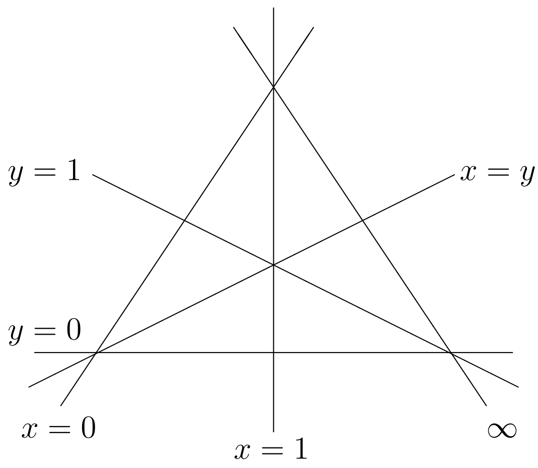
Arrangements of hyperplanes arising from sections of rigid local systems

Yoshishige Haraoka
Department of Mathematics, Kumamoto University

Arrangements of hyperplanes in the thoery of differential equations

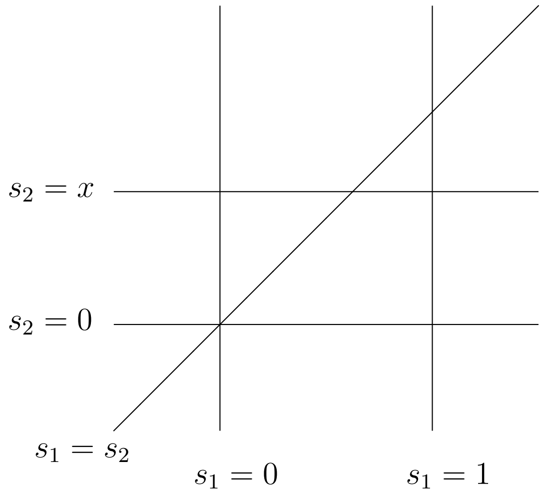
Arrangements of hyperplanes appear:

- as a singular locus of a completely integrable system



Singular locus of the system for Appell's $F_1(\alpha, \beta, \beta', \gamma; x, y)$

- as a set of branching points for an integral representation of solutions of a linear differential equation

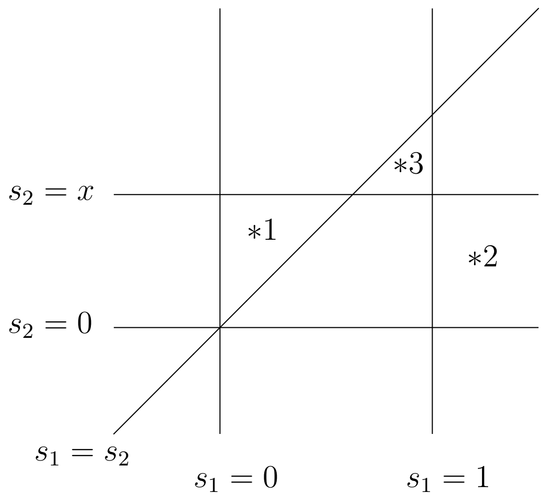


Branching points for the integral representation of the generalized hypergeometric function

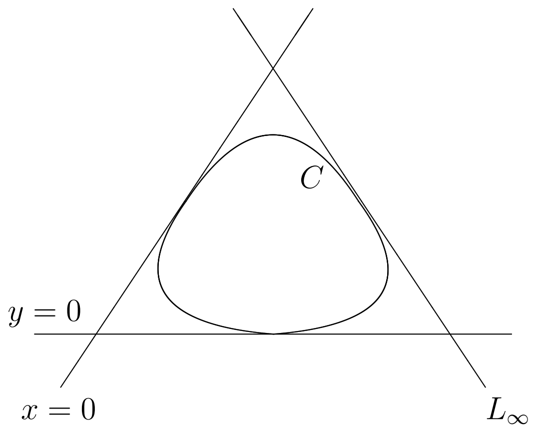
$$ {}_3F_2\left(\begin{matrix}\alpha_1\ \alpha_2\ \alpha_3\\ \beta_1\ \beta_2\end{matrix};x\right) = C\iint_\Delta {s_1}^{\alpha_2-\beta_1}(s_1-1)^{\beta_1-\alpha_1-1}{s_2}^{\alpha_3-\beta_2}(s_2-s_1)^{\beta_2-\alpha_2-1}(s_2-x)^{-\alpha_3}\,ds_1\,ds_2 $$

