An introduction to toric arrangements

MSJ SI 2009 on Arrangements of Hyperplanes

Luca Moci

University of Roma Tre (Italy)

August, 9 2009

1. Introduction

Natural questions

- What are toric arrangements?
- Why are they interesting objects?
- What is it known about them?

Natural questions

- What are toric arrangements?
- Why are they interesting objects?
- What is it known about them?

Natural questions

- What are toric arrangements?
- Why are they interesting objects?
- What is it known about them?

An example

Let be $V = \mathbb{C}^2$ with coordinates (z_1, z_2),
$T = (\mathbb{C}^*)^2$ with coordinates (t_1, t_2), and

$$X = \{2z_1^*, 2z_2^*, z_1^* + z_2^*, z_1^* - z_2^*\} \subset V^*.$$

We associate to X 3 arrangements:

1. a central h.a. $\mathcal{H} = \{H_\chi\}_{\chi \in X}$ in V, defined by the equations $\chi(v) = 0$ (e.g. $2z_1 = 0$)
2. a periodic affine h.a. $\mathcal{A} = \{H_{\chi,m}\}_{\chi \in X, m \in \mathbb{Z}}$ in V, defined by the equations $\chi(v) = m$ (e.g. $2z_1 = 7$)
3. a toric arrangement $\mathcal{T} = \{U_\chi\}_{\chi \in X}$ in T, defined by the equations: $t_1^2 = 1$, $t_2^2 = 1$, $t_1 t_2 = 1$, $t_1 t_2^{-1} = 1$.

How does $\mathcal{T}$ arise, and how is it related with $\mathcal{A}$?

An example

Let be $V = \mathbb{C}^2$ with coordinates (z_1, z_2),
$T = (\mathbb{C}^*)^2$ with coordinates (t_1, t_2), and

$$X = \{2z_1^*, 2z_2^*, z_1^* + z_2^*, z_1^* - z_2^*\} \subset V^*.$$

We associate to X 3 arrangements:

1. a central h.a. $\mathcal{H} = \{H_\chi\}_{\chi \in X}$ in V,
defined by the equations $\chi(v) = 0$ (e.g. $2z_1 = 0$)
2. a periodic affine h.a. $\mathcal{A} = \{H_{\chi,m}\}_{\chi \in X, m \in \mathbb{Z}}$ in V,
defined by the equations $\chi(v) = m$ (e.g. $2z_1 = 7$)
3. a toric arrangement $\mathcal{T} = \{U_\chi\}_{\chi \in X}$ in T,
defined by the equations: $t_1^2 = 1$, $t_2^2 = 1$, $t_1 t_2 = 1$, $t_1 t_2^{-1} = 1$.

How does $\mathcal{T}$ arise, and how is it related with $\mathcal{A}$?

An example

Let be $V = \mathbb{C}^2$ with coordinates (z_1, z_2),
$T = (\mathbb{C}^*)^2$ with coordinates (t_1, t_2), and

$$X = \{2z_1^*, 2z_2^*, z_1^* + z_2^*, z_1^* - z_2^*\} \subset V^*.$$

We associate to X 3 arrangements:

1. a central h.a. $\mathcal{H} = \{H_\chi\}_{\chi \in X}$ in V,
   defined by the equations $\chi(v) = 0$ (e.g. $2z_1 = 0$)
2. a periodic affine h.a. $\mathcal{A} = \{H_{\chi,m}\}_{\chi \in X, m \in \mathbb{Z}}$ in V,
   defined by the equations $\chi(v) = m$ (e.g. $2z_1 = 7$)
3. a toric arrangement $\mathcal{T} = \{U_\chi\}_{\chi \in X}$ in T,
   defined by the equations: $t_1^2 = 1$, $t_2^2 = 1$, $t_1 t_2 = 1$, $t_1 t_2^{-1} = 1$.

How does $\mathcal{T}$ arise, and how is it related with $\mathcal{A}$?

An example

Let be $V = \mathbb{C}^2$ with coordinates (z_1, z_2),
$T = (\mathbb{C}^*)^2$ with coordinates (t_1, t_2), and

$$X = \{2z_1^*, 2z_2^*, z_1^* + z_2^*, z_1^* - z_2^*\} \subset V^*.$$

We associate to X 3 arrangements:

1. a central h.a. $\mathcal{H} = \{H_\chi\}_{\chi \in X}$ in V,
   defined by the equations $\chi(v) = 0$ (e.g. $2z_1 = 0$)
2. a periodic affine h.a. $\mathcal{A} = \{H_{\chi,m}\}_{\chi \in X, m \in \mathbb{Z}}$ in V,
   defined by the equations $\chi(v) = m$ (e.g. $2z_1 = 7$)
3. a toric arrangement $\mathcal{T} = \{U_\chi\}_{\chi \in X}$ in T,
   defined by the equations: $t_1^2 = 1,\ t_2^2 = 1,\ t_1 t_2 = 1,\ t_1 t_2^{-1} = 1.$

How does $\mathcal{T}$ arise, and how is it related with $\mathcal{A}$?

An example

Let be $V = \mathbb{C}^2$ with coordinates (z_1, z_2),
$T = (\mathbb{C}^*)^2$ with coordinates (t_1, t_2), and

$$X = \{2z_1^*, 2z_2^*, z_1^* + z_2^*, z_1^* - z_2^*\} \subset V^*.$$

We associate to X 3 arrangements:

1. a central h.a. $\mathcal{H} = \{H_\chi\}_{\chi \in X}$ in V,
defined by the equations $\chi(v) = 0$ (e.g. $2z_1 = 0$)
2. a periodic affine h.a. $\mathcal{A} = \{H_{\chi,m}\}_{\chi \in X, m \in \mathbb{Z}}$ in V,
defined by the equations $\chi(v) = m$ (e.g. $2z_1 = 7$)
3. a toric arrangement $\mathcal{T} = \{U_\chi\}_{\chi \in X}$ in T,
defined by the equations: $t_1^2 = 1,\ t_2^2 = 1,\ t_1 t_2 = 1,\ t_1 t_2^{-1} = 1.$

How does $\mathcal{T}$ arise, and how is it related with $\mathcal{A}$?

Affine and toric arrangements

Let Λ be the $\mathbb{Z}-$span of X. We have a natural map

$$exp : V \to V/\Lambda \simeq T$$

(topologically $exp : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}^2 \simeq (\mathbb{C}^*)^2$)

$$\underline{z} \mapsto \underline{t} \doteq e^{2\pi i \underline{z}}.$$

exp maps $\{H_{\chi,m}\}_{m\in\mathbb{Z}}$ onto U_χ, and the complement of $\mathcal{A}$ onto the complement of $\mathcal{T}$ (covering with fiber Λ).

Switching from $\mathcal{A}$ to $\mathcal{T}$ we loose linearity, but we earn finiteness!

Affine and toric arrangements

Let Λ be the $\mathbb{Z}-$span of X. We have a natural map

$$exp : V \to V/\Lambda \simeq T$$

$$(\text{topologically } \; exp : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}^2 \simeq (\mathbb{C}^*)^2)$$

$$\underline{z} \mapsto \underline{t} \doteq e^{2\pi i \underline{z}}.$$

exp maps $\{H_{\chi,m}\}_{m\in\mathbb{Z}}$ onto U_χ, and the complement of $\mathcal{A}$ onto the complement of $\mathcal{T}$ (covering with fiber Λ).

Switching from $\mathcal{A}$ to $\mathcal{T}$ we loose linearity, but we earn finiteness!

Let Λ be the $\mathbb{Z}-$span of X. We have a natural map

$$exp : V \to V/\Lambda \simeq T$$

$$(\text{topologically} \;\; exp : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}^2 \simeq (\mathbb{C}^*)^2)$$

$$\underline{z} \mapsto \underline{t} \doteq e^{2\pi i \underline{z}}.$$

exp maps $\{H_{\chi,m}\}_{m\in\mathbb{Z}}$ onto U_χ, and the complement of $\mathcal{A}$ onto the complement of $\mathcal{T}$ (covering with fiber Λ).

Switching from $\mathcal{A}$ to $\mathcal{T}$ we loose linearity, but we earn finiteness!

Let Λ be the $\mathbb{Z}-$span of X. We have a natural map

$$exp : V \to V/\Lambda \simeq T$$

$$(\text{topologically} \;\; exp : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}^2 \simeq (\mathbb{C}^*)^2)$$

$$\underline{z} \mapsto \underline{t} \doteq e^{2\pi i \underline{z}}.$$

exp maps $\{H_{\chi,m}\}_{m\in\mathbb{Z}}$ onto U_χ, and the complement of $\mathcal{A}$ onto the complement of $\mathcal{T}$ (covering with fiber Λ).

Switching from $\mathcal{A}$ to $\mathcal{T}$ we loose linearity, but we earn finiteness!

Affine and toric arrangements

Let Λ be the $\mathbb{Z}-$span of X. We have a natural map

$$exp : V \to V/\Lambda \simeq T$$

$$\text{(topologically } \; exp : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}^2 \simeq (\mathbb{C}^*)^2)$$

$$\underline{z} \mapsto \underline{t} \doteq e^{2\pi i \underline{z}}.$$

exp maps $\{H_{\chi,m}\}_{m\in\mathbb{Z}}$ onto U_χ, and the complement of $\mathcal{A}$ onto the complement of $\mathcal{T}$ (covering with fiber Λ).

Switching from $\mathcal{A}$ to $\mathcal{T}$ we loose linearity, but we earn finiteness!

Let Λ be the $\mathbb{Z}-$span of X. We have a natural map

$$exp : V \to V/\Lambda \simeq T$$

$$(\text{topologically } \ exp : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}^2 \simeq (\mathbb{C}^*)^2)$$

$$\underline{z} \mapsto \underline{t} \doteq e^{2\pi i \underline{z}}.$$

exp maps $\{H_{\chi,m}\}_{m\in\mathbb{Z}}$ onto U_χ, and the complement of $\mathcal{A}$ onto the complement of $\mathcal{T}$ (covering with fiber Λ).

Switching from $\mathcal{A}$ to $\mathcal{T}$ we loose linearity, but we earn finiteness!

