On Rybnikov's example

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[1]Consejo Superior de Investigaciones Cientificas

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Summary

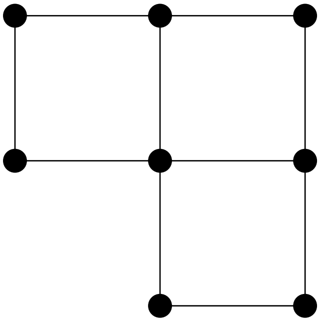
- Constructing the arrangements and the presentations of the groups.
- Distinguishing up to homologically trivial isomorphism: Alexander invariant.
- Showing that every isomorphism is homologically trivial: homological rigidity.

McLane arrangements

$$x \cdot y \cdot (x-y) \cdot z \cdot (x-z) \cdot (z+\omega^{\pm}y) \cdot (z+\omega^{\pm}y-(\omega^{\pm}+1)x) \cdot (z+(\omega^{\pm}+1)y-x)$$

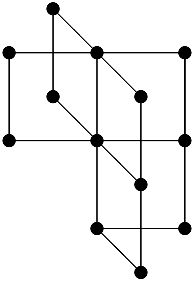
where $\omega^{\pm} = e^{\frac{\pm 2\pi i}{3}}$.

Are the two realizations of the dualization of the configuration of points in $\mathbb{F}_3\mathbb{P}^2$.

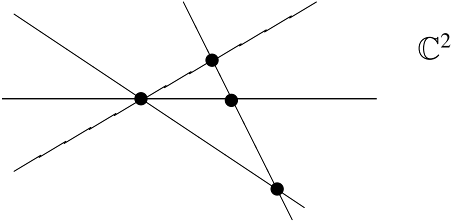


Rybnikov's arrangements

Take a generic projective transformation ρ that fixes the lines $x, y, x-y$, and glue ML^+ with $\rho(ML^{\pm})$



Fundamental group of the complement.



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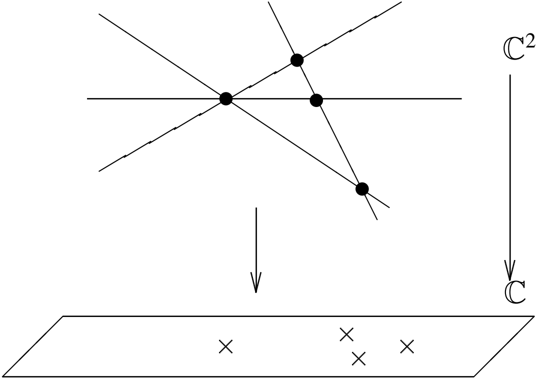

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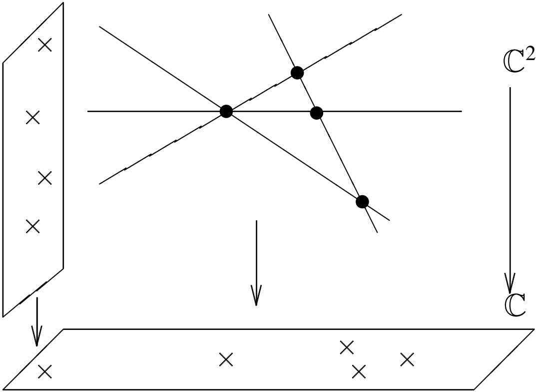

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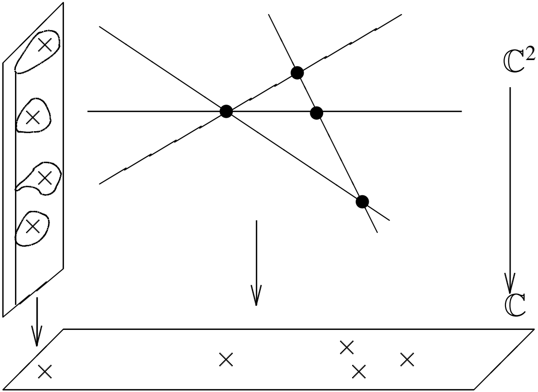

