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Root Systems
Lie Algebras

and

Their Representations

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Lecture 1: Root Systems

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1.1 Axiom of a root system

Let (F, I) be a pair of

a vector space F over $\mathbf{R}$ =(real number field)

a symmetric bilnear form $I : F \times F \to \mathbf{R}$ with rank$(I) < \infty$.

For an non-isotropic $\alpha \in F$ (i.e. $I(\alpha, \alpha) \neq 0$), we put

$$\alpha^\vee := 2\alpha / I(\alpha, \alpha)$$

and define a reflection $w_\alpha \in O(F, I)$ by

$$w_\alpha(u) := u - \alpha I(\alpha^\vee, u).$$

The reflection hyperplane H_α (in the dual space F^*) is defined by

$$H_\alpha := \{x \in F^* = \mathrm{Hom}_{\mathbf{R}}(F, \mathbf{R}) \mid \langle \alpha, x \rangle = 0\}.$$

Definition. A non-empty subset R of F is called a (generalized) *root system* belonging to (F, I) if it satisfies following 1.-5.

1. For any $\alpha \in R$, one has $I(\alpha, \alpha) > 0$.
2. The subgroup of F generated by R (the *weigt lattice* of R):
$$Q(R) := \mathbf{Z}R$$
is a full-lattice of F, i.e. one has a natural isomorphism
$$Q(R) \otimes_{\mathbf{Z}} \mathbf{R} \simeq F.$$
3. For $\forall \alpha, \beta \in R$, one has $I(\alpha, \beta^{\vee}) \in \mathbf{Z}$.
4. For $\forall \alpha \in R$, one has $w_\alpha R = R$.
5. *Irreducibility*: if there is a decomposition $R = R_1 \amalg R_2$ with $R_1 \perp R_2$, then either $R_1 = \emptyset$ or $R_2 = \emptyset$.

The *Weyl group* of the root system R is the group:
$$W(R) := \langle w_\alpha \mid \alpha \in R \rangle.$$

Two root systems R and R' belonging to (F, I) and (F', I') are called *isomorphic*, if there is a linear isomorphism $\varphi : F \equiv F'$ s.t. $\varphi(R) = R'$. Then, automatically, there exists $c > 0$ such that $I' \circ \varphi = c \cdot I$ (that is, I is determined from R up to constant).

For any root system R there exists a constant $c > 0$ s.t. $c \cdot I :$ $Q(R) \times Q(R) \to \mathbf{Z}$ defines an *even lattice structure* on $Q(R)$. We choose minimal such c and put $I_R := c \cdot I$.

A subspace G of F is called to be *defined over* $\mathbf{Z}$, if we have

$$\text{rank}_{\mathbf{R}}\, G = \text{rank}_{\mathbf{Z}}\, G \cap Q(R).$$

The radical $\text{rad}(I) := \{x \in F \mid I(x, y) = 0 \ \forall y \in F\}$ is defined over $\mathbf{Z}$.

A subspace G of $\text{rad}(I)$ is called a *marking* of R if it is defined over $\mathbf{Z}$. For any marking G, we can define the quotient root system $\bar{R} := R/G = R \bmod G$ as the image of R in $\bar{F} := F/G$.

1.2 Signature of a root system

Signature (or, sign) of a root system R is defined by

$$\mathrm{sign}(R) := \mathrm{sign}(I) := (\mu_+, \mu_0, \mu_-)$$

where $\mu_\pm$ (resp. μ_0) is the number of positive, negative (resp. 0) eigenvalues of the bilinear form I.

Definition.

1. R is called *finite* (or, *classical*) if sign(R) $= (\mu_+, 0, 0)$
   $\Leftrightarrow \#R < \infty \Leftrightarrow \#W(R) < \infty$.
2. R is called *affine* if $\mathrm{sign}(R) = (\mu_+, 1, 0)$, and
   μ-*extended-affine* if $\mathrm{sign}(R) = (\mu_+, 1 + \mu, 0)$ $(\mu \in \mathbf{Z}_{\geq 0})$.
3. R is called *hyperbolic* if $\mathrm{sign}(R) = (\mu_+, 0, 1)$, and
   μ-*extended-hyperbolic* if $\mathrm{sign}(R) = (\mu_+, \mu, 1)$ $(\mu \in \mathbf{Z}_{\geq 0})$.

In particular, 1-extended affine =:elliptic, and 1-extended hyperbolic=:cuspidal.

Figures 1, 2, 3, 4, 5 and 6.

1.3 Positive cone of a root system

Let R be a root system belonging to (F, I). Put $q(x) := I(x,x)/2$.

Definition. 1. A *sign decomposition* of R is a decomposition:

$$R \;=\; R^{+} \;\amalg\; R^{-} \tag{1}$$

such that there exists a linear form $l : F \to \mathbf{R}$ satisfying relations:

i) $l^{-1}(0) \cap q^{-1}(\mathbf{R}_{\leq 0}) \subset \mathrm{rad}(I)$,

ii) $\ker(l) \cap \mathrm{rad}(I)$ is a marking (i.e. is defined over $\mathbf{Z}$),

iii) $l^{-1}(0) \cap R = \emptyset$ and $R^{\pm} := \{\alpha \in R \mid \pm l(\alpha) > 0\}$.

An element of R^{+} is called a *positive* root.

2. The *positive cone* Q^{+} with respect to the sign decomposition is the cone in F spanned over $\mathbf{R}_{\geq 0}$ by the set of positive roots:

$$Q^{\pm} := \sum \mathbf{R}_{\geq 0} R^{\pm} := \sum_{\alpha \in R^{\pm}} \mathbf{R}_{\geq 0}\ \alpha. \tag{2}$$

We define the *radical* of the positive cone Q^+ by

$$\mathrm{rad}(Q^+) := \overline{Q^+} \cap \overline{Q^-} \cap \mathrm{rad}(I) = \ker(l) \cap \mathrm{rad}(I).$$

Theore m. *The radical of a cone is a marking of R such that the quotient root system $\bar{R} := R \bmod \mathrm{rad}(Q^+)$ is either a finite, an affine or a hyperbolic root system.* Then, the datum of the positive cone Q^+ further chooses a *chamber* C of the quotient root system $\bar{R}$ as follows.

