A conjugation-free geometric presentation of fundamental groups of arrangements

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Importance and Applications

- Used by Chisini, Kulikov and Kulikov-Teicher in order to distinguish between connected components of the moduli space of surfaces.

- The Zariski-Lefschetz hyperplane section theorem:

$$\pi_1(\mathbb{CP}^N \setminus S) \cong \pi_1(H \setminus H \cap S),$$

  where S is an hypersurface and H is a generic 2-plane. This invariant can be used also for computing the fundamental group of complements of hypersurfaces in $\mathbb{CP}^N$.

- Getting more examples of Zariski pairs: A pair of plane curves is called a *Zariski pair* if they have the same combinatorics, but their complements are not homeomorphic.

- Exploring new finite non-abelian groups which are serving as fundamental groups of complements of plane curves in general.

- Computing the fundamental group of the Galois cover of a surface: By the fundamental group of a complement of a branch curve of a surface, we can find the fundamental group of the Galois cover of the surface, with respect to a generic projection of the surface onto $\mathbb{CP}^2$.

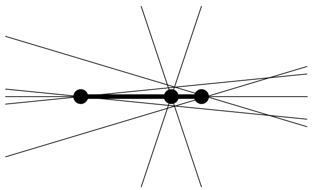
Graph of multiple points

Line arrangement in $\mathbb{CP}^2$: An algebraic curve in $\mathbb{CP}^2$ which is a union of projective lines. An arrangement is called *real* if its defining equations can be written with real coefficients.

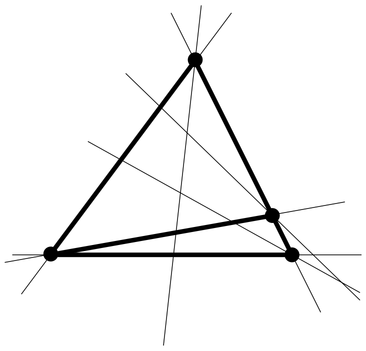
$G(\mathcal{L})$:

Vertices: Multiple points

Edges: Segments on lines with more than two multiple points.



(a)



(b)

Fan (1997): Let $\mathcal{L}$ be an arrangement of n lines and $S = \{a_1, \cdots, a_p\}$ be the set of multiple points of $\mathcal{L}$ (multiplicity ≥ 3). Suppose that $\beta(\mathcal{L}) = 0$ (i.e. the graph $G(\mathcal{L})$ has no cycles). Then:

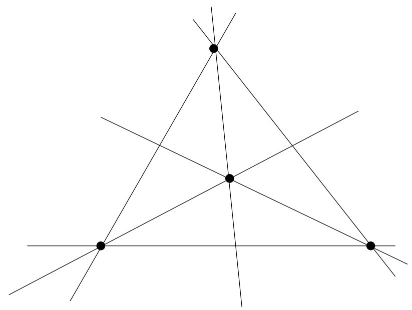
$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^r \oplus \mathbb{F}_{m(a_1)-1} \oplus \cdots \oplus \mathbb{F}_{m(a_p)-1}$$

where $r = n + p - 1 - m(a_1) - \cdots - m(a_p)$.

G-Teicher: Part of this result by braid monodromy techniques.

Eliyahu-Liberman-Schaps-Teicher (2009): If the fundamental group is a sum of free groups, then $G(\mathcal{L})$ has no cycles.

Ceva arrangement



Fadell-Neuwirth (1962): If $\mathcal{L}$ is the Ceva arrangement, then:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \mathbb{F} \ltimes \mathbb{F}_2 \ltimes \mathbb{F}_3$$

Eliyahu-G-Teicher (2008): Let $\mathcal{L}$ be a real arrangement of 6 lines whose graph is a cycle of length 3, then:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \mathbb{F} \ltimes \mathbb{F}_2 \ltimes (\mathbb{Z} \star \mathbb{Z}^2).$$

Idea of proof:

Cohen-Suciu (1998): Presentation of $\mathbb{F}_3 \rtimes_{\alpha_3} \mathbb{F}_2 \rtimes_{\alpha_2} \mathbb{F}_1$:

$$\mathbb{F}_1 = \langle u \rangle, \qquad \mathbb{F}_2 = \langle t, s \rangle, \qquad \mathbb{F}_3 = \langle x, y, z \rangle$$

