Characteristic varieties of arrangements with isolated non normal crossings

A.Libgober

August 8, Sapporo

1. Charactersitic varieties of arrangements.

2. Hodge structures on cohomology of local systems and corresponding decomposition of characteristic varieties.

3. INNC

4. Calculation of characteristic varieties using position of singularities.

5. Examples.

Abelian covers of the complements to arrangements of hyperplanes:

$$H_1(\mathbf{P}^{n+1} - \mathcal{A}, \mathbf{Z}) = \mathbf{Z}^{\mathrm{Card}\mathcal{A}-1}$$

Let $\widetilde{\mathbf{P}^{n+1} - \mathcal{A}}$ be universal abelian cover.

$H^i(\widetilde{\mathbf{P}^{n+1} - \mathcal{A}}, \mathbf{C})$ is module over

$$\mathbf{C}[H_1(\mathbf{P}^{n+1} - \mathcal{A}, \mathbf{Z})]$$

(isomorphic to the ring of Laurent polynomials)

Definition(A.Libgober, 1992)

Characteristic variety is $S^k = Supp(H^k(\widetilde{\mathbf{P}^{n+1} - \mathcal{A}}$
(or S^k_l **for its** l**-th exterior power)**

It is affine subvariety of $Spec\mathbf{C}[H_1(\mathbf{P}^{n+1} - \mathcal{A}, \mathbf{Z})]$

Can replace $\mathbf{P}^{n+1} - \mathcal{A}$ by a CW complex (in case of Alexander-Fox CW-complexes are complements to links in S^3).

Invariant of homotopy type determining other invariants

Homotopy groups and characteristic varieties:

if X is a CW complex, $\pi_1(X)$ is abelian and $\pi_i(X) = 0, i = 2, ...n-1$ then $H_n(\tilde{X}, \mathbf{C}) = \pi_n(X) \otimes \mathbf{C}$ as modules over $\mathbf{C}[H_1(X)]$.

If $\pi_1(X)$ is not abelian then

$$H_1(\tilde{X}, \mathbf{Z}) = \pi_1(X)'/\pi_1(X)''$$

and characteristic variety depends on

$$\pi_1(X)/\pi_1(X)''$$

Milnor fiber $M_{\mathcal{A}}$:

$$\Pi_k(l_k(x_0, ..., x_{n+1})) = 1$$

where $l_k(x_0, ..., x_n), k = 1, ..., \text{card}\mathcal{A}$ are equations of hyperplanes.

$M_{\mathcal{A}}$ is a cyclic cover of $\mathbf{P}^{n+1} - \mathcal{A}$ (considering affine coordinates as homogeneous). Galois group: $\mathbf{Z}/\text{Card}\mathcal{A}$. Map:

$$\pi_1(\mathbf{P}^{n+1} - \mathcal{A}) \to \mathbf{Z}/\text{Card}\mathcal{A}$$

sends each generator to 1 mod Card$\mathcal{A}$.

(Co)-Homology of $M_{\mathcal{A}}$ can be determined in terms of characteristic varieties.

Rank one local systems: $\chi : \pi_1(X) \to \mathbf{C}^*$.

Homology of local system obtianed from complex:

$$... \to C_i(\tilde{X}) \otimes_{H_1(X,\mathbf{Z})} \mathbf{C} \to ...$$

Charactersitic varieties determine the cohomology of local systems via spectral sequence.

There is canonical identification:

$$Spec\mathbf{C}[H_1(\mathbf{P}^{n+1} - \mathcal{A}, \mathbf{Z})] = Char\pi_1(X)$$

In the case H^1 and $\pi_1(X)$ is abelian or H^n and $\pi_i(X) = 0$ for $2 \le i < n$ one has

$$1 \neq \chi \in Supp\Lambda^j H^n(\tilde{X}) \Longleftrightarrow rkH^n(X, \chi) \ge j$$

Structure results:

Theorem(D.Arapura, 1996) Jumping loci of local systems on quasi-projective varieties:

X is quasiprojective $\Rightarrow$ jumping loci are unions of cosets (translated tori).