First, recall what is a chamber for a finite, an affine or a hyperbolic root system $\bar{R}$ belonging in $(\bar{F}, \bar{I})$. In each case, let B be a domain in the dual space $\bar{F}^* := \mathrm{Hom}_{\mathbf{R}}(\bar{F}, \mathbf{R})$ given as follows.

$B_0 := \bar{F}^*$ if $\bar{R}$ is finite,
$B_0 :=$ a connected component of $\bar{F}^* \setminus \{x \in \bar{F}^* \mid q^*(x) = 0\}$ if $\bar{R}$ is affine,
$B_0 :=$ a connected component of $\{x \in \bar{F}^* \mid q^*(x) < 0\}$ if $\bar{R}$ is hyperbolic.

A connected component of $B_0 \setminus \cup_{\alpha \in R} H_\alpha$ is called a chamber.

Figures 7, 8 and 9.

1.4 Chamber associated with a positive cone

For a positive cone Q^+, we associate a chamber C as follows:

Let l *be a linear form on* F *defining the sign decomposition* (1) *of a root system* R. *Then the induced linear form* $\bar{l}$ *on* $\overline{F} := F/\operatorname{rad}(Q^+)$ *belongs to a chamber* C *of the root system* $\overline{R} = R/\operatorname{rad}(Q^+)$ *such that we have the relations:*

$*)\quad R^+ = \{\alpha \in R \mid \overline{\alpha}|_C > 0\}$ and $C = \{x \in B_0 \mid \langle \bar{\alpha}, x \rangle > 0 \ \forall \alpha \in R^+\}$.

Here, we denote by $\bar{\alpha}$ the image in $\bar{R}$ of $\alpha \in R$.

Theorem. *Let* G *be a marking of a root system* R *such that the quotient root system* $R \bmod G$ *is either finite, affine or hyperbolic. Then,* $*)$ *gives a one to one correspondence:*

$\{$ positive cones of R whose radical is equal to $G\}$
$\simeq \{$ chambers of the root system $\bar{R} := R \bmod G\}$.

1.5 Tits cone of a root system

A chamber C for a finite or an affine root system is a cone over a simplex. This is not the case for a hyperbolic root system, since

1. The number of walls of C may be more than rank(F),
2. The "light cone" $\bar{B}_0$ may cut the polyhedral cone:
$$\widehat{C} := \{x \in \bar{F}^* \mid \langle \bar{\alpha}, x \rangle \geq 0 \;\; \forall \alpha \in R^+\}.$$

Definition. We call $\widehat{C}$ the *hull* of the chamber C. Put

$$T(\bar{R}, B_0) := \cup_{C\text{: chamber of } \bar{R}} \widehat{C}$$

and call it the *Tits cone* of a finite, an affine or a hyperbolic root system $\bar{R}$ (w.r.t. B_0).

The Tits cone of a finite, an affine or a hyperbolic root system is a convex set in $\bar{F}^*$ (which may neither be open nor closed).

Figure 10.

1.6 Simple Root basis

Using a chamber C, we introduce *simple root basis* $\bar{\Gamma}$ for a finite, an affine or a hyperbolic root system $\bar{R}$. To the root basis, we assign a "Dynkin" diagram, where all datum of the root system $\bar{R}$ is encoded. For simplicity, we assume that $\bar{R}$ is reduced (i.e. $\mathbf{R}\alpha \cap R = \{\pm\alpha\}$ for any $\alpha \in R$).

Definition. For a finite, an affine or a hyperbolic root system $\bar{R}$ with a chamber C, put

$$\bar{\Gamma} := \{\bar{\alpha} \in \bar{R} \mid \bar{\alpha}|_C > 0 \ \ \& \ \ H_{\bar{\alpha}} \text{ is a wall of } C\}.$$

Here, $H_{\bar{\alpha}}$ is a wall of C iff $H_{\bar{\alpha}} \cap \bar{C}$ contains an open subset of $H_{\bar{\alpha}}$.

Lemma. 1. *The set $\bar{\Gamma}$ is a weak root basis of $\bar{R}$, i.e. i) $Q(R) = \sum_{\bar{\alpha}\in\bar{\Gamma}} \mathbf{Z}\bar{\alpha}$, ii) $W(R) = \langle w_{\bar{\alpha}} \mid \bar{\alpha} \in \bar{\Gamma}\rangle$ and iii) $R = \cup_{\bar{\alpha}\in\bar{\Gamma}} W(R)\bar{\alpha}$.*

2. *One has* $I(\bar\alpha^\vee,\bar\beta)\le 0$ *and* $\bar\alpha-\bar\beta\notin\bar R\ \ \forall\bar\alpha,\bar\beta\in\bar\Gamma$ *with* $\bar\alpha\neq\bar\beta$.

3. *Any element of* $\bar R^+ = \{\bar\alpha\in\bar R \mid \bar\alpha(C)>0\}$ *is a non-negative integral linear combination of elements of* $\bar\Gamma$.

Corollary. *Let* R *be a root system with a positive cone* Q^+ *so that a chamber* C *of* $\bar R := R \bmod Q^+$ *is naturally assigned, and let* $\pi: R\to\bar R$ *be the natural projection. Then,*

i) *The set* $\pi^{-1}(\bar\Gamma)\cap R$ *is a weak root basis of the root system* R.

ii) *The cone* Q^+ *is spanned over* $\mathbf{R}_{\ge 0}$ *by the set* $\pi^{-1}(\bar\Gamma)\cap R$.

Finally, we assign a diagram to the root basis $\bar{\Gamma}$.

Definition of the diagram for $\bar{\Gamma}$.