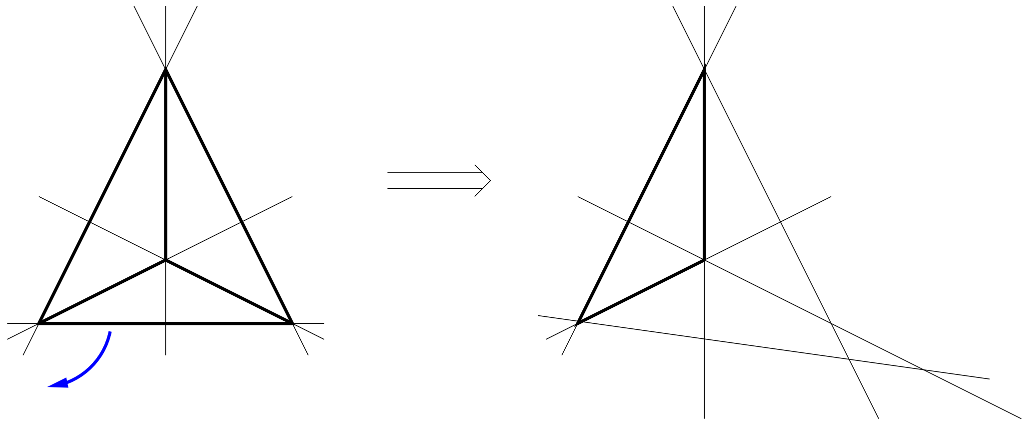
The action of the automorphism α_i is defined as follows:

$$\begin{array}{ll}
(\alpha_2(u))(t) = sts^{-1}, & (\alpha_2(u))(s) = stst^{-1}s^{-1}, \\
(\alpha_2(u))(x) = x, & (\alpha_2(u))(y) = zyz^{-1}, \\
(\alpha_2(u))(z) = zyzy^{-1}z^{-1} &
\end{array}$$

$$\begin{array}{ll}
(\alpha_3(s))(x) = zxz^{-1}, & (\alpha_3(s))(y) = zxz^{-1}x^{-1}yxzx^{-1}z^{-1}, \\
(\alpha_3(s))(z) = zxzx^{-1}z^{-1} &
\end{array}$$

$$\begin{array}{ll}
(\alpha_3(t))(x) = yxy^{-1}, & (\alpha_3(t))(y) = yxyx^{-1}y^{-1}, \\
(\alpha_3(t))(z) = z &
\end{array}$$

By rotation, Ceva arrangement becomes an arrangement $\mathcal{L}$ whose graph is a triangle:



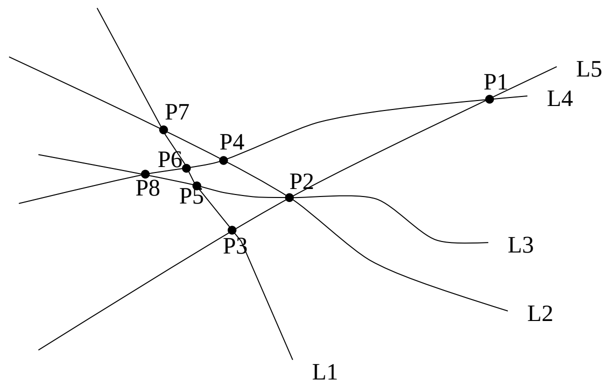
Its effect is one new relation $[x, z] = e$. Hence, the change is:

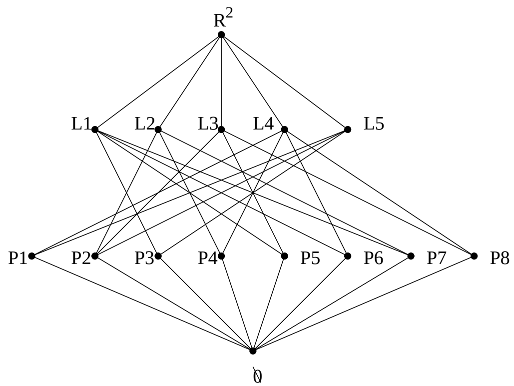
$$(\alpha_3(s))(x) = x, \qquad (\alpha_3(s))(y) = y, \qquad (\alpha_3(s))(z) = z$$

Note that $\langle x, y, z | xz = zx \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$. Hence, we get the group structure: $(\mathbb{Z}^2 * \mathbb{Z}) \rtimes_{\alpha_3} \mathbb{F}_2 \rtimes_{\alpha_2} \mathbb{F}$.

