Webs and Arrangements

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Outline

Unachieved goal

Study the first resonance variety $R^1(A)$.

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Notation

$A = \{H_1, \dots, H_m\}$ arrangement in $\mathbb{P}^n = \mathbb{P}(V)$
$M = \mathbb{P}^n \setminus A$
$R^1(A)$ = maximal isotropic subspaces of $H^1(A, \mathbb{C})$ of dimension at least two.

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Irreducible components are well understood

The irreducible components of $R^1(A)$ of dimension d are in one to one correspondence with the pencil of hypersurfaces having irreducible generic member and $d + 1$ completely decomposable fibers with support contained in $|A|$.

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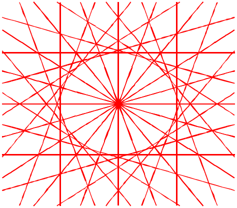

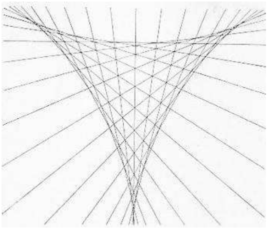

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$$\int \sum_{i=1}^{k} \eta_i \quad \Longrightarrow \quad \sum_{i=1}^{k} g_i(u_i) = 0$$

Bounds for the rank

Theorem (Bol ($n = 2$) Chern ($n \geq 3$))

If $\mathcal{W}$ is a smooth k-web on $(\mathbb{C}^n, 0)$ then

$$\dim \mathcal{A}(\mathcal{W}) = \mathrm{rank}(\mathcal{W}) \leq \pi(n,k) = \sum_{j=1}^{\infty} \max(0, k - j(n-1) - 1)\,.$$

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Idea of the proof

$$F^\bullet \mathcal{A}(\mathcal{W}) : \quad F^j \mathcal{A}(\mathcal{W}) = \ker \left\{ \mathcal{A}(\mathcal{W}) \longrightarrow \left(\frac{\Omega^1(\mathbb{C}^n, 0)}{\mathfrak{m}^j \cdot \Omega^1(\mathbb{C}^n, 0)} \right)^k \right\}.$$

$$\begin{aligned} \dim \frac{F^j \mathcal{A}(\mathcal{W})}{F^{j+1} \mathcal{A}(\mathcal{W})} \quad & \leq \quad k - \dim \left(\mathbb{C} \cdot \ell_1^{j+1} + \cdots + \mathbb{C} \cdot \ell_k^{j+1} \right) \\ & \leq \quad \max(0, k - (j+1)(n-1) + 1) \end{aligned}$$

where ℓ_i is a linear form defining $T_0 \mathcal{F}_i$.

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- $\pi(n, k)$ is Castelnuovo's bound for the genus of irreducible non-degenerate degree k curves in $\mathbb{P}^n$.

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Remarks

- $\pi(n,k)$ is Castelnuovo's bound for the genus of irreducible non-degenerate degree k curves in $\mathbb{P}^n$.
- the proof shows how to bound the rank of quasi-smooth webs. One has to know the dimension of the space generated by powers of linear forms determining $T_0\mathcal{F}_i$.

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Questions

- Is the bound sharp ?
- Is there a characterization of webs of maximal rank ?

Algebraic Webs

$C \subset \mathbb{P}^n$ reduced curve. $H_0 \in \check{\mathbb{P}}^n$ transverse to C.
$H_0 \cap C = p_1 + \cdots + p_k$.

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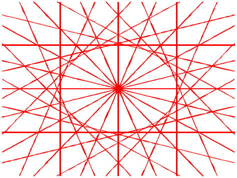
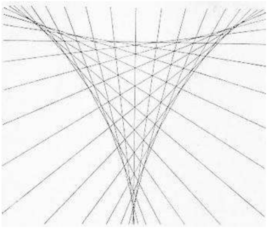
These functions define the k-web $\mathcal{W}_C = \mathcal{W}_C(H_0)$.





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Theorem (Abel's Addition Theorem)

$$(p_1 \oplus \cdots \oplus p_k)^* \mathrm{H}^0(C, \omega_C) \hookrightarrow \mathcal{A}(\mathcal{W}_C).$$

In particular, $\mathrm{rank}(\mathcal{W}_C) \geq h^0(C, \omega_C)$.

Algebraization results

Theorem (Lie)

If $\mathcal{W}$ is a quasi-smooth (= smooth in dimension two) 4-web on the plane with one abelian relation then $\mathcal{W}$ is algebraizable.