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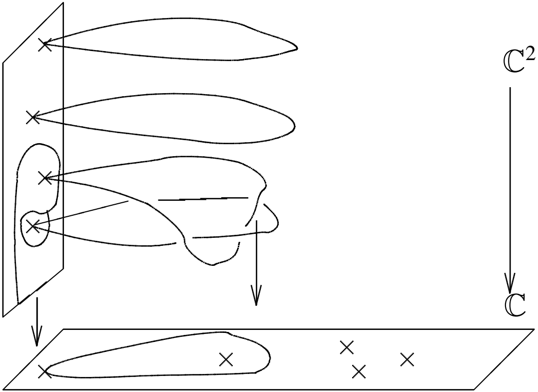

Presentation of the fundamental group.

Theorem (Zariski-Van Kampen)

The group $\pi_1(\mathbb{C}^2 \setminus \bigcup \mathscr{L})$ *admits a presentation as follows:*

- *Generators: the meridians around the lines,* $\{x_1, \ldots, x_n\}$.
- *Relations: the actions of the braids image of* $\pi_1(\mathbb{C} \setminus \pi(\Delta))$ *by the braid monodromy.*
- *In* $\mathbb{CP}^2$*: the meridian at infinity* $x_0 x_1 x_2 \cdots x_n$.

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The relations are always of the form $[x_{i_1}^{t_{i_1,p}}, \ldots, x_{i_m}^{t_{i_m,p}}]$; where

- $[a_1, \ldots, a_m]$ represents $a_1 \cdots a_m = a_2 \cdots a_m a_1 = \cdots = a_m a_1 \cdots a_{m-1}$.

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- p is the intersection point of $l_{i_1}, \ldots, l_{i_m}$.
- $t_{i_j,p}$ is some word in $x_1, \ldots, x_n$.

Alexander Invariant

Let X be a topological space. $G = \pi_1(X)$.

$$
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Hence, G'/G'' has a module structure over $\Lambda := \mathbb{Z}[H] = \mathbb{Z}[\mathbb{Z}^n]$. This module will be denoted M_G.

Algebraic setting

$$\begin{array}{ccc} G/G' \times G'/G'' & \to & G'/G'' \\ (g,[a,b]) & \longmapsto & g * [a,b] \bmod G'' \quad = [g,[a,b]] + [a,b] \end{array}$$

where $a * b := a \cdot b \cdot a^{-1}$.

Presentation of M_G

Theorem

A presentation of M_G as Λ-module can be obtained as follows:

- *The generators $\{[x_i, x_j] \mid 1 \leq i < j \leq n\}$.*
- *The relations of G expressed in terms of the generators.*
- *The Jacobi relations:*

$$(t_{x_i} - 1)[x_j, x_k] + (t_{x_j} - 1)[x_k, x_i] + (t_{x_k} - 1)[x_i, x_j]$$

Properties

Lema

The following relations hold in M_G:

- $[x,p] = (t_x - 1)p \ \forall p \in G'$.
- $[x^{-1}, y] = -t_x^{-1}[x,y]$.
- $[x_1 \cdots x_m, y_1 \cdots y_k] = \sum_{i=1}^m \sum_{j=1}^k T_{ij}[x_i, y_i]$ *where* $T_{ij} = \prod_{k=1}^{i-1} t_{x_k} \cdot \prod_{l=1}^{j-1} t_{y_l}$.
- $[p_1 \cdots p_m, x] = -(t_x - 1)(p_1 + \cdots p_m) \ \forall p_i \in G'$.
- $[p_x x, p_y y] = [x,y] + (t_x - 1)p_y - (t_y - 1)p_x \ \forall p_x, p_y \in G'$.
- $[x_1^{\alpha_1} \cdots x_m^{\alpha_m}, y_1^{\beta_1} \cdots y_k^{\beta_k}] = \sum_{i=1}^m \sum_{j=1}^k T_{ij}([x_i, x_j] + \delta(i,j))$, *where* $\delta(i,j) = -(t_{y_j} - 1)[\alpha_i^{-1}, x_i] + (t_{x_i} - 1)[\beta_j^{-1}, y_j]$.

