On a problem of arrangements related to the hypergeometric integrals of confluent type

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Purpose

- Explanation of general HGI from the view point of Radon transform using Gauss HGF and its confluent functions.

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- Compute the rational de Rham cohomology groups for the general HGI.

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- Compute the rational de Rham cohomology groups for the general HGI.
- Provide a working example to the hyperplane arragement theory.

Classical HGF

Gauss HGF:

$$
{}_2F_1(a,b,c;x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b} du
$$

holomorphic solution at $x = 0$ of

$$
x(1-x)y'' + \{c - (a+b+1)x\}y' - aby = 0.
$$

with $y(0) = 1$.

Confluent type functions considered here:

$$ {}_2F_1(a,b,c;x) = C\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}d $$

(Gauss)

$$ {}_1F_1(a,c;x) = C\int_0^1 e^{xu}u^{a-1}(1-u)^{c-a-1}du $$

(Kummer)

$$ J_a(x) = C\int_\gamma e^{x(u-1/u)}u^{-a-1}du \qquad \text{(Bessel)} $$

$$ H_a(x) = C\int_\gamma e^{xu-\frac{1}{2}u^2}u^{-a-1}du \qquad \text{(Hermite)} $$

$$ \mathrm{Ai}(x) = C\int_\gamma e^{xu-\frac{1}{3}u^3}du \qquad \text{(Airy)} $$

The differential equations

$$x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0,$$

$$xy'' + (c-x)y' - ay = 0, \qquad \text{(Kummer)}$$

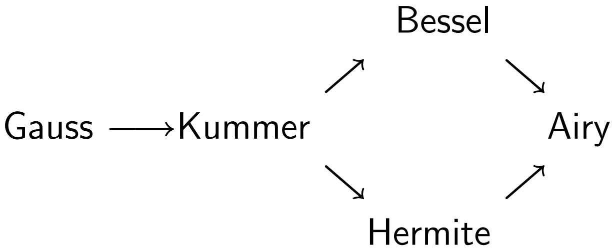
$$x^2y'' + xy' + (x^2-a^2)y = 0, \qquad \text{(Bessel)}$$

$$y'' - xy' + ay = 0, \qquad \text{(Hermite)}$$

$$y'' - xy = 0 \qquad \text{(Airy)}$$

HGF and partition of 4

Arrange these functions as



To these functions we associate the partitions of $\mathbf{4}$:

$$\begin{array}{ccccccc} & & & & 2+2 & & \\ & & & \nearrow & & \searrow & \\ 1{+}1{+}1{+}1 & \longrightarrow & 2+1+1 & & & & 4 \\ & & & \searrow & & \nearrow & \\ & & & & 3+1 & & \end{array}$$

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Definition

$\boldsymbol{a} \in \mathbf{GL}_4(\mathbb{C})$ *a regular element.*
$\Leftrightarrow$ $\boldsymbol{O(a)} = \{\boldsymbol{gag^{-1}} \mid \boldsymbol{g} \in \mathbf{GL}_4(\mathbb{C})\}$ *is of maximum dimension.*
$\Leftrightarrow$ *any two Jordan cells of* $\boldsymbol{a}$ *have different eigenvalues.*

If $\boldsymbol{a} \in \mathbf{GL}_4(\mathbb{C})$ is regular , then $\boldsymbol{a}$ is similar to

$$\begin{pmatrix} a_0 & & & \\ & a_1 & & \\ & & a_2 & \\ & & & a_3 \end{pmatrix} \longleftrightarrow (1+1+1+1)$$

$$\begin{pmatrix} a_0 & 1 & & \\ & a_0 & & \\ & & a_2 & \\ & & & a_3 \end{pmatrix} \longleftrightarrow (2+1+1)$$

$$\begin{pmatrix} a_0 & 1 & & \\ & a_0 & & \\ & & a_2 & 1 \\ & & & a_2 \end{pmatrix} \longleftrightarrow (2+2)$$

$$\begin{pmatrix} a_0 & 1 & & \\ & a_0 & 1 & \\ & & a_0 & \\ & & & a_3 \end{pmatrix} \longleftrightarrow (3+1)$$

$$\begin{pmatrix} a_0 & 1 & & \\ & a_0 & 1 & \\ & & a_0 & 1 \\ & & & a_0 \end{pmatrix} \longleftrightarrow (4)$$

where $a_i \neq a_j$ $(i \neq j)$.