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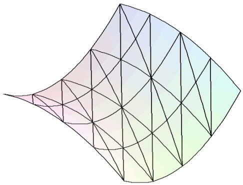

A **double translation surface** $S \subset \mathbb{R}^3$ that admits two independent parametrizations of the form $(x, y) \mapsto f(x) + g(y)$. S carries a natural 4-web $\mathcal{W}$. The leaves tangents of $\mathcal{W}$ cuts the hyperplane at infinity at 4 germs of curves. Lie's Theorem says that these 4 curves are contained in a degree 4 algebraic curve. Latter generalized by Wirtinger to arbitrary translation manifolds.

Algebraization results II

Theorem (Bol(n=3), Chern-Griffiths(hypothesis), Trépreau)

Let $n \geq 3$ and $k \geq 2n$. If $\mathcal{W}$ is a **smooth** *k-web on $(\mathbb{C}^n, 0)$ of maximal rank then $\mathcal{W}$ is algebraizable.*

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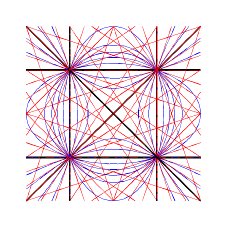
Questions

- What happens when $n = 2$ and $k \geq 5$?
- When $n \geq 3$ and $k \geq 2n$, are **quasi-smooth** k-webs of maximal rank algebraizable ?

Exceptional Webs

Bol's 5-web



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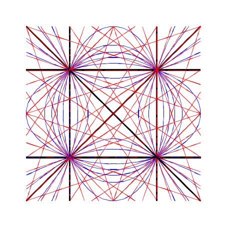

5 l.i. abelian relations of log type

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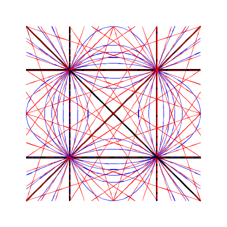

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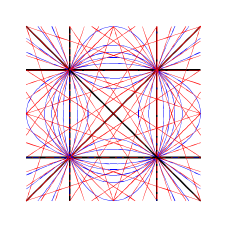

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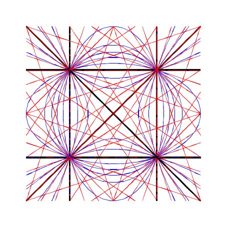

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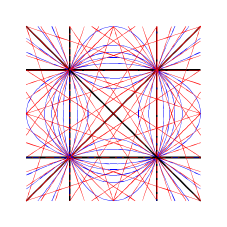

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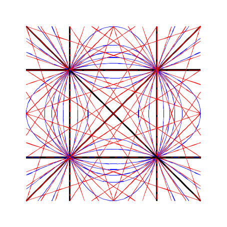

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Related to Spence-Kummer's functional equation for the trilog.

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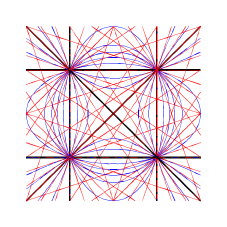

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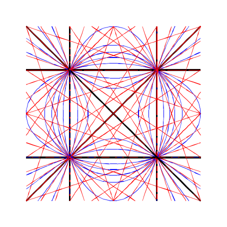

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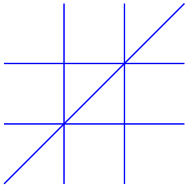
Bol’s 5-web revisited



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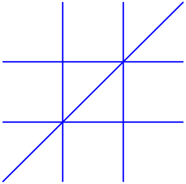

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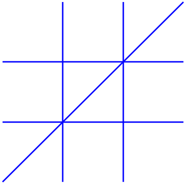
Bol's 5-web revisited



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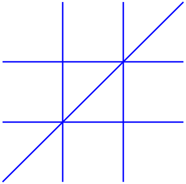

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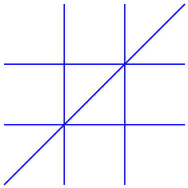

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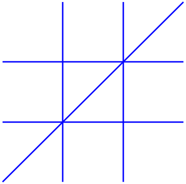
4 pencils of lines
1 pencil of conics

Bol's 5-web revisited



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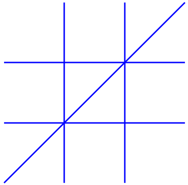

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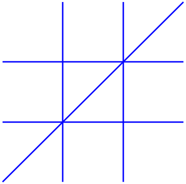

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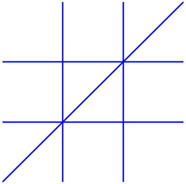
Bol's 5-web has maximal rank

$$\bigoplus_{\Sigma \subset \mathcal{R}^1(\mathcal{A})} H^1(C_\Sigma) \longrightarrow H^1(M_{0,5})$$

$$(\eta_\Sigma) \longmapsto \sum_\Sigma f_\Sigma^* \eta_\Sigma$$

Bol's 5-web revisited



Call it $\mathcal{A}_{0,5}$

$R^1(\mathcal{A}_{0,5})$ has 5 irred. components

4 pencils of lines

1 pencil of conics

Each with dimension two

The associated web is Bol's 5-web

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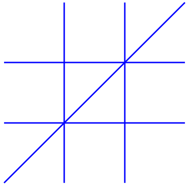
$$\bigoplus_{\Sigma \subset \mathcal{R}^1(\mathcal{A})} H^1(C_\Sigma) \longrightarrow H^1(M_{0,5}) \to 0$$