1. The set of vertices of the diagrams are in one to one correspondence with the set of roots $\overline{\Gamma}$.

2. Between two distinct vertices α, β, we either equipp or do not equipp with a laveled edge.

a) If $I(\alpha, \beta) = 0$, we put no edge between the vertices.

b) If $I(\alpha, \beta) < 0$, we put a real labeled edge $\circ\!\underset{M}{\overset{r}{\text{———}}}\!\circ$ where $r := I(\alpha, \alpha) : I(\beta, \beta) \in \mathbf{Q}_{>0}$ and $M := I(\alpha, \beta^\vee) I(\alpha^\vee, \beta) \in \mathbf{Z}_{\geq 1}$.

c) If $I(\alpha, \beta) > 0$, we put a dotted labeled edge $\circ - \frac{r}{M} - \circ$ where $r := I(\alpha, \alpha) : I(\beta, \beta) \in \mathbf{Q}_{>0}$ and $M := I(\alpha, \beta^\vee) I(\alpha^\vee, \beta) \in \mathbf{Z}_{\geq 1}$.

All edges with labels.

Case $I(\alpha, \beta) < 0$.

Sign	Name	Dynkin	Coxeter	New	
$(2,0,0)$	$A_1 \times A_1$	○ ○	○ ○	○ ○	
$(2,0,0)$	A_2	○—○	○—○	○ $\frac{1}{1}$ ○	
$(2,0,0)$	$B_2 = C_2$	○⇐○	○—○ (4)	○ $\frac{1:2}{2}$ ○	
$(2,0,0)$	G_2	○⇚○	○—○ (6)	○ $\frac{1:3}{3}$ ○	
$(1,1,0)$	$\tilde{A}_1$	○ ⟺ ○	○—○ (∞)	○ $\frac{1:1}{4}$ ○	
$(1,1,0)$	$\widetilde{BC}_1$	○⇐○	○—○ (∞)	○ $\frac{1:4}{4}$ ○	
$(1,0,1)$			○—○ (∞)	○ $\frac{r}{M}$ ○	$M > 4$

Case $I(\alpha, \beta) > 0$.

Sign	Name	Dynkin	Coxeter	New
$(1,1,0)$	$\tilde{A}_1$	○===○	○—○ (∞)	○ – $\frac{1:1}{4}$ – ○
$(1,1,0)$	$\widetilde{BC}_1$		○—○ (∞)	○ – $\frac{1:4}{4}$ – ○

The labels r and M on edges are necessary when we study hyperbolic root systems.

Figures 11, 12.

Lecture 2: Lie algebras over a root system

1. Axiom of a Lie algebra $\tilde{\mathfrak{g}}(R)$ over a root system R

2. Elementary properties of R-algebras $\tilde{\mathfrak{g}}(R)$

3. Subalgebra $\mathsf{sl}_{2,\alpha}$ for a real root $\alpha \in R$

4. Sub-quotient of $\tilde{\mathfrak{g}}(R)$

5. Triangular decomposition of $\tilde{\mathfrak{g}}(R)$

6. Integral (Chevalley) basis of $\tilde{\mathfrak{g}}(R)$

2.1 Axiom of a Lie algebra over a root system

R: a root system, K: a coefficient field with $ch(K)=0$.

A K-vector space $\mathfrak{g}$ equipped with a K-bilinear map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ and $[x,x]=0$ $\forall x,y,z\in\mathfrak{g}$ is called a Lie algebra. We denote ad$(x)(y):=[x,y]$ and call the *adjoint action* of x on y: $\mathfrak{g}\to End(\mathfrak{g})$, $x\mapsto$ ad(x).

Definition. A Lie algebra $\tilde{\mathfrak{g}}(R)$ is called an R-*algebra* , or, a *Lie algebra defined over* R , if there exists a tuple $(\tilde{\mathfrak{g}},\tilde{\mathfrak{h}},\tilde{I}^*,\omega,\iota)$ where

1. $\tilde{\mathfrak{h}}$ is an abelian subalgebra of $\tilde{\mathfrak{g}}$,
2. $\tilde{I}^*$ is a $\tilde{\mathfrak{g}}$-invariant bilinear form on $\tilde{\mathfrak{g}}$,
3. ω is a Chevalley involution on $\tilde{\mathfrak{g}}$,
4. ι is an identification of the set of real roots of $\tilde{\mathfrak{g}}$ with R,

satisfying Axioms 1-4. in the next pages.

Axiom 1. $\tilde{\mathfrak{h}}$ is an abelian Lie subalgebra of $\tilde{\mathfrak{g}}$ satisfying

i) The normalizer of $\tilde{\mathfrak{h}}$ in $\tilde{\mathfrak{g}}$ is $\tilde{\mathfrak{h}}$ itself.

$$\tilde{\mathfrak{h}} = \{x \in \tilde{\mathfrak{g}} \ \mid \ [x, \tilde{\mathfrak{h}}] \subset \tilde{\mathfrak{h}}\}. \tag{3}$$

ii) The adjoint action of $\tilde{\mathfrak{h}}$ on $\tilde{\mathfrak{g}}$ is diagonalizable.

$$\tilde{\mathfrak{g}}(R) = \tilde{\mathfrak{h}} \oplus \oplus_{\alpha \in \tilde{\mathfrak{h}}^* \setminus \{0\}} \mathfrak{g}_\alpha, \tag{4}$$

where $\tilde{\mathfrak{h}}^* := \mathrm{Hom}_k(\tilde{\mathfrak{h}}, K)$, and, for $\alpha \in \tilde{\mathfrak{h}}^* \setminus \{0\}$, we put

$$\mathfrak{g}_\alpha := \{x \in \tilde{\mathfrak{g}} \mid \mathrm{ad}(h)(x) = \langle \alpha, h \rangle x \text{ for } \ \forall h \in \tilde{\mathfrak{h}}\}. \tag{5}$$

If $\mathfrak{g}_\alpha \neq 0$, we call $\alpha \in \tilde{\mathfrak{h}}^*$ a *root* and $\mathfrak{g}_\alpha$ the *root space*. A non-zero element of $\mathfrak{g}_\alpha$ is called a root vector of the algebra $\tilde{\mathfrak{g}}$.
We define the set of roots of the algebra $\tilde{\mathfrak{g}}$:

$$\Delta := \{\alpha \in \tilde{\mathfrak{h}}^* \mid \mathfrak{g}_\alpha \neq 0\}. \tag{6}$$