Hyperplane and toric arrangements

A hyperplane arrangement in a vector space V is a family of hyperplanes $\mathcal{H} = \{H_\chi\}_{\chi \in X}$, where $X \subset Hom(V, \mathbb{C})$ and $H_\chi \doteq \{v \in V | \chi(v) = 0\}$.
A toric (toral) arrangement in a torus T is a family of hypersurfaces $\mathcal{T} = \{U_\chi\}_{\chi \in X}$, where $X \subset Hom(T, \mathbb{C}^*)$ and $U_\chi \doteq \{t \in T | \chi(t) = 1\}$.

(Lehrer and others in the '90s; De Concini and Procesi, 2005).

With $\mathcal{H}$ is associated the poset $\mathcal{L}(X)$ of the intersections of the $\{H_\chi\}_{\chi \in X}$.
With $\mathcal{T}$ is associated the poset $\mathcal{C}(X)$ of the components: connected components of the intersections of the $\{U_\chi\}_{\chi \in X}$.

Hyperplane and toric arrangements

A hyperplane arrangement in a vector space V is a family of hyperplanes $\mathcal{H} = \{H_\chi\}_{\chi \in X}$, where $X \subset Hom(V, \mathbb{C})$ and $H_\chi \doteq \{v \in V | \chi(v) = 0\}$.
A toric (toral) arrangement in a torus T is a family of hypersurfaces $\mathcal{T} = \{U_\chi\}_{\chi \in X}$, where $X \subset Hom(T, \mathbb{C}^*)$ and $U_\chi \doteq \{t \in T | \chi(t) = 1\}$.

(Lehrer and others in the '90s; De Concini and Procesi, 2005).

With $\mathcal{H}$ is associated the poset $\mathcal{L}(X)$ of the intersections of the $\{H_\chi\}_{\chi \in X}$.
With $\mathcal{T}$ is associated the poset $\mathcal{C}(X)$ of the components: connected components of the intersections of the $\{U_\chi\}_{\chi \in X}$.

A hyperplane arrangement in a vector space V is a family of hyperplanes $\mathcal{H} = \{H_\chi\}_{\chi \in X}$, where $X \subset Hom(V, \mathbb{C})$ and $H_\chi \doteq \{v \in V | \chi(v) = 0\}$.
A toric (toral) arrangement in a torus T is a family of hypersurfaces $\mathcal{T} = \{U_\chi\}_{\chi \in X}$, where $X \subset Hom(T, \mathbb{C}^*)$ and $U_\chi \doteq \{t \in T | \chi(t) = 1\}$.

(Lehrer and others in the '90s; De Concini and Procesi, 2005).

With $\mathcal{H}$ is associated the poset $\mathcal{L}(X)$ of the intersections of the $\{H_\chi\}_{\chi \in X}$.
With $\mathcal{T}$ is associated the poset $\mathcal{C}(X)$ of the components: connected components of the intersections of the $\{U_\chi\}_{\chi \in X}$.

Hyperplane and toric arrangements

A hyperplane arrangement in a vector space V is a family of hyperplanes $\mathcal{H} = \{H_\chi\}_{\chi \in X}$, where $X \subset Hom(V, \mathbb{C})$ and $H_\chi \doteq \{v \in V | \chi(v) = 0\}$.
A toric (toral) arrangement in a torus T is a family of hypersurfaces $\mathcal{T} = \{U_\chi\}_{\chi \in X}$, where $X \subset Hom(T, \mathbb{C}^*)$ and $U_\chi \doteq \{t \in T | \chi(t) = 1\}$.

(Lehrer and others in the '90s; De Concini and Procesi, 2005).

With $\mathcal{H}$ is associated the poset $\mathcal{L}(X)$ of the intersections of the $\{H_\chi\}_{\chi \in X}$.
With $\mathcal{T}$ is associated the poset $\mathcal{C}(X)$ of the components: connected components of the intersections of the $\{U_\chi\}_{\chi \in X}$.

Hyperplane versus toric arrangements

If in the previous example we replace $2z_1^*$ by z_1^* or $5z_1^*$, we get the same $\mathcal{H}$, but different $\mathcal{T}$. So $\mathcal{H}$ depends only on the linear algebra of X, whereas $\mathcal{T}$ also depends on its arithmetics. In fact

- $\mathcal{H}$ is more related with splines, $\mathcal{T}$ with partition functions;
- $\mathcal{H}$ with differential equations, $\mathcal{T}$ with difference equations;
- $\mathcal{H}$ with volume of polytopes, $\mathcal{T}$ with integral points in polytopes;

(see De Concini and Procesi's forthcoming book "Topics in Hyperplane Arrangements, Polytopes, and Box Splines").

Partition function: counts in how many ways a vector of a lattice Λ can be written as a (repeated) sum of given elements.

Hyperplane versus toric arrangements

If in the previous example we replace $2z_1^*$ by z_1^* or $5z_1^*$, we get the same $\mathcal{H}$, but different $\mathcal{T}$. So $\mathcal{H}$ depends only on the linear algebra of X, whereas $\mathcal{T}$ also depends on its arithmetics. In fact

- $\mathcal{H}$ is more related with splines, $\mathcal{T}$ with partition functions;
- $\mathcal{H}$ with differential equations, $\mathcal{T}$ with difference equations;
- $\mathcal{H}$ with volume of polytopes, $\mathcal{T}$ with integral points in polytopes;

(see De Concini and Procesi's forthcoming book "Topics in Hyperplane Arrangements, Polytopes, and Box Splines").

Partition function: counts in how many ways a vector of a lattice Λ can be written as a (repeated) sum of given elements.

Hyperplane versus toric arrangements

If in the previous example we replace $2z_1^*$ by z_1^* or $5z_1^*$, we get the same $\mathcal{H}$, but different $\mathcal{T}$. So $\mathcal{H}$ depends only on the linear algebra of X, whereas $\mathcal{T}$ also depends on its arithmetics. In fact

- $\mathcal{H}$ is more related with splines, $\mathcal{T}$ with partition functions;
- $\mathcal{H}$ with differential equations, $\mathcal{T}$ with difference equations;
- $\mathcal{H}$ with volume of polytopes, $\mathcal{T}$ with integral points in polytopes;

(see De Concini and Procesi's forthcoming book "Topics in Hyperplane Arrangements, Polytopes, and Box Splines").

Partition function: counts in how many ways a vector of a lattice Λ can be written as a (repeated) sum of given elements.

Hyperplane versus toric arrangements

If in the previous example we replace $2z_1^*$ by z_1^* or $5z_1^*$, we get the same $\mathcal{H}$, but different $\mathcal{T}$. So $\mathcal{H}$ depends only on the linear algebra of X, whereas $\mathcal{T}$ also depends on its arithmetics. In fact

- $\mathcal{H}$ is more related with splines, $\mathcal{T}$ with partition functions;
- $\mathcal{H}$ with differential equations, $\mathcal{T}$ with difference equations;
- $\mathcal{H}$ with volume of polytopes, $\mathcal{T}$ with integral points in polytopes;

(see De Concini and Procesi's forthcoming book "Topics in Hyperplane Arrangements, Polytopes, and Box Splines").

Partition function: counts in how many ways a vector of a lattice Λ can be written as a (repeated) sum of given elements.

Hyperplane versus toric arrangements

If in the previous example we replace $2z_1^*$ by z_1^* or $5z_1^*$, we get the same $\mathcal{H}$, but different $\mathcal{T}$. So $\mathcal{H}$ depends only on the linear algebra of X, whereas $\mathcal{T}$ also depends on its arithmetics. In fact

- $\mathcal{H}$ is more related with splines, $\mathcal{T}$ with partition functions;
- $\mathcal{H}$ with differential equations, $\mathcal{T}$ with difference equations;
- $\mathcal{H}$ with volume of polytopes, $\mathcal{T}$ with integral points in polytopes;

(see De Concini and Procesi's forthcoming book "Topics in Hyperplane Arrangements, Polytopes, and Box Splines").

Partition function: counts in how many ways a vector of a lattice Λ can be written as a (repeated) sum of given elements.

Toric arrangements and partition functions

One-dimensional problem: to count in how many ways an integer m can be written as a sum of given positive integers m_i. This amounts to compute the coefficient of x^m in the generating function

$$\prod_i \left(\sum_{k=0}^{\infty} x^{k\, m_i} \right) = \prod_i \frac{1}{1 - x^{m_i}}$$

i.e. to compute the residue at 0 of the function $\prod_i \frac{x^{-m-1}}{1-x^{m_i}}$ which is the opposite of the sum of the residues at the other poles, that are the $d-$th roots of 1, where $d = GCD\{m_i\}$.

In the general problem:

- m_i are replaced with vectors α_i in a $n-$dimensional lattice;
- the generating function has n variables, and its poles are the points of a toric arrangement.

Toric arrangements and partition functions

One-dimensional problem: to count in how many ways an integer m can be written as a sum of given positive integers m_i. This amounts to compute the coefficient of x^m in the generating function

$$\prod_i \left(\sum_{k=0}^{\infty} x^{k\, m_i} \right) = \prod_i \frac{1}{1 - x^{m_i}}$$

i.e. to compute the residue at 0 of the function $\prod_i \frac{x^{-m-1}}{1-x^{m_i}}$ which is the opposite of the sum of the residues at the other poles, that are the $d-$th roots of 1, where $d = GCD\{m_i\}$.

In the general problem:

- m_i are replaced with vectors α_i in a $n-$dimensional lattice;
- the generating function has n variables, and its poles are the points of a toric arrangement.

Toric arrangements and partition functions

One-dimensional problem: to count in how many ways an integer m can be written as a sum of given positive integers m_i. This amounts to compute the coefficient of x^m in the generating function

$$\prod_i \left(\sum_{k=0}^{\infty} x^{k\, m_i} \right) = \prod_i \frac{1}{1 - x^{m_i}}$$

i.e. to compute the residue at 0 of the function $\prod_i \frac{x^{-m-1}}{1-x^{m_i}}$ which is the opposite of the sum of the residues at the other poles, that are the $d-$th roots of 1, where $d = GCD\{m_i\}$.

In the general problem:

- m_i are replaced with vectors α_i in a $n-$dimensional lattice;
- the generating function has n variables, and its poles are the points of a toric arrangement.

One-dimensional problem: to count in how many ways an integer m can be written as a sum of given positive integers m_i. This amounts to compute the coefficient of x^m in the generating function

$$\prod_i \left(\sum_{k=0}^{\infty} x^{k\, m_i} \right) = \prod_i \frac{1}{1 - x^{m_i}}$$

i.e. to compute the residue at 0 of the function $\prod_i \frac{x^{-m-1}}{1-x^{m_i}}$ which is the opposite of the sum of the residues at the other poles, that are the $d-$th roots of 1, where $d = GCD\{m_i\}$.