Regular elements $\longrightarrow$ the centralizers.

$$H_{(1,1,1,1)} = \left\{ \begin{pmatrix} h_0 & & & \\ & h_1 & & \\ & & h_2 & \\ & & & h_3 \end{pmatrix} \right\} \qquad \text{(Gauss)}$$

$$H_{(2,1,1)} = \left\{ \begin{pmatrix} h_0 & h_1 & & \\ & h_0 & & \\ & & h_2 & \\ & & & h_3 \end{pmatrix} \right\} \qquad \text{(Kummer)}$$

$$H_{(2,2)} = \left\{ \begin{pmatrix} h_0 & h_1 & & \\ & h_0 & & \\ & & h_2 & h_3 \\ & & & h_2 \end{pmatrix} \right\} \qquad \text{(Bessel)}$$

$$H_{(3,1)} = \left\{ \begin{pmatrix} h_0 & h_1 & h_2 & \\ & h_0 & h_1 & \\ & & h_0 & \\ & & & h_3 \end{pmatrix} \right\} \qquad \text{(Hermite)}$$

$$H_{(4)} = \left\{ \begin{pmatrix} h_0 & h_1 & h_2 & h_3 \\ & h_0 & h_1 & h_2 \\ & & h_0 & h_1 \\ & & & h_0 \end{pmatrix} \right\} \qquad \text{(Airy)}$$

Gauss case

$$
{}_2F_1(a,b,c;x) = C\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du.
$$

- Zeros of u, $1-u$, $1-xu$ is important.
- The information at $u=\infty$ is also important!
- Make $u=\infty$ visible.
- $t=(t_0,t_1)$: homog. coord. of $\mathbb{P}^1$.
- $u\in\mathbb{C}\subset\mathbb{P}^1$ is related by $u=t_1/t_0$.
- Put $(\alpha_1,\alpha_2,\alpha_3) := (a-1, c-a-1, -b)$

integrand

$$= \left(\frac{t_1}{t_0}\right)^{\alpha_1}\left(1-\frac{t_1}{t_0}\right)^{\alpha_2}\left(1-x\frac{t_1}{t_0}\right)^{\alpha_3} d\left(\frac{t_1}{t_0}\right)$$

$$= t_0^{-2-\alpha_1-\alpha_2-\alpha_3} t_1^{\alpha_1}(t_0-t_1)^{\alpha_2}(t_0-xt_1)^{\alpha_3} \times (t_0 dt_1 - t_1 dt_0).$$

- Put $\alpha_0 = -2-\alpha_1-\alpha_2-\alpha_3$.
- The behavior at $u=\infty$ is visible as the term $t_0^{\alpha_0}$.

We think the integrand is constructed as follows:

- $\boldsymbol{\lambda} = (1, 1, 1, 1)$,
- $$\boldsymbol{H} = \left\{ \boldsymbol{h} = \begin{pmatrix} h_0 & & & \\ & h_1 & & \\ & & h_2 & \\ & & & h_3 \end{pmatrix} \right\} \subset \mathbf{GL}_4(\mathbb{C}).$$

- $\chi : \tilde{H} \to \mathbb{C}^\times$: a character

$$\chi(h;\alpha) = h_0^{\alpha_0} \cdots h_3^{\alpha_3},$$

with $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ such that

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = -2.$$

- Substitute into χ the linear functions of t_0, t_1:

$$h_0(t) = t_0,\ h_1(t) = t_1,$$
$$h_2(t) = t_0 - t_1,\ h_3(t) = t_0 - xt_1.$$

-

$$_2F_1(a,b,c;x) = C \int \chi(h(t);\alpha)\,(t_0 dt_1 - t_1 dt_0).$$

$h_i(t)$ are determined by the columns of

$$\begin{aligned} h(t) &= (t_0, t_1, t_0 - t_1, t_0 - xt_1) \\ &= (t_0, t_1) \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix} \end{aligned} \tag{1}$$

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Question: Why the particular linear polynomials specified by (1) are chosen?

Gelfand's idea: Replace the matrix (1) by a general 2×4 matrix.