Axiom 2. The $\tilde{I}^*$ is a symmetric K-bilinear form on $\tilde{\mathfrak{g}}$ satisfying

i) The form $\tilde{I}^*$ is $\tilde{\mathfrak{g}}$-invariant, i.e.

$$\tilde{I}^*([x,y],z) = \tilde{I}^*(x,[y,z]) \quad \forall x,y,z \in \tilde{\mathfrak{g}}. \tag{7}$$

ii) The restriction $\tilde{I}^* : \tilde{\mathfrak{h}} \times \tilde{\mathfrak{h}} \to K$ is a perfect pairing.

iii) There does not exists a proper linear subspace $\tilde{\mathfrak{h}}'$ of $\tilde{\mathfrak{h}}$, which satisfies: a) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \tilde{\mathfrak{h}}'$ for all $\alpha \in \Delta$, b) $\tilde{I}^*|_{\tilde{\mathfrak{h}}' \times \tilde{\mathfrak{h}}'}$ is perfect.

Definition. Due to ii), we have an injection: $\tilde{I}^* : \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}^*$. The image, denoted by

$$\tilde{F}_K \ := \ \tilde{I}^*(\tilde{\mathfrak{h}}) \ = \ \text{Image of } \tilde{I}^*|_{\tilde{\mathfrak{h}}} \ , \tag{8}$$

carries a perfect symmetric bilinear form, denoted by $\tilde{I}$. The $\tilde{I}$

induces the inverse isomorphism $\tilde{I} : \tilde{F}_K \simeq \tilde{\mathfrak{h}}$. Put

$$h_x \ := \ \tilde{I}(x) := (\tilde{I}^*)^{-1}(x) \qquad \text{for } x \in \tilde{F}_K \tag{9}$$

such that $\tilde{I}(x,y) = \langle h_x, y\rangle = \langle x, h_y\rangle$ for $x, y \in \tilde{F}_K$.

Actually, we shall see that

$$\Delta \subset \tilde{F}_K$$

Axiom 3. The ω is an involution of $\tilde{\mathfrak{g}}$ (i.e. a Lie algebra automorphism of $\tilde{\mathfrak{g}}$ with $\omega^2 = id$) such that $\omega|\tilde{\mathfrak{h}} = -\mathrm{id}_{\tilde{\mathfrak{h}}}$.

Axiom 4. There exists an injective K-linear map

$$\iota \ : \ Q(R) \otimes_{\mathbf{Z}} K \ \longrightarrow \ \tilde{F}_K, \tag{10}$$

satisfying i)-v) (we shall regard $Q(R)$ as a subset of $\tilde{F}_K$).

i) The restriction of the form $\tilde{I}$ on $\tilde{F}_K$ to the lattice $Q(R)$ coincides with the form I_R on $Q(R)$ for the root system R,

ii) The inclusion map ι induces an inclusion of the set of roots:

$$R \ \subset \ \Delta, \tag{11}$$

iii) The algebra $\tilde{\mathfrak{g}}$ is generated by $\tilde{\mathfrak{h}}$ and $\mathfrak{g}_\alpha$ for $\alpha \in R$.

iv) The inclusion map (11) induces a bijection:

$$R \ \simeq \ \{\delta \in \Delta \mid \tilde{I}(\delta, \delta) > 0\}. \tag{12}$$

v) The pairing $\tilde{I}^* : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to K$ is non-zero for any $\alpha \in R$.

2.2 Elementary properties of R-algebras

1. $$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \qquad \text{for } \alpha, \beta \in \Delta,\ \alpha+\beta \neq 0,$$
$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \tilde{\mathfrak{h}} \qquad \text{for } \alpha \in \Delta.$$

2. $$\tilde{I}^*(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \qquad \text{for } \alpha, \beta \in \Delta,\ \alpha+\beta \neq 0.$$

3. $$[x, y] = \tilde{I}^*(x, y) h_\alpha \qquad \text{for } \alpha \in \Delta \text{ and } x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\alpha}.$$

5. $$\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha} \qquad \text{for } \alpha \in \Delta.$$

6. $$\tilde{I}^*(\omega(x), \omega(y)) = \tilde{I}^*(x, y) \qquad \text{for } x, y \in \tilde{\mathfrak{g}}.$$

20. *The center of the R-algebra $\tilde{\mathfrak{g}}$ is given by*

$$z(\tilde{\mathfrak{g}}) \ = \ h_{\mathrm{rad}(I)_K} \ = \ \mathrm{rad}(I^*|_{\tilde{\mathfrak{h}}})_K. \qquad (13)$$

2.3 Subalgebra $\mathfrak{sl}_{2,\alpha}$ for a real root $\alpha \in R$

Definition. An element of R is called a *real root* of $\tilde{\mathfrak{g}}$ and an element of $\Delta \setminus R$ is called an *imaginary root* of $\tilde{\mathfrak{g}}$.

13. $$K\alpha \cap \Delta = \{\pm\alpha\} \qquad \text{for any real root } \alpha \in R$$

14. $$\mathrm{rank}_K\, \mathfrak{g}_{\pm\alpha} = 1 \qquad \text{for any real root } \alpha \in R.$$

Then,

$$\mathfrak{sl}_{2,\alpha} := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus Kh_{\alpha^\vee}$$

is a Lie-subalgebra of $\tilde{\mathfrak{g}}$ naturally isomorphic to $\mathfrak{sl}_2 = Ke \oplus Kf \oplus Kh$.

8. *The adjoint action of* $\mathfrak{sl}_{2,\alpha}$ *for* $\alpha \in R$ *on* $\tilde{\mathfrak{g}}$ *is integrable: for any* $z \in \tilde{\mathfrak{g}}$, *there exist a finite dimensional subspace of* $\tilde{\mathfrak{g}}$, *which contains* z, *and is invariant under the adjoint action of* $\mathfrak{sl}_{2,\alpha}$.