In the general problem:

- m_i are replaced with vectors α_i in a $n-$dimensional lattice;
- the generating function has n variables, and its poles are the points of a toric arrangement.

Toric arrangements and partition functions

One-dimensional problem: to count in how many ways an integer m can be written as a sum of given positive integers m_i. This amounts to compute the coefficient of x^m in the generating function

$$\prod_i \left(\sum_{k=0}^{\infty} x^{k\, m_i} \right) = \prod_i \frac{1}{1 - x^{m_i}}$$

i.e. to compute the residue at 0 of the function $\prod_i \frac{x^{-m-1}}{1-x^{m_i}}$ which is the opposite of the sum of the residues at the other poles, that are the $d-$th roots of 1, where $d = GCD\{m_i\}$.

In the general problem:

- m_i are replaced with vectors α_i in a $n-$dimensional lattice;
- the generating function has n variables, and its poles are the points of a toric arrangement.

Toric arrangements and partition functions

One-dimensional problem: to count in how many ways an integer m can be written as a sum of given positive integers m_i. This amounts to compute the coefficient of x^m in the generating function

$$\prod_i \left(\sum_{k=0}^{\infty} x^{k\, m_i} \right) = \prod_i \frac{1}{1 - x^{m_i}}$$

i.e. to compute the residue at 0 of the function $\prod_i \frac{x^{-m-1}}{1-x^{m_i}}$ which is the opposite of the sum of the residues at the other poles, that are the $d-$th roots of 1, where $d = GCD\{m_i\}$.

In the general problem:

- m_i are replaced with vectors α_i in a $n-$dimensional lattice;
- the generating function has n variables, and its poles are the points of a toric arrangement.

Toric arrangements and partition functions

One-dimensional problem: to count in how many ways an integer m can be written as a sum of given positive integers m_i. This amounts to compute the coefficient of x^m in the generating function

$$\prod_i \left(\sum_{k=0}^{\infty} x^{k\, m_i} \right) = \prod_i \frac{1}{1 - x^{m_i}}$$

i.e. to compute the residue at 0 of the function $\prod_i \frac{x^{-m-1}}{1-x^{m_i}}$ which is the opposite of the sum of the residues at the other poles, that are the $d-$th roots of 1, where $d = GCD\{m_i\}$.

In the general problem:

- m_i are replaced with vectors α_i in a $n-$dimensional lattice;
- the generating function has n variables, and its poles are the points of a toric arrangement.

Toric arrangements and partition functions

One-dimensional problem: to count in how many ways an integer m can be written as a sum of given positive integers m_i. This amounts to compute the coefficient of x^m in the generating function

$$\prod_i \left(\sum_{k=0}^{\infty} x^{k\, m_i} \right) = \prod_i \frac{1}{1 - x^{m_i}}$$

i.e. to compute the residue at 0 of the function $\prod_i \frac{x^{-m-1}}{1-x^{m_i}}$ which is the opposite of the sum of the residues at the other poles, that are the $d-$th roots of 1, where $d = GCD\{m_i\}$.

In the general problem:

- m_i are replaced with vectors α_i in a $n-$dimensional lattice;
- the generating function has n variables, and its poles are the points of a toric arrangement.

2. General results

Tutte polynomial for $\mathcal{H}$

We recall that the Tutte polynomial associated to a list of vectors X is

$$T(x, y) \doteq \sum_{A \subseteq X} (x-1)^{r(X)-r(A)} (y-1)^{|A|-r(A)}.$$

This is an important invariant of the matroid...
In particular it specializes to the characteristic polynomial of $\mathcal{L}(X)$:

$$(-1)^n T(1-q, 0) = \chi(q).$$

This reflects the fact that $\mathcal{L}(X)$ only depends on the matroid defined by X.
The same is not true for $\mathcal{C}(X)$: we need to add to the matroid some "arithmetic data".

Tutte polynomial for $\mathcal{H}$

We recall that the Tutte polynomial associated to a list of vectors X is

$$T(x,y) \doteq \sum_{A \subseteq X} (x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

This is an important invariant of the matroid...
In particular it specializes to the characteristic polynomial of $\mathcal{L}(X)$:

$$(-1)^n T(1-q,0) = \chi(q).$$

This reflects the fact that $\mathcal{L}(X)$ only depends on the matroid defined by X.
The same is not true for $\mathcal{C}(X)$: we need to add to the matroid some "arithmetic data".

Tutte polynomial for $\mathcal{H}$

We recall that the Tutte polynomial associated to a list of vectors X is

$$T(x,y) \doteq \sum_{A \subseteq X} (x-1)^{r(X)-r(A)} (y-1)^{|A|-r(A)}.$$

This is an important invariant of the matroid...
In particular it specializes to the characteristic polynomial of $\mathcal{L}(X)$:

$$(-1)^n T(1-q, 0) = \chi(q).$$

This reflects the fact that $\mathcal{L}(X)$ only depends on the matroid defined by X.
The same is not true for $\mathcal{C}(X)$: we need to add to the matroid some "arithmetic data".

Tutte polynomial for $\mathcal{H}$

We recall that the Tutte polynomial associated to a list of vectors X is

$$T(x,y) \doteq \sum_{A \subseteq X} (x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

This is an important invariant of the matroid...
In particular it specializes to the characteristic polynomial of $\mathcal{L}(X)$:

$$(-1)^n T(1-q,0) = \chi(q).$$

This reflects the fact that $\mathcal{L}(X)$ only depends on the matroid defined by X.
The same is not true for $\mathcal{C}(X)$: we need to add to the matroid some "arithmetic data".

Tutte polynomial for $\mathcal{H}$

We recall that the Tutte polynomial associated to a list of vectors X is

$$T(x,y) \doteq \sum_{A \subseteq X} (x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

This is an important invariant of the matroid...
In particular it specializes to the characteristic polynomial of $\mathcal{L}(X)$:

$$(-1)^n T(1-q,0) = \chi(q).$$

This reflects the fact that $\mathcal{L}(X)$ only depends on the matroid defined by X. The same is not true for $\mathcal{C}(X)$: we need to add to the matroid some "arithmetic data".

Analogous of Tutte polynomial for $\mathcal{T}$

Let be $X \subset \mathbb{Z}^n$. For every $A \subseteq X$ let us define

$$m(A) \doteq [\mathbb{Z}^n \cap \langle A \rangle_{\mathbb{Q}} : \langle A \rangle_{\mathbb{Z}}].$$

We can then define a polynomial $\widetilde{T}(x, y)$ depending only on the matroid and on the multiplicity function m:

$$\widetilde{T}(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

This seems to be the right analogous of the Tutte polynomial; in particular $\widetilde{T}(1, 1)$ equals the volume of the zonotope associated to X, and

Theorem (M.)

$$(-1)^n \widetilde{T}(1-q, 0) = \chi(q)$$

where $\chi(q)$ is the characteristic polynomial of $\mathcal{C}(X)$.

Analogous of Tutte polynomial for $\mathcal{T}$

Let be $X \subset \mathbb{Z}^n$. For every $A \subseteq X$ let us define

$$m(A) \doteq [\mathbb{Z}^n \cap \langle A \rangle_{\mathbb{Q}} : \langle A \rangle_{\mathbb{Z}}] .$$

We can then define a polynomial $\widetilde{T}(x, y)$ depending only on the matroid and on the multiplicity function m:

$$\widetilde{T}(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

This seems to be the right analogous of the Tutte polynomial; in particular $\widetilde{T}(1, 1)$ equals the volume of the zonotope associated to X, and

Theorem (M.)

$$(-1)^n \widetilde{T}(1-q, 0) = \chi(q)$$

where $\chi(q)$ is the characteristic polynomial of $\mathcal{C}(X)$.

Analogous of Tutte polynomial for $\mathcal{T}$

Let be $X \subset \mathbb{Z}^n$. For every $A \subseteq X$ let us define

$$m(A) \doteq [\mathbb{Z}^n \cap \langle A \rangle_{\mathbb{Q}} : \langle A \rangle_{\mathbb{Z}}] .$$

We can then define a polynomial $\widetilde{T}(x,y)$ depending only on the matroid and on the multiplicity function m:

$$\widetilde{T}(x,y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

This seems to be the right analogous of the Tutte polynomial; in particular $\widetilde{T}(1,1)$ equals the volume of the zonotope associated to X, and

Theorem (M.)

$$(-1)^n \widetilde{T}(1-q,0) = \chi(q)$$

where $\chi(q)$ is the characteristic polynomial of $\mathcal{C}(X)$.

Analogous of Tutte polynomial for $\mathcal{T}$

Let be $X \subset \mathbb{Z}^n$. For every $A \subseteq X$ let us define

$$m(A) \doteq [\mathbb{Z}^n \cap \langle A \rangle_{\mathbb{Q}} : \langle A \rangle_{\mathbb{Z}}] .$$

We can then define a polynomial $\widetilde{T}(x,y)$ depending only on the matroid and on the multiplicity function m:

$$\widetilde{T}(x,y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

This seems to be the right analogous of the Tutte polynomial; in particular $\widetilde{T}(1,1)$ equals the volume of the zonotope associated to X, and

Theorem (M.)

$$(-1)^n \widetilde{T}(1-q,0) = \chi(q)$$

where $\chi(q)$ is the characteristic polynomial of $\mathcal{C}(X)$.

Analogous of Tutte polynomial for $\mathcal{T}$

Let be $X \subset \mathbb{Z}^n$. For every $A \subseteq X$ let us define

$$m(A) \doteq [\mathbb{Z}^n \cap \langle A \rangle_{\mathbb{Q}} : \langle A \rangle_{\mathbb{Z}}] \, .$$

We can then define a polynomial $\widetilde{T}(x, y)$ depending only on the matroid and on the multiplicity function m:

$$\widetilde{T}(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

This seems to be the right analogous of the Tutte polynomial; in particular $\widetilde{T}(1, 1)$ equals the volume of the zonotope associated to X, and

Theorem (M.)

$$(-1)^n \widetilde{T}(1-q, 0) = \chi(q)$$

where $\chi(q)$ is the characteristic polynomial of $\mathcal{C}(X)$.