Theorem(L, 2007) Jumping loci for complements to germs of hypersurfaces:

D is hypersurface in $\mathbf{C}^{n+1}$ then

$$\{\chi \in Char\pi_1(\mathbf{C}^{n+1}-X) | dim H^k(\mathbf{C}^{n+1}-X, \chi) \geq j\}$$

is a union of finite order cosets (in quasi-projective case cosets have finite order also).

Example 1:

$$\mathbf{C}^2 \supset D = \{(x_1, x_2) | \Pi_{i=1}^{i=r} l_i(x_1, x_2) = 0\}$$

Then

$$\{\chi | dim H^1(\mathbf{C}^2 - D, \chi) \geq 1\} = \{(t_1, ..., t_r) | t_1 \cdot ... \cdot t_r = 1\}$$

More generally: jumping loci for complement to algebraic link has form $\Pi(t_1^{n_1} \cdot ... \cdot t_r^{n_r} - 1) = 0$

Example 2:

$$\mathbf{C}^{n+1} \supset D = \{(x_1, ..., x_{n+1}) | \Pi_{i=1}^{i=r} l_i(.., x_i, ...) = 0\}$$

(l_i are generic forms). Then

$$\{\chi | dim H^n(\mathbf{C}^{n+1} - D, \chi) \geq 1\} =$$

$$\{(t_1, ..., t_r) | t_1 \cdot ... \cdot t_r = 1\}$$

Problems Calculate the characteristic varieties which are translated tori.

Components of characteristic varieties and holomorphic maps.

Theorem: For each component $\mathcal{C}$ of characteristic variety S^1_1 which has form $\rho\mathcal{C}_0$ where $\mathcal{C}_0 \subset Char\pi_1$ is a subgroup there exist map

$$\mathbf{P}^{n+1} - \mathcal{A} \to \mathbf{P}^1 - n\ pts$$

such that for any

$$\chi = \pi^*(\chi_0) \quad \chi_0 \in Char\pi_1(\mathbf{P}^1 - n\ pts)$$

Example 1: $\mathbf{C}^2 - r\ lines \to \mathbf{P}^1 - r\ points$

Example 2 $\mathbf{C}^2 - \{(x-1)y(y-1)(x-y)\} \to \mathbf{P}^1 - 3\ points$

On $\mathbf{P}^2$ one has: $x(y-z)+z(x-y)+y(z-x) = 0$ and pencil gives map $\mathbf{C}^2 - 5\ lines \to \mathbf{P}^1 - 3\ pts$.

Theorem(Deligne-Timmerscheidt) Let χ be a unitary local system on a quasi-projective variety X. Then there exist mixed Hodge structure on the cohomology $H^i(X, \chi)$.

More precisely, if $\bar{X}$ is compactification ($\bar{X} - X$ is NCD), V is locally constant vector bundle with connection ∇_χ such that horizontal sectoins define the local system χ there is Hodge-deRham spectral sequence:

$$H^p(\bar{X}, \Omega^q(logD) \otimes V) \to H^{p+q}(X, \chi)$$

($\chi = 1$ is classical). It degenerates in term E_1 and induced filtration is Hodge filtration on $H^{p+q}(\chi)$.

In particular

$$Gr_F^p H^{p+q}(X, \chi) = H^p(\bar{X}, \Omega^q(logD) \otimes V)$$

Refined Problem Calculate jumping loci for

$$Gr_F^p H^{p+q}(X, \chi) = H^p(\bar{X}, \Omega^q(logD) \otimes V)$$

Then chracteristic varieties are Zariski closures of jumping loci for $Gr_F^p H^{p+q}(X, \chi)$.

Local version of existence of Hodge structure on cohomology of local systems.

