# The 2nd MSJ-SI

**The Mathematical Society of Japan, Seasonal Institute**

# *Arrangements of Hyperplanes*

## August 1 - 13, 2009

**Conference Hall, Hokkaido University, Sapporo, Japan**

**Survey Lecturers:**

# From splines to the index theorem

Claudio Procesi: joint work with C. De Concini, M. Vergne

Sapporo August 9, 2009

# Basic input

I want to explain the connection between four different parts of Mathematics.

1. Hyperlane arrangements
2. Splines and approximation Theory
3. The arithmetic of partition functions
4. Transversally elliptic operators.

# Basic input

**The common ground of these four fields, is**

a list $X := (a_1, \dots, a_m)$ of real or integer vectors.

1. They are the *linear equations* of the hyperplanes, in the theory of arrangements.
2. In approximation Theory they build the *box spline*.
3. In arithmetic they are *integral vectors* and we study the partition function they generate.
4. Finally in the theory of elliptic operators the vectors are integral and appear as *characters of a torus*.

# Toric arrangements

A list of integral vectors in $\mathbb{Z}^s$ gives rise to a periodic arrangement or a *toric arrangement*.
The arrangement of subgroups of a torus $(\mathbb{C}^*)^s$ of equations $x^\chi = 1$ for $\chi \in X$.

# Toric arrangements

In coordinates $\chi = (m_1, \dots, m_s)$ gives

$$\prod_i x_i^{m_i} = 1, \; x_i = e^{2\pi i \theta_i} \implies \sum_i m_i \theta_i = k \in \mathbb{Z}.$$

So a list $\chi_j = (m_{j,1}, \dots, m_{j,s}) \in \mathbb{Z}^s$ gives rise to the periodic arrangement generated by the hyperplanes

$$\sum_i m_{j,i} \theta_i = k \in \mathbb{Z}.$$

# Some function spaces

In all these Theories appear certain remarkable spaces of functions which are building blocks of the Theory.

They are all described through equations associated to *cocircuits*.

# Cocircuits

I will explain more these equations, let me recall that a cocircuit $Y$ in $X$ is an idea from matroids. Assume $X$ spans the ambient space.

**A cocircuit in $X$ is:**

a sublist $Y$ minimal with the property that, when we remove $Y$ from $X$, the vectors in the remaing list $X \setminus Y$ do not span the ambient space.

# Cocircuits

In other words. Assume $X$ spans the ambient space.

**A cocircuit in $X$ is:**

a sublist $Y := X \setminus \mathbb{H}$, where $\mathbb{H}$ is a hyperplane generated by a sublist of $X$.

# TWO SPACES OF FUNCTIONS $D(X), DM(X)$.

## The space $D(X)$

if a space of polynomials. For a vector $v$ denote by $D_v$ the usual directional derivative

For a list $Y$ of vectors denote by $D_Y := \prod_{a \in Y} D_a$.

$D(X) := \{p \mid D_Y p = 0, \ \ Y$ runs over all cocircuits$\}$.

# THE SPACE $D(X)$.

## Remark

a function $p$ or even a distribution $p$ satisfying
$D_Y p = 0, \ Y \quad$ all cocircuits is necessarily a polynomial.

# THE SPACE $DM(X)$.

## The space $DM(X)$

if defined when $X$ is a list of integral vectors. It is a space of integer values functions on $\mathbb{Z}^s$.

Denote by

$$\nabla_a f(x) := f(x) - f(x-a)$$

the usual difference operator

For a list $Y$ of vectors denote by $\nabla_Y := \prod_{a \in Y} \nabla_a$.

$DM(X) := \{p \mid \nabla_Y p = 0, \ \ Y \quad \text{runs over all cocircuits}\}$.

# Why these spaces

- $D(X)$ appears in approximation Theory and the Theory of splines
- $DM(X)$ in the Theory of the *partition* functions.

## Essentially you ask

They both appear in equivariant cohomology and index Theory.

In how many ways can you write a vector $v$ as a sum $\sum_{a\in X} t_a a$ with $t_a$ positive?

# Why these spaces

You make sense of the previous question as follows:

**in how many ways**

can you write a vector $v$ as a sum $\sum_{a\in X} t_a a$ with $t_a$ positive?
Means ($m = |X|$).

1. In approximation theory: the volume $T_X(v)$ of the polytope $\Pi_X(v)$ formed of the $(t_a) \in (\mathbb{R}^+)^m$ with $\sum_{a\in X} t_a a = v$.
2. In arithmetic: the number $P_X(v)$ of solutions $(t_a) \in (\mathbb{N})^m$ with $\sum_{a\in X} t_a a = v$ (i.e. in $\Pi_X(v) \cap \mathbb{Z}^m$ ).

In order for these numbers to be finite you need a

***positivity* condition:**

there is a linear form which is positive on all the vectors in $X$.

# Two basic Theorems of Dahmen–Micchelli

The two spaces $D(X), DM(X)$ are made of functions that describe in a piecewise way the previous functions $T_X(v), P_X(v)$.

We have several Theorems describing the spaces $D(X)$ and $DM(X)$ mostly due to Dahmen–Micchelli.

# FIRST THEOREM OF DAHMEN–MICCHELLI

## Dimension

The dimension of $D(X)$ is the number of bases one can extract from $X$.