Problem: Describe the relation between the asymptotic behaviors and the cycles

$$\left\{ \begin{matrix} x=0 & x=1 & x=\infty \\ 0 & 0 & \alpha_1 \\ 1-\beta_1{}^{(*1)} & 1 & \alpha_2 \\ 1-\beta_2{}^{(*2)} & \rho^{(*3)} & \alpha_3 \end{matrix} \right\} \qquad \left(\rho = \sum \beta_i - \sum \alpha_i\right)$$

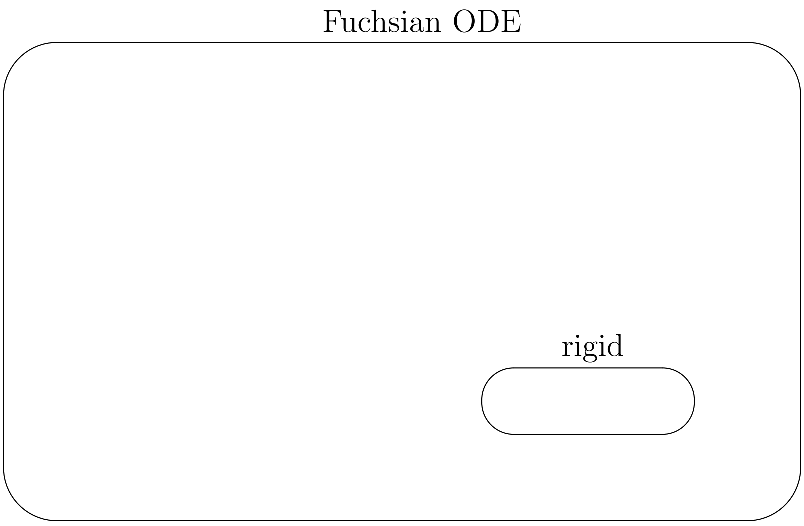


Remark: Hypersurfaces may appear



Singular locus of the system for Appell's $F_4(\alpha, \beta, \gamma, \gamma'; x, y)$

Rigid Fuchsian systems



rigid = free of accessory parameters

Theorem (N. M. Katz) Any irreducible rigid Fuchsian equation is connected to rank 1 Fuchsian equation by additions and middle convolutions.

We consider a Fuchsian system

$$\frac{dU}{dx} = \left(\sum_{j=1}^{p} \frac{A_j}{x - a_j} \right) U$$

A_j: $n \times n$-constant matrix

singularities: $a_0 := \infty, a_1, \dots, a_p$

We denote the system by $(A_1, \dots, A_p)$

Addition: $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{C}^p$

$$a_\alpha : (A_1, \dots, A_p) \mapsto (A_1 + \alpha_1, \dots, A_p + \alpha_p)$$

analytically:

$$U(x) \mapsto \prod_{j=1}^{p} (x - a_j)^{\alpha_j} \cdot U(x)$$

Middle convolution: $\mu \in \mathbb{C}$
We first define **convolution** c_μ with parameter μ

$$c_\mu : (A_1, \ldots, A_p) \mapsto (G_1, \ldots . G_p)$$

where

$$G_j = \begin{pmatrix} O & \cdots & O & \cdots & O \\ \vdots & & \vdots & & \vdots \\ O & \cdots & O & \cdots & O \\ A_1 & \cdots & A_j + \mu & \cdots & A_p \\ O & \cdots & O & \cdots & O \\ \vdots & & \vdots & & \vdots \\ O & \cdots & O & \cdots & O \end{pmatrix} \quad (j$$

We can find in an explicit way an invariant subspace $\mathcal{W} \subset \mathbb{C}^{pn}$ for $(G_1, \ldots, G_p)$.

$$\hat{G}_j \ : \ \text{action of } G_j \text{ on } \mathbb{C}^{pn}/\mathcal{W}$$

Middle convolution mc_μ with parameter μ is defined by

$$mc_\mu : (A_1, \ldots, A_p) \mapsto (\hat{G}_1, \ldots, \hat{G}_p)$$

analytically (Dettwiler-Reiter):

$$U(x) \mapsto V(x) := \begin{pmatrix} \frac{U(x)}{x-a_1} \\ \frac{U(x)}{x-a_2} \\ \vdots \\ \frac{U(x)}{x-a_p} \end{pmatrix} \mapsto \tilde{V}(x) := \int_\Delta V(s)(s-x)^\mu \, ds$$