Remark. Using exp(x) for $\alpha \in R$ and $x \in \mathfrak{g}_\alpha$, adjoint group $\tilde{G}^{ad}$ can be constructed.

Some consequences of $\mathfrak{sl}_{2,\alpha}$ structure.

12. *Let* R *be a root system with a sign decomposition. Let* $\pi^{-1}(\bar{\Gamma})$ *be its root basis introduced in* §1.5. *Then, for* $\alpha, \beta \in \pi^{-1}(\bar{\Gamma})$ *with* $\bar{\alpha} \neq \bar{\beta}$,

$$\text{i)}\quad \mathrm{ad}(\mathfrak{g}_\alpha)^{1-I(\alpha^\vee,\beta)}\mathfrak{g}_\beta = 0 \quad \textit{and} \quad \text{ii)}\quad [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\beta] = 0.$$

17. *Let* R *be either a finite, an affine or a hyperbolic root system, and let* $\Gamma = \bar{\Gamma}$ *be its root basis introduce in §1.5. Then,* $\tilde{\mathfrak{g}}$ *is isomorphic to the Kac-Moody algebra assoicated with the Cartan matrix* $\{I(\alpha^\vee, \beta)\}_{\alpha,\beta\in\Gamma}$.

Remark. If the root system R is neither finite, affine nor hyperbolic, then the Lie algebra $\tilde{\mathfrak{g}}$ is no longer a Kac-Moody algbra. **Eg. elliptic algebras and cuspidal algebras.** We do not have a general criterion (yet) for a root sytem R to have an R-algebra.

Existence of R-algebras

1. If R is a finite, affine or hyperbolic root system, then, correspondingly, the classical simple Lie algebra, affine algebra and kac-Moody algebra plays the role of R-algebra.

2. If R is an elliptic root system, then the elliptic algebra plays the role of R-algebra, where the elliptic algebra is consructed by 4 different means: 1. vertex algebra defined over $Q(R)$, 2. Chevalley generators and a generalization of Serre relations, 3. amalgamation of an affine algebra and a Heisenberg algebra, 4. universal central extension of a troidal algebra, all of which give the same algebra (joint work with D.Yoshii).

3. If R is a cuspidal root system, then a similar construction as the elliptic algebra works and we call the constructed algebra a cuspidal algebra (joint work with Xiao and Xu).

4. For further root sytems, the construction using vertex algebra seems to work. However, we do not know yet exactly, for which class of root systems, the construction works.

5. Using the integral basis sgiven in 2.6, we, conjecturally, may be able to define R-algebras by generators and relations.

2.4 Sub-quotient of the algebra $\tilde{\mathfrak{g}}(R)$

For a marking G of the root system R, we shrink the Cartan subalgebra $\tilde{\mathfrak{h}}$ to $\hat{\mathfrak{h}} := \{h \in \tilde{\mathfrak{h}} \mid \langle G, h \rangle = 0\}$. In this section, we describe $\bar{R} := R/G$-algebras as a quotient of $\hat{\mathfrak{g}}(R) := \hat{\mathfrak{h}} \oplus \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

Definition. An ideal $\mathcal{I}$ of the algebra $\hat{\mathfrak{g}}$ is called a G-*ideal* if it satisfies the following four condidtions.

I. i) $\mathcal{I}$ is diagonalizable with resepct to the adjoint action of $\hat{\mathfrak{h}}$.
I. ii) The intersection $\mathcal{I} \cap \hat{\mathfrak{h}}$ is equal to h_G.
I. iii) The sum $\hat{\mathfrak{h}} + \mathcal{I}$ contains the weight space $\mathfrak{g}_\delta$ for all $\delta \in G \cap \Delta$.
I. iv) $\mathcal{I}$ is invariant under the action of ω.

Theorem. *Let G be a $\tilde{I}$-closed marking (recall §1.2) and $\mathcal{I}$ a G-ideal. Then, $\hat{\mathfrak{g}}/\mathcal{I}$ carries an $\bar{R} := R/G$-algebra structure.*

Corollary. We have the following comparisons of root spaces:

a) *For any* $\delta \in \Delta \cap G$:

$$P_\delta : \ \mathfrak{g}_\delta \ \longrightarrow \ \widehat{\mathfrak{h}}/h_G. \tag{14}$$

b) *For any* $\alpha, \alpha + \delta \in R$ *with* $\delta \in G$:

$$Q_{\alpha,\delta} : \ \mathfrak{g}_{\alpha+\delta} \ \simeq \ \mathfrak{g}_\alpha. \tag{15}$$

2.5 Triangular decomposition of $\tilde{\mathfrak{g}}(R)$

Theorem. (Triangular Decompsosition.) *Let* R *be a root system with a positive cone* Q^+*, and let* $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(R)$ *be a Lie algebra defined over* R*. Then, the algebra* $\tilde{\mathfrak{g}}$ *decomposes into the direct sum of three subalgebras:*

$$\tilde{\mathfrak{g}}(R) \;=\; \mathfrak{n}_{Q^-} \oplus \tilde{\mathfrak{h}}^{\mathrm{rad}(Q^+)} \oplus \mathfrak{n}_{Q^+}, \tag{16}$$

where the subalgebras are given as follows:

$$\mathfrak{n}_{Q^\pm} := \textit{the subalgebra of } \tilde{\mathfrak{g}} \textit{ generated by } \mathfrak{g}_\alpha \textit{ for } \alpha \in \pm\pi^{-1}(\bar{\Gamma}) \tag{17}$$

and

$$\tilde{\mathfrak{h}}^{\mathrm{rad}(Q^+)} := \tilde{\mathfrak{h}} \oplus \oplus_{\alpha\in \mathrm{rad}(Q^+)\setminus\{0\}} \mathfrak{g}_\alpha. \tag{18}$$