Distinguishing the modules

- We have presentations of both modules.
- An isomorphism is given by a matrix corresponding to the generating systems, with entries in the ring Λ.
- Such a matrix induces an isomorphism if and only if and only if the image of the relations is in the submodule generated by the relations.
- To check the existence of such a matrix can be very difficult.
- Solution: truncate by powers of the augmentation ideal $(t_1 - 1, \ldots, t_n - 1)$.
- Then the problem becomes solving a system of equations over $\mathbb{Z}$.
- They have no solution!

Situation

$$G_1 \longrightarrow G_2$$

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Isomorphisms of fundamental groups induce **twisted** isomorphisms of the alexander invariants We need to study which are the possible isomorphisms of H.

Lower central series

- $\gamma_1 G := G$
- $\gamma_{i+1} G := [G, \gamma_i G]$
- $gr_i G := \gamma_i G / \gamma_{i+1} G$

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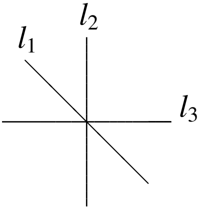
When $G = \pi_1(\mathbb{CP}^2 \setminus \bigcup \mathscr{L})$ we have the following:

- $gr_1 G = \mathbb{Z}\bar{x}_1 \oplus \ldots \oplus \mathbb{Z}\bar{x}_n =: H = \mathbb{Z}\bar{x}_0 \oplus \ldots \oplus \mathbb{Z}\bar{x}_n / x_0 + \cdots + x_n$
- $gr_2 G = H \bigwedge H / R$, where $R = \langle \sum_{l_j \in p} x_j \wedge x_i \mid p \in \mathscr{P}, l_i \in p \rangle$.

Example



gives the following relations: $\{(x_1 + x_2 + x_3) \wedge x_i\}_{i=1,2,3}$

So a basis of the quotient is $\{x_i \wedge x_j \mid p(i,j)_1 < i < j\}$, where $p(i,j)_1$ is the index of the first line that goes through $l_i \cap l_j$.

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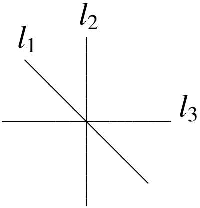

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Conditions on the isomorphism

A relation should be mapped to zero in the quotient:

$$R_1 \xrightarrow{i} H \wedge H \xrightarrow{\phi \wedge \phi} H \wedge H \xrightarrow{p} \frac{H \wedge H}{R_2}$$

Conditions on the isomorphism

Let $A = (a_{i,j})$ be a matrix that represents ϕ on the canonical generating system. Since it is not a basis, the comlumns of the matrix are defined modulo $(1, \ldots, 1)$.

The coordinates of the relations corresponding to the point $\{l_{i_1}, \ldots l_{i_m}\}$ on the elements of the basis of the quotient coming from $\{l_{j_1}, l_{j_2}, l_{j_3}\}$ are

$$\begin{vmatrix} a_{j_1,i_k} & a_{j_1,i_1} + \cdots + a_{j_1,i_m} & 1 \\ a_{j_2,i_k} & a_{j_2,i_1} + \cdots + a_{j_2,i_m} & 1 \\ a_{j_3,i_k} & a_{j_3,i_1} + \cdots + a_{j_3,i_m} & 1 \end{vmatrix}$$

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Conditions on the isomorphism

That is, for each point of multiplicity m, we have a map $\alpha : \mathcal{L} : \mathbb{Z}^{m-1}$ satisfying:

- For every point $p = \{l_{i_1}, \ldots, l_{i_m}\}$, and each line $l_{i_j} \in p$, the vectors $\alpha(l_{i_j})$ and $\alpha(l_{i_1}) + \cdots + \alpha(l_{i_m})$ are linearly deppendent.
- The images span $\mathbb{Z}^{m-1}$.

If we write this map in the form of a matrix, the conditions are equivalent to the rows belonging to a component of the resonance variety.
This is actually expected, since R and the subspace of the relations of the OS algebra are orthogonal. So in fact, H induces a permutation of the components of the resonance variety.