- $Z = \{z \in \mathrm{Mat}_{2,4}(\mathbb{C}) \mid \text{any 2-minor} \neq 0\}$
- For $z = (z_0, z_1, z_2, z_3) \in Z$, put

$$h(t) = tz = (tz_0, tz_1, tz_2, tz_3)$$

- Gelfand HGF:

$$F(z;\gamma) = \int_\gamma \chi(h(t);\alpha) \cdot (t_0 dt_1 - t_1 dt_0)$$

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Answer: Essentially the same!

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We will explain why essentially the same.

Consider the action of $\mathrm{GL}_2(\mathbb{C}) \times H$ on Z:

$$\mathrm{GL}_2(\mathbb{C}) \times Z \times H \ni (g, z, h) \mapsto gzh \in Z. \qquad (2)$$

Proposition

1) For $g \in \mathrm{GL}_2(\mathbb{C})$

$$F(gz; \gamma') = (\det g)^{-1} F(z; \gamma).$$

where $\gamma' = (g^{-1})_* \gamma$ *is the image of* γ *by* $t \mapsto tg^{-1}$.
2) For $h \in \tilde{H}$,

$$F(zh; \gamma) = F(z; \gamma) \chi(h; \alpha).$$

The above proposition says

1. the values of F on the orbit $O(z)$ is determined by the value at z.
2. If we can take $X \subset Z$ which intersect once with each orbit, the restriction of F on X determines F.
3. As a realization $X \subset Z$ of $\mathrm{GL}_2(\mathbb{C})\backslash Z/H$, we can take

$$X = \left\{ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix} \right\} \subset Z.$$

Conclusion

Gauss HGF
$\Leftrightarrow$ Radon transform of $\chi(h,\alpha)$ of $\tilde{H}_{(1,1,1,1)}$.

Airy's case

We can understand the Airy integral

$$\mathbf{Ai}(x) = \int_{\gamma} \exp(xu - \frac{1}{3}u^3)du$$

in a similar way.

- $\boldsymbol{\lambda} = (4)$: a partition of 4.

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$$\boldsymbol{H}_{(4)} = \left\{ \boldsymbol{h} = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 \\ & h_0 & h_1 & h_2 \\ & & h_0 & h_1 \\ & & & h_0 \end{pmatrix} \right\} \subset \mathrm{GL}_4(\mathbb{C})$$
$$= \{ h_0 I + h_1 \Lambda + h_2 \Lambda^2 + h_3 \Lambda^3 \mid h_0 \neq 0 \}$$
$$= \left(\mathbb{C}[T]/(T^4) \right)^{\times}.$$

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- $\chi : \tilde{H} \to \mathbb{C}^{\times}$: a character.

Explicit form of χ.

Let $\theta_j(h)$ $(j = 0, \dots, 3)$ be defined by

$$\log(h_0 I + h_1 \Lambda + h_2 \Lambda^2 + h_3 \Lambda^3) = (\log h_0) I + \theta_1(h) \Lambda + \theta_2(h) \Lambda^2 + \theta_3(h) \Lambda^3 \quad (3)$$

The Taylor expansion of $\log$ gives

$$\theta_1(h) = \frac{h_1}{h_0},$$

$$\theta_2(h) = \frac{h_2}{h_0} - \frac{1}{2} \left(\frac{h_1}{h_0} \right)^2,$$

$$\theta_3(h) = \frac{h_3}{h_0} - \left(\frac{h_1}{h_0} \right) \left(\frac{h_2}{h_0} \right) + \frac{1}{3} \left(\frac{h_1}{h_0} \right)^3.$$

- The map $h \mapsto (h_0, \theta_1(h), \dots, \theta_3(h))$ gives an isomorphism $H_{(4)} \simeq \mathbb{C}^\times \times \mathbb{C}^3$.

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$$\chi(h;\alpha) = h_0^{\alpha_0} \exp(\alpha_1\theta_1(h)+\alpha_2\theta_2(h)+\alpha_3\theta_3(h))$$

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Question: Why we choose particluar linear polynomials specified by

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The answer is again given by considering the generalized Airy function.