Then $\tilde{V}(x)$ satisfies the system $(G_1, \ldots, G_p)$. Take P such that $P^{-1}G_jP = \left(\begin{array}{c|c} * & * \\ \hline O & \hat{G}_j \end{array}\right)$, and define $\tilde{Z}(x)$ by $\tilde{V}(x) = P\tilde{Z}(x)$. Set $\tilde{Z}(x) =: \begin{pmatrix} Z_1(x) \\ Z(x) \end{pmatrix}$. Then $Z(x)$ satisfies

$$\frac{dZ}{dx} = \left(\sum_{j=1}^{p} \frac{\hat{G}_j}{x-a_j} \right) Z.$$

Thus

$$mc_\mu : U(x) \mapsto Z(x)$$

Example: We start from a rank 1 equation

$$\frac{du}{dx} = \left(\frac{\alpha_1}{x} + \frac{\beta_1}{x-1}\right) u,$$

which has a solution $u(x) = x^{\alpha_1}(x-1)^{\beta_1}$.

$$\begin{aligned}
u(x) &\mapsto \begin{pmatrix} x^{\alpha_1-1}(x-1)^{\beta_1} \\ x^{\alpha_1}(x-1)^{\beta_1-1} \end{pmatrix} \\
&\mapsto \begin{pmatrix} \int_\delta s^{\alpha_1-1}(s-1)^{\beta_1}(s-x)^{\gamma_1}\,ds \\ \int_\delta s^{\alpha_1}(s-1)^{\beta_1-1}(s-x)^{\gamma_1}\,ds \end{pmatrix} \\
&\mapsto x^{\alpha_2}(x-1)^{\beta_2} \begin{pmatrix} \int_\delta s^{\alpha_1-1}(s-1)^{\beta_1}(s-x)^{\gamma_1}\,ds \\ \int_\delta s^{\alpha_1}(s-1)^{\beta_1-1}(s-x)^{\gamma_1}\,ds \end{pmatrix} \\
&\mapsto \begin{pmatrix} \iint_\Delta {s_1}^{\alpha_1-1}(s_1-1)^{\beta_1}(s_1-s_2)^{\gamma_1}{s_2}^{\alpha_2-1}(s_2-1)^{\beta_2}(s_2-x)^{\gamma_2}\,ds_1\,ds_2 \\ \iint_\Delta {s_1}^{\alpha_1}(s_1-1)^{\beta_1-1}(s_1-s_2)^{\gamma_1}{s_2}^{\alpha_2-1}(s_2-1)^{\beta_2}(s_2-x)^{\gamma_2}\,ds_1\,ds_2 \\ \iint_\Delta {s_1}^{\alpha_1-1}(s_1-1)^{\beta_1}(s_1-s_2)^{\gamma_1}{s_2}^{\alpha_2}(s_2-1)^{\beta_2-1}(s_2-x)^{\gamma_2}\,ds_1\,ds_2 \\ \iint_\Delta {s_1}^{\alpha_1}(s_1-1)^{\beta_1-1}(s_1-s_2)^{\gamma_1}{s_2}^{\alpha_2}(s_2-1)^{\beta_2-1}(s_2-x)^{\gamma_2}\,ds_1\,ds_2 \end{pmatrix}
\end{aligned}$$

$$mc_{\mu_1} \to a_{\alpha^{(1)}} \to mc_{\mu_2} \to \cdots \to a_{\alpha^{(\ell-1)}} \to mc_{\mu_\ell}$$

$$mc_{\mu_j} = L_j \circ c_{\mu_j}$$

$$c_{\mu_1} \to a_{\alpha^{(1)}} \to c_{\mu_2} \to \cdots \to a_{\alpha^{(\ell-1)}} \to c_{\mu_\ell} \to L$$

Starting from a rank 1 equation

$$\frac{du}{dx} = \left(\sum_{j=1}^{p} \frac{\alpha^{(1)}}{x - a_j} \right) u$$

Solution: $u(x) = \prod_{j=1}^{p} (x - a_j)^{\alpha_j^{(1)}}$

$$\xrightarrow{mc_{\mu_1}} \quad \xrightarrow{a_{\alpha^{(2)}}} \quad \xrightarrow{mc_{\mu_2}} \quad \cdots \quad \xrightarrow{a_{\alpha^{(\ell)}}} \quad \xrightarrow{mc_{\mu_\ell}}$$

$$\xrightarrow{c_{\mu_1}} \quad \xrightarrow{a_{\alpha^{(2)}}} \quad \xrightarrow{c_{\mu_2}} \quad \cdots \quad \xrightarrow{a_{\alpha^{(\ell)}}} \quad \xrightarrow{c_{\mu_\ell}} \quad \xrightarrow{L}$$