Trivial local system case: Durfee (1986)

Theorem Let X be a germ of an algebraic space having an isolated normal singularity and let D be a divisor on X. Denote by χ a *unitary* representation of $\pi_1(X - D)$ and let χ be the corresponding local system. Then the cohomology groups

$$H^i(X - D, \chi)$$

support the canonical ($\mathbf{C}$)-mixed Hodge structure compatible with the holomorphic maps of pairs (X, D) endowed with a local system on the complement $X - D$

Corollary There is well defined local invariant of singularities:

$$S_l^{n,p} = \{\chi \in Char_{unitary}\pi_1(X - D)$$

$$|dimGr_F^p H^(X - D, \chi) \geq l\}$$

Structure of the jumping loci for

$$dimGr_F^pH^n(X,\chi)$$

Theorem(L-2007) Let U be the fundamental domain of the covering group of $Char_u(\pi_1(X))$ acting on the universal cover of the latter. (if $Char_u(\pi_1(X)) = (S^1)^r$ then universal cover is $\mathbf{R}^r$ and U is the unit cube.

Then preimage of $S_l^{n,p}$ in U is a union of rational polytopes i.e. set of solution of a finite set of inequalities with integer coefficients.

N.B. Here X can be quasi-projective or complement to a divisor in a germ of singularity.

Remark: this implies translated subgroup theorem in both local and global case since Zariski closure of exponent of polytope (i.e. image of $\mathbf{R}^r \to (S^1)^r$) is translated subgroup.

Remark: $S_l^{n,p}$ in the case $X = \mathbf{C}^n - (f = 0)$, $f = 0$ is isolated singularity is the spectrum of f.

Polytopes can be calculated using identification of Deligne extension (L-2007).

Some of jumping polytopes can be calculated using top degree differential forms (case of curves L-1998).

Calculation of $h^{n,0}H^n(L_\chi) = dim Gr_F^n Gr_n^W$

Case of cone in $\mathbf{C}^n$ over a generic arrangement of $n+l$ hyperplanes in $\mathbf{P}^{n-1}$.

Consider complete intersection:

$$V_{m_1,...,m_{n+l}}:$$

$$z_1^{m_1} = l_1(u_1, ..., u_n), \;\; .... \;\; , z_{n+l}^{m_{n+l}} = l_{n+l}(u_1, ..., u_n)$$

and extension problem of differential form:

$$\omega_\phi = \frac{z_1^{i_1} .... z_{n+l}^{i_{n+l}} \phi(u_1, ..., u_n) du_1 \wedge .... \wedge du_n}{z_1^{m_1} .... z_{n+l}^{m_{n+l}}}$$

For fixed $\phi \in \mathbf{C}\{u_1, ..., u_n\}$ ω_ϕ extends iff $x_j = \frac{i_j+1}{m_j}$ satisfies:

$$x_1 + ... + x_{n+l} = k \quad 1 \le k \le n + l - 2$$

(**polytope of quasi-adjunction**)

Theorem: $dim Gr^n_F H^n(\mathbf{C}^n - \mathcal{A})$ is constant for all χ in the polytope.

Moreovere, for fixed k the set of ϕ such that for $(i_1, ..i_{n+l}, m_1, .., m_{n+l})$ extends form an ideal (**ideal of quasi-adjucntion**) (power of maximal ideal).

Example Triple point in $\mathbf{C}^2$

Preimage of $t_1 t_2 t_3 = 1$ is

$$x_1 + x_2 + x_3 = 1, 2$$

Polytopes of quasiadjunction is correspond to $RHS = 1$. Ideal of quasi-adjunciton is $\mathcal{M}$.

Example Quadruple point in $\mathbf{C}^2$.

Preimage of $t_1 t_2 t_3 t_4 = 1$ in unit cube is union of 3 hyperplanes:

$$x_1 + x_2 + x_3 + x_4 = 1, 2, 3$$

These are polytopes of quasiadjunction for $RHS = 1, 2$. The ideal of quasi-adjunction corresponding to $RHS = 1$ is $\mathcal{M}^2$ and to $RHS = 2$ corresponds $\mathcal{M}$.