- $\boldsymbol{Z_{(4)} = \{(z_0, \dots, z_3) \in \mathrm{Mat}_{2,4}(\mathbb{C}); [0,1] \neq 0\}}.$

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- Define

$$F(z, \gamma) = \int_\gamma \chi(h(t); \alpha)(t_0 dt_1 - t_1 dt_0)$$

with $h(t) = tz, \; z \in Z_{(4)}$

- We can show $F(z,\gamma)$ satisfies

$$F(gz;\gamma') = (\det g)^{-1} F(z;\gamma),\ g \in \mathrm{GL}_2(\mathbb{C})$$
$$F(zh;\gamma) = F(z;\gamma)\chi(h;\alpha), \tilde{h} \in H_{(4)}.$$

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-

$$\mathrm{GL}_2(\mathbb{C}) \backslash Z_{(4)} / H_{(4)} \simeq \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x \end{pmatrix} ; x \in \mathbb{C} \right\}.$$

This explain that the generalized Airy function is essentially the same as the classical one.

Remark

The reason for choosing $\boldsymbol{\alpha}$ *in* $\boldsymbol{\chi}$ *as*

$$\boldsymbol{\alpha = (-2, 0, 0, -1).}$$

$\Rightarrow$ *the group of symmetry for GAI which is an analogue of Weyl group:* $\boldsymbol{N_{\mathrm{GL}_4(\mathbb{C})}(H_{(4)})/H_{(4)}}$.

General Hypergeometric integrals

Consider a generalization of the above examples.

Maximal abelian group

- $N \geq 3$ integer.
- $\lambda = (n_1, \ldots, n_\ell)$, a partition of N, i.e.,

$$n_1 \geq n_2 \geq \cdots \geq n_\ell > 0, \quad n_1 + \cdots + n_\ell = N.$$

- A maximal abelian subgroup

$$H_\lambda = J(n_1) \times \cdots \times J(n_\ell) \subset \mathrm{GL}_N(\mathbb{C}),$$

where

$$J(n) = \left\{ h = \begin{pmatrix} h_0 & h_1 & \dots & h_{n-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & h_1 \\ & & & h_0 \end{pmatrix} ; h_0 \neq 0 \right\}$$

$$\simeq \left(\mathbb{C}[T]/(T^n)\right)^\times \text{ Jordan group.}$$

- An element $h \in H_\lambda$ is denoted as

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- Isomorphism $J(n) \simeq \mathbb{C}^\times \times \mathbb{C}^{n-1}$ is given by

$$h \mapsto (h_0, \theta_1(h), \dots, \theta_{n-1}(h)),$$

where $\theta_m(h)$ $(m = 1, 2, \dots)$ is defined by

$$\log h = \log(h_0 I + h_1 \Lambda + \cdots + h_{n-1}\Lambda^{n-1})$$
$$= (\log h_0) I + \sum_{m=1}^{n-1} \theta_m(h) \Lambda^m$$

where $\Lambda = (\delta_{i+1,j}) \in \mathrm{Mat}_n(\mathbb{C})$ the shift matrix.

Put $\theta_0(h) = \log h_0$. $\theta_m(h)$ is explicitely given as

$$\theta_m(h) = \sum (-1)^{|k|-1} \frac{(|k|-1)!}{k_1! \cdots k_m!} \left(\frac{h_1}{h_0}\right)^{k_1} \cdots \left(\frac{h_m}{h_0}\right)^{k_m}.$$

where the sum is taken for k s.t.
$k_1 + 2k_2 + \cdots + mk_m = m$.

Character of H_λ

- $\chi_n : \tilde{J}(n) \longrightarrow \mathbb{C}^\times$ is given by

$$\chi_n(h;\alpha) = \exp\left(\alpha_0\theta_0(h) + \cdots + \alpha_{n-1}\theta_{n-1}(h)\right)$$

- $\chi : \tilde{H}_\lambda \longrightarrow \mathbb{C}^\times$ is

$$\begin{aligned} \chi(h;\alpha) &= \prod_{k=1}^{\ell} \chi_{n_k}(h^{(k)};\alpha^{(k)}) \\ &= \prod_{k=1}^{\ell} \exp\left(\sum_{m=0}^{n_k-1} \alpha_m^{(k)}\theta_m(h^{(k)})\right), \end{aligned}$$

where $\alpha^{(k)} = (\alpha_0^{(k)}, \ldots, \alpha_{n_k-1}^{(k)}) \in \mathbb{C}^{n_k}$.

- Assumption:

$$\sum_{k=1}^{\ell} \alpha_0^{(k)} = -r - 1, \quad \alpha_{n_k-1}^{(k)} \neq 0 \ (\forall k). \qquad (4)$$

Radon transform Consider the Radon transform of the character $\chi(\cdot\,;\alpha)$ of $\tilde{H}_\lambda$.