Thus we have a general form of solutions of rigid Fuchsian equations:

$$U(x) = L \int_{\Delta} \prod_{i=1}^{\ell} \prod_{j=1}^{p} (s_i - a_j)^{\alpha_j^{(i)}} \prod_{i=1}^{\ell-1} (s_i - s_{i+1})^{\mu_i} (s_\ell - x)^{\mu_\ell} \left(\frac{ds_1 \wedge \cdots \wedge ds_\ell}{\prod_{i=1}^{\ell} (s_i - a_{j_i})} \right)_{1 \le j_1, \dots, j_\ell \le p}$$

Generating system (of rank p^ℓ)

$$\frac{dU}{dx} = \left(\sum_{j=1}^{p} \frac{G_j}{x - a_j} \right) U$$

where

$$(G_1, \dots, G_p) = c_{\mu_\ell} \circ a_{\alpha^{(\ell)}} \circ \dots \circ a_{\alpha^{(2)}} \circ c_{\mu_1}(\alpha_1^{(1)}, \dots, \alpha_p^{(1)})$$

eigenvalue	multiplicity
0	$p^\ell - p^{\ell-1}$
$\alpha_j^{(\ell)} + \mu_\ell$	$p^{\ell-1} - p^{\ell-2}$
$(\alpha_j^{(\ell-1)} + \mu_{\ell-1}) + (\alpha_j^{(\ell)} + \mu_\ell)$	$p^{\ell-2} - p^{\ell-3}$
$\vdots$	$\vdots$
$(\alpha_j^{(q)} + \mu_q) + \dots + (\alpha_j^{(\ell)} + \mu_\ell)$	$p^{q-1} - p^{q-2}$
$\vdots$	$\vdots$
$(\alpha_j^{(1)} + \mu_1) + (\alpha_j^{(2)} + \mu_2) + \dots + (\alpha_j^{(\ell)} + \mu_\ell)$	1

Table Eigenvalues of G_j

Solution:

$$U(x) = \int_{\Delta} \prod_{i=1}^{\ell} \prod_{j=1}^{p} (s_i - a_j)^{\alpha_j^{(i)}} \prod_{i=1}^{\ell-1} (s_i - s_{i+1})^{\mu_i} (s_\ell - x)^{\mu_\ell} \left(\frac{ds_1 \wedge \cdots \wedge ds_\ell}{\prod_{i=1}^{\ell} (s_i - a_{j_i})} \right)_{1 \le j_1, \ldots, j_\ell \le p}$$

In the following we take the arrnagement of hyperplanes

$$D = \bigcup_{\substack{1 \le i \le \ell \\ 1 \le j \le p}} \{s_i = a_j\} \cup \bigcup_{1 \le i \le \ell - 1} \{s_i = s_{i+1}\} \cup \{s_\ell = x\}$$

in $\mathbb{C}^\ell$, and set $X = \mathbb{C}^\ell \setminus D$. We consider the multi-valued function

$$\Phi(s) = \prod_{i=1}^{\ell} \prod_{j=1}^{p} (s_i - a_j)^{\alpha_j^{(i)}} \prod_{i=1}^{\ell-1} (s_i - s_{i+1})^{\mu_i} (s_\ell - x)^{\mu_\ell}$$

on X.

Remark: The solution of a generating system is a special kind of Selberg integral. A general one contains $\prod_{i<j}(s_i - s_j)^{\mu_{ij}}$ instead of $\prod_{i=1}^{\ell-1}(s_i - s_{i+1})^{\mu_i}$.

$$\Phi(s) = \prod_{i=1}^{\ell}\prod_{j=1}^{p}(s_i - a_j)^{\alpha_j^{(i)}} \boxed{\prod_{i=1}^{\ell-1}(s_i - s_{i+1})^{\mu_i}} (s_\ell - x)^{\mu_\ell}$$

Topological Theory

$\mathcal{L}$: the local system on X of local solutions of

$$\nabla z := dz + d\log\Phi(s) \wedge z = 0.$$

where we have defined

$$\Phi(s) := \prod_{i=1}^{\ell}\prod_{j=1}^{p}(s_i - a_j)^{\alpha_j^{(i)}} \prod_{i=1}^{\ell-1}(s_i - s_{i+1})^{\mu_i}(s_\ell - x)^{\mu_\ell}.$$