Combinatorial pencils

Theorem

Let S be a k-dimensional component of the resonance variety and $\mathscr{L}'$ the subarrangement formed by the lines in its support. Then there exists $\Pi_0, \ldots, \Pi_k$ a partition of $\mathscr{L}'$ and a map $m : \mathscr{L}' \to \mathbb{Z}^+$ such that, at every intersection point p, one of the following conditions hold:

- *All the lines in p are in the same Π_i.*
- *$\sum_{l \in \Pi_i} m(l)$ is independent of i.*

The converse is also true.

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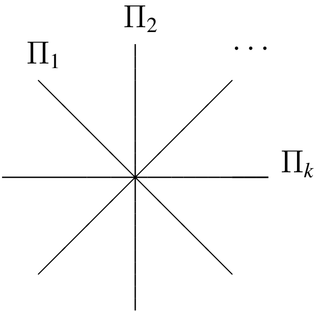
Definition

A triple $(\mathscr{L}, \Pi, m)$ as before is called a ***combinatorial pencil***

Examples of combinatorial pencils

- Multiple points
- Ceva arrangement
- Double cover branched along Ceva
- Finite fields.



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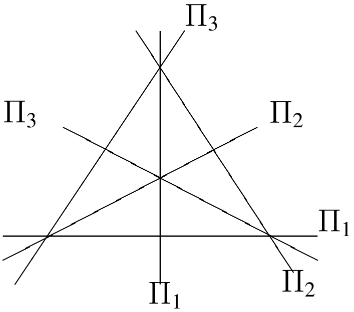

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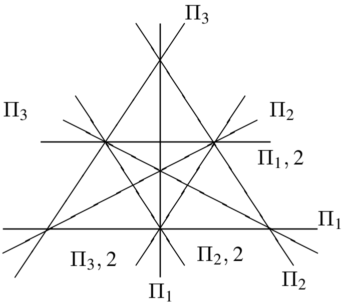

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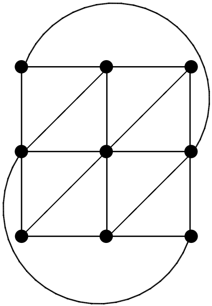

Existing combinatorial pencils

lines	combinatorics	pencils
3	2	1
4	3	1
5	5	1
6	10	2
7	24	1
8	69	1
9	384	6
10	5250	1
11	232929	3

Triangles of combinatorial pencils

Let $\alpha_1, \alpha_2, \alpha_3$ be components of the resonance variety.

Definition

We will say that $\{\alpha_1, \alpha_2, \alpha_3\}$ *form a* ***triangle*** *if*
$\sum_{i=1}^{3} dim(\{\alpha_i\}) - dim(\langle \alpha_1, \alpha_2, \alpha_3 \rangle) = 1.$

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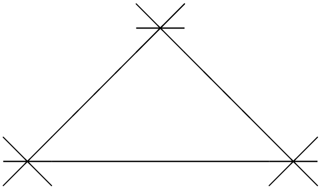
In the case that they correspond to point subarrangements, they are in triangle if and only if they are in the following disposition:



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- Using the triangular structure, we can bound the group of such permutations.
- This group must contain $Aut(\mathscr{L}, \mathscr{P}) \times \pm Id$.
- If the previous inclusion is an identity, the combinatorics is said to be **homologically rigid**.

Homological rigidity.

Definition

We will say that a combinatorcs $(\mathscr{L}, \mathscr{P})$ is ***strongly connected*** *if, given three distinct lines, two of them can be connected by multiple points not belonging to the thirthd.*

Theorem

Let $(\mathscr{L}, \mathscr{P})$ be a strongly connected combinatorics, $\sigma \in Aut(\mathscr{L}, \mathscr{P})$, and $\tau \in Aut_{(\mathscr{L}, \mathscr{P})}(H)$ such that the induced permutation of components of the resonance variety coincides with the one determined by σ. Then $\tau = \pm\sigma$.

Some results.

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Rybnikov combinatorics homologically rigid.

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The fundamental groups of Rybnikov's arrangements are non isomorphic.

Thank you!

Questions?