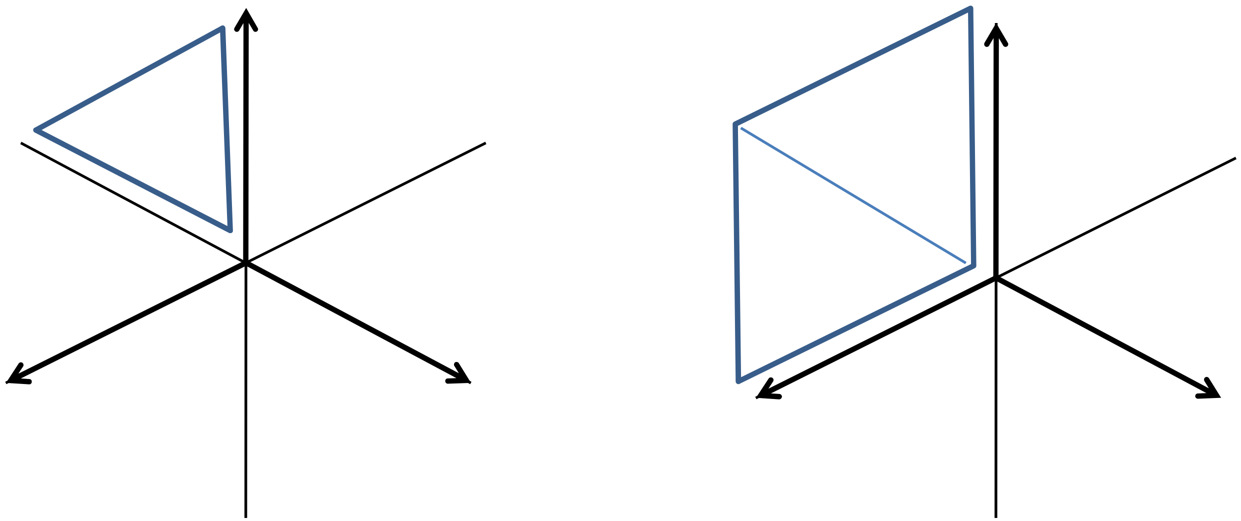
a chamber (an A_n type Weyl chamber)

= a connected component of

$\mathbb{R}^n - \bigcup_{H \in B_n} H$

Problem

- Since $A(\Gamma)$ is a subarrangement of B_n, each chamber is contained in some Γ-cone.



- Problem 1:

  How many chambers are in each Γ-cone?

Γ-cones and orientations

AO(Γ) := { all acyclic orientations of Γ }

$= \{\, o = \{\, i < j \mid ij \in E(\Gamma) \,\} \mid \text{acyclic orientation} \,\}$

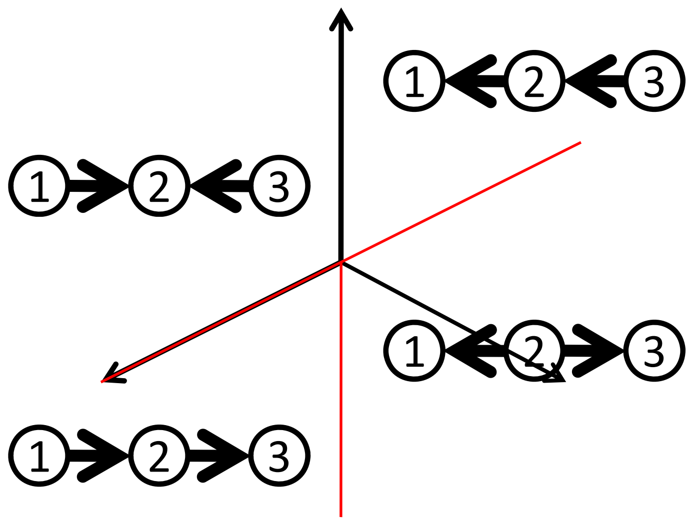
<u>Prop.</u> **AO**(Γ) → { Γ-cones }

$o = \{\, i < j \mid ij \in E(\Gamma) \,\} \rightarrow E_o := \bigcap_{ij \in E(\Gamma)} H_{ij}^{+}$

is bijective.