Let $\mathcal{B}(X)$ denote the set of all bases extracted from $X$

$$\dim D(X) = |\mathcal{B}(X)|$$

# SECOND THEOREM OF DAHMEN–MICCHELLI

## Weighted dimension

The dimension of $DM(X)$ is

$$\dim DM(X) := \sum_{\underline{b} \in \mathcal{B}(X)} |\det(\underline{b})|.$$

This formula has a strict connection with an important polytope that we call *the box* $B(X)$.

# THE BOX

An important object is the compact polytope:

***THE BOX*** $B(X)$

## The Box $B(X)$

is the compact convex polytope called *zonotope*

$$B(X) := \{\sum_{i=1}^{m} t_i a_i\}, 0 \leq t_i \leq 1, \ \forall i.$$

It is the *Minkowsky sum* of the segments $[0, a], \ a \in X$.

The box $B(X)$ has a nice combinatorial structure, and can be paved by a set of parallelepipeds indexed by:
all the bases which one can extract from $X$!

**Example**

In the next example

$$X = \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix}$$

we have 15 bases and 15 parallelograms.

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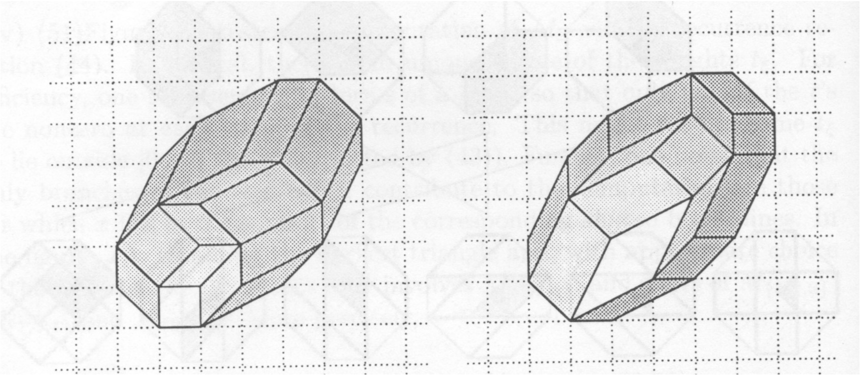
# EXAMPLE *paving the box*

$$X = \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix}$$

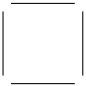


# *Step–wise paving of the box*

$$= \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix}$$

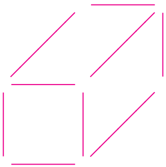
START WITH

$$X = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$



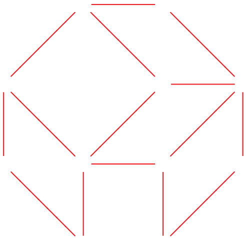
# Step–wise paving of the box

$$X = \begin{vmatrix} 1 & 0 & 1 & \\ 0 & 1 & 1 & \end{vmatrix}$$



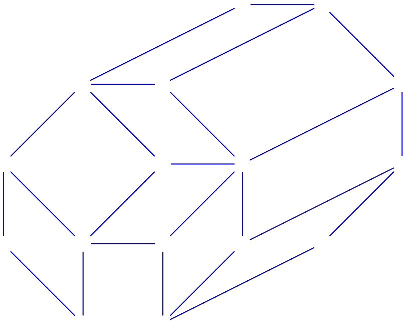
# Step–wise paving of the box

$$X = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$



# Step–wise paving of the box

$$X = \begin{vmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{vmatrix}$$



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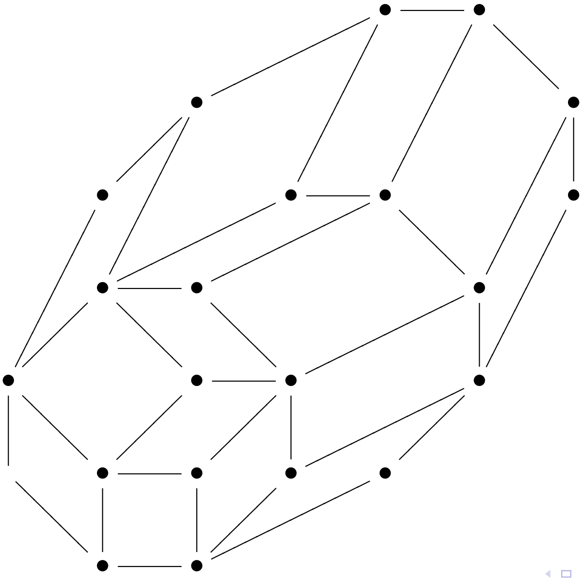


# Root systems.

$X$ a root system

$$\rho = 1/2 \sum_{a \in X} a$$

**The zonotope associated to a root system is**

up to translation by $\rho$ the convex hull of the Weyl group orbit of $\rho$.

# Two zonotopes.

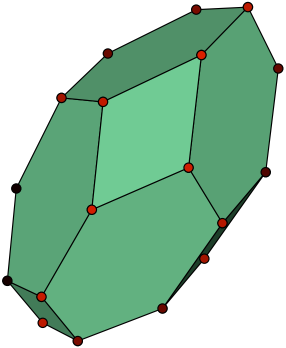


Figure: The zonotope associated to the root system $A_3$.