Two theorems and their analogues

Two famous results for $\mathcal{H}$ are:

1. Cohomology of the complement (Orlik and Solomon)
2. Wonderful models (De Concini and Procesi)

There are some analogues for $\mathcal{T}$ are:

1. Cohomology of the complement (Looijenga; De Concini and Procesi)
2. Wonderful models (M.)

In both cases, results are in terms of the poset $\mathcal{C}(X)$.

Two theorems and their analogues

Two famous results for $\mathcal{H}$ are:

1. Cohomology of the complement (Orlik and Solomon)
2. Wonderful models (De Concini and Procesi)

There are some analogues for $\mathcal{T}$ are:

1. Cohomology of the complement (Looijenga; De Concini and Procesi)
2. Wonderful models (M.)

In both cases, results are in terms of the poset $\mathcal{C}(X)$.

Two theorems and their analogues

Two famous results for $\mathcal{H}$ are:

1. Cohomology of the complement (Orlik and Solomon)
2. Wonderful models (De Concini and Procesi)

There are some analogues for $\mathcal{T}$ are:

1. Cohomology of the complement (Looijenga; De Concini and Procesi)
2. Wonderful models (M.)

In both cases, results are in terms of the poset $\mathcal{C}(X)$.

Two theorems and their analogues

Two famous results for $\mathcal{H}$ are:

1. Cohomology of the complement (Orlik and Solomon)
2. Wonderful models (De Concini and Procesi)

There are some analogues for $\mathcal{T}$ are:

1. Cohomology of the complement (Looijenga; De Concini and Procesi)
2. Wonderful models (M.)

In both cases, results are in terms of the poset $\mathcal{C}(X)$.

Two theorems and their analogues

Two famous results for $\mathcal{H}$ are:

1. Cohomology of the complement (Orlik and Solomon)
2. Wonderful models (De Concini and Procesi)

There are some analogues for $\mathcal{T}$ are:

1. Cohomology of the complement (Looijenga; De Concini and Procesi)
2. Wonderful models (M.)

In both cases, results are in terms of the poset $\mathcal{C}(X)$.

Two theorems and their analogues

Two famous results for $\mathcal{H}$ are:

1. Cohomology of the complement (Orlik and Solomon)
2. Wonderful models (De Concini and Procesi)

There are some analogues for $\mathcal{T}$ are:

1. Cohomology of the complement (Looijenga; De Concini and Procesi)
2. Wonderful models (M.)

In both cases, results are in terms of the poset $\mathcal{C}(X)$.

Cohomology and wonderful model for $\mathcal{T}$

For every $C \in \mathcal{C}(X)$, let us define $X_C \doteq \{\chi \in X | \chi(t) = 1 \forall t \in C\}$.

1. The cohomology of the complement of $\mathcal{T}$ can be expressed as direct sum over $\mathcal{C}(X)$, the contribution of every $C \in \mathcal{C}(X)$ depending (on its dimension and) on the number of unbroken bases which can be extracted by X_C.
2. The wonderful model of $\mathcal{T}$ is obtained by blowing up along those components $C \in \mathcal{C}(X)$ (of codimension > 1 and) such that X_C is an irreducible set of vectors.

Then to make both results explicit, we need an enumeration of the components, together with a description of the sets X_C.

Cohomology and wonderful model for $\mathcal{T}$

For every $C \in \mathcal{C}(X)$, let us define $X_C \doteq \{\chi \in X | \chi(t) = 1 \forall t \in C\}$.

1. The cohomology of the complement of $\mathcal{T}$ can be expressed as direct sum over $\mathcal{C}(X)$, the contribution of every $C \in \mathcal{C}(X)$ depending (on its dimension and) on the number of unbroken bases which can be extracted by X_C.
2. The wonderful model of $\mathcal{T}$ is obtained by blowing up along those components $C \in \mathcal{C}(X)$ (of codimension > 1 and) such that X_C is an irreducible set of vectors.

Then to make both results explicit, we need an enumeration of the components, together with a description of the sets X_C.

Cohomology and wonderful model for $\mathcal{T}$

For every $C \in \mathcal{C}(X)$, let us define $X_C \doteq \{\chi \in X | \chi(t) = 1 \forall t \in C\}$.

1. The cohomology of the complement of $\mathcal{T}$ can be expressed as direct sum over $\mathcal{C}(X)$, the contribution of every $C \in \mathcal{C}(X)$ depending (on its dimension and) on the number of unbroken bases which can be extracted by X_C.
2. The wonderful model of $\mathcal{T}$ is obtained by blowing up along those components $C \in \mathcal{C}(X)$ (of codimension > 1 and) such that X_C is an irreducible set of vectors.

Then to make both results explicit, we need an enumeration of the components, together with a description of the sets X_C.

Cohomology and wonderful model for $\mathcal{T}$

For every $C \in \mathcal{C}(X)$, let us define $X_C \doteq \{\chi \in X | \chi(t) = 1 \forall t \in C\}$.

1. The cohomology of the complement of $\mathcal{T}$ can be expressed as direct sum over $\mathcal{C}(X)$, the contribution of every $C \in \mathcal{C}(X)$ depending (on its dimension and) on the number of unbroken bases which can be extracted by X_C.
2. The wonderful model of $\mathcal{T}$ is obtained by blowing up along those components $C \in \mathcal{C}(X)$ (of codimension > 1 and) such that X_C is an irreducible set of vectors.

Then to make both results explicit, we need an enumeration of the components, together with a description of the sets X_C.

3. Lie case : combinatorics

Hyperplane arrangements defined by root systems

Notations:

- $\mathfrak{g}$ a simple Lie algebra of rank n over $\mathbb{C}$
- $\mathfrak{h}$ a Cartan subalgebra
- $\Phi \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$
- $\Phi^\vee \subset \mathfrak{h}$ the coroot system
- W be the Weyl group of Φ

Φ defines in $V = \mathfrak{h}$ the hyperplane arrangement $\mathcal{H} = \{H_\alpha\}_{\alpha \in \Phi^+}$, where $H_\alpha = \{v \in V | \alpha(v) = 0\}$

Hyperplane arrangements defined by root systems

Notations:

- $\mathfrak{g}$ a simple Lie algebra of rank n over $\mathbb{C}$
- $\mathfrak{h}$ a Cartan subalgebra
- $\Phi \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$
- $\Phi^\vee \subset \mathfrak{h}$ the coroot system
- W be the Weyl group of Φ

Φ defines in $V = \mathfrak{h}$ the hyperplane arrangement
$\mathcal{H} = \{H_\alpha\}_{\alpha \in \Phi^+}$, where $H_\alpha = \{v \in V | \alpha(v) = 0\}$

The coroot system $\Phi^\vee$ spans a lattice $\langle\Phi^\vee\rangle$ in $\mathfrak{h}$.
$T \doteq \mathfrak{h}/\langle\Phi^\vee\rangle$ is a complex torus of rank n.
Each root α is a linear map $\mathfrak{h} \to \mathbb{C}$ taking integer values on $\langle\Phi^\vee\rangle$.
So it induces a homomorphism $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote e^α.

$\{e^\alpha(t) = 1\}_{\alpha\in\Phi}$ defines in T a finite family $\mathcal{T}$ of hypersurfaces.
this is the toric arrangement defined by Φ.
Let $\mathcal{C}(\Phi)$ be the poset of the components .
W acts naturally on $\mathcal{T}$ and on $\mathcal{C}(\Phi)$.

Toric arrangements defined by root systems

The coroot system $\Phi^\vee$ spans a lattice $\langle\Phi^\vee\rangle$ in $\mathfrak{h}$.
$T \doteq \mathfrak{h}/\langle\Phi^\vee\rangle$ is a complex torus of rank n.
Each root α is a linear map $\mathfrak{h} \to \mathbb{C}$ taking integer values on $\langle\Phi^\vee\rangle$.
So it induces a homomorphism $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote e^α.

$\{e^\alpha(t) = 1\}_{\alpha\in\Phi}$ defines in T a finite family $\mathcal{T}$ of hypersurfaces.
this is the toric arrangement defined by Φ.
Let $\mathcal{C}(\Phi)$ be the poset of the components .
W acts naturally on $\mathcal{T}$ and on $\mathcal{C}(\Phi)$.

Toric arrangements defined by root systems

The coroot system $\Phi^\vee$ spans a lattice $\langle\Phi^\vee\rangle$ in $\mathfrak{h}$.
$T \doteq \mathfrak{h}/\langle\Phi^\vee\rangle$ is a complex torus of rank n.
Each root α is a linear map $\mathfrak{h} \to \mathbb{C}$ taking integer values on $\langle\Phi^\vee\rangle$.
So it induces a homomorphism $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote e^α.

$\{e^\alpha(t) = 1\}_{\alpha\in\Phi}$ defines in T a finite family $\mathcal{T}$ of hypersurfaces.
this is the toric arrangement defined by Φ.
Let $\mathcal{C}(\Phi)$ be the poset of the components .
W acts naturally on $\mathcal{T}$ and on $\mathcal{C}(\Phi)$.

The coroot system $\Phi^\vee$ spans a lattice $\langle\Phi^\vee\rangle$ in $\mathfrak{h}$.
$T \doteq \mathfrak{h}/\langle\Phi^\vee\rangle$ is a complex torus of rank n.
Each root α is a linear map $\mathfrak{h} \to \mathbb{C}$ taking integer values on $\langle\Phi^\vee\rangle$.
So it induces a homomorphism $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote e^α.

$\{e^\alpha(t) = 1\}_{\alpha\in\Phi}$ defines in T a finite family $\mathcal{T}$ of hypersurfaces.
this is the toric arrangement defined by Φ.
Let $\mathcal{C}(\Phi)$ be the poset of the components .
W acts naturally on $\mathcal{T}$ and on $\mathcal{C}(\Phi)$.

Toric arrangements defined by root systems

The coroot system $\Phi^\vee$ spans a lattice $\langle \Phi^\vee \rangle$ in $\mathfrak{h}$.
$T \doteq \mathfrak{h}/\langle \Phi^\vee \rangle$ is a complex torus of rank n.
Each root α is a linear map $\mathfrak{h} \to \mathbb{C}$ taking integer values on $\langle \Phi^\vee \rangle$.
So it induces a homomorphism $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote e^α.