- $\vec{u} = (1, u_1, \ldots, u_r)$: variables of integration: $\mathbb{C}^r \subset \mathbb{P}^r$.
- Space of coefficients of linear polynomials:

$$Z_{r,N}^{\lambda} = \{z = (z^{(1)}, \ldots, z^{(\ell)}) \in \mathbf{Mat}_{r+1,N}(\mathbb{C}) \mid (*$$

where $z^{(k)} = (z_0^{(k)}, \ldots, z_{n_k-1}^{(k)}) \in \mathbf{Mat}_{r+1,n_k}(\mathbb{C})$

- The condition (*): for any $(m_1, \ldots, m_\ell)$ s.t.
  1. $0 \leq m_k \leq n_k \quad (k = 1, \ldots, \ell)$,
  2. $m_1 + \cdots + m_\ell = r + 1$,

  $$\det(z_0^{(1)}, \ldots, z_{m_1-1}^{(1)}, \ldots, z_0^{(\ell)}, \ldots, z_{m_\ell-1}^{(\ell)}) \neq 0.$$

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  $$\det(z_0^{(1)}, \dots, z_{m_1-1}^{(1)}, \dots, z_0^{(\ell)}, \dots, z_{m_\ell-1}^{(\ell)}) \neq 0.$$

- For a character of $\chi(\cdot\,;\alpha)$ of $\tilde{H}_\lambda$ with (4), GHGI is

  $$I(z, \alpha, c) = \int_c \chi(\vec{u}z; \alpha) du,$$

  where $du = du_1 \wedge \cdots \wedge du_r$ and c is a cycle of some homology group defined by using $\chi(\vec{u}z; \alpha)$.

Remark

From the explicit form of $\boldsymbol{\theta}_m$, $\Rightarrow$

- *the integrand* $\chi(\vec{u}z;\alpha)$ *is a multivalued holo. function of* $u \in \mathbb{C}^r$.

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- *the integrand* $\chi(\vec{u}z;\alpha)$ *is a multivalued holo. function of* $u \in \mathbb{C}^r$.
- *the branch locus is the arrangement* $\mathcal{A} = \{H_1, \dots, H_\ell\}$, *where* $H_k = \{u \in \mathbb{C}^r \mid \vec{u} \cdot z_0^{(k)} = 0\}$.

Twisted de Rham cohomology

We want to compute explicitely the twisted de Rham cohomology group for GHGI of type $\boldsymbol{\lambda} = (\boldsymbol{n}_1, \ldots, \boldsymbol{n}_\ell)$.

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- $\chi = \chi(\vec{u}z; \alpha)$.
- $\mathcal{A} = \{H_1, \ldots, H_\ell\}$: arrangement in $\mathbb{C}^r$ where $H_k = \{\vec{u} \cdot z_0^{(k)} = 0\} \subset \mathbb{C}^r$.

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- $\chi = \chi(\vec{u}z; \alpha)$.
- $\mathcal{A} = \{H_1, \ldots, H_\ell\}$: arrangement in $\mathbb{C}^r$ where $H_k = \{\vec{u} \cdot z_0^{(k)} = 0\} \subset \mathbb{C}^r$.
- $\Omega^p(*\mathcal{A})$: the set of rational p-forms having poles at most on $\bigcup_{k=1}^{\ell} H_k$.

- twisted differentiation $\nabla : \Omega^p(*\mathcal{A}) \to \Omega^{p+1}(*\mathcal{A})$:

$$\begin{aligned}\nabla(\eta) &= (\chi^{-1} \cdot d \cdot \chi)(\eta) \\ &= d\eta + \Big(d \log \chi(\vec{u}z; \alpha)\Big) \wedge \eta.\end{aligned}$$

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- $\nabla \circ \nabla = 0$.
- Twisted rational de Rham complex:

$$C_{z,\alpha}(*\mathcal{A}) : \Omega^0(*\mathcal{A}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^r(*\mathcal{A}) \longrightarrow 0$$

We sometimes write $C_{z,\alpha}$ instead of $C_{z,\alpha}(*\mathcal{A})$.

Twisted de Rham cohomology group:

$$\boldsymbol{H}^{p}(\boldsymbol{C}_{z,\alpha}(*\mathcal{A})) := \frac{\mathbf{Ker}\,\{\nabla : \Omega^{p}(*\mathcal{A}) \longrightarrow \Omega^{p+1}(*\mathcal{A})\}}{\mathbf{Im}\,\{\nabla : \Omega^{p-1}(*\mathcal{A}) \longrightarrow \Omega^{p}(*\mathcal{A})\}}.$$

We know the following cases about the computation of the cohomology groups.