Corollaries. $\Delta \subset Q^- \amalg \mathrm{rad}(Q^+) \amalg Q^+.$

$$\mathfrak{n}_{Q^\pm} = \oplus_{\alpha\in Q^\pm} \mathfrak{g}_\alpha.$$

$$\cap_{n\in\mathbf{Z}_{>0}}[\mathfrak{n}_{Q^\pm},[\mathfrak{n}_{Q^\pm},\cdots,[\mathfrak{n}_{Q^\pm},\mathfrak{n}_{Q^\pm}]\cdots]] = 0.$$

2.6 Integral basis of $\tilde{\mathfrak{g}}(R)$

Theorem. Let $\tilde{\mathfrak{g}}$ be an R-algebra defined over $\mathbf{C}$ with a positive cone Q^+ and a rad(Q^+)-ideal $\mathcal{I}$. There exists

(i) a system of root vectors,

$$e_\alpha \in \mathfrak{g}_\alpha \qquad \text{for} \quad \alpha \in \pm\pi^{-1}(\bar{\Gamma}), \tag{19}$$

(ii) a system of $\mathbf{Z}$-linear maps

$$H^{(\delta)}: \ Q(R^\vee)/(Q(R^\vee) \cap \mathbf{R}\delta) \ \to \ \mathfrak{g}_\delta \quad \text{for} \quad \delta \in \Delta \cap \mathrm{rad}(Q^+) \tag{20}$$

satisfying the following **1.**, **2.**, **3.** and **4.**.

1.

(ii)

2. (i) $$\omega(e_\alpha) = e_{-\alpha} \qquad \text{for } \alpha \in \pm\pi^{-1}(\Gamma).$$

(ii)
$$\begin{array}{ccc} Q(R^\vee)/(Q(R^\vee)\cap \mathbf{R}\delta) & \xrightarrow{H^{(\delta)}} & \mathfrak{g}_\delta \\ \downarrow\ -1 & & \downarrow \omega \\ Q(R^\vee)/(Q(R^\vee)\cap \mathbf{R}\delta) & \xrightarrow{H^{(-\delta)}} & \mathfrak{g}_{-\delta} \end{array} \qquad \text{for } \delta \in \Delta \cap \mathrm{rad}(Q^+)$$

3. For all $\alpha, \beta \in \pi^{-1}(\bar\Gamma)$

(i) $$[e_\alpha, e_{-\alpha}] = h_{\alpha^\vee}$$

(ii) $$\mathrm{ad}(e_\alpha)^{-I(\alpha^\vee,\beta)+1} e_\beta = 0 \qquad \text{if } \bar\alpha \neq \bar\beta,$$
$$[e_\alpha, e_\beta] = 0 \qquad \text{if } \bar\alpha = \bar\beta.$$

(iii) $$[e_\alpha, e_{-\beta}] = 0 \qquad \text{if } \bar\alpha \neq \bar\beta,$$

(iv) $$[e_\alpha, e_{-\beta}] = H^{(\alpha-\beta)}(\alpha^\vee) \qquad \text{if } \bar\alpha = \bar\beta.$$

4.

$$\begin{aligned}[H^{(\delta)}(\alpha^\vee), H^{(\gamma)}(\beta^\vee)] &= \quad I(\alpha, \beta^\vee)\ H^{(\delta+\gamma)}(\tfrac{2}{I(\alpha,\alpha)}\delta) \\ &= -I(\alpha^\vee, \beta)\ H^{(\delta+\gamma)}(\tfrac{2}{I(\beta,\beta)}\gamma),\end{aligned}$$

for $\alpha, \beta \in R$, $\delta\gamma \in \mathrm{rad}(Q^+) \cap \Delta$.

5.

$$[H^{(\delta)}(h), e_\alpha] = I(h, \alpha)\ e_{\alpha+\delta}$$

for $\alpha \in R$, $\delta \in \mathrm{rad}(Q^+) \cap \Delta$ and $h \in Q(R^\vee)$.

Lecture 3: Integrable Representations

1. Integrable modules

2. Highest weight module

3. The dominant integral weight Λ

4. The integrable module $L(\Lambda)$

5. The highest part $A(\Lambda)$

6. The block decomposition

3.1 Integrable modules

Let $\mathfrak{g}$ be a Lie algebra over a field K of characteristic 0. A pair (V, π) of a K-vector space V and a Lie algebra homomorphism $\pi : \mathfrak{g} \to End_K(V)$ is called a *representation*. One has the left action $(x, v) \in \mathfrak{g} \times V \mapsto \pi(x)v \in V$. We, sometimes, write xv instead of $\pi(x)v$. The vector space V with the action of $\mathfrak{g}$ is called a $\mathfrak{g}$-*module*.

Definition. 1. A $\tilde{\mathfrak{g}}(R)$-module V is *diagonalizable* with respect to the Cartan algebra $\tilde{\mathfrak{h}}$, if it admits an equi-eigenspace decomposition:

$$V = \oplus_{\lambda \in \tilde{F}} V^{\lambda}, \tag{21}$$

where one put

$$V^{\lambda} = \{v \in V \mid \pi(h)v = \langle \lambda, h \rangle v \quad \forall h \in \tilde{\mathfrak{h}}\}. \tag{22}$$

An element $\lambda \in \tilde{F}$ is called a *weight* of the module V, if $V^\lambda \neq 0$, where V^λ may not necessarily be of finite dimensional. The set of all weigts of V is denoted by

$$P(V) := \{\lambda \in \tilde{F} \mid V^\lambda \neq \{0\}\}.$$

2. Let V be a diagonalizable $\tilde{\mathfrak{g}}(R)$-module. A vector $v \in V$ is called *integrable*, if for all real root vector $x \in \mathfrak{g}(R)_\alpha$ $(\alpha \in R)$, there exists a positive integer $n \in Z_{\geq 0}$ such that $\pi(x)^n v = 0$. A $\tilde{\mathfrak{g}}(R)$-module V is called *integrable* if every element of V is integrable.