where $H_{ij}^{+} := \{\, x \in \mathbb{R}^n \mid x_i < x_j \,\}$

$\Gamma = P_3$



Chambers and linear orderings

Ord([n]) := { all linear orderings of [n] }

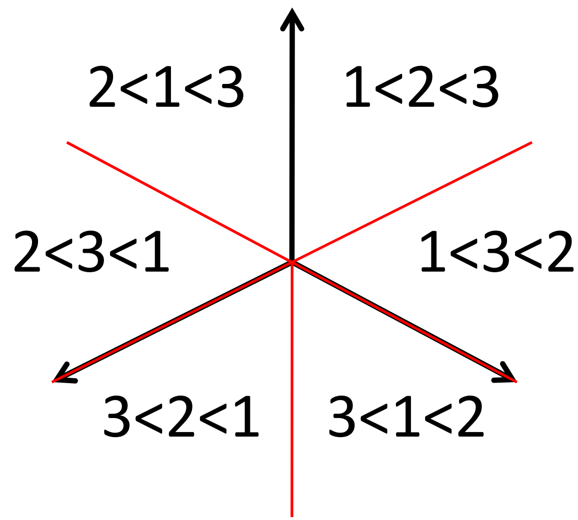
= { $c = \{ i_1 < i_2 < \ldots < i_n \}$ | linear ordering }

Prop. **Ord**([n]) → { chambers in $\mathbb{R}^n$ }

$c = \{ i_1 < i_2 < \ldots < i_n \}$ → $C_c := \bigcap_{k=1}^{n-1} H_{i_k i_{k+1}}^{+}$

is bijective.

where $H_{ij}^{+} := \{ x \in \mathbb{R}^n \mid x_i < x_j \}$

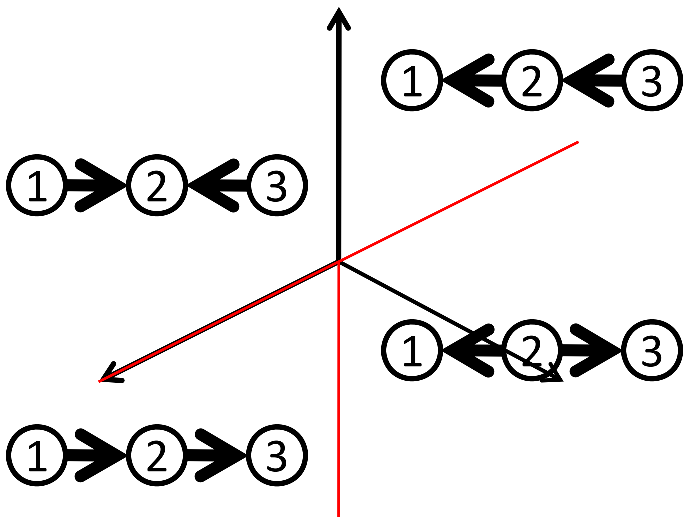


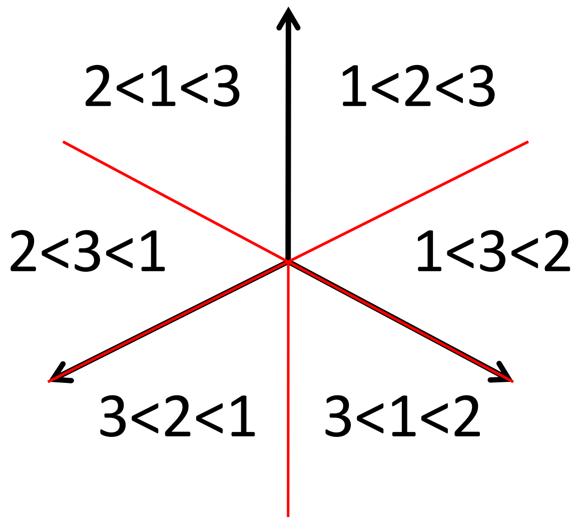
Linear extensions

- $o \in \mathbf{AO}(\Gamma) \rightarrow (V(\Gamma), <_o)$: a partially ordered set
- For $o \in \mathbf{AO}(\Gamma)$ & $c \in \mathbf{Ord}([n])$,

  the Γ-cone E_o contains the chamber C_c (i.e. $C_c \subseteq E_o$)

  ⇔ c is a linear extension of o (i.e. $c|_{E(\Gamma)} = o$)





Problem (again)

Def. For $o \in \mathbf{AO}(\Gamma)$,

$$\sigma(\Gamma,o) := \#\{\, C\text{: chamber in } \mathbb{R}^n \mid C \subseteq E_o \,\}$$
$$= \#\{\, c \in \mathbf{Ord}([n]) \mid c|_{E(\Gamma)} = o \,\}$$

- Problem 1 (again):