1) $r = 1$. i.e. the HGI is 1-dimensional.
2) r is general, and $\lambda = (1, \ldots, 1)$. The case of Aomoto-Gelfand.
3) r is general and $\lambda = (N)$. The case of generalized Airy integral.
4) r is general and $\lambda = (q + 1, 1, \ldots, 1)$.

1-dimensional case

Proposition

For $z \in Z^{\lambda}_{1,N}$, *we have*

1. $H^p(C_{z,\alpha}) = 0$ *for* $p \neq 1$.
2. $\dim_{\mathbb{C}} H^1(C_{z,\alpha}) = N - 2$.
3. *As a basis of* $H^1(C_{z,\alpha})$ *we can take*

$$d\theta_1(\vec{u}z^{(1)}), \dots, d\theta_{n_1-2}(\vec{u}z^{(1)}),$$
$$d\theta_0(\vec{u}z^{(k)}), \dots, d\theta_{n_k-1}(\vec{u}z^{(k)}), \quad (2 \leq k \leq \ell).$$

Generalized Airy case

- $\lambda = (N)$: a partition of N.
- $Z = Z^{\lambda}_{r,N} = \{z = (z_0, \dots, z_{N-1}) \in \mathrm{Mat}_{r+1,N}(\mathbb{C}) \mid \det(z_0, \dots, z_r) \neq 0\}$,
- Assumption:

$$z_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \alpha_{N-1} \neq 0.$$

- Put
  $\chi(\vec{u}z;\alpha) = e^{f(u)}, \quad f(u) = \sum_{m=1}^{N-1} \alpha_m \theta_m(\vec{u}z),$
- $f(u) \in \mathbb{C}[u_1, \ldots, u_r]$, has isolated critical point,
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Proposition

For the generalized Airy integral,

1. $H^p(C_{z,\alpha}) = 0$ *for* $p \neq r$.
2. $\dim_{\mathbb{C}} H^r(C_{z,\alpha}) = \binom{N-2}{r}$.

To state the result on a basis of $H^r(C_{z,\alpha})$, we prepare some notations.

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- $s_Y(v)$: Schur polynomial of $v_1, \dots, v_r$ for $Y \in \mathcal{Y}(r,l)$.
- $S_Y(u)$: polynomial of u s.t.

$$s_Y(v) = S_Y(e(v)),$$

where $e_1(v), \dots, e_r(v)$ are elementary symmetric functions of v.

Proposition

For the generalized Airy integral, we can take a basis of $H^r(C_{z,\alpha})$ as

$$S_Y(u)du, \quad Y \in \mathcal{Y}(r, N-r-2). \tag{5}$$

Remark

1. *In the case* $r = 1$*, namely the integral is one dimensional, the basis above is*

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2. *Another choice of a basis is given in Proposition 5.*

$$d(\theta_1(\vec{u}z)), \ldots, d(\theta_{N-2}(\vec{u}z)).$$

It is an analogue of flat basis of the Jacobi ring of singularity of A_{N-2} type.

$\lambda = (q+1, 1, \dots, 1)$ case

In this case the integral has the form

$$F(z) = \int \chi(\vec{u}z; \alpha) du$$

with

$$\chi(\vec{u}z; \alpha) = e^{g(u,z)} \prod_{j=q+1}^{N-1} f_j^{\alpha_j},$$

where

$$f_j = \vec{u}z_j, \quad (0 \le j < N)$$
$$g = \sum_{k=1}^{q} \alpha_k \theta_k(f_0, f_2, \dots, f_q).$$

Exterior power structure

A partition λ of N is general,
Compute the cohomology group at Veronese points $\tilde{z} \in Z^{\lambda}_{r,N}$.
At Veronese points, $H^r(C_{\tilde{z},\tilde{\alpha}}) \simeq \bigwedge^r H^1(C_{z,\alpha})$

Veronese map for $\lambda = (1, \ldots, 1)$

- Consider the map

$$\psi : \mathbb{C}^2 \ni \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \mapsto \begin{pmatrix} v_0^r \\ v_0^{r-1} v_1 \\ \vdots \\ v_1^r \end{pmatrix} \in \mathbb{C}^{r+1}. \qquad (6)$$