Assertion. *If V is integrable then $\tilde{I}(\lambda, \alpha^\vee) \in \mathbf{Z}$ for any weight $\lambda \in P(V)$ and any root $\alpha \in R$. Therefore, $\tilde{I}(\lambda, Q(R^\vee)) \subset \mathbf{Z}$.*

3.2 Highest weight module

Definition. 1. Let (V, π) be a $\tilde{\mathfrak{g}}(R)$-module. A nonzero vector $v \in V$ is called a *highest weight vector* with respect to a positive cone Q^+ (2), if

i) v is a weight vector, i.e. $\exists \Lambda \in \tilde{F}$ s.t. $h \cdot v = \langle \Lambda, h \rangle v \ \forall h \in \tilde{\mathfrak{h}}$,

ii) $\pi(\mathfrak{n}_{Q^+}) \cdot v = 0$.

2. A $\tilde{\mathfrak{g}}(R)$-module (V, π) is called a *highest weight module* if there is a highest weight vector which generates V over $\tilde{\mathfrak{g}}(R)$.

The linear form on the center $\Lambda|_{\mathfrak{z}(\tilde{\mathfrak{g}}(R))}$ (where Λ is the weight of a generator v of V) is independent of a choice of a generating highest weight vector v, which we call the *level* of V.

3.3 The dominant integral weight Λ

We, first, state necessary conditions on an element $\Lambda \in \tilde{F}$ so that there exists an integrable highest weight module of the highest weight Λ.

Theorem. *Consider the R-algebra $\tilde{\mathfrak{g}}(R)$ over a root system R. Let Λ be an element of $\tilde{F} = \mathrm{Hom}(\tilde{\mathfrak{h}}, K)$. Suppose that there exists a positive cone Q^+ for R and an integrable highest weight vector v_+ of the weight Λ with respect to the cone Q^+. Then the following i)- iv) hold.*

i) The Λ is integral in the sense: $\quad \tilde{I}(\Lambda, \gamma) \in \mathbf{Z} \quad \forall \gamma \in Q(R^\vee)$.

ii) One has either a) $\Lambda \in \mathrm{rad}(I)$ or b) $\mathrm{rad}(\Lambda) = \mathrm{rad}(Q^+)$, where $\mathrm{rad}(\Lambda) := \mathrm{rad}(I) \cap \Lambda^\perp = \{\gamma \in \mathrm{rad}(I) \mid \tilde{I}(\Lambda, \gamma) = 0\}$.

In particular, the quotient root system $R_\Lambda := R \,\mathrm{mod}\, \mathrm{rad}(\Lambda)$ *is either finite, affine or hyperbolic.*

iii) *The dual weight* $h_\Lambda = \tilde{I}(\Lambda) \in \tilde{\mathfrak{h}}$ *projects to a vector* $\bar{h}_\Lambda$ *in the Tits cone of the root system* R_Λ *in* F^*_Λ*.*

$$\bar{h}_\Lambda \in T(R_\Lambda, B_0(Q^+)). \tag{23}$$

iv) *The* v_+ *satisfies the following equations:*

$$(e_\alpha)^{\max\{0, -\tilde{I}(\Lambda, \alpha^\vee)\}+1} v_+ = 0 \tag{24}$$

for all $\alpha \in R$*.*

3.4 The integrable module $L(\Lambda)$

We show the converse of the previous theorem.

Theorem. *Let* R *be a root system. Let* $\Lambda \in \tilde{F}$ *satisfy*

i) Integrality: $\tilde{I}(\Lambda, \alpha^\vee) \in \mathbf{Z}$ *for* $\alpha \in R$,

ii) The quotient $R_\Lambda := R/\operatorname{rad}(\Lambda)$ *is a root system of Witt index* ≤ 1 *(i.e. either finite, affine or hyperbolic).*

iii) The dual weight $h_\Lambda := \tilde{I}(\Lambda)$ *projects into the Tits cone* $T(R_\Lambda)$.

Then, there exists, up to $\tilde{\mathfrak{g}}(R)$*-isomorphism, a unique pair* $(L(\Lambda), v_+)$ *satisfying the following* 1. *and* 2.

1. $L(\Lambda)$ *is an integrable* $\tilde{\mathfrak{g}}(R)$*-module and* v_+ *is a weight vector of* $L(\Lambda)$ *of the weight* Λ *which generates* $L(\Lambda)$.

v_+ is highest with respect to any positive cone Q^+ where Λ is dominant ($\underset{\text{def}}{\Leftrightarrow}$ the dual weight h_Λ projects into the hull of the chamber C of the cone Q^+).

2. Let v be an integrable weight vector of weight Λ in a $\tilde{\mathfrak{g}}(R)$ module L which is highest with respect to a positive cone of R. Then there exists a unique $\tilde{\mathfrak{g}}(R)$-homomorphism $L(\Lambda) \to L$ with $v_+ \mapsto v$.

Plan of the proof for the case $\Lambda \notin \mathrm{rad}(I)$.

Consider the left $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ generated by a single element $[1]_\Lambda$ which satisfies the relations:

$$h[1]_\Lambda = \langle \Lambda, h \rangle [1]_\Lambda \qquad \text{for} \qquad h \in \tilde{\mathfrak{h}}$$
$$(e_\alpha)^{\max\{0, -\tilde{I}(\Lambda, \alpha^\vee)\}+1}[1]_\Lambda = 0 \qquad \text{for} \qquad \alpha \in R$$

which is a diagnalizable and integrable left $\tilde{\mathfrak{g}}$-module depending only on Λ.