  Calculate $\sigma(\Gamma,o)$ for each $o \in \mathbf{AO}(\Gamma)$ *!*

  But … it seems to be difficult in general …

The principal number of Γ

Def. $\sigma(\Gamma) := \max\{ \sigma(\Gamma,o) \mid o \in \mathbf{AO}(\Gamma) \}$

the principal number of Γ

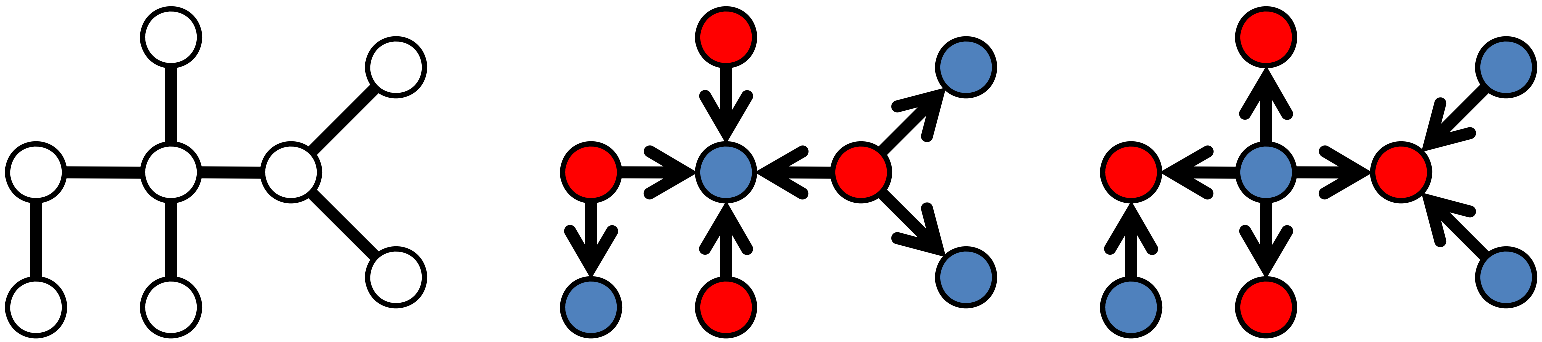
- Problem 2 (subproblem):

  (1) Find $o \in \mathbf{AO}(\Gamma)$ s.t. $\sigma(\Gamma,o) = \sigma(\Gamma)$.

  (2) Calculate $\sigma(\Gamma)$.

K. Saito's result

- Theorem [K. Saito, 2007]
  If Γ is a tree (= a conn. graph without cycle) with n>1,
  then the principal number σ(Γ) is attained by
  two "principal" orientations.



Γ $(\Gamma, o_{V,V})$ $(\Gamma, o_{V,V})$

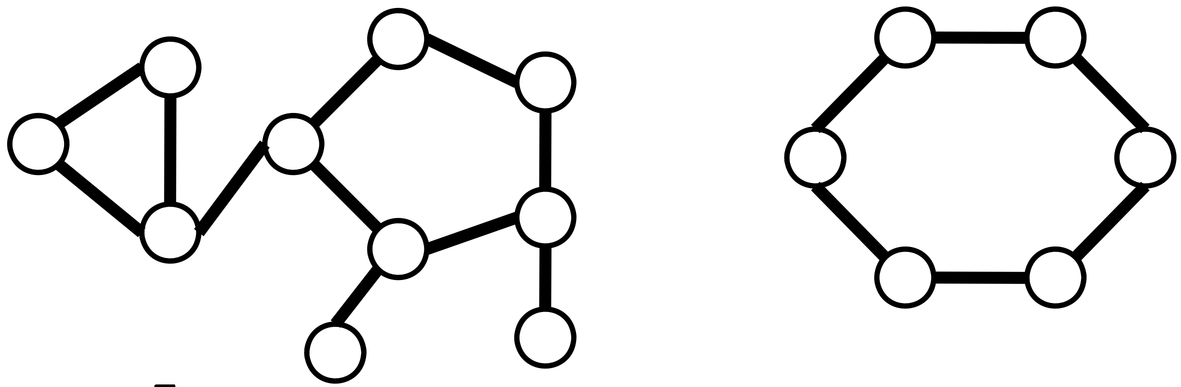
Open Question 1

- Open Question 1:

  For a graph Γ which is not a tree,

  which orientation gives the “largest” Γ-cone ?

(or, which o∈**AO**(Γ) satisfies σ(Γ,o)=σ(Γ)?)