# Two zonotopes.

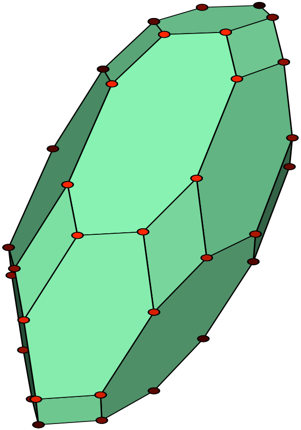


Figure: The zonotope associated to the root system $B_3$.

# Duality

The zonotope $B(X)$ is combinatorially a picture dual to the hyperplane arrangement associated to $X$ as system of linear equations.

To a face $F$ of $B(X)$ associate the set $C_F$ of linear functions which take their maximum in $B(X)$ on $F$. You get an order reversing bijection with the facets of the arrangement.

# SECOND THEOREM OF DAHMEN–MICCHELLI

## Weighted dimension

The dimension of $DM(X)$ (as free $\mathbb{Z}$ module) is the *Volume* $\delta(X)$ *of the box* $B(X)$*!* we have:

$$\delta(X) := \sum_{\underline{b} \in \mathcal{B}(X)} |\det(\underline{b})|.$$

This formula has a strict connection with the paving of the box.

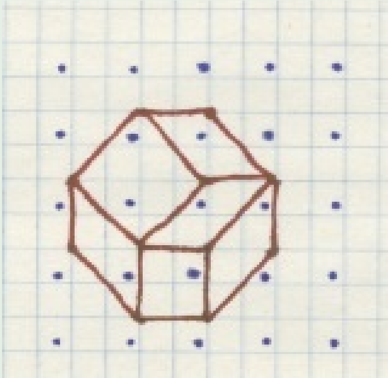
# The initial conditions

1. The fact is that the difference equations are *recursions* and
2. a function in $DM(X)$ is determined by the values it takes on a gives special set of $\delta(X)$ points.
3. This is obtained by taking a sufficiently generic vector $v$ and taking the set of points in which $v - B(X)$, intersects the lattice $\mathbb{Z}^s$!

## Example

Let us take

$$X = \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$



See that $\delta(X) = 1+1+1+1+1+2 = 7$ is the number of points in which the box $B(X)$, shifted generically a little, intersects the lattice!

In this example there are 4 cocircuits, $DM(X)$ is 7 dimensional and can be identified to the $\mathbb{Z}$ valued functions on the 7 points of the picture.

The first reason to study $DM(X)$ is that the partition function associated to $X$ is described explicitly through elements of $DM(X)$. There is a strong connection between partition functions and $DM(X)$.

The second reason is the connection with the index theorem

# A THIRD SPACE

Recall the Partition function associated to a list $X = (a_1, a_2, \dots, a_m)$ of integral vectors.
Assume first that there is a linear form $\phi$ so that $\langle \phi \mid a \rangle > 0,\ \forall a \in X$. Then we have a $\mathbb{Z}$ valued function $P_X(b)$ on $\mathbb{Z}^s$ defined as:

$$P_X(b) := \#\{t_1, \dots, t_m\} \in \mathbb{N}^m \mid \sum_{i=1}^{m} t_i a_i = b$$

$P_X(b)$ is best expressed via its generating series

$$\sum_{b \in \mathbb{Z}^s} P_X(b) e^b = \prod_{a \in X} \frac{1}{1 - e^a}.$$

For a general $X$ consider a linear function $\phi$ which is non–zero on each element of $X$.

- then $X$ is divided into two parts $A, B$ where $\phi$ is positive, respectively negative.
- We may thus consider the partition function $P_{A,-B}$,
- it depends only on the chamber of the hyperplane arrangement defined by $X$ in the dual space, in which $\phi$ lies.

# The space $\tilde{\mathcal{F}}_X$

Integer valued functions on $\mathbb{Z}^s$ are a module (using translation) over the group algebra $\mathbb{Z}[\mathbb{Z}^s]$.

**We can thus consider**

the $\mathbb{Z}[\mathbb{Z}^s]$ module $\tilde{\mathcal{F}}_X$ of functions on $\mathbb{Z}^s$ generated by all the partition functions $P_{A,-B}$ for all the chambers.

# The role of $\tilde{\mathcal{F}}_X$

- The space $\tilde{\mathcal{F}}_X$ can be completely described
- it is built naturally from the space $DM(X)$ and the spaces $DM(Y)$ from sublists of $X$.
- It describes certain coefficients in equivariant $K-$theory and in the index Theorem for transversally elliptic operators.
- It can be defined by difference equations.

# The Atiyah–Singer index theorem

**In 1968 appears the fundamental work of Atiyah and Singer**

on the index theorem of elliptic operators, a theorem formulated in successive steps of generality.
This theorem is the crowning point of a long sequence of ideas from Riemann, De Rham, Hodge, Hirzebruch on cohomology on manifolds

# What is an index?

Recall that the index $i(A)$ of a linear operator, when defined is

$$i(A) := \dim(\ker A) - \dim(\ker A^*),$$

where $\ker(A)$ is the space of solutions of the homogeneous equation $Au = 0$. $A^*$ denotes the adjoint.

- The linear operators consider by Atiyah and Singer are differential (or rather pseudodifferential) operators on manifolds.
- Precisely they operate not simply on functions but as usual in global geometry on *sections of vector bundles*.