Example Quadruple point in $\mathbf{C}^3$ (cone over generic arrangement of 4 lines in $\mathbf{P}^2$.

Polytopes of quasiadjunction is

$$x_1 + x_2 + x_3 + x_4 = 1$$

and corresponding ideal of quasi-adjunction is $\mathcal{M}$

In some case local polytopes of quasi-adjunction determine global ones. Case of arrangments of lines (and arbitrary reducible algebraic curves: L-1998). This is always the case for INNC.

Isolated Non Normal crossings.

Definition: Arrangement $\mathcal{A} \subset \mathbf{P}^{n+1}$ is called INNC if the set of points where fails to be be a normal crossing divisor has dimension zero.

Examples: Any arrangement in $\mathbf{P}^2$

Arrangements in $\mathbf{P}^3$ such that no 3 planes contain a line.

Theorem (L-2004) $\mathcal{A} \subset \mathbf{P}^{n+1}$ is INNC and $n > 1 \Rightarrow$

$$\pi_1(\mathbf{P}^{n+1} - \mathcal{A}) = \mathbf{Z}^{Card\mathcal{A}-1}$$

$$\pi_i(\mathbf{P}^{n+1} - \mathcal{A}) = 0 \quad i < n$$

Problem: calculate $Supp\pi_n(\mathbf{P}^{n+1} - \mathcal{A}) \subset Char\pi_1$

Support of $\pi_n(\mathbf{P}^{n+1} - \mathcal{A})$ is union of translated subgroups.

Essential components.

Theorem: Each component of subarrangement is component of $\mathcal{A}$.

Example: If INNC arrangement in $\mathbf{P}^3$ has only quadruple points then to each quadruple point correspond component of $Supp\pi_n$.

Definition: Essential component is component which is not a component of subarrangment.

Remark: This is NOT equivalent to being coordinate component (case of curves: Artal-Cogolludo).

Definition A polytope is essential if its Zariski closure is an essential component of characteristic variety.

Definition Let $\mathcal{A}$ be arrangement in $\mathbf{P}^{n+1}$. $r = Card\mathcal{A}$. Polytope in $\mathcal{U} \subset \mathbf{R}^r$ consisoting of $(x_1, .., x_r)$ such that $exp(x_1, ..., x_r) \subset$ Char$\pi_1(\mathbf{P}^{n+1} - \mathcal{A} - H_\infty)$ belongs to set

$$\{\chi | dim Gr_F^n H^n(\mathbf{P}^{n+1} - \mathcal{A}, \chi) \geq l\}$$

is called global polytope of quasi-adjunction.

$S_l^{n,p}$ is called global jumping polytope.

Each point non NCD point P of INNC having multiplicity grater than n and defining subarrangement $\mathcal{A}_P$ yields local polytope of quasi-adjunction or jumping polytope which using projection:

$$\text{Char}\pi_1(\mathbf{P}^{n+1} - \mathcal{A}) \to \text{Char}\pi_1(\mathbf{C}^{n+1} - \mathcal{A}_P)$$

and

$$\mathcal{U} \to \mathcal{U}_{\mathcal{A}_P}$$

Global polytope corresponding to collection $\mathcal{S}$ of non NCD points is intersection in $\mathcal{U}$ of preimages of local polytopes corresponding to points P in the collection.

Global polytope coresponding to collection $\mathcal{S}$ is called contributing if it belongs to global polytope of quasi-adjunction.

Theorem Let $\mathcal{A} \subset \mathbf{P}^{n+1}$ be INNC arrangement and $r = Card\mathcal{A}$. $\mathcal{U}$ is the unit cube in $\mathbf{R}^r$. Let $\mathcal{F} \subset \mathcal{U}$ be the essential global polytope $Gr_F^n H^n(\mathbf{P}^{n+1} - \mathcal{A}, \chi)$ i.e. a polytope which is an intersection of polytopes of quasi-adjunction corresponding to a collection $\mathcal{S}$ of non-normal crossings of $\mathcal{A}$.