Then the main task is to show $L(\Lambda) \neq \{0\}$, which is equivalent to sow the non-vanishing of the vector $[1]_\Lambda$ in $L(\Lambda)$. It suffices to show the non-vanishing of the highest weight part of $L(\Lambda)$, say $A(\Lambda)[1]_\Lambda$. Actually, $A(\Lambda)$ is the image of the universal envelopping algebra $\mathfrak{U}(\mathfrak{h}^{\mathrm{rad}(\Lambda)'}\mathbb{Z})$ of (shrinked) diagonal part

$$\mathfrak{h}^{\mathrm{rad}(Q^+)'} := h_{\mathrm{rad}(Q^+)} \oplus \oplus_{\gamma\in \mathrm{rad}(Q^+)\setminus\{0\}} \mathfrak{g}_\gamma$$

of the triagnular decomposition (18). Actually, the "shrinked" algebra $\mathfrak{U}(\mathfrak{h}^{\mathrm{rad}(\Lambda)'}\mathbb{Z})$ acts on $L(\Lambda)$ also from the right, $A(\Lambda)$ is its quotient algebra by a both-sided ideal.

$$A(\Lambda) := \mathfrak{U}(\mathfrak{h}^{\mathrm{rad}(\Lambda)'}\mathbb{Z}) / \Big(\mathfrak{U}(\mathfrak{h}^{\mathrm{rad}(\Lambda)'}\mathbb{Z}) \cap \sum_{\alpha\in R\cap Q^+} \mathfrak{U}(\tilde{\mathfrak{g}}(R))\cdot(e_{-\alpha})^{\tilde{I}(\Lambda,\alpha^\vee)+1}[1]_\Lambda \Big).$$

3.5 The highest part $A(\Lambda)$

Theorem-added. *The algebra* $A(\Lambda)$ *acts on* $L(\Lambda)$ *from the right, where the action commutes with the left action of* $\mathfrak{U}(\tilde{\mathfrak{g}}(R))$*.*

We give two proofs of non vanishing of the algebra $A(\Lambda)$.

1. The first proof is to use a specialization of the module $L(\Lambda)$ to a module over $\tilde{\mathfrak{g}}(R_\Lambda)$ where the image of $[1]_\Lambda$ is non-trivial.

Assertion. Recall the integral basis $H^\delta(\alpha^\vee)$ of the radical root space $\mathfrak{g}_\delta$ (2.6 Theorem ii)). The correspondence $H^\delta(\alpha^\vee) \mapsto \langle \Lambda, \alpha^\vee \rangle \, e^\delta$ induces a surjective homomorphism from $A(\Lambda)$ to the group-ring $K \cdot \mathrm{rad}(\Lambda)_{\mathbf{Z}}$.

2. The second proof is to use the explicite description of the defining ideal of $A(\Lambda)$ determined by a use the datum Q^+.

Definition. We shall call a surjective algebra homomorphism: $A(\Lambda) \to \mathbf{C}$ a *spector*.

The previous result assert the existence of a spector. In fact, using a spector, we obatain an irreducible integrable representation:

$$L(\Lambda) \otimes \mathbf{C}. \;]$$

Therefore, our next main task is to determine the structure of the algebra $A(\Lambda)$.

3.6 The block decomposition

Let Γ_Λ be a simple root basis of R with respect to a positive cone Q^+ compatible with Λ (recall **??** and **??**). For each vertex $\bar{\alpha} \in \Gamma_\Lambda$, let us introduce the $z(\mathrm{rad}(\Lambda)_{\mathbf{Z}})$-subalgebra of $A(\Lambda)$:

$$\begin{aligned} & B(\mathrm{rad}(\Lambda)_{\mathbf{Z}}, \tilde{I}(\Lambda, \bar{\alpha}^\vee)) \\ := \ & \text{the } z(\mathrm{rad}(\Lambda)_{\mathbf{Z}})\text{-subalgebra of } A(\Lambda) \text{ generated by } H_\delta^{(\gamma)} \\ & \text{for } \gamma \in \mathrm{rad}(\Lambda)_{\mathbf{Z}} \text{ and } \alpha + \delta \in \pi^{-1}(\bar{\alpha}). \end{aligned} \tag{25}$$

The generators of the defining ideal $\widehat{\mathcal{I}}(\Lambda)$ split into groups according to the index α. So we have the following decomposition.

Assertion. *1. The algebra $A(\Lambda)$ is isomorphic to the tensor product*

$$A(\Lambda) \simeq \otimes_{\alpha \in \Gamma_\Lambda} B(\mathrm{rad}(\Lambda)_{\mathbf{Z}}, \tilde{I}(\Lambda, \alpha^\vee)) \tag{26}$$

over the central algebra $z(\mathrm{rad}(\Lambda)_{\mathbf{Z}})$. *Here, the product structure in the tensor expression is that for non-commutative algebras where the generators in the different factor satisfy a commutation relation* (**??**). *That is:*

$$(H_{\alpha^\vee}^{(\delta)} \otimes 1)(1 \otimes H_{\beta^\vee}^{(\gamma)}) - (1 \otimes H_{\beta^\vee}^{(\gamma)})(H_{\alpha^\vee}^{(\delta)} \otimes 1) = I(\alpha^\vee, \beta^\vee) H_\delta^{(\gamma+\delta)}$$

2. The tensor decomposition (26) *does not depend on a choice of the root basis* Γ_Λ *and is unique up to an isomorphism.*

Concluding remarks.

Starting from a (generalized) root system R, we have constructed Lie algebras $\tilde{\mathfrak{g}}(R)$ defined over R and their highest integrable representations $L(\Lambda)$ for dominant integral weights Λ of R.

These supply sufficient data to construct Lie groups over R:

$$G(R) \ := \ G^{\mathrm{ad}}(R) \ * \ \lim_{\longleftarrow} G(\Lambda).$$

The next subject to be studied is the adjoint quotient morphism

$$\tilde{\mathfrak{g}}(R) \longrightarrow \tilde{\mathfrak{g}}(R)/\,\mathrm{Ad}(G(R)) \simeq \tilde{\mathfrak{h}}/\tilde{W}(R).$$

For certain good cases (elliptic and cuspidal), some partial results are obtained and the work is in progress. We hope that the understanding of the adjoint quotient morphim should supply supply sufficient materials to understand the arrangement $\{H_{\alpha,\mathbf{C}}\}_{\alpha\in R}$ on $\tilde{\mathfrak{h}}$ whose understanding was one of the motivation and the starting point of the present lectures.

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Thank you very much !

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