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# The condition of being *elliptic*

(like the Laplace operator) is needed to insure that the index is well defined and:

**What is the index theorem?**

the index theorem produces a formula for the index through cohomological data associated to the operator, the manifold and the bundles involved.
(Chern character and Todd class).

# The need for $K-$theory

An essential step in the index theory is given by the construction of a generalized cohomology theory called $K - theory$.

Then the index theorem passes through several steps:

- to an operator one associates the symbol, which is a (matrix valued) function on the cotangent bundle.
- to the symbol an element of a suitable group of $K - theory$, the index can be computed this way.
- Alternatively from $K - theory$ one passes to cohomology.
- The final formula is proved by showing enough properties of all these steps which reduce the formula to some basic cases.

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# A deep generalization

**One general and useful setting is for operators on a manifold $M$ which:**

1. satisfy a symmetry with respect to a compact Lie group $G$
2. are elliptic in directions transverse to the $G$-orbits.

**In this case:**

The values of the index are generalized functions on $G$.

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# The appearance of distributions

- In this case the kernel of $A$ or of $A^*$ is no more finite dimensional, but it is of trace class as representation.
- $\rho : G \longrightarrow U(\mathbb{H})$ from the group of symmetries $G$ to unitary operators on a Hilbert space.

**This means that**

we can integrate the $C^\infty$ functions of $G$ to operators $T_f := \int_G f(g)\rho(g)$ which have a trace and

$f \mapsto Tr(T_f)$ is a distribution.

# The analytic index

## The analytic index of the operator $A$

is the distribution on $G$ obtained as difference of the trace on the spaces of solutions of $A$ and its adjoint $A^*$ (in an appropriate Sobolev space).

This is equivalently described by Fourier coefficients, which are an integral valued function on the set of irreducible representations, *the multiplicity*.

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# An open question was

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Elliptic operators and compact groups,
Springer L.N.M., n. 401, 1974.

Atiyah explains how to reduce general computations to the case in which

- $G$ is a torus,
- the manifold $M$ is a complex linear representation $M_X = \oplus_{a \in X} L_a$,
- $X \subset \hat{G}$ is a finite list of characters and $L_a$ the one dimensional complex line where $G$ acts by the character $a \in X$.

He then computes explicitly in several cases and ends his introduction saying

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# An open question

" ... for a circle (with any action) the results are also quite explicit. However for the general case we give only a reduction process and one might hope for something explicit. This probably requires the development of an appropriate algebraic machinery, involving cohomology but going beyond it."

# Solution to this open question

**Reference**

Vector partition functions and index of transversally elliptic operators
Authors: Corrado De Concini, Claudio C. Procesi, Michele Vergne
arXiv:0808.2545

This algebraic machinery turns out to be a spinoff of the theory of splines and partition functions, and provides a complete solution to this question.

# The main topological ingredient of the theory is

**the equivariant $K-$theory group**

$K^0_G(T^*_G M)$ which is defined in a topological fashion.

**Where:**

$T^*M$ is the cotangent bundle.
We denote by $T^*_G M$ the closed subset of $T^*M$, union of the conormals to the $G$ orbits.

This index depends only of the class defined by $\sigma$ in $K^0_G(T^*_G M)$, so that in the end:

# Combinatorics of Fourier coefficients

the index defines a $R(G)$ module homomorphism from $K^0_G(T^*_G M)$ to virtual trace class representations of $G$.

## The main goal

Describe explicitly the map induced by the index, from $K^0_G(T^*_G M)$ to virtual trace class representations of $G$.

# Combinatorics of Fourier coefficients

This is described combinatorially, if we identify the character group of $G$ with the free abelian group $\mathbb{Z}^s$ we have as values of the index a space of integer valued functions on $\mathbb{Z}^s$.

## The result

- We identify the space of functions image of the index for $M = M_X$.
- We prove that the index is an isomorphism between $K^0_G(T^*_G M_X)$ and its image in the functions.

The MAIN THEOREM is that

$K^1_G(T^*_G M_X) = 0$, and the index induces an isomorphism between $K^0_G(T^*_G M_X)$ and $\tilde{\mathcal{F}}_X$.

Moreover there is a very precise description of the group $\tilde{\mathcal{F}}_X$.

SEE the last part of this presentation.

# An example

**Example (of $T^*_G M$)**

$M = \mathbb{R}^2$, $G = S^1$ acting by rotations on the plane $\mathbb{R}^2$, the orbits are the circles centered at the origin, we identify $T^*M = \mathbb{R}^4$ with pairs of vectors.

A pair of vectors $(a, b) \in \mathbb{R}^2 \times \mathbb{R}^2$ is in $T^*_G M$ if either $a = 0$ its orbit reduces to 0 and $b$ arbitrary or $a \neq 0$ its orbit is a circle and $b$ is orthogonal to the orbit if it is proportional to $a$.

# An example

We write generalized functions by their generating series $\sum_{n\in\mathbb{Z}} a_n t^n$.

## Example (of values of the index)

Consider the case where $G = S^1$ acts by homotheties on $M_X = \mathbb{C}^{k+1}$.
We denote by $t$ the basic character of $S^1 := \{t \mid |t| = 1\}$, so that $R(G) = \mathbb{Z}[t, t^{-1}]$ and $X = [t, t, \dots, t]$ , $k+1$ times.