Question: Can one generalize it to a cycle of length n?

Lattice of an arrangement





Generic presentation of the fundamental group

(Orlik-Terao, Arvola, Randell, Cohen-Suciu, ...)

Let $\mathcal{L}$ be an arrangement of n lines.

Then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ is generated by $x_1, \ldots, x_n$ - the natural topological generators.

The relations: for each intersection point of multiplicity k:

$$x_{i_k}^{s_k} x_{i_{k-1}}^{s_{k-1}} \cdots x_{i_1}^{s_1} = x_{i_{k-1}}^{s_{k-1}} \cdots x_{i_1}^{s_1} x_{i_k}^{s_k} = \cdots = x_{i_1}^{s_1} x_{i_k}^{s_k} \cdots x_{i_2}^{s_2}$$

where $a^b = b^{-1}ab$ and s_i are words in $\langle x_1, \ldots, x_n \rangle$ $(1 \leq i \leq k)$.

Conjugation-free geometric presentation of fundamental group

A *conjugation-free geometric presentation* of a fundamental group is a presentation with the natural topological generators $x_1, \dots, x_n$ and the cyclic relations:

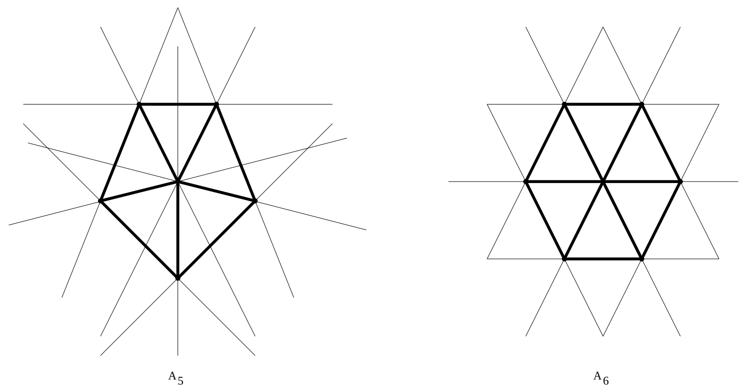
$$x_{i_k} x_{i_{k-1}} \cdots x_{i_1} = x_{i_{k-1}} \cdots x_{i_1} x_{i_k} = \cdots = x_{i_1} x_{i_k} \cdots x_{i_2}$$

with no conjugations on the generators.

Main importance: For this family the lattice determines the fundamental group. Moreover, one can read this presentation directly from the arrangement.

Eliyahu-G-Teicher (2008): if $G(\mathcal{L})$ is a union of cycles, then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation.

Family A_n:



Computationally proved: A_5, A_6 have a conjugation-free geometric presentation. A_3 (Ceva) and A_7 have no conjugation-free geometric presentation.

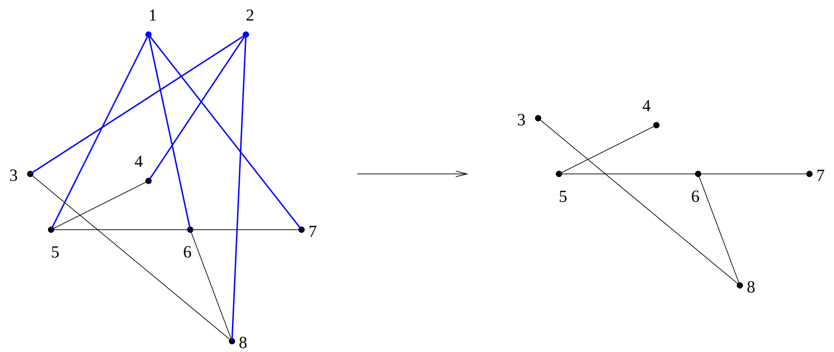
Conjecture (Eliyahu-G-Teicher, 2008): If $G(\mathcal{L})$ is a A_n*-free graph*, then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation.