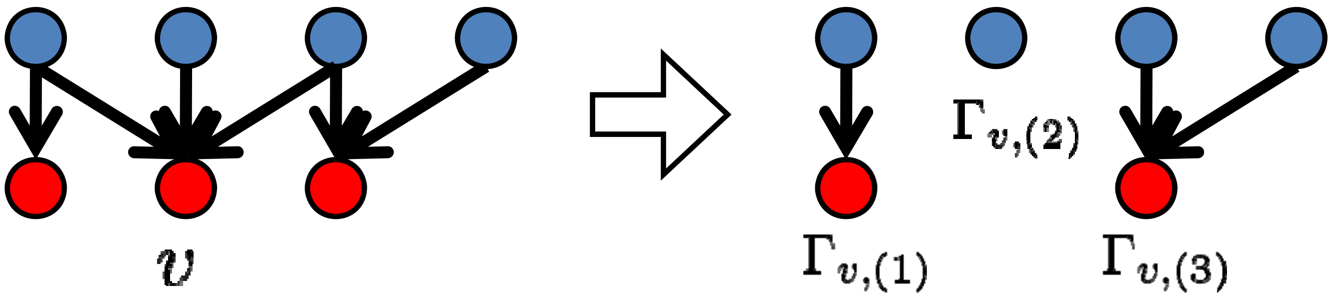
Γ

An induction formula for σ(Γ)

- Γ : a tree V(Γ) = V ∪ V

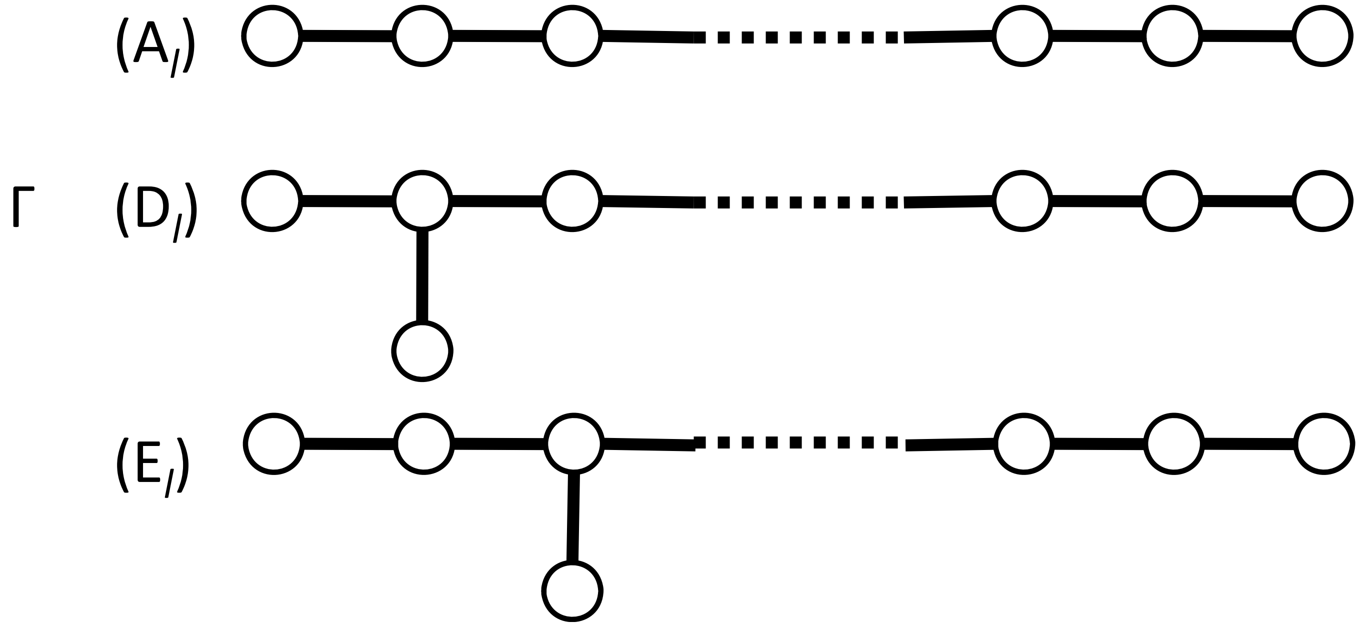
$$\sigma(\Gamma) = \sum_{v \in V} |\Gamma_v|! \prod_{k=1}^{t_v} \frac{\sigma(\Gamma_{v,(k)})}{|\Gamma_{v,(k)}|!} = \sum_{v \in V} |\Gamma_v|! \prod_{k=1}^{t_v} \frac{\sigma(\Gamma_{v,(k)})}{|\Gamma_{v,(k)}|!}$$

where $\Gamma_{v,(k)} (k = 1, \ldots, t_v)$ are the connected components of $\Gamma_v := \Gamma - v$