$\{e^\alpha(t) = 1\}_{\alpha \in \Phi}$ defines in T a finite family $\mathcal{T}$ of hypersurfaces.
this is the toric arrangement defined by Φ.
Let $\mathcal{C}(\Phi)$ be the poset of the components .
W acts naturally on $\mathcal{T}$ and on $\mathcal{C}(\Phi)$.

Kostant partition function

In this case the partition function we are computing is the Kostant partition function, that counts in how many ways an element of the lattice $\langle \Phi \rangle$ can be written as sum of positive roots.

It is involved in:

- Kostant's formula for weight multiplicities c_μ^λ (c_μ^λ is the multiplicity of the weight λ in the representation $V(\mu)$ of $\mathfrak{g}$ of highest weight μ);
- Steinberg's formula for Littlewood-Richardson coefficients $c_{\mu,\nu}^\lambda$ ($c_{\mu,\nu}^\lambda$ is the multiplicity of $V(\lambda)$ in $V(\mu) \otimes V(\nu)$).

The previous formulae have been implemented in computer programs, which are the most efficient for "large" weights.

Kostant partition function

In this case the partition function we are computing is the Kostant partition function, that counts in how many ways an element of the lattice $\langle\Phi\rangle$ can be written as sum of positive roots.
It is involved in:

- Kostant's formula for weight multiplicities c_μ^λ
  (c_μ^λ is the multiplicity of the weight λ
  in the representation $V(\mu)$ of $\mathfrak{g}$ of highest weight μ);
- Steinberg's formula for Littlewood-Richardson coefficients $c_{\mu,\nu}^\lambda$
  ($c_{\mu,\nu}^\lambda$ is the multiplicity of $V(\lambda)$ in $V(\mu)\otimes V(\nu)$).

The previous formulae have been implemented in computer programs, which are the most efficient for "large" weights.

Kostant partition function

In this case the partition function we are computing is the Kostant partition function, that counts in how many ways an element of the lattice $\langle \Phi \rangle$ can be written as sum of positive roots.
It is involved in:

- Kostant's formula for weight multiplicities c_μ^λ
  (c_μ^λ is the multiplicity of the weight λ
  in the representation $V(\mu)$ of $\mathfrak{g}$ of highest weight μ);
- Steinberg's formula for Littlewood-Richardson coefficients $c_{\mu,\nu}^\lambda$
  ($c_{\mu,\nu}^\lambda$ is the multiplicity of $V(\lambda)$ in $V(\mu) \otimes V(\nu)$).

The previous formulae have been implemented in computer programs, which are the most efficient for "large" weights.

Kostant partition function

In this case the partition function we are computing is the Kostant partition function, that counts in how many ways an element of the lattice $\langle\Phi\rangle$ can be written as sum of positive roots.
It is involved in:

- Kostant's formula for weight multiplicities c_μ^λ
  (c_μ^λ is the multiplicity of the weight λ
  in the representation $V(\mu)$ of $\mathfrak{g}$ of highest weight μ);
- Steinberg's formula for Littlewood-Richardson coefficients $c_{\mu,\nu}^\lambda$
  ($c_{\mu,\nu}^\lambda$ is the multiplicity of $V(\lambda)$ in $V(\mu)\otimes V(\nu)$).

The previous formulae have been implemented in computer programs, which are the most efficient for "large" weights.

Points of the arrangement

We say that a subset Θ of Φ is a subsystem if:

1. $\alpha \in \Theta \Rightarrow -\alpha \in \Theta$
2. $\alpha, \beta \in \Theta$ and $\alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Theta$.

We start from the set $\mathcal{C}_0(\Phi)$ of the 0-dimensional components, that we call the points of the arrangement.

For every $t \in \mathcal{C}_0(\Phi)$ we will describe its stabilizer $W(t)$ in W and the subsystem of Φ

$$\Phi(t) \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1\}.$$

Points of the arrangement

We say that a subset Θ of Φ is a subsystem if:

1. $\alpha \in \Theta \Rightarrow -\alpha \in \Theta$
2. $\alpha, \beta \in \Theta$ and $\alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Theta$.

We start from the set $\mathcal{C}_0(\Phi)$ of the 0-dimensional components, that we call the points of the arrangement.

For every $t \in \mathcal{C}_0(\Phi)$ we will describe its stabilizer $W(t)$ in W and the subsystem of Φ

$$\Phi(t) \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1\}.$$

Points of the arrangement

We say that a subset Θ of Φ is a subsystem if:

1. $\alpha \in \Theta \Rightarrow -\alpha \in \Theta$
2. $\alpha, \beta \in \Theta$ and $\alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Theta$.

We start from the set $\mathcal{C}_0(\Phi)$ of the 0-dimensional components, that we call the points of the arrangement.

For every $t \in \mathcal{C}_0(\Phi)$ we will describe its stabilizer $W(t)$ in W and the subsystem of Φ

$$\Phi(t) \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1\}.$$

Affine Dynkin diagrams

Let $\alpha_1, \ldots, \alpha_n$ be simple roots of Φ and α_0 the lowest root.
Let Γ be the affine Dynkin diagram of Φ (see picture).
The set of its vertices $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$.

Let Φ_p be the subsystem of Φ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$,
and let W_p be its Weyl group.
The (ordinary) Dynkin diagram Γ_p of Φ_p (and of W_p) is obtained by removing from Γ its vertex p.

Affine Dynkin diagrams

Let $\alpha_1, \ldots, \alpha_n$ be simple roots of Φ and α_0 the lowest root.
Let Γ be the affine Dynkin diagram of Φ (see picture).
The set of its vertices $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$.

Let Φ_p be the subsystem of Φ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$,
and let W_p be its Weyl group.
The (ordinary) Dynkin diagram Γ_p of Φ_p (and of W_p) is obtained by removing from Γ its vertex p.

Affine Dynkin diagrams

Let $\alpha_1, \ldots, \alpha_n$ be simple roots of Φ and α_0 the lowest root.
Let Γ be the affine Dynkin diagram of Φ (see picture).
The set of its vertices $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$.

Let Φ_p be the subsystem of Φ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$,
and let W_p be its Weyl group.
The (ordinary) Dynkin diagram Γ_p of Φ_p (and of W_p) is obtained by removing from Γ its vertex p.

Affine Dynkin diagrams

Let $\alpha_1, \dots, \alpha_n$ be simple roots of Φ and α_0 the lowest root.
Let Γ be the affine Dynkin diagram of Φ (see picture).
The set of its vertices $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$.

Let Φ_p be the subsystem of Φ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$,
and let W_p be its Weyl group.
The (ordinary) Dynkin diagram Γ_p of Φ_p (and of W_p) is obtained by removing from Γ its vertex p.

Points theorem

Theorem (M.)

There is a bijection $V(\Gamma) \leftrightarrow \mathcal{C}_0(\Phi)/W$*, having the property that given a vertex* p *and a point* t *in the corresponding orbit* $\mathcal{O}_p$ *, then:*

- $\Phi(t)$ *is* $W-$*conjugated to* Φ_p*;*
- $W(t)$ *is* $W-$*conjugated to* W_p*.*

Then we have:

$$|\mathcal{C}_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}.$$

Points theorem

Theorem (M.)

There is a bijection $V(\Gamma) \leftrightarrow \mathcal{C}_0(\Phi)/W$, having the property that given a vertex p and a point t in the corresponding orbit $\mathcal{O}_p$, then:

- *$\Phi(t)$ is $W-$conjugated to Φ_p;*
- *$W(t)$ is $W-$conjugated to W_p.*

Then we have:

$$|\mathcal{C}_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}.$$

Points theorem

Theorem (M.)

There is a bijection $V(\Gamma) \leftrightarrow \mathcal{C}_0(\Phi)/W$, *having the property that given a vertex p and a point t in the corresponding orbit* $\mathcal{O}_p$, *then:*

- $\Phi(t)$ *is* $W-$*conjugated to* Φ_p;
- $W(t)$ *is* $W-$*conjugated to* W_p.

Then we have:

$$|\mathcal{C}_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}.$$

Points theorem

Theorem (M.)

There is a bijection $V(\Gamma) \leftrightarrow \mathcal{C}_0(\Phi)/W$, *having the property that given a vertex* p *and a point* t *in the corresponding orbit* $\mathcal{O}_p$, *then:*

- $\Phi(t)$ *is* $W-$*conjugated to* Φ_p;
- $W(t)$ *is* $W-$*conjugated to* W_p.

Then we have:

$$|\mathcal{C}_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}.$$

Example: Case C_n

$$\Phi^+ = \{z_i^* - z_j^*\}_{i<j} \cup \{z_i^* + z_j^*\} \cup \{2z_i^*\}$$

Then on the torus $T = \{(t_1, \dots, t_n), t_i \in \mathbb{C}^*\}$ the equations $e^\alpha(t) = 1$ are:

$$\{t_i t_j^{-1} = 1\} \cup \{t_i t_j = 1\} \cup \{t_i^2 = 1\}.$$

The system of n independent equations

$$t_1^2 = 1, \dots, t_n^2 = 1$$

has 2^n solutions: $(\pm 1, \dots, \pm 1)$
and all other systems do not have other solutions.

Example: Case C_n

$$\Phi^+ = \{z_i^* - z_j^*\}_{i<j} \cup \{z_i^* + z_j^*\} \cup \{2z_i^*\}$$

Then on the torus $T = \{(t_1, \dots, t_n), t_i \in \mathbb{C}^*\}$ the equations $e^\alpha(t) = 1$ are:

$$\{t_i t_j^{-1} = 1\} \cup \{t_i t_j = 1\} \cup \{t_i^2 = 1\}.$$

The system of n independent equations

$$t_1^2 = 1, \dots, t_n^2 = 1$$

has 2^n solutions: $(\pm 1, \dots, \pm 1)$
and all other systems do not have other solutions.

Example: Case C_n

$$\Phi^+ = \{z_i^* - z_j^*\}_{i<j} \cup \{z_i^* + z_j^*\} \cup \{2z_i^*\}$$

Then on the torus $T = \{(t_1, \ldots, t_n), t_i \in \mathbb{C}^*\}$ the equations $e^\alpha(t) = 1$ are:

$$\{t_i t_j^{-1} = 1\} \cup \{t_i t_j = 1\} \cup \{t_i^2 = 1\}.$$

The system of n independent equations

$$t_1^2 = 1, \ldots, t_n^2 = 1$$

has 2^n solutions: $(\pm 1, \ldots, \pm 1)$
and all other systems do not have other solutions.