Nice arrangements (Jiang-Yau)

For $\mathcal{L}$, define a graph $G(V,E)$: The vertices are the multiple points of $\mathcal{L}$.
u,v are connected if there exists $\ell \in \mathcal{L}$ such that $u,v \in \ell$.

For $v \in V$, define a subgraph $G_{\mathcal{L}}(v)$: The vertex set is v and all his neighbors from G.
u,v are connected if there exists $\ell \in \mathcal{L}$ such that $u,v \in \ell$.

$\mathcal{L}$ is *nice* if there is $V' \subset V$ such that $G_{\mathcal{L}}(v) \cap G_{\mathcal{L}}(u) = \emptyset$ for all $u,v \in V'$, and if we delete the vertex v and the edges of its subgraph $G_{\mathcal{L}}(v)$ from G, for all $v \in V'$, we get a forest (a graph without cycles).



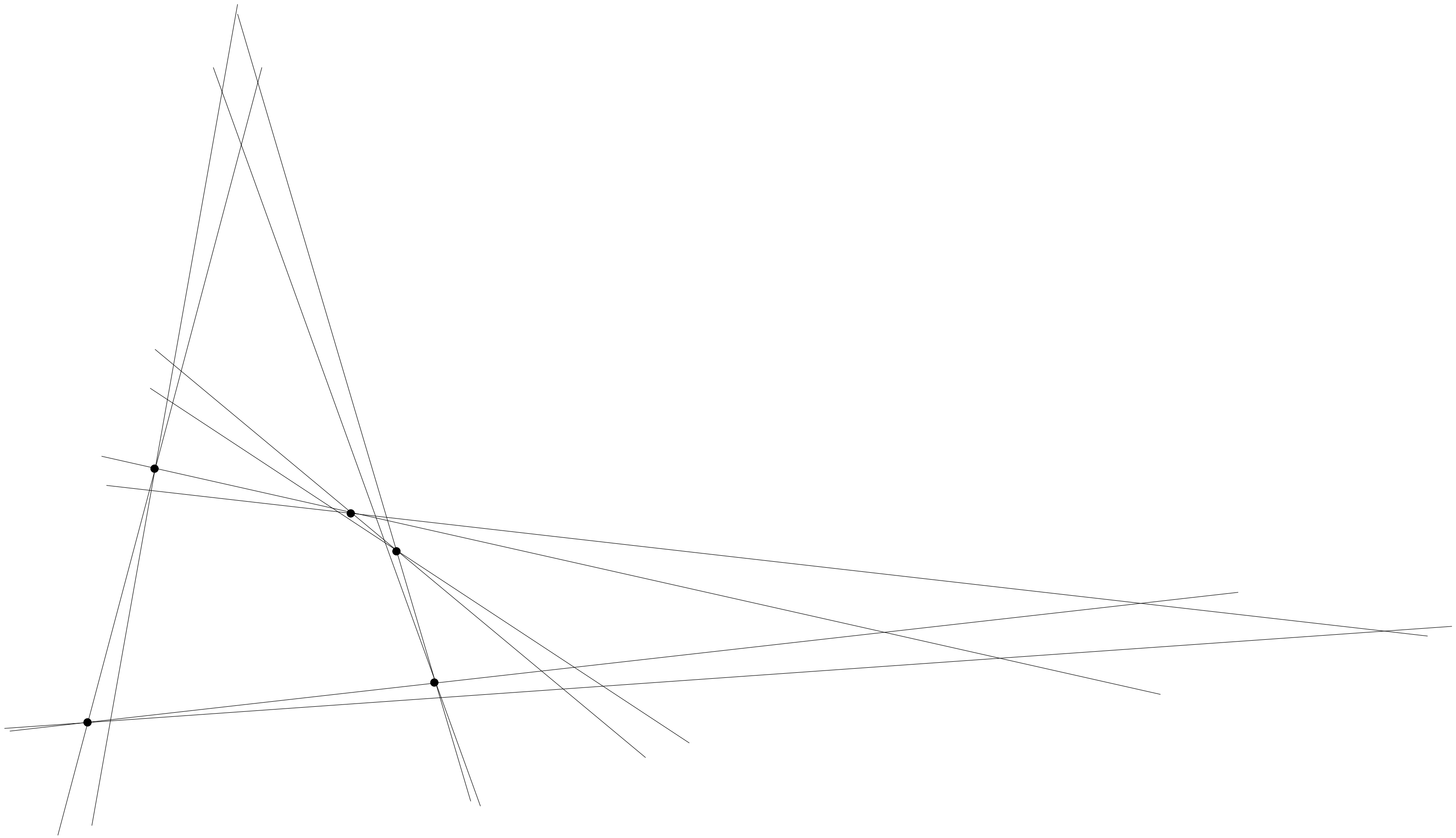
Jiang-Yau (1994): Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two nice projective arrangements in $\mathbb{CP}^2$. If their lattices are isomorphic, then their complements are diffeomorphic. In particular,