- It induces the Veronese map $\bar{\psi}$:

$$\mathbb{P}^1 \ni [v_0, v_1] \mapsto [v_0^r, v_0^{r-1} v_1, \ldots, v_1^r] \in \mathbb{P}^r.$$

- For $\lambda = (1, \ldots, 1)$, the map $\Psi_{(1,\ldots,1)}$:

$$Z_{1,N} \ni \begin{pmatrix} z_{00} & \cdots & z_{0,N-1} \\ z_{10} & \cdots & z_{1,N-1} \end{pmatrix}$$

$$\mapsto \begin{pmatrix} (z_{00})^r & \cdots & (z_{0,N-1})^r \\ (z_{00})^{r-1} z_{10} & \cdots & (z_{0,N-1})^{r-1} z_{1,N-1} \\ \vdots & & \vdots \\ (z_{10})^r & \cdots & (z_{1,N-1})^r \end{pmatrix} \in Z_r$$

which we call also the Veronese map.

Veronese map for $\lambda = (n_1, \ldots, n_\ell)$

Want to define the analogous map Ψ_λ to $\Psi_{(1,\ldots,1)}$.
Recall the usual Veronese map ψ is stated as follows.

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Recall the usual Veronese map ψ is stated as follows.

- V: vector space of $\dim_{\mathbb{C}} V = 2$.
- $S^r V$: r-th symmetric tensor product. $\dim_{\mathbb{C}} S^r V = r + 1$.
- $$\psi : V \ni v \mapsto \overbrace{v \otimes \cdots \otimes v} \in S^r V.$$

- Let e_0, e_1 be a basis of V, and let $\mathrm{e}_0, \ldots, \mathrm{e}_r$ be a basis of $S^r V$ defined by

$$\mathrm{e}_k = \sum_{i_1+i_2+\cdots+i_r=k} e_{i_1} \otimes \cdots \otimes e_{i_r}.$$

-

$$(v_0 e_0 + v_1 e_1)^{\otimes r} = \sum_{k=0}^{r} v_0^{r-k} v_1^k \mathrm{e}_k$$

we see that ψ is the same as

$$\psi : \mathbb{C}^2 \ni \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \mapsto \begin{pmatrix} v_0^r \\ v_0^{r-1} v_1 \\ \vdots \\ v_1^r \end{pmatrix} \in \mathbb{C}^{r+1}.$$

Veronese map for $\lambda = (n_1, \ldots, n_\ell)$ again

- V: a vector space of $\dim_{\mathbb{C}} V = 2$,
- $R_n = \mathbb{C}[T]/(T^n)$, with T indeterminate,
- $V_n := V \otimes R_n$: R_n-module.
- $S^r V_n$: r-th symmetric tensor product as R_n-module.
- Define the map

$$\psi_n : V_n \ni v \mapsto \overbrace{v \otimes \cdots \otimes v}^{r} \in S^r V_n.$$

- Let us express ψ_n using $\mathbb{C}$-base

$$\boldsymbol{e}_i \otimes \boldsymbol{T}^j \text{ for } \boldsymbol{V}_n, \quad \mathrm{e}_i \otimes \boldsymbol{T}^j \text{ for } \boldsymbol{S}^r\boldsymbol{V}_n.$$

-

$$\boldsymbol{V}_n \ni \sum_{i,j} v_{ij} \boldsymbol{e}_i \otimes \boldsymbol{T}^j$$

$$\leftrightarrow \begin{pmatrix} v_{00} & v_{01} & \cdots & v_{0,n-1} \\ v_{10} & v_{11} & \cdots & v_{1,n-1} \end{pmatrix} \in \mathbf{Mat}_{2,n}(\mathbb{C}),$$

$$\boldsymbol{S}^r\boldsymbol{V}_n \ni \sum_{i,j} w_{ij} \mathrm{e}_i \otimes \boldsymbol{T}^j$$

$$\leftrightarrow \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0,n-1} \\ \vdots & \vdots & & \vdots \\ w_{r0} & w_{r1} & \cdots & w_{r,n-1} \end{pmatrix} \in \mathbf{Mat}_{r+1,n}(\mathbb{C})$$