Example: Case C_n, 2

$W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$ acts on T by permutations and inversions thus the second factor acts trivially on $\mathcal{C}_0(\Phi)$.
Then orbits are given by the number of negative coordinates.
Let $\mathcal{O}_p$ be the set of points with p negative coordinates.

Clearly the stabilizer of a such point is

$$\mathfrak{S}_p \times \mathfrak{S}_{n-p} \ltimes (\mathfrak{C}_2)^n$$

thus $|\mathcal{O}_p| = \binom{n}{p}$ and our formula is checked:

$$|\mathcal{C}_0(\Phi)| = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

(see example).

Example: Case C_n, 2

$W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$ acts on T by permutations and inversions thus the second factor acts trivially on $\mathcal{C}_0(\Phi)$.
Then orbits are given by the number of negative coordinates.
Let $\mathcal{O}_p$ be the set of points with p negative coordinates.

Clearly the stabilizer of a such point is

$$\mathfrak{S}_p \times \mathfrak{S}_{n-p} \ltimes (\mathfrak{C}_2)^n$$

thus $|\mathcal{O}_p| = \binom{n}{p}$ and our formula is checked:

$$|\mathcal{C}_0(\Phi)| = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

(see example).

Example: Case C_n, 2

$W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$ acts on T by permutations and inversions thus the second factor acts trivially on $\mathcal{C}_0(\Phi)$.
Then orbits are given by the number of negative coordinates.
Let $\mathcal{O}_p$ be the set of points with p negative coordinates.

Clearly the stabilizer of a such point is

$$\mathfrak{S}_p \times \mathfrak{S}_{n-p} \ltimes (\mathfrak{C}_2)^n$$

thus $|\mathcal{O}_p| = \binom{n}{p}$ and our formula is checked:

$$|\mathcal{C}_0(\Phi)| = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

(see example).

Example: Case C_n, 2

$W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$ acts on T by permutations and inversions thus the second factor acts trivially on $\mathcal{C}_0(\Phi)$.
Then orbits are given by the number of negative coordinates.
Let $\mathcal{O}_p$ be the set of points with p negative coordinates.

Clearly the stabilizer of a such point is

$$\mathfrak{S}_p \times \mathfrak{S}_{n-p} \ltimes (\mathfrak{C}_2)^n$$

thus $|\mathcal{O}_p| = \binom{n}{p}$ and our formula is checked:

$$|\mathcal{C}_0(\Phi)| = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

(see example).

Example: Case C_n, 2

$W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$ acts on T by permutations and inversions thus the second factor acts trivially on $\mathcal{C}_0(\Phi)$.
Then orbits are given by the number of negative coordinates.
Let $\mathcal{O}_p$ be the set of points with p negative coordinates.

Clearly the stabilizer of a such point is

$$\mathfrak{S}_p \times \mathfrak{S}_{n-p} \ltimes (\mathfrak{C}_2)^n$$

thus $|\mathcal{O}_p| = \binom{n}{p}$ and our formula is checked:

$$|\mathcal{C}_0(\Phi)| = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

(see example).

Example: Case C_n, 3

The previous choice is not canonical!
(we could define as well $\mathcal{O}_p$ as the set of points with p positive coordinates)

Observation:

- Γ has a symmetry exchanging the vertices p and $n-p$.
- Multiplication by -1 exchanges the corresponding orbits.

Example: Case C_n, 3

The previous choice is not canonical!
(we could define as well $\mathcal{O}_p$ as the set of points with p positive coordinates)

Observation:

- Γ has a symmetry exchanging the vertices p and $n-p$.
- Multiplication by -1 exchanges the corresponding orbits.

The center

Given the coweight lattice

$$\Lambda(\Phi) \doteq \{h \in \mathfrak{h} | \alpha(h) \in \mathbb{Z} \forall \alpha \in \Phi\}$$

we define the center

$$Z(\Phi) \doteq \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle} = \{t \in T | \Phi(t) = \Phi\}.$$

Thus:

- $Z(\Phi) \subseteq \mathcal{C}_0(\Phi)$;
- $Z(\Phi)$ acts by multiplication on $\mathcal{C}_0(\Phi)$.

The center

Given the coweight lattice

$$\Lambda(\Phi) \doteq \{h \in \mathfrak{h} | \alpha(h) \in \mathbb{Z} \forall \alpha \in \Phi\}$$

we define the center

$$Z(\Phi) \doteq \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle} = \{t \in T | \Phi(t) = \Phi\}.$$

Thus:

- $Z(\Phi) \subseteq \mathcal{C}_0(\Phi)$;
- $Z(\Phi)$ acts by multiplication on $\mathcal{C}_0(\Phi)$.

The center

Given the coweight lattice

$$\Lambda(\Phi) \doteq \{h \in \mathfrak{h} | \alpha(h) \in \mathbb{Z} \forall \alpha \in \Phi\}$$

we define the center

$$Z(\Phi) \doteq \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle} = \{t \in T | \Phi(t) = \Phi\}.$$

Thus:

- $Z(\Phi) \subseteq \mathcal{C}_0(\Phi)$;
- $Z(\Phi)$ acts by multiplication on $\mathcal{C}_0(\Phi)$.

Canonical bijection

We can make canonical the bijection between vertices and $W-$orbits by identifying:

- $Aut(\Gamma)-$conjugated vertices
- $Z(\Phi)-$conjugated orbits

Complete subsystems

We define the completion of a subsystem Θ as

$$\overline{\Theta} \doteq \langle \Theta \rangle_{\mathbb{R}} \cap \Phi$$

and we say that Θ is complete if $\Theta = \overline{\Theta}$. (see example).

Let $\mathcal{K}_d$ be the set of complete subsystems of Φ of rank $n-d$: they are in natural bijection with the $d-$dimensional elements of $\mathcal{L}(\Phi)$ (the intersection poset of $\mathcal{H}$).

Complete subsystems

We define the completion of a subsystem Θ as

$$\overline{\Theta} \doteq \langle \Theta \rangle_{\mathbb{R}} \cap \Phi$$

and we say that Θ is complete if $\Theta = \overline{\Theta}$. (see example).

Let $\mathcal{K}_d$ be the set of complete subsystems of Φ of rank $n - d$: they are in natural bijection with the $d-$dimensional elements of $\mathcal{L}(\Phi)$ (the intersection poset of $\mathcal{H}$).

Complete subsystems

We define the completion of a subsystem Θ as

$$\overline{\Theta} \doteq \langle \Theta \rangle_{\mathbb{R}} \cap \Phi$$

and we say that Θ is complete if $\Theta = \overline{\Theta}$. (see example).

Let $\mathcal{K}_d$ be the set of complete subsystems of Φ of rank $n - d$: they are in natural bijection with the $d-$dimensional elements of $\mathcal{L}(\Phi)$ (the intersection poset of $\mathcal{H}$).

The poset $\mathcal{L}(\Phi)$ has been completely described for every Φ, computing how many elements (and $W-$orbits) there are for each type of subsystem.
This was done in 1980 by Orlik and Solomon case-by-case according to the type of Φ.

We now show a case-free way to extend this analysis to the poset $\mathcal{C}(\Phi)$.

Spaces and components

The poset $\mathcal{L}(\Phi)$ has been completely described for every Φ, computing how many elements (and $W-$orbits) there are for each type of subsystem.
This was done in 1980 by Orlik and Solomon
case-by-case according to the type of Φ.

We now show a case-free way to extend this analysis
to the poset $\mathcal{C}(\Phi)$.

Spaces and components

The poset $\mathcal{L}(\Phi)$ has been completely described for every Φ, computing how many elements (and $W-$orbits) there are for each type of subsystem. This was done in 1980 by Orlik and Solomon case-by-case according to the type of Φ.

We now show a case-free way to extend this analysis to the poset $\mathcal{C}(\Phi)$.

Components and subsystems

Given a component C of $\mathcal{T}$ let us consider

$$\Theta_C \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1 \, \forall t \in C\}.$$

In general Θ_C is not complete (see example).
Then for each complete subsystem Θ let us define $\mathcal{C}_{\Theta}^{\Phi}$ as the set of components C such that $\overline{\Theta_C} = \Theta$.
This is clearly a partition of the set of $d-$dimensional components of $\mathcal{T}$:

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}_{\Theta}^{\Phi}$$

Then we just have to describe every $\mathcal{C}_{\Theta}^{\Phi}$, that is the set of the elements of $\mathcal{C}(\Phi)$ corresponding to a given element of $\mathcal{L}(\Phi)$.

Components and subsystems

Given a component C of $\mathcal{T}$ let us consider

$$\Theta_C \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1 \,\forall t \in C\}.$$

In general Θ_C is not complete (see example).
Then for each complete subsystem Θ let us define $\mathcal{C}^{\Phi}_{\Theta}$ as the set of components C such that $\overline{\Theta_C} = \Theta$.
This is clearly a partition of the set of $d-$dimensional components of $\mathcal{T}$:

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}^{\Phi}_{\Theta}$$

Then we just have to describe every $\mathcal{C}^{\Phi}_{\Theta}$, that is the set of the elements of $\mathcal{C}(\Phi)$ corresponding to a given element of $\mathcal{L}(\Phi)$.

Components and subsystems

Given a component C of $\mathcal{T}$ let us consider

$$\Theta_C \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1 \, \forall t \in C\}.$$

In general Θ_C is not complete (see example).
Then for each complete subsystem Θ let us define $\mathcal{C}_{\Theta}^{\Phi}$ as the set of components C such that $\overline{\Theta_C} = \Theta$.
This is clearly a partition of the set of $d-$dimensional components of $\mathcal{T}$:

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}_{\Theta}^{\Phi}$$

Then we just have to describe every $\mathcal{C}_{\Theta}^{\Phi}$, that is the set of the elements of $\mathcal{C}(\Phi)$ corresponding to a given element of $\mathcal{L}(\Phi)$.