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}_1) \cong \pi_1(\mathbb{CP}^2 - \mathcal{L}_2)$$

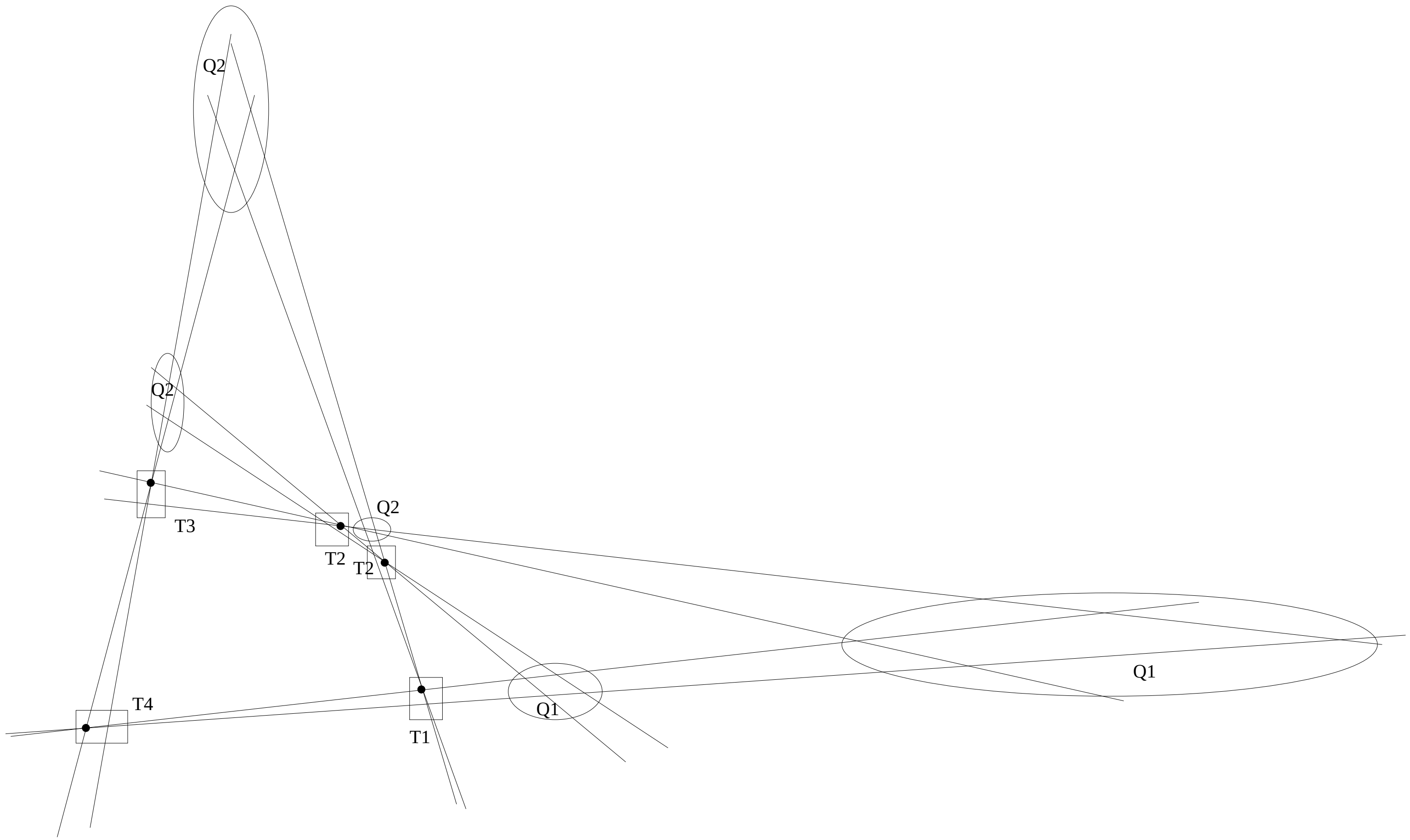
Remark (Eliyahu-G-Teicher): A_5 has a conjugation-free geometric presentation, but is not nice and simple.