Geometric meanings of σ(Γ)

- Γ: Coxeter-Dynkin graphs of types A_l, D_l and E_l



- σ(Γ) = # (topological types of Morsification of a simple polynomial)

Generating functions

- Theorem [YS, 2007]

  The exponential generating functions of the series $\{\sigma(A_l)\}_{l=1}^{\infty}$, $\{\sigma(D_l)\}_{l=3}^{\infty}$ and $\{\sigma(E_l)\}_{l=4}^{\infty}$ are given by

$$\sum_{l=1}^{\infty} \sigma(A_l) \frac{x^l}{l!} = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - 1.$$

$$\sum_{l=3}^{\infty} \sigma(D_l) \frac{x^l}{l!} = 2(x-1)\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - x^2 + 2.$$

$$\sum_{l=4}^{\infty} \sigma(E_l) \frac{x^l}{l!} = \left(\frac{1}{2}x^2 - 2x + 3\right)\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - 3x^3 - x - 3.$$

the principal numbers			
l	$\sigma(A_l)$	$\sigma(D_l)$	$\sigma(E_l)$
1	1	-	-
2	1	-	-
3	2	2	-
4	5	6	5
5	16	18	18
6	61	70	66
7	272	310	298
8	1385	1582	1511
9	7936	9058	8670
10	50521	57678	55168
11	353792	403878	386394
12	2702765	3085478	2951673
13	22368256	25535378	24428657
14	199360981	227589206	217723390
15	1903757312	2173314806	2079109386

An induction formula for σ(Γ,o)

- Γ : a graph

$$\sigma(\Gamma, o) = \sum_{v \in \mathrm{Min}(o)} |\Gamma_v|! \prod_{k-1}^{t_v} \frac{\sigma(\Gamma_{v,(k)}, o|_{E(\Gamma_{v,(k)})})}{|\Gamma_{v,(k)}|!}$$

$$= \sum_{v \in \mathrm{Max}(o)} |\Gamma_v|! \prod_{k=1}^{t_v} \frac{\sigma(\Gamma_{v,(k)}, o|_{E(\Gamma_{v,(k)})})}{|\Gamma_{v,(k)}|!}$$

where $\Gamma_{v,(k)} (k = 1, \ldots, t_v)$ are the connected components of $\Gamma_v := \Gamma - v$

An example

$\sigma(1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \rightarrow 6)$ o = " 1+1+1+2 "

$= (5!/5!)\ \sigma(2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \rightarrow 6)$

$+ (5!/2!3!)\ \sigma(1 \leftarrow 2)\ \sigma(4 \rightarrow 5 \rightarrow 6)$

$+ (5!/5!)\ \sigma(1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5)$

$= \sigma(2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \rightarrow 6) + 10 \cdot 1 \cdot 1 + \sigma(A_5)$

$= (4!/1!3!)\ \sigma(2)\ \sigma(4 \rightarrow 5 \rightarrow 6)$

$+ (4!/4!)\ \sigma(2 \rightarrow 3 \leftarrow 4 \rightarrow 5) + 10 + 16$

$= 4 \cdot 1 \cdot 1 + \sigma(A_4) + 26$

$= 4 + 5 + 26 = 35$

A_3		
composition	$\sigma(o)$	$\times$
1+1	2	2
2	1	2

A_4		
composition	$\sigma(o)$	$\times$
1+1+1	5	2
1+2	3	4
3	1	2

A_5		
composition	$\sigma(o)$	$\times$
1+1+1+1	16	2
1+2+1	11	2
1+1+2	9	4
2+2	6	2
1+3	4	4
4	1	2

A_6		
composition	$\sigma(o)$	$\times$
1+1+1+1+1	61	2
1+1+2+1	40	4
1+1+1+2	35	4
1+2+2	26	4
2+1+2	19	2
1+3+1	19	2
1+1+3	14	4
2+3	10	4
1+4	5	4
5	1	2

A_7		
composition	$\sigma(o)$	$\times$
1+1+1+1+1+1	272	2
1+1+1+2+1	181	4
1+1+2+1+1	169	2
1+1+1+1+2	155	4
1+2+2+1	132	2
1+1+2+2	111	4
1+2+1+2	99	4
2+1+1+2	90	2
1+1+3+1	78	4
2+2+2	71	2
1+1+1+3	64	4
1+3+2	55	4
1+2+3	50	4
2+1+3	34	4
1+4+1	29	2
1+1+4	20	4
3+3	20	2
2+4	15	4
1+5	6	4
6	1	2