- The map $\psi_n : V_n \longrightarrow S^r V_n$ induces the map $\psi_n : \mathrm{Mat}_{2,n}(\mathbb{C}) \longrightarrow \mathrm{Mat}_{r+1,n}(\mathbb{C})$.
- For $\lambda = (n_1, \ldots, n_\ell)$, define $\Psi_\lambda : Z^{\lambda}_{1,N} \longrightarrow Z^{\lambda}_{r,N}$ by

$$z = (z^{(1)}, \ldots, z^{(\ell)}) \mapsto (\psi_{n_1}(z^{(1)}), \ldots, \psi_{n_\ell}(z^{(\ell)})).$$

We call this map the Veronese map of type λ and the set $\Psi_\lambda(Z^{\lambda}_{1,N})$ the Veronese image.

Example

Let $r = 2$. Then the map
$\psi_3 : \mathbf{Mat}_{2,3}(\mathbb{C}) \longrightarrow \mathbf{Mat}_{3,3}(\mathbb{C})$ is given by as follows:

$$v \mapsto \begin{pmatrix} v_{00}^2 & 2v_{00}v_{01} & 2v_{00}v_{02} + v_{01}^2 \\ v_{00}v_{01} & v_{00}v_{11} + v_{01}v_{10} & v_{00}v_{12} + v_{02}v_{10} + v_{01} \\ v_{01}^2 & 2v_{01}v_{11} & 2v_{10}v_{12} + v_{11}^2 \end{pmatrix}$$

Power structure of cohomology group

Theorem

Let $z \in Z^{\lambda}_{1,N}$ *be such that* $z^{(1)}_0 = {}^t(1,0)$ *and let*

$$\tilde{z} = \Psi(z) \in Z^{\lambda}_{r,N}, \quad \tilde{\alpha} = \alpha + (-r+1, 0, \dots, 0).$$

Then we have

$$H^r(C_{\tilde{z},\tilde{\alpha}}) \simeq \bigwedge^r H^1(C_{z,\alpha}),$$

and

$$\dim_{\mathbb{C}} H^r(C_{\tilde{z},\tilde{\alpha}}) = \binom{N-2}{r}.$$

Examples

Generalized Airy case $\lambda = (N)$

Let $z \in Z_{1,N}$ and $\tilde{z} = \Psi_\lambda(z)$. Theorem says:
if we take a basis of $H^1(C_{z,\alpha})$ as

$$\varphi_i = u^i du, \quad (0 \le i \le N-3),$$

then

$$\varphi_{i_1} \square \cdots \square \varphi_{i_r} \mapsto S_Y(v) dv_1 \wedge \cdots \wedge dv_r.$$

Here $i_1 > i_2 > \cdots > i_r \ge 0$ and

$$Y = (i_1 - r + 1, i_2 - r + 2, \dots, i_r) \in \mathcal{Y}(r, N - r - 2)$$

If one take a basis of $H^1(C_{z,\alpha})$ as

$$\varphi_i = d\theta_i(\vec{u}z), \quad (1 \leq i \leq N-2),$$

then

$$\varphi_{i_1}\Box \cdots \Box\varphi_{i_r} \mapsto d\theta_{i_1}(\vec{v}\tilde{z}) \wedge \cdots \wedge d\theta_{i_r}(\vec{v}\tilde{z}).$$

Example ($\lambda = (n_1, \dots, n_\ell)$ case)

Let $z \in Z^{\lambda}_{1,N}$ then as a basis of $H^1(C_{z,\alpha})$ we can take

$$d\theta_1(\vec{u}z^{(1)}), \dots, d\theta_{n_1-2}(\vec{u}z^{(1)})$$
$$d\theta_0(\vec{u}z^{(k)}), d\theta_1(\vec{u}z^{(k)}), \dots, d\theta_{n_k-1}(\vec{u}z^{(k)}), \quad (2 \leq k$$

Put $\tilde{z} = \Psi_\lambda(z)$. Then as a basis of $H^r(C_{\tilde{z},\tilde{\alpha}})$ we can take the r-forms obtained by choosing r forms $d(\theta_j(\vec{v}\tilde{z}^{(k)}))$ and taking exterior product of them.

Question:

In the case $\boldsymbol{\lambda} = (n_1, \ldots, n_\ell)$, do the r-forms, constructed above in Example, give a basis for $H^r(C_{z,\alpha})$ at any point $z \in Z^{\boldsymbol{\lambda}}_{r,N}$.?