First Atiyah-Singer constructed a "pushed" $\overline{\partial}$ operator on $M_X$, with index the trace of the representation of $G$ in the symmetric algebra $S(M_X)$.
We get the generalized function

$$\Theta_X(t) := \sum_{n=0}^{\infty} \binom{n+k}{k} t^n.$$

# An example

Notice that

$$\binom{n+k}{k} = \frac{(n+k)(n+k-1)\dots(n+1)}{k!}, \ n \geq 0$$

is the partition function for $X = \underbrace{\{1, \dots, 1\}}_{k+1}$.

In this case we only have two chambers and the other partition function

$$\binom{-n+k}{k}, n \leq 0$$

for $X = \underbrace{\{-1, \dots, -1\}}_{k+1}$.

# An example

**Remark**

*$n \mapsto \binom{n+k}{k}$ extends to a polynomial function on $\mathbb{Z}$.*

For any $n$ positive or negative, the function $n \mapsto \binom{n+k}{k}$ represents the dimension of a *virtual space*, the alternate sum of the cohomology spaces of the sheaf $\mathcal{O}(n)$ on $k-$ dimensional projective space.

# An example

**In particular, the tangential Cauchy-Riemann operator**

on the unit sphere $S_{2k+1}$ of $\mathbb{C}^{k+1}$ is a transversally elliptic operator with index

$$\theta_k(t) := \sum_{n=-\infty}^{\infty} \binom{n+k}{k} t^n,$$

a generalized function on $G$ supported at $t = 1$.

# An example

**Theorem**

*The index map is an isomorphism from $K^0_G(T^*_G M_X)$ to the space $\tilde{\mathcal{F}}(X)$ of generalized functions on $G$ generated by $\Theta_X$ and $\theta_X$ under multiplication by elements of $R(G) = \mathbb{Z}[t, t^{-1}]$.*

In fact the $R(G)$ module generated by $\Theta_X$ is free over $R(G)$.

**The $R(G)$ module generated by $\theta_X$ is the torsion submodule.**

- This submodule is the module of polynomial functions on $\mathbb{Z}$ of degree at most $k$ so it is a free $\mathbb{Z}$-module of rank $k+1$.
- It corresponds to indices of operators on $\mathbb{C}^{k+1} - \{0\}$, the set where $S^1$ acts freely
- is the space of solutions of the difference equation $\nabla^k f = 0$ where we define the operator $\nabla$ by $(\nabla f)(n) = f(n) - f(n-1)$.

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# WHY SPLINES

*THE SPLINES*

# SPLINES $X := \{a_1, \dots, a_m\},\ a_i \in \mathbb{R}^s$

Given a list $X := \{a_1, \dots, a_m\}$, of vectors $a_i \in \mathbb{R}^s$, the

## *multivariate spline*

is the function $T_X(x)$ characterized by the formula:

$$\int_{\mathbb{R}^n} f(x) T_X(x) dx = \int_{\mathbb{R}^m_+} f(\sum_{i=1}^m t_i a_i) dt,$$

where $f(x)$ is any continuous function with compact support.
It represents the volume of a suitable variable polytope.
$T_X$ is the convolution of the tempered distributions $T_a$ corresponding to the half lines generated by the elements $a \in X$.

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# BOX SPLINES

While the function $T_X(x)$ is the basic object, the more interesting object for numerical analysis is the

## *box spline*

that is the function $B_X(x)$ characterized by the formula:

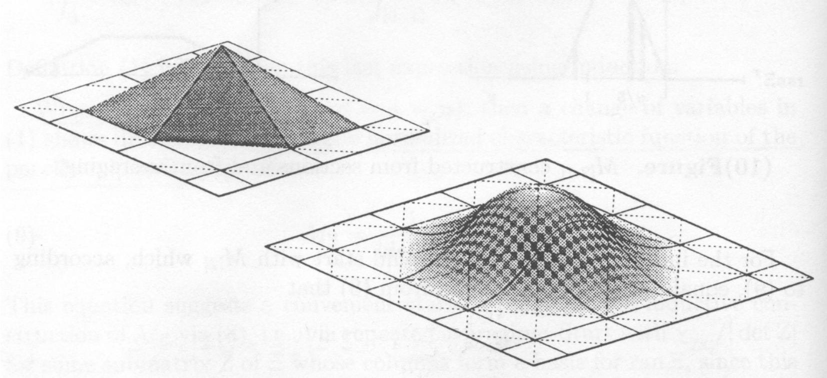
$$\int_{\mathbb{R}^n} f(x)B_X(x)dx = \int_{[0,1]^m} f(\sum_{i=1}^{m} t_i a_i)dt,$$

where $f(x)$ is any continuous function.
$B_X$ is the convolution of the tempered distributions $B_a$ corresponding to thesegments $[0, a]$ with $a \in X$.

# two box splines of class $C^0$, $h = 2$ and $C^1$, $h = 3$

$X = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ Hat function or Courant element

$X = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$ Zwart-Powell element



# BASIC REFERENCE BOOKS

**The book:**

C. De Boor, K. Höllig, S. Riemenschneider,

*Box splines*

Applied Mathematical Sciences 98 (1993).