Let $x_1 + ... + x_r = l$ be a hyperplane containing the global polytope $\mathcal{F}$. If $H^1(\mathcal{A}_{\mathcal{F}} \otimes \mathcal{O}(r - n - 2 - l)) = k$, then $\mathcal{F}$ is contributing i.e. $\mathcal{F}$ is a global polytope of quasi-adjunction and the Zariski closure of $exp(\mathcal{F}) \subset \mathrm{Char} H_1(\mathbf{P}^{n+1} - D)$ is a component of characteristic variety of $\pi_n(\mathbf{P}^{n+1} - \mathcal{A})$

Example *Hesse arrangement.* Let $\mathcal{A}$ be arrangement in $\mathbf{P}^2$ formed by 12 lines containing nine inflection points of a non-singular cubic. These 12 lines also can be viewed as union of 4 cubics each being union of three transversal lines. The arrangement contains 12 quadruple points. The ideal corresponding to the face of quasi-adjunction $\sum x_i = 2$ is the maximal ideal. If we label lines $x_i, i = 1, ..., 12$ and $\mathcal{V}_P$ is collection of indices of lines containing a quadruple point P then polytope of quasi-adjunction corresponding to full collection of all nine quadruple points

$$\sum_{i \in \mathcal{V}_P} x_i = 2$$

This polytopes is contributing since it belongs to hyperplane:

$$3 \sum_{i=1}^{12} x_i = 2 \cdot 9$$

(each line contains three quadruple points) and

$$dim H^1(\mathbf{P}^2, \mathcal{J}_{P_1,...,P_9}(12 - 3 - 6)) = 1$$

(by Cayley-Bacharach).

Example Consider the arrangement of nine lines in $\mathbf{P}^2$ given by equations:

$$L_v^1: \; x = -z, \; L_v^2 : x = 0, \; L_v^3 : x = z$$

$$L_h^1 : y = -z, \; L_h^2 : y - 0, \; L_h^3 : \; y = -z, \quad (1)$$

$$L_s^1 : \; y = x \; L_s^2 : \; y = -x \; L_\infty : z = 0$$

It has 4 triple and 3 quadruple points.

$$x_v^1 + x_h^1 + x_s^1 = 1 \quad x_v^3 + x_h^1 + x_s^2 = 1$$

$$x_s^1 + x_h^3 + x_v^3 = 1 \quad x_v^1 + x_h^3 + x_s^2 = 1$$

$$x_v^2 + x_h^2 + x_s^1 + x_s^2 = 2 \quad x_\infty + x_v^1 + x_v^2 + x_v^3 = 2$$

$$x_\infty + x_h^1 + x_h^2 + x_h^3 = 2$$

Intersection of these hyperplanes gives polytope which belongs to hyperplane (assigning weight 1 (resp. 2) to triple (resp. quadruple) points:

$$x_v^1 + x_v^2 + x_v^3 + x_h^1 + x_h^2 + x_h^3 + x_s^1 + x_s^2 + x_\infty = 4$$

The collection of all seven multiple points is contributing since clearly $dim H^1(\mathbf{P}^2, \mathcal{J}(9-3-4)) = 1$ as follows from the cohomology sequence corresponding to the sequence

$$0 \rightarrow \mathcal{J}(2) \rightarrow \mathcal{O}_{\mathbf{P}^2}(2) \rightarrow \mathcal{O}_{7P} \rightarrow 0$$

(here $\mathcal{O}_{7P}$ is the skyscraper sheaf with support at seven singular points of the arrangement).