Components and subsystems

Given a component C of $\mathcal{T}$ let us consider

$$\Theta_C \doteq \{\alpha \in \Phi | e^\alpha(t) = 1\ \forall t \in C\}.$$

In general Θ_C is not complete (see example).
Then for each complete subsystem Θ let us define $\mathcal{C}_\Theta^\Phi$ as the set of components C such that $\overline{\Theta_C} = \Theta$.
This is clearly a partition of the set of $d-$dimensional components of $\mathcal{T}$:

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}_\Theta^\Phi$$

Then we just have to describe every $\mathcal{C}_\Theta^\Phi$, that is the set of the elements of $\mathcal{C}(\Phi)$ corresponding to a given element of $\mathcal{L}(\Phi)$.

Example: 1-dimensional components of C_3

There are 3 conjugation classes of 1-dimensional spaces of $\mathcal{H}$, having representatives

$$(h, h, h)\ ,\ (h, h, 0)\ ,\ (h, 0, 0)\ ,\ h \in \mathbb{C}$$

corresponding respectively to 1, 2, 4 components of $\mathcal{T}$:

$$(t, t, t)\ ,\ (t, t, \pm 1)\ ,\ (t, \pm 1, \pm 1)\ ,\ t \in \mathbb{C}^*$$

This suggests to relate the components of $\mathcal{C}_\Theta^\Phi$ to the points of a smaller toric arrangement.

Example: 1-dimensional components of C_3

There are 3 conjugation classes of 1-dimensional spaces of $\mathcal{H}$, having representatives

$$(h, h, h)\ ,\ (h, h, 0)\ ,\ (h, 0, 0)\ ,\ h \in \mathbb{C}$$

corresponding respectively to 1, 2, 4 components of $\mathcal{T}$:

$$(t, t, t)\ ,\ (t, t, \pm 1)\ ,\ (t, \pm 1, \pm 1)\ ,\ t \in \mathbb{C}^*$$

This suggests to relate the components of $\mathcal{C}_{\Theta}^{\Phi}$ to the points of a smaller toric arrangement.

Example: 1-dimensional components of C_3

There are 3 conjugation classes of 1-dimensional spaces of $\mathcal{H}$, having representatives

$$(h, h, h)\,,\ (h, h, 0)\,,\ (h, 0, 0)\,,\ h \in \mathbb{C}$$

corresponding respectively to 1, 2, 4 components of $\mathcal{T}$:

$$(t, t, t)\,,\ (t, t, \pm 1)\,,\ (t, \pm 1, \pm 1)\,,\ t \in \mathbb{C}^*$$

This suggests to relate the components of $\mathcal{C}_{\Theta}^{\Phi}$ to the points of a smaller toric arrangement.

Example: 1-dimensional components of C_3

There are 3 conjugation classes of 1-dimensional spaces of $\mathcal{H}$, having representatives

$$(h, h, h)\,,\ (h, h, 0)\,,\ (h, 0, 0)\,,\ h \in \mathbb{C}$$

corresponding respectively to 1, 2, 4 components of $\mathcal{T}$:

$$(t, t, t)\,,\ (t, t, \pm 1)\,,\ (t, \pm 1, \pm 1)\,,\ t \in \mathbb{C}^*$$

This suggests to relate the components of $\mathcal{C}_\Theta^\Phi$ to the points of a smaller toric arrangement.

Example: 1-dimensional components of C_3

There are 3 conjugation classes of 1-dimensional spaces of $\mathcal{H}$, having representatives

$$(h, h, h)\ ,\ (h, h, 0)\ ,\ (h, 0, 0)\ ,\ h \in \mathbb{C}$$

corresponding respectively to 1, 2, 4 components of $\mathcal{T}$:

$$(t, t, t)\ ,\ (t, t, \pm 1)\ ,\ (t, \pm 1, \pm 1)\ ,\ t \in \mathbb{C}^*$$

This suggests to relate the components of $\mathcal{C}_\Theta^\Phi$ to the points of a smaller toric arrangement.

Reduction theorem

Notations:

- Θ be a complete subsystem of Φ
- W^{Θ} its Weyl group
- $Z(\Theta) \doteq \frac{\Lambda(\Theta)}{\langle\Theta^{\vee}\rangle}$ the center
- $\mathcal{D}$ the toric arrangement defined by Θ on the torus D
- $\mathcal{C}_0(\Theta)$ the set of points of $\mathcal{D}$

Theorem (M.)

There is a $W^{\Theta}-$equivariant surjective map

$$\varphi : \mathcal{C}^{\Phi}_{\Theta} \to \mathcal{C}_0(\Theta)/Z(\Theta)$$

such that $\ker \varphi \simeq Z(\Phi) \cap Z(\Theta)$ *and* $\Theta_U = \Theta(\varphi(U))$.

Reduction theorem

Notations:

- Θ be a complete subsystem of Φ
- W^Θ its Weyl group
- $Z(\Theta) \doteq \frac{\Lambda(\Theta)}{\langle \Theta^\vee \rangle}$ the center
- $\mathcal{D}$ the toric arrangement defined by Θ on the torus D
- $\mathcal{C}_0(\Theta)$ the set of points of $\mathcal{D}$

Theorem (M.)

There is a $W^\Theta-$equivariant surjective map

$$\varphi : \mathcal{C}_\Theta^\Phi \to \mathcal{C}_0(\Theta)/Z(\Theta)$$

such that $\ker \varphi \simeq Z(\Phi) \cap Z(\Theta)$ *and* $\Theta_U = \Theta(\varphi(U))$.

The number of the components

Corollary

$$|\mathcal{C}_{\Theta}^{\Phi}| = n_{\Theta}^{-1}|\mathcal{C}_0(\Theta)|$$

where $n_{\Theta} \doteq \frac{|Z(\Theta)|}{|Z(\Phi) \cap Z(\Theta)|}$.

Then

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_{\Theta}^{-1}|\mathcal{C}_0(\Theta)|.$$

Moreover the reduction theorem yields a description of the action of W on $\mathcal{C}(\Phi)$.

Then we get a $W-$equivariant decomposition of the cohomology of R.

The number of the components

Corollary

$$|\mathcal{C}_\Theta^\Phi| = n_\Theta^{-1} |\mathcal{C}_0(\Theta)|$$

where $n_\Theta \doteq \frac{|Z(\Theta)|}{|Z(\Phi) \cap Z(\Theta)|}$.
Then

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |\mathcal{C}_0(\Theta)|.$$

Moreover the reduction theorem yields a description
of the action of W on $\mathcal{C}(\Phi)$.
Then we get a $W-$equivariant decomposition of the cohomology of R.

The number of the components

Corollary

$$|\mathcal{C}^{\Phi}_{\Theta}| = n_{\Theta}^{-1}|\mathcal{C}_0(\Theta)|$$

where $n_{\Theta} \doteq \frac{|Z(\Theta)|}{|Z(\Phi)\cap Z(\Theta)|}$.
Then

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta\in\mathcal{K}_d} n_{\Theta}^{-1}|\mathcal{C}_0(\Theta)|.$$

Moreover the reduction theorem yields a description of the action of W on $\mathcal{C}(\Phi)$.
Then we get a $W-$equivariant decomposition of the cohomology of R.

The number of the components

Corollary

$$|\mathcal{C}_\Theta^\Phi| = n_\Theta^{-1}|\mathcal{C}_0(\Theta)|$$

where $n_\Theta \doteq \frac{|Z(\Theta)|}{|Z(\Phi)\cap Z(\Theta)|}$.
Then

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta\in\mathcal{K}_d} n_\Theta^{-1}|\mathcal{C}_0(\Theta)|.$$

Moreover the reduction theorem yields a description of the action of W on $\mathcal{C}(\Phi)$.
Then we get a $W-$equivariant decomposition of the cohomology of R.

4. Lie case : applications

Degrees, components and Poincaré polynomial

Our results about the components yield more explicit description of the cohomology of the complement $\mathcal{R}$ of $\bigcup_{U \in \mathcal{T}} U$ in T.

Let $d_1, \ldots, d_n$ be the degrees of W
(i.e. the degrees of the generators of the ring of $W-$invariant regular functions on $\mathfrak{h}$).
It is well known that $d_1 \ldots d_n = |W|$.
We define $\mathcal{B}(\Phi) \doteq (d_1 - 1) \ldots (d_n - 1)$.
By De Concini-Procesi formula for cohomology, the Poincaré polynomial is

$$P_\Phi(q) = \sum_C \mathcal{B}(\Theta_C)(q+1)^{d(C)} q^{n-d(C)}$$

where C varies on all the components of $\mathcal{T}$ and $d(C)$ is its dimension.

Degrees, components and Poincaré polynomial

Our results about the components yield more explicit description of the cohomology of the complement $\mathcal{R}$ of $\bigcup_{U \in \mathcal{T}} U$ in T.

Let $d_1, \ldots, d_n$ be the degrees of W
(i.e. the degrees of the generators of the ring of $W-$invariant regular functions on $\mathfrak{h}$).
It is well known that $d_1 \ldots d_n = |W|$.
We define $\mathcal{B}(\Phi) \doteq (d_1 - 1) \ldots (d_n - 1)$.
By De Concini-Procesi formula for cohomology, the Poincaré polynomial is

$$P_\Phi(q) = \sum_C \mathcal{B}(\Theta_C)(q+1)^{d(C)} q^{n-d(C)}$$

where C varies on all the components of $\mathcal{T}$ and $d(C)$ is its dimension.

Degrees, components and Poincaré polynomial

Our results about the components yield more explicit description of the cohomology of the complement $\mathcal{R}$ of $\bigcup_{U \in \mathcal{T}} U$ in T.

Let $d_1, \ldots, d_n$ be the degrees of W
(i.e. the degrees of the generators of the ring of $W-$invariant regular functions on $\mathfrak{h}$).
It is well known that $d_1 \ldots d_n = |W|$.
We define $\mathcal{B}(\Phi) \doteq (d_1 - 1) \ldots (d_n - 1)$.
By De Concini-Procesi formula for cohomology, the Poincaré polynomial is

$$P_\Phi(q) = \sum_C \mathcal{B}(\Theta_C)(q+1)^{d(C)} q^{n-d(C)}$$

where C varies on all the components of $\mathcal{T}$ and $d(C)$ is its dimension.

Degrees, components and Poincaré polynomial

Our results about the components yield more explicit description of the cohomology of the complement $\mathcal{R}$ of $\bigcup_{U \in \mathcal{T}} U$ in T.