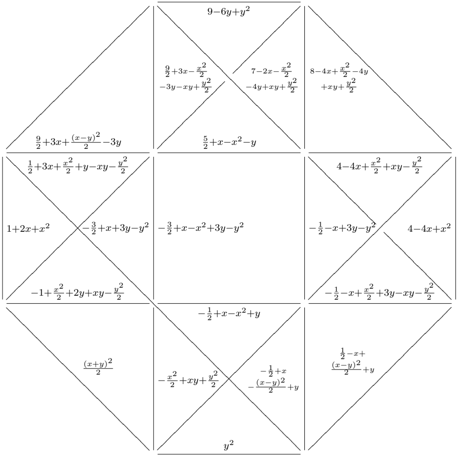
**Forthcoming book**

Topics in hyperplane arrangements, polytopes and box–splines

De Concini C., Procesi C.

http://www.mat.uniroma1.it/~procesi/dida.html

# The box–spline for type $B_2$



# 6 reasons WHY the BOX SPLINE ?

$$\int_{\mathbb{R}^n} B_X(x)dx = 1$$

**recursive definition**

$$B_{[X,v]}(x) = \int_0^1 B_X(x - tv)dt$$

in the case of integral vectors, we have

**THE CARDINAL SPLINE SPACE $\mathcal{S}_X$**

The space $\mathcal{S}_X$ of linear combinations of translates

$$\mathcal{S}_X := \{\sum_{\lambda \in \Lambda} B_X(x - \lambda)a(\lambda)\}$$

$a(\lambda)$ is any function on $\Lambda$ a mesh function.

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# 6 reasons WHY the BOX SPLINE ?

**A polynomial $p$ is in the cardinal space $\mathcal{S}_X$**

if and only if it belongs to the space

$$D(X) = \{p \mid D_Y p = 0, \ \ Y \quad \text{a cocircuit}\} \ D_Y = \prod_{a \in Y} D_a$$

$D_a$ is the usual directional derivative

**PARTITION OF UNITY $X$ integral vectors**

The translates $B_X(x - \lambda)$, $\lambda$ runs over the integral vectors form a *partition of 1*.

# 6 reasons WHY the BOX SPLINE ?

## EVEN MORE

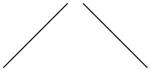
The convolution $\sum_{\lambda\in\Lambda} B_X(x-\lambda)p(\lambda)$, as $p$ runs over the remarkable space of polynomials $D(X)$ is again a polynomial in $D(X)$.

## The box–spline $B_X$ is refinable:

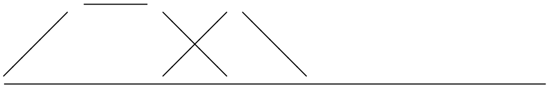
$$B_X(x) = \frac{1}{2^{|X|-s}} \sum_{S\subset X} B_X(2x - a_S).$$

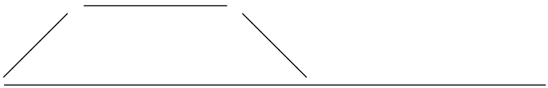
# TRIVIAL EXAMPLE

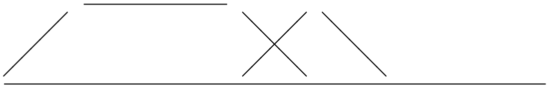
$X = \{1, 1\}$ we have for the box spline

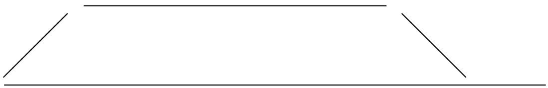


Now let us add to it its translates!









# APPROXIMATION THEORY

## *APPROXIMATION POWER*

## *THE STRANG-FIX CONDITIONS*

# THE CARDINAL SPLINE SPACE

The box spline, when the $a_i$ are integral vectors, can be effectively used in the *finite element method* to approximate functions through function in the

**Cardinal spline space $\mathcal{S}_X$**

$$\mathcal{S}_X := \{\sum_{i \in \mathbb{Z}^s} B_X(x-i) f(i), \ f : \mathbb{Z}^s \to \mathbb{R}\}$$

and its RESCALED functions.

This means that for every positive integer $n$ we want to find ***weights*** $c_{\underline{i},n}$ and approximate $f$ by $f(x) \mapsto \sum_{\underline{i}\in\mathbb{Z}^s} B_X(nx-\underline{i})c_{\underline{i},n}$ and determine a constant $k \in \mathbb{N}$ so that (on some bounded region):

$$|f(x) - \sum_{\underline{i}\in\mathbb{Z}^s} B_X(nx-\underline{i})c_{\underline{i},n}| \leq Cn^{-k}$$

The maximum $k$ is the

*approximation power of $B_X$.*

# Strang–Fix conditions

The Strang–Fix conditions is a general statement:

*The approximation power of a function $M$ is the maximum $r$ such that the space of all polynomials of degree $\leq r$ is contained in the cardinal space $\mathcal{S}_M$.*

# The cocircuits

We can thus apply the property of the cardinal spline space is that it contains many polynomials.

**Theorem**

*A polynomial p belongs to $\mathcal{S}_X$ if and only if it lies in $D(X)$.*

# THE THEORY OF DHAMEN–MICCHELLI

From the definitions it follows that $D(X)$ contains all polynomials of degree $< m(X)$ where $m(X)$ is the minimum number of elements in a cocircuit.