A_8		
composition	$\sigma(o)$	$\times$
1+1+1+1+1+1+1	1385	2
1+1+1+1+2+1	917	4
1+1+1+2+1+1	875	4
1+1+1+1+1+2	791	4
1+1+2+2+1	643	4
1+2+1+2+1	589	2
1+1+1+2+2	573	4
1+2+1+1+2	531	4
1+1+2+1+2	477	4
2+1+1+1+2	449	2
1+2+2+2	413	4
1+1+1+3+1	407	4
1+1+3+1+1	365	2
1+1+1+1+3	323	4
2+1+2+2	315	4
1+2+3+1	315	4
1+1+3+2	259	4
1+1+2+3	245	4
1+3+1+2	217	4
1+2+1+3	203	4
2+1+1+3	189	4
2+3+2	181	2
2+2+3	155	4
1+1+4+1	133	4
1+3+3	125	4
1+1+1+4	105	4
1+4+2	99	4
1+2+4	85	4
3+1+3	69	2
2+1+4	55	4
1+5+1	41	2
3+4	35	4
1+1+5	27	4
2+5	21	4
1+6	7	4
7	1	2

A_9		
composition	$\sigma(o)$	$\times$
1+1+1+1+1+1+1+1+1	7936	2
1+2+1+1+1+1+1+1	5263	4
1+1+1+1+2+1+1+1	5095	2
1+1+2+1+1+1+1+1	4985	4
2+1+1+1+1+1+1+1	4529	4
1+2+2+1+1+1+1	3736	4
1+2+1+1+2+1+1	3526	2
1+1+2+2+1+1+1	3526	2
2+2+1+1+1+1+1	3268	4
1+2+1+2+1+1+1	3196	4
2+1+1+1+2+1+1	2990	4
2+1+1+2+1+1+1	2890	4
2+1+2+1+1+1+1	2780	4
1+2+2+2+1+1	2701	2
2+1+1+1+1+1+2	2590	2
1+3+1+1+1+1+1	2312	4
2+2+2+1+1+1	2261	4
1+1+3+1+1+1+1	2144	4
2+2+1+2+1+1	2107	4
2+1+2+2+1+1	2051	4
2+2+1+1+2+1	1889	4
3+1+1+1+1+1+1	1856	4
1+3+2+1+1+1	1735	4
1+2+3+1+1+1	1667	4
2+3+1+1+1+1	1519	4
2+1+2+1+2+1	1513	2
1+3+1+2+1+1	1457	4
2+2+2+2+1	1456	2
3+2+1+1+1+1	1421	4
2+1+1+3+1+1	1351	4
3+1+1+2+1+1	1253	4
2+3+2+1+1	1168	4
2+1+3+1+1+1	1141	4
1+3+2+2+1	1100	4
3+1+2+1+1+1	1099	4
3+1+1+1+1+2	1051	4

3+2+2+1	1016	4
1+3+3+1	880	2
2+3+1+2	812	4
1+4+1+1+1	785	4
3+2+1+2	784	4
3+1+2+2	728	4
1+1+4+1+1	685	2
3+3+1+1	664	4
1+4+2+1	632	4
4+1+1+1+1	595	4
2+4+1+1	512	4
3+1+3+1	496	4
4+2+1+1	470	4
3+3+2	461	4
3+1+1+3	448	2
1+4+1+2	412	4
3+2+3	379	2
2+4+2	379	2

4+1+2+1	370	4
4+1+1+2	350	4
4+2+2	295	4
3+4+1	259	4
4+3+1	245	4
1+5+1+1	208	4
2+5+1	161	4
5+1+1+1	160	4
5+2+1	133	4
4+1+3	125	4
5+1+2	83	4
4+4	70	2
5+3	56	4
1+6+1	55	2
6+1+1	35	4
6+2	28	4
7+1	7	4
8	1	2

Open Question 2

- Although we have an induction formula,

  it seems to be difficult to compare the numbers …

- Open Question 2:

  Which orientation of a path Γ gives the "2^{nd}, 3^{rd}, … largest" Γ-cone ?

- Conjecture:

  The second largest Γ-cone for a path Γ

  is given by an orientation " 1+2+1+ … +1 ".

References

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Thank you for your attention !!