Bol's 5-web revisited



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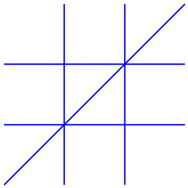
$$0 \to AR^1_{log}(\mathcal{A}_{0,5}) \longrightarrow \bigoplus_{\Sigma \subset \mathcal{R}^1(\mathcal{A})} H^1(C_\Sigma) \longrightarrow H^1(M_{0,5}) \to 0$$

$\dim AR^1_{log}(\mathcal{A}_{0,5}) = 5$

Bol's 5-web revisited



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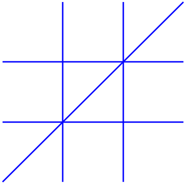
$$\begin{tikzcd}
 & & & N^2(M_{0,5}) \arrow[d] \\
0 \arrow[r] & AR^2_{log}(\mathcal{A}_{0,5}) \arrow[r] & \bigoplus_l H^1(C_\Sigma)^{\otimes 2} \arrow[r] \arrow[ur] & H^1(M_{0,5})^{\otimes 2} \arrow[d] \\
 & & & H^2(M_{0,5})
\end{tikzcd}$$

Bol's 5-web revisited



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Bol's 5-web has maximal rank

$$\dim AR^2_{log}(\mathcal{A}_{0,5}) = 1$$

$$\dim AR^1_{log}(\mathcal{A}_{0,5}) = 5$$

$$\dim \mathcal{A}(\mathcal{A}_{0,5}) = 6$$

Natural web on $M_{0,n+3}$

Let $\mathcal{A}_{0,n+3}$ be the arrangement defined by

$$\prod_{i=1}^{n} x_i \prod_{i=1}^{n}(x_i - 1)\prod_{i<j}(x_i - x_j)$$

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Theorem (P.)

For every $n \geq 2$ the equality

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holds true.

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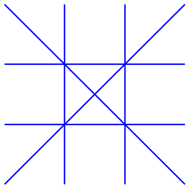
Examples of quasi-smooth webs with $k > 2n$, maximal rank and non-algebraizable.

Spence-Kummer's 9-web revisited



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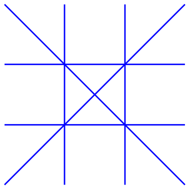

Call it $\mathcal{A}_{SK}$

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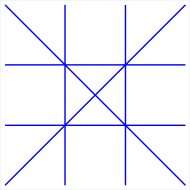

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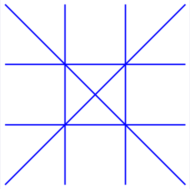

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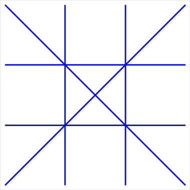

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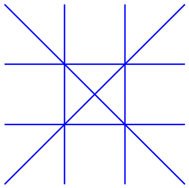

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All of them have dimension two

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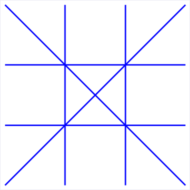

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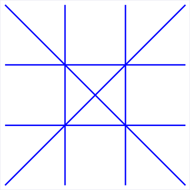

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$\dim AR^1_{log}(\mathcal{A}_{SK}) = 12$

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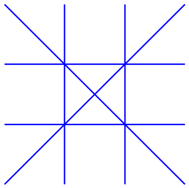

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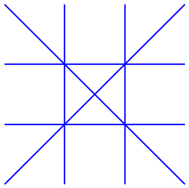

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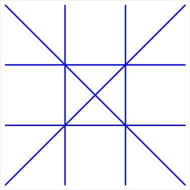

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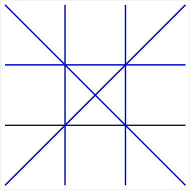

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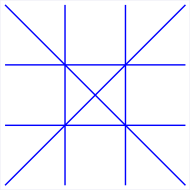

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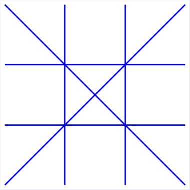
There is one missing.

Spence-Kummer's 9-web revisited



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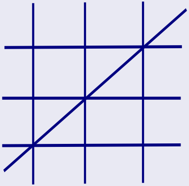
There is one missing. Intersection of characteristic varieties.