**Strang-Fix conditions**

The power of approximation of $B_X$ is $m(X)$.

# SUPERFUNCTIONS

Consider the following algorithm applied to a function $g$:

$$g_h := \sum_{\underline{i} \in \Lambda} F(x/h - \underline{i}) g(h\underline{i}), \;\; h \to 0$$

**super–functions**

There are functions $F$ in the cardinal spline space such that this transformation is the identity on polynomials in $D(X)$, these are the super–functions.

For such functions the previous algorithm satisfies the requirements of the Strang–Fix approximation

# SUPERFUNCTIONS

## Theorem

*We have, under the explicit algorithm previously constructed that, for any domain G:*

$$||f_h - f||_{L^\infty(G)} = O(h^{m(X)}).$$

*For every multi–index* $\alpha \in \mathbb{N}^s$ *with* $|\alpha| \leq m(X) - 1$*, we have:*

$$||\partial^\alpha f_h - \partial^\alpha f||_{L^\infty(G)} = ||\partial^\alpha (f_h - f)||_{L^\infty(G)} = O(h^{m(X)-|\alpha|}).$$

# TWO SUPERFUNCTIONS

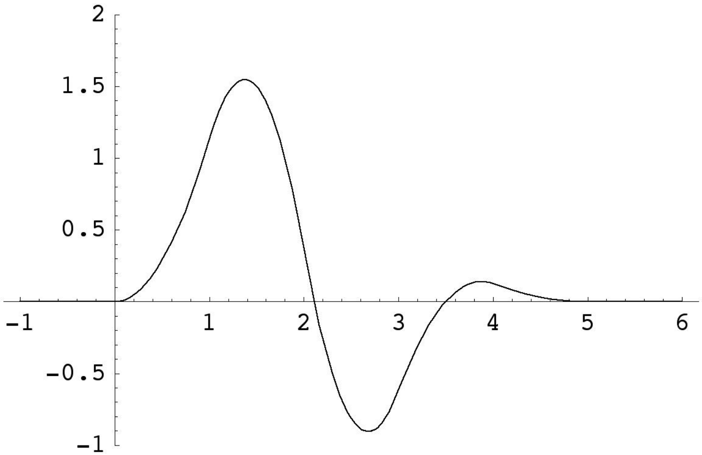



Figure: The superfunction associated to 1, 1, 1, 1.

# TWO SUPERFUNCTIONS

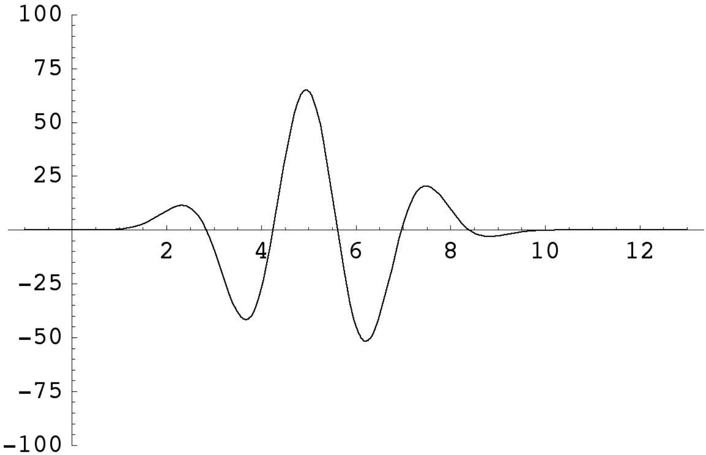



Figure: The superfunction associated to $1^7$.

# THE DISCRETE CASE

*PARTITION FUNCTIONS*

*THE DISCRETE CASE*

# THE PARTITION FUNCTION

Let $A := \{a_1, \dots, a_m\}, b$ be a list of vectors with integer coordinates.
It is natural to think of an expression like:
$b = t_1 a_1 + \cdots + t_m a_m$ with $t_i$ not negative integers as a:

**partition of $b$ with the vectors $a_i$,**

in $t_1 + t_2 + \cdots + t_m$ parts, hence the name partition function for the number

$$P_X(b) = \#\{t_1, \dots, t_m \in \mathbb{N} \mid \sum_{i=1}^{m} t_i a_i = b\},$$

thought of as a function of the vector $b$.

SIMPLE EXAMPLE $m = 2,\ n = 1,\ A = \{2, 3\}$

## Parts are 2 and 3

In how many ways can you write a number $b$ as:

$$b = 2x + 3y, \quad x, y \in \mathbb{N} \ ?$$

## ANSWER (Quasi polynomial!)

It depends on the class of $n$ modulo 6.

$$n \cong 0 \qquad \frac{n}{6} + 1$$

$$n \cong 1 \qquad \frac{n}{6} - \frac{1}{6}$$

$$n \cong 2 \qquad \frac{n}{6} + \frac{2}{3}$$

$$n \cong 3 \qquad \frac{n}{6} + \frac{1}{2}$$

$$n \cong 4 \qquad \frac{n}{6} + \frac{1}{3}$$

$$n \cong 5 \qquad \frac{n}{6} + \frac{1}{6}$$

# THE DISCRETE CASE

## multivariate spline and partition function

1. The partition function is the natural discrete analogue of the multivariate spline. They depend both on a list of vectors.
2. The multivariate spline describes the volume of a variable polytope.
3. The partition function describes the number of integral points in a variable polytope.