We shall study

- cohomology group $H^i(X, \mathcal{L})$ and contiguity relations
- homology group $H_i(X, \mathcal{L}^\vee)$ and asymptotic behaviros

Twisted Cohomology

Theorem 1. For generic values of parmaeters,

$$H^i(X, \mathcal{L}) = 0 \qquad (i \neq \ell)$$
$$H^\ell(X, \mathcal{L}) = \left\langle \left. \frac{ds_1 \wedge \cdots \wedge ds_\ell}{\prod_{i=1}^{\ell}(s_i - a_{j_i})} \right| j_1, j_2, \ldots, j_\ell \in \{1, 2, \ldots, p\} \right\rangle.$$

In the proof we use [Esnault-Schechtman-Vieweg], [Schechtman-Terao-Varchenko], and

$$\chi(X) = (-p)^\ell.$$

Contiguity Relations

$$U_{j_1\cdots j_\ell}(x) \quad \leftrightarrow \quad \frac{ds_1 \wedge \cdots \wedge ds_\ell}{\prod_{i=1}^{\ell}(s_i - a_{j_i})}$$

Let $\bar{U}_{j_1\cdots j_\ell}(x)$ be obtained by increasing $\alpha_k^{(m)}$ by 1.

$$\begin{aligned}
\frac{s_m - a_k}{\prod_{i=1}^{\ell}(s_i - a_{j_i})} &= \frac{s_m - a_{j_m}}{\prod_{i=1}^{\ell}(s_i - a_{j_i})} + \frac{a_{j_m} - a_k}{\prod_{i=1}^{\ell}(s_i - a_{j_i})} \\
&= \frac{1}{\prod_{i\neq m}(s_i - a_{j_i})} + \frac{a_{j_m} - a_k}{\prod_{i=1}^{\ell}(s_i - a_{j_i})} \\
&\equiv \sum_{j_1',\dots,j_\ell'} c_{j_1,\dots,j_\ell}^{j_1',\dots,j_\ell'} \frac{1}{\prod_{i=1}^{\ell}(s_i - a_{j_i'})} + \frac{a_{j_m} - a_k}{\prod_{i=1}^{\ell}(s_i - a_{j_i})}
\end{aligned}$$

From which we obtain

$$\bar{U}_{j_1\cdots j_\ell}(x) = \sum_{j_1',\dots,j_\ell'} c_{j_1,\dots,j_\ell}^{j_1',\dots,j_\ell'} U_{j_1'\cdots j_\ell'}(x) + (a_{j_m} - a_k) U_{j_1\cdots j_\ell}(x)$$

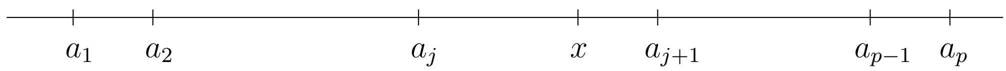
Twisted Homology and Asymptotic Behavior

We assume $a_1, \dots, a_p \in \mathbb{R}$ and moreover

$$a_1 < a_2 < \cdots < a_p$$

We are interested in the asymptotic behavior as $x \to a_j$, and then we assume

$$a_j < x < a_{j+1}$$



Recall:

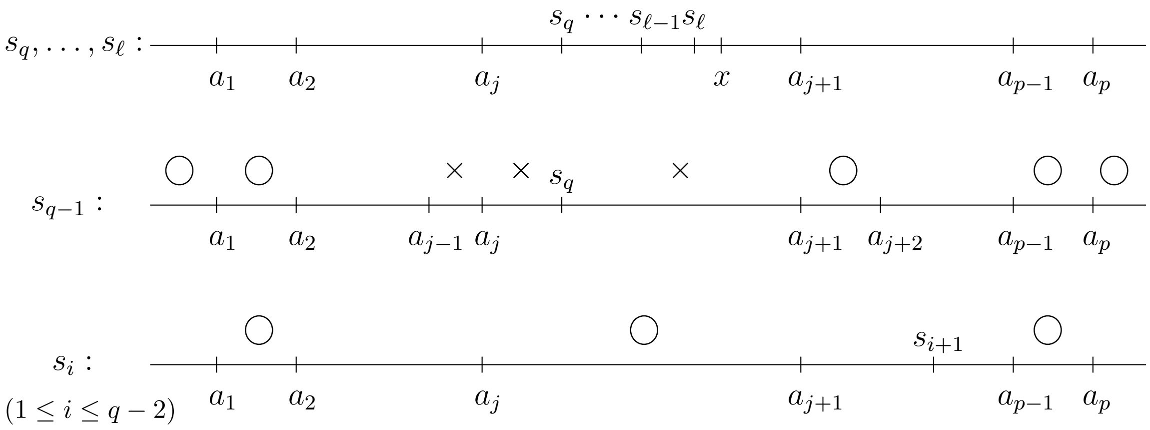
eigenvalues of G_j	multiplicity
0	$p^{\ell} - p^{\ell-1}$
$\alpha_j^{(\ell)} + \mu_\ell$	$p^{\ell-1} - p^{\ell-2}$
$\vdots$	$\vdots$
$(\alpha_j^{(q)} + \mu_q) + \cdots + (\alpha_j^{(\ell)} + \mu_\ell)$	$p^{q-1} - p^{q-2}$
$\vdots$	$\vdots$
$(\alpha_j^{(1)} + \mu_1) + (\alpha_j^{(2)} + \mu_2) + \cdots + (\alpha_j^{(\ell)} + \mu_\ell)$	1