One parameter family of 8-webs



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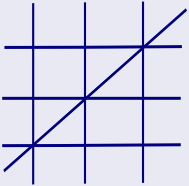

Call it $\mathcal{A}_P(a)$, $a \in \mathbb{C} \setminus \{0, 1\}$

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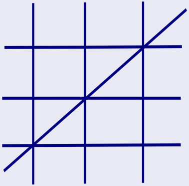

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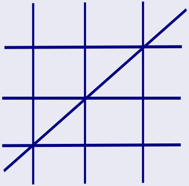

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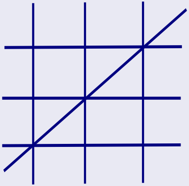

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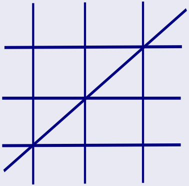

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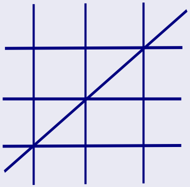

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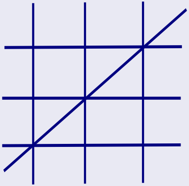

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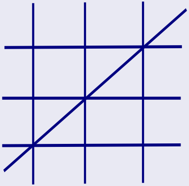

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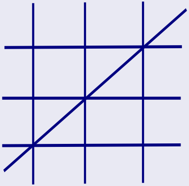

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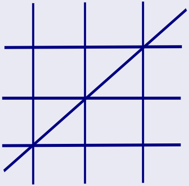

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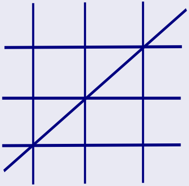

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There is one missing. Mixed iterated integrals.

Outline

Completely Decomposable Quasi-Linear Webs

The classification of exceptional planar webs is wide open.

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At some point, I needed to know the number of completely decomposible fibers in a pencil. $\Longrightarrow$ Google told me about Falk-Yuzvinsky's work on multi-nets.

Select Examples

The infinity families

$\mathcal{A}_I^k = \left[(dx^k - dy^k)\right] \boxtimes \left[d(xy)\right]$ where $k \geq 4$;

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Select Examples

The infinity families

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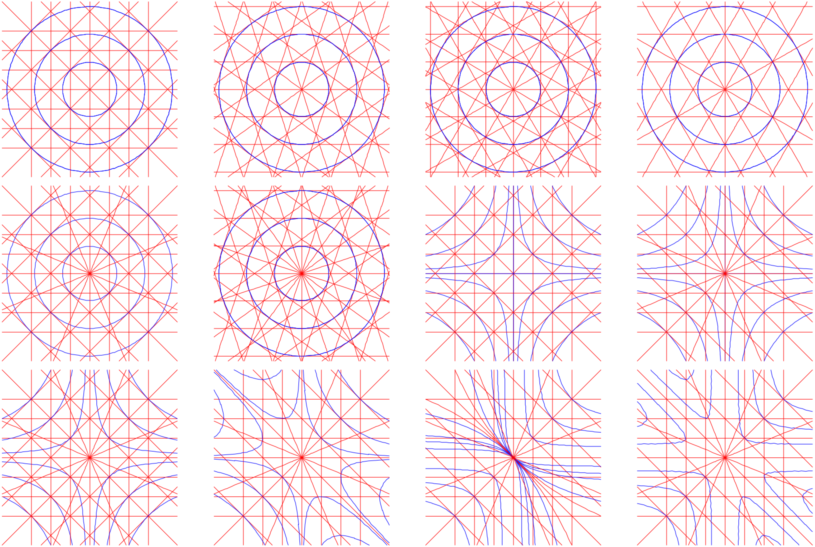
Pictures



Classification on tori (compact)

Theorem (P.,Pirio)

Up to isogenies, there are exactly three sporadic exceptional CDQL k-webs (one for each $k \in \{5, 6, 7\}$) and one continuous family of exceptional CDQL 5-webs on complex tori.

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Theorem

If T is a two-dimension complex tori and $f : T \dashrightarrow \mathbb{P}^1$ a meromorphic map then the number of linear fibers of f, when finite, is at most six. Moreover, the bound is sharp.

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Remark

Although linear fibers are rigid the bound is worst than for the projective plane (4 after Stipins-Yuzvinsky result).

The list

Infinity family

The elements of the continuous family are

$$\mathcal{E}_\tau = \left[dx\, dy\, (dx^2 - dy^2) \right] \boxtimes \left[d \left(\frac{\vartheta_1(x,\tau)\vartheta_1(y,\tau)}{\vartheta_4(x,\tau)\vartheta_4(y,\tau)} \right)^2 \right].$$

on E_τ^2, $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$. The functions ϑ_i are the Jacobi theta functions. (Pirio - Trépreau)

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