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# THE DISCRETE CASE

In general the cone generated by $A$ decomposes into regions called

**Big cells**

$P_X(b)$ is a quasi polynomial on each big cell and $T_X(b)$ is a polynomial on each big cell.
The quasi polynomials describing $P_X(b)$ belongs to the remarkable space $DM(X)$ introduced by Dahmen–Micchelli.

# Some intuitive pictures

We want to decompose the cone $C(A)$ into *big cells* and define its singular and regular points.

We do everything on a transversal section, where the cone looks like a bounded convex polytope and then project.

**Example**

Let us start with some pictures where $A$ is the list of positive roots for type $A_3$.

$$\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3$$

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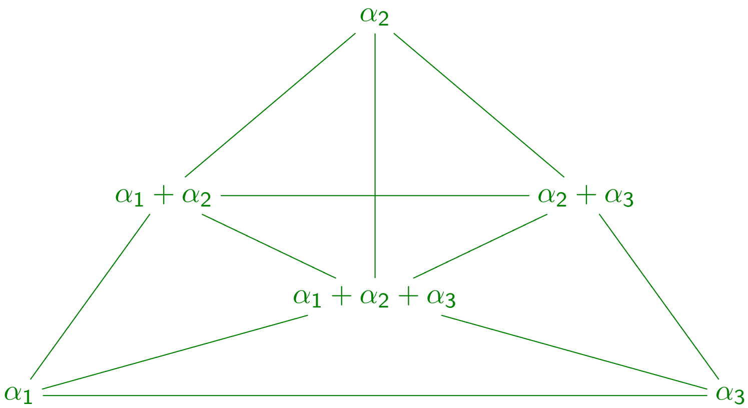
# EXAMPLE Type $A_3$ in section (big cells):




We have 7 big cells.

In fact take a vector $v$ in the big cell $\mathfrak{c}$ so that $v - B(X) \cap C(X) = \{0\}$ then the partition function on $\mathfrak{c}$ coincides with the element of $DM(X)$ which is 0 on $[v - B(X) \cap C(X)] \setminus \{0\}$ and 1 at 0.

# BACK TO THE INDEX

*INDEX THEORY*

*THE VALUES OF THE INDEX*

# BACK TO THE INDEX

THE SPACE $DM(X)$

*IS THE KEY TO UNDERSTAND THE INDEX*

# Recall The MAIN THEOREM

## Recall

the $\mathbb{Z}[\mathbb{Z}^s]$ module $\tilde{\mathcal{F}}_X$ of functions on $\mathbb{Z}^s$ generated by all the partition functions $P_{A,-B}$ for all the chambers.

## Theorem

*the index induces an isomorphism between $K^0_G(T^*_G M_X)$ and $\tilde{\mathcal{F}}_X$.*

Moreover there is a very precise description of the group $\tilde{\mathcal{F}}_X$.

# The values of the index

The main result is about the open set $M_X^f$ of $M_X$ where $G$ acts with finite stabilizers.

**Theorem**

*The index map induces an isomorphism between $K_G^0(T_G^* M_X^f)$ and the space $DM(X)$.*

For the general case we need some associated spaces of functions, generalizing $DM(X)$.

The spaces $\tilde{\mathcal{F}}_i(X)$ associated to the orbit types in $M_X$.

Any linear subspace $\underline{r}$ generated by vectors in $X$ is called a rational subspace

## An auxiliary space

$$\mathcal{F}(X) := \{f \mid \nabla_{X \setminus \underline{r}} f \text{ is supported on } \underline{r} \text{ for every rational subspace } \underline{r}\}.$$

Clearly $DM(X)$ is contained in $\mathcal{F}(X)$.

One can show that $\tilde{\mathcal{F}}(X)$ is the space of functions generated by $\mathcal{F}(X)$ under translations by integral vectors.

Define the spaces

$$\mathcal{F}_i(X) := \cap_{\underline{t} \in S_X^{(i-1)}} \ker \nabla_{X \setminus \underline{t}} \cap \mathcal{F}(X).$$

We denote by $\tilde{\mathcal{F}}_i(X)$ the space of functions generated by $\mathcal{F}_i(X)$ under translations by integral vectors.

Set $M_{\geq i}$ as the open set of points in $M_X$ with the property that the orbit has dimension $\geq i$.

**Theorem**

*For each $s \geq i \geq 0$, the index multiplicity map $ind_m$ gives an isomorphism between $K^0_G(T^*_G M_{\geq i})$ and the space $\tilde{\mathcal{F}}_i(X)$.*

We have a decomposition

$$\mathcal{F}(X) = DM(X) \oplus \Big( \oplus_{\underline{r} \in S_X | \underline{r} \neq V} \mathcal{P}_{X \setminus \underline{r}}^{F_{\underline{r}}} * DM(X \cap \underline{r}) \Big).$$

$S_X$ is the set of rational subspaces.

Finally one has very explicit descriptions of $DM(X)$.

From these one describes completely the spaces

$\tilde{\mathcal{F}}_i(X)$ and hence the values of the index.

# Beautiful Hokkaido

*THE END*

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