Let $d_1, \dots, d_n$ be the degrees of W
(i.e. the degrees of the generators of the ring of $W-$invariant regular functions on $\mathfrak{h}$).
It is well known that $d_1 \dots d_n = |W|$.
We define $\mathcal{B}(\Phi) \doteq (d_1 - 1) \dots (d_n - 1)$.
By De Concini-Procesi formula for cohomology, the Poincaré polynomial is

$$P_\Phi(q) = \sum_C \mathcal{B}(\Theta_C)(q+1)^{d(C)} q^{n-d(C)}$$

where C varies on all the components of $\mathcal{T}$ and $d(C)$ is its dimension.

Degrees, components and Poincaré polynomial

Our results about the components yield more explicit description of the cohomology of the complement $\mathcal{R}$ of $\bigcup_{U \in \mathcal{T}} U$ in T.

Let $d_1, \ldots, d_n$ be the degrees of W
(i.e. the degrees of the generators of the ring
of $W-$invariant regular functions on $\mathfrak{h}$).
It is well known that $d_1 \ldots d_n = |W|$.
We define $\mathcal{B}(\Phi) \doteq (d_1 - 1) \ldots (d_n - 1)$.
By De Concini-Procesi formula for cohomology, the Poincaré polynomial is

$$P_\Phi(q) = \sum_C \mathcal{B}(\Theta_C)(q+1)^{d(C)} q^{n-d(C)}$$

where C varies on all the components of $\mathcal{T}$ and $d(C)$ is its dimension.

The Euler characteristic

Theorem

The Euler characteristic χ_Φ *of* $\mathcal{R}$ *is equal to* $(-1)^n|W|$

Proof.

1. When we evaluate the Poincaré polynomial in $q = -1$ all the contributions vanish except for those of the points.
2. Applying our "points theorem" theorem we get

$$\chi_\Phi = (-1)^n \sum_{p=0}^{n} \frac{|W|}{|W_p|} \mathcal{B}(\Phi_p).$$

3. The equivalence between this expression and the claimed one is the "curious identity" $\sum_{p=0}^{n} \frac{(d_1^p - 1)\ldots(d_n^p - 1)}{d_1^p \ldots d_n^p} = 1$ (where $d_1^p, \ldots, d_n^p$ are the degrees of W_p) (De Concini and Procesi; Stembridge; Denham).

The Euler characteristic

Theorem

The Euler characteristic χ_Φ *of* $\mathcal{R}$ *is equal to* $(-1)^n|W|$

Proof.

1. When we evaluate the Poincaré polynomial in $q = -1$ all the contributions vanish except for those of the points.
2. Applying our "points theorem" theorem we get

$$\chi_\Phi = (-1)^n \sum_{p=0}^{n} \frac{|W|}{|W_p|} \mathcal{B}(\Phi_p).$$

3. The equivalence between this expression and the claimed one is the "curious identity" $\sum_{p=0}^{n} \frac{(d_1^p - 1)\dots(d_n^p - 1)}{d_1^p \dots d_n^p} = 1$ (where $d_1^p, \dots, d_n^p$ are the degrees of W_p) (De Concini and Procesi; Stembridge; Denham).

The Euler characteristic

Theorem

The Euler characteristic χ_Φ *of* $\mathcal{R}$ *is equal to* $(-1)^n|W|$

Proof.

1. When we evaluate the Poincaré polynomial in $q = -1$ all the contributions vanish except for those of the points.
2. Applying our "points theorem" theorem we get

$$\chi_\Phi = (-1)^n \sum_{p=0}^{n} \frac{|W|}{|W_p|} \mathcal{B}(\Phi_p).$$

3. The equivalence between this expression and the claimed one is the "curious identity" $\sum_{p=0}^{n} \frac{(d_1^p - 1)\dots(d_n^p - 1)}{d_1^p \dots d_n^p} = 1$ (where $d_1^p, \dots, d_n^p$ are the degrees of W_p) (De Concini and Procesi; Stembridge; Denham).

The Euler characteristic

Theorem

The Euler characteristic χ_Φ of $\mathcal{R}$ is equal to $(-1)^n|W|$

Proof.

1. When we evaluate the Poincaré polynomial in $q = -1$ all the contributions vanish except for those of the points.
2. Applying our "points theorem" theorem we get

$$\chi_\Phi = (-1)^n \sum_{p=0}^{n} \frac{|W|}{|W_p|} \mathcal{B}(\Phi_p).$$

3. The equivalence between this expression and the claimed one is the "curious identity" $\sum_{p=0}^{n} \frac{(d_1^p - 1)\dots(d_n^p - 1)}{d_1^p \dots d_n^p} = 1$ (where $d_1^p, \dots, d_n^p$ are the degrees of W_p) (De Concini and Procesi; Stembridge; Denham).

The Poincaré polynomial

Moreover we get a formula which allows to compute explicitly the Poincaré polynomial $P_\Phi(q)$ of $\mathcal{R}$:

Corollary

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |W^\Theta|$$

Wonderful models for toric arrangements

Let be

$$\mathcal{I}(\Phi) \doteq \{C \in \mathcal{C}(\Phi) | \Theta_C \text{ is irreducible}\}$$

where we recall that

$$\Theta_C \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1 \, \forall t \in C\}.$$

Corollary

$\mathcal{T}$ has a wonderful model, which is obtained blowing-up T along all the components $C \in \mathcal{I}(\Phi)$ of codimension > 1 (in any dimension-increasing order). The irreducible components of the NCD are in bijection with the elements of $\mathcal{I}(\Phi)$. Moreover this model is minimal among all the wonderful models obtained by blow-ups.

Let be

$$\mathcal{I}(\Phi) \doteq \{C \in \mathcal{C}(\Phi) | \Theta_C \text{ is irreducible}\}$$

where we recall that

$$\Theta_C \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1 \, \forall t \in C\}.$$

Corollary

$\mathcal{T}$ has a wonderful model, which is obtained blowing-up T along all the components $C \in \mathcal{I}(\Phi)$ of codimension > 1 (in any dimension-increasing order). The irreducible components of the NCD are in bijection with the elements of $\mathcal{I}(\Phi)$. Moreover this model is minimal among all the wonderful models obtained by blow-ups.

Wonderful models for toric arrangements

Let be

$$\mathcal{I}(\Phi) \doteq \{C \in \mathcal{C}(\Phi) | \Theta_C \text{ is irreducible}\}$$

where we recall that

$$\Theta_C \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1 \, \forall t \in C\}.$$

Corollary

$\mathcal{T}$ has a wonderful model, which is obtained blowing-up T along all the components $C \in \mathcal{I}(\Phi)$ of codimension > 1 (in any dimension-increasing order). The irreducible components of the NCD are in bijection with the elements of $\mathcal{I}(\Phi)$. Moreover this model is minimal among all the wonderful models obtained by blow-ups.

Let be

$$\mathcal{I}(\Phi) \doteq \{C \in \mathcal{C}(\Phi) | \Theta_C \text{ is irreducible}\}$$

where we recall that

$$\Theta_C \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1 \, \forall t \in C\}.$$

Corollary

$\mathcal{T}$ has a wonderful model, which is obtained blowing-up T along all the components $C \in \mathcal{I}(\Phi)$ of codimension > 1 (in any dimension-increasing order). The irreducible components of the NCD are in bijection with the elements of $\mathcal{I}(\Phi)$. Moreover this model is minimal among all the wonderful models obtained by blow-ups.

Description of $\mathcal{I}(\Phi)$

The proof follows from a general theorem of Procesi and MacPherson on "conical stratifications". However, now we know exactly who $\mathcal{I}(\Phi)$ is: if subsystem Θ is irreducible, then also its completion $\overline{\Theta}$ is, and the Dynkin diagram of Θ is connected and is obtained by removing a vertex from the affine Dynkin diagram of $\overline{\Theta}$.
Then we just need the list of complete irreducible subsystems of Φ, that we can get by Orlik and Solomon's tables.
Finally, by the "reduction theorem" we know how many components correspond to a given subsystem.

Description of $\mathcal{I}(\Phi)$

The proof follows from a general theorem of Procesi and MacPherson on "conical stratifications". However, now we know exactly who $\mathcal{I}(\Phi)$ is:
if subsystem Θ is irreducible, then also its completion $\overline{\Theta}$ is, and the Dynkin diagram of Θ is connected and is obtained by removing a vertex from the affine Dynkin diagram of $\overline{\Theta}$.
Then we just need the list of complete irreducible subsystems of Φ, that we can get by Orlik and Solomon's tables.
Finally, by the "reduction theorem" we know how many components correspond to a given subsystem.

Description of $\mathcal{I}(\Phi)$

The proof follows from a general theorem of Procesi and MacPherson on "conical stratifications". However, now we know exactly who $\mathcal{I}(\Phi)$ is: if subsystem Θ is irreducible, then also its completion $\overline{\Theta}$ is, and the Dynkin diagram of Θ is connected and is obtained by removing a vertex from the affine Dynkin diagram of $\overline{\Theta}$.

Then we just need the list of complete irreducible subsystems of Φ, that we can get by Orlik and Solomon's tables.

Finally, by the "reduction theorem" we know how many components correspond to a given subsystem.

Description of $\mathcal{I}(\Phi)$

The proof follows from a general theorem of Procesi and MacPherson on "conical stratifications". However, now we know exactly who $\mathcal{I}(\Phi)$ is: if subsystem Θ is irreducible, then also its completion $\overline{\Theta}$ is, and the Dynkin diagram of Θ is connected and is obtained by removing a vertex from the affine Dynkin diagram of $\overline{\Theta}$.
Then we just need the list of complete irreducible subsystems of Φ, that we can get by Orlik and Solomon's tables.
Finally, by the "reduction theorem" we know how many components correspond to a given subsystem.

Description of $\mathcal{I}(\Phi)$

The proof follows from a general theorem of Procesi and MacPherson on "conical stratifications". However, now we know exactly who $\mathcal{I}(\Phi)$ is: if subsystem Θ is irreducible, then also its completion $\overline{\Theta}$ is, and the Dynkin diagram of Θ is connected and is obtained by removing a vertex from the affine Dynkin diagram of $\overline{\Theta}$.
Then we just need the list of complete irreducible subsystems of Φ, that we can get by Orlik and Solomon's tables.
Finally, by the "reduction theorem" we know how many components correspond to a given subsystem.

The End

An introduction to toric arrangements

MSJ SI 2009 on Arrangements of Hyperplanes

Luca Moci

University of Roma Tre (Italy)

August, 9 2009