The corresponding component of characteristic variety is given by:

$$t_v^1 t_h^1 t_s^1 = 1 \quad t_v^3 t_h^1 t_s^2 = 1 \quad t_s^1 t_h^3 t_v^3 = 1 \quad t_v^1 t_h^3 t_s^2 = 1$$

$$t_v^2 t_h^2 t_s^1 t_s^2 = 1 \quad t_\infty t_v^1 t_v^2 t_v^3 = 1 \quad t_\infty t_h^1 t_h^2 t_h^3 = 1$$

(here $t_b^a = exp(2\pi i x_b^a)$)

Example Consider web of quadrics in $\mathbf{P}^n$. given by

$$a_0x_0^2 + ... + a_nx_n^2 = 0 \quad \sum a_i = 0$$

Among $\binom{n+1}{2}$ reducible quadrics in this web we select $n+1$ which form INNC. Selection of these $n+1$ quadrics can be done as follows. The $\binom{n+1}{2}$ points in $\mathbf{P}^{n-1}$ corresponding to reducible quadrics are the 0-dimensional strata of generic arrangement of $n+1$ hyperplanes in $\mathbf{P}^{n-1}$ since each zero-dimensional stratum is an intersection of $n-1$ hyperplanes since arrangement is generic. If we select among these $\binom{n+1}{2}$ points such that no $n-1$ belong to hyperplane (we call such collection of points *generic*) we get reducible quadrics with hyperplanes forming INNC. An explicite example of generic collection of $n+1$ point can be given as follows. Suppose that $n+1$ hyperplanes in $\mathbf{P}^{n-1}$ in coordintates $u_1, ..., u_n$ are given by

$u_i = 0, i = 1, ..., n$ and $u_1 + ... + u_n = 0$. Then the points

$$(1, -1, 0, 0, ..., 0)$$

$$(0, 1, -1, 0, ..., 0)$$

$$(\quad ... \quad)$$

$$(0, 0, ..., 0, 1, -1)$$

$$(1, 0, ..., 0)$$

$$(0, ..., 1)$$

form a generic collection In the case $n = 3$ we get four quadrics in this web:

$$Q_0 : \; x_0^2 - x_1^2 = 0 \quad Q_1 : \; x_1^2 - x_2^2 = 0$$

$$Q_2 : \; x_2^2 - x_3^2 = 0 \quad Q_3 : \; x_0^2 - x_3^2 = 0$$

We have eight planes: $Q_i = L_i^+ \cup L_i^-$. The eight base points are:

$$(\pm 1, \pm 1, \pm 1 \pm 1)/((1, 1, 1, 1), -(1, 1, 1, 1)$$

or explicitely:

$$(1,1,1,1)$$

$$(1,1,1,-1)$$

$$(1,1,-1,1)$$

$$(1,1,-1,-1)$$

$$(1,-1,1,1)$$

$$(1,-1,1,-1)$$

$$(1,-1,-1,1)$$

$$(1,-1,-1,-1)$$

Denoting element of $H_1(\mathbf{P}^3 - \mathcal{A}, \mathbf{Z}) = \mathbf{Z}^7$ corresponding $L_i^{\pm}$ by $x_i^{\pm}$ we obatin the following components of charactersitic varieties. There are eight non essential components and one essential one corresponding to the web of quadrics and given by eight equations corresponding to each base point:

$$x_1^+ + x_2^+ + x_3^+ + x_4^+ = k_1$$

$$x_1^+ + x_2^+ + x_3^- + x_4^- = k$$

$$x_1^+ + x_2^- + x_3^- + x_4^+ = k$$

$$x_1^+ + x_2^- + x_3^+ + x_4^- = k$$

$$x_1^- + x_2^- + x_3^+ + x_4^+ = k$$

$$x_1^- + x_2^- + x_3^- + x_4^- = k$$

$$x_1^- + x_2^+ + x_3^- + x_4^+ = k$$

$$x_1^- + x_2^+ + x_3^+ + x_4^- = k$$

The ideal of quasiadjunction corresponding to hyperplanes $k_i = 2$ is $\mathcal{M}$. Hence we obtain that hyperplanes with $k = 2$ gives subset of 3-dimensional torus.