Question: Is there a nice or a simple arrangement which has no conjugation-free geometric presentation?

Proof of an arrangement whose graph is a cycle



Break into blocks



Presentation

Generators: $\{x_1, \ldots, x_{2n}\}$
Relations:

- Quadruples of type Q1:

1. $[x_{2i}, x_{2n-3}] = e$ where $2 \leq i \leq n-2$
2. $[x_{2i-1}, x_{2n-3}] = e$ where $2 \leq i \leq n-2$
3. $[x_{2i}, x_{2n-3}^{-1} x_{2n-2} x_{2n-3}] = e$ where $2 \leq i \leq n-2$
4. $[x_{2i-1}, x_{2n-3}^{-1} x_{2n-2} x_{2n-3}] = e$ where $2 \leq i \leq n-2$

- Quadruples of type Q2: for $i, j \neq n-1$, $|i-j| > 1$, $(i,j) \neq (n-2, n)$:

1. $[x_{2i}, x_{2i+1}^{-1} \cdots x_{2j-1}^{-1} x_{2j} x_{2j-1} \cdots x_{2i+1}] = e$
2. $[x_{2i-1}, x_{2i}^{-1} x_{2i+1}^{-1} \cdots x_{2j-1}^{-1} x_{2j} x_{2j-1} \cdots x_{2i+1} x_{2i}] = e$
3. $[x_{2i}, x_{2i+1}^{-1} \cdots x_{2j-2}^{-1} x_{2j-1} x_{2j-2} \cdots x_{2i+1}] = e$
4. $[x_{2i-1}, x_{2i}^{-1} x_{2i+1}^{-1} \cdots x_{2j-2}^{-1} x_{2j-1} x_{2j-2} \cdots x_{2i+1} x_{2i}] = e$

Presentation (cont.)

- A triple of type T1:

1. $[x_2, x_{2n-3}] = e$
2. $[x_1, x_{2n-3}] = e$
3. $x_{2n-2}x_2x_1 = x_2x_1x_{2n-2} = x_1x_{2n-2}x_2$

- Triples of type T2:

1. $x_{2i+2}x_{2i+1}x_{2i-1} = x_{2i+1}x_{2i-1}x_{2i+2} = x_{2i-1}x_{2i+2}x_{2i+1}$ where $1 \leq i \leq n-3$
2. $[x_{2i}, x_{2i+2}] = e$ where $1 \leq i \leq n-3$
3. $[x_{2i}, x_{2i+1}] = e$ where $1 \leq i \leq n-3$

Presentation (cont.)

- A triple of type T3:

1. $x_{2n}x_{2n-1}x_{2n-5} = x_{2n-1}x_{2n-5}x_{2n} = x_{2n-5}x_{2n}x_{2n-1}$
2. $[x_{2n-4}, x_{2n}] = e$
3. $[x_{2n-4}, x_{2n-1}] = e$

- A triple of type T4:

1. $[x_{2n-2}, x_{2n}] = e$
2. $[x_{2n-3}, x_{2n}] = e$
3. $x_{2n-1}x_{2n-2}x_{2n-3} = x_{2n-2}x_{2n-3}x_{2n-1} = x_{2n-3}x_{2n-1}x_{2n-2}$

Proof of an arrangement whose graph is a union of cycles

We use:

Oka-Sakamoto (1978): Let C_1 and C_2 be algebraic plane curves in $\mathbb{C}^2$. Assume that the intersection $C_1 \cap C_2$ consists of distinct $d_1 \cdot d_2$ points, where d_i $(i = 1, 2)$ are the respective degrees of C_1 and C_2. Then:

$$\pi_1(\mathbb{C}^2 - (C_1 \cup C_2)) \cong \pi_1(\mathbb{C}^2 - C_1) \oplus \pi_1(\mathbb{C}^2 - C_2)$$

THE END

Thank you!!!