at $x = a_j$, the generating system has the exponent

$$\rho_{jq} := \sum_{k=q}^{\ell} (\alpha_j^{(k)} + \mu_k)$$

of multiplicity $p^{q-1} - p^{q-2}$.

Theorem 2. The exponent ρ_{jq} is realized by the following twisted cycles:



$$\# \text{ of cycles} = (p-1) \times p^{q-2} = p^{q-1} - p^{q-2}$$

We call them **asymptotic cycles**.

cf.

$$\Phi(s) = \prod_{i=1}^{\ell} \prod_{j=1}^{p} (s_i - a_j)^{\alpha_j^{(i)}} \prod_{i=1}^{\ell-1} (s_i - s_{i+1})^{\mu_i} (s_\ell - x)^{\mu_\ell}$$

These cycles are independent:

$$\int_{\Delta} \Phi(s) \frac{ds_1 \wedge \cdots \wedge ds_\ell}{\prod_{i=1}^{\ell}(s_i - a_{j_i})} \approx C(x - a_j)^{\rho_{jq}} \int_{\Delta'} \Phi'(s') \frac{ds_1 \wedge \cdots \wedge ds_{q-1}}{\prod_{i=1}^{q-1}(s_i - a_{j_i})},$$

where

$$\begin{aligned} \pi : \mathbb{C}^\ell &\to \mathbb{C}^{q-1}, \\ (s_1, \ldots, s_\ell) &\mapsto (s_1, \ldots, s_{q-1}) \end{aligned} \qquad \Delta' = \pi(\Delta),$$

$$\Phi'(s') := \prod_{i=1}^{q-1} \prod_{k=1}^{p} (s_i - a_k)^{\alpha_k^{(i)}} \prod_{i=1}^{q-2} (s_i - s_{i+1})^{\mu_i} (s_{q-1} - a_j)^{\mu_{q-1}}.$$

We can show that $\left(\int_{\Delta'} \Phi'(s') \frac{ds_1 \wedge \cdots \wedge ds_{q-1}}{\prod_{i=1}^{q-1}(s_i - a_{j_i})} \right)$ is of full rank.

Problems.

• Specify the asymptotic cycles when the parameters are resonant. (cf. K. Mimachi's talk)
Can we characterize such cycles using the terminology of arrangements of hyperplanes?

• Middle convolution may be defined also for completely integrable systems.
We are interested in the change of the singular locus under middle convolution,

or

arrangements of hyperplanes which are invariant under an appropriate middle convolution.

$$dU = \left[\sum_{i=1}^{q} \sum_{j=1}^{p} A_{ij} \frac{dx_i}{x_i - a_j} + \sum_{i \neq i'} C_{ii'} \frac{d(x_i - x_{i'})}{x_i - x_{i'}} \right] U$$

We fix i. Then we can define a convolution w.r.t. x_i by

$$\begin{aligned} & \qquad\qquad\qquad\qquad i \\ V_j &= \int_\Delta \frac{1}{s - a_j} U(x_1, \ldots, s, \ldots, x_q)(s - x_i)^\mu \, ds \quad (1 \leq j \leq p) \\ V_{p+i'} &= \int_\Delta \frac{1}{s - x_{i'}} U(x_1, \ldots, s, \ldots, x_q)(s - x_i)^\mu \, ds \quad (i' \neq i) \end{aligned}$$

$$V = {}^t(V_1, \ldots, V_{p+q})$$

Then we can define the middle convolution in a similar way as ODE case, and get a completely integrable system with the **same** singular locus.