Vanishing products in Orlik-Solomon algebras

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MSJ-SI Arrangements of Hyperplanes
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Goal

To describe the most natural generalization of the “neighborly partition/incidence matrix" description of the degree-one resonance variety to degrees ≥ 2.

Sources

- M. Falk, Geometry and combinatorics of resonant weights, in *Lecture notes of CIMPA Summer School on Arrangements, Local Systems, and Singularities, Galatasaray University, Istanbul, 2007*, (M. Uludağ *et. al.*, eds.), Progress in Mathematics Series, Birkhäuser, in press.
- D. Cohen, G. Denham, M. Falk, A. Varchenko, Resonant one-forms and critical loci of products of powers of linear forms, in preparation.
- C. Bibby, M. Falk, I. Williams, Vanishing products in the Orlik-Solomon algebras of k-generic arrangements, in preparation.

Notation

- $\mathcal{A} = \{H_1, \ldots, H_n\}$: a fixed central hyperplane arrangement in $\mathbb{C}^\ell$.
- $H_i = \ker(f_i \colon \mathbb{C}^\ell \to \mathbb{C})$.
- $A = \bigoplus_{p=1}^{\ell} A^p$: the OS algebra of $\mathcal{A}$ over $\mathbb{C}$.
- $e_1, \ldots e_n \in A^1$: the canonical generators of A.

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For this problem we can assume that $\mathcal{A}$ has rank $p+1$.

(Note: If $\{a_1, \ldots, a_p\}$ is linearly dependent then the product vanishes automatically.)

The case $p = 2$

Linear independence of $\{a_1, a_2\}$ is equivalent to $a_1 \neq 0$ and $a_2 \notin \mathbb{C}a_1$. Then $a_1 a_2 = 0$ means a_2 represents a nontrivial class in the first cohomology of the complex

$$0 \longrightarrow A^0 \xrightarrow{a_1} A^1 \xrightarrow{a_1} A^2 \xrightarrow{a_1} \cdots .$$

That means $a_1 \in \mathcal{R}^1(\mathcal{A})$, the degree-one resonance variety of $\mathcal{A}$.

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Conclusion: The $p = 2$ case is quite interesting.

Remark: $a_2 \in \mathcal{R}^1(\mathcal{A})$ as well. In fact the whole two-dimensional subspace spanned by a_1 and a_2 is in $\mathcal{R}^1$: $\mathcal{R}^1(\mathcal{A})$ is (projectively) ruled by lines.

Characterizations in case $p = 2$

Write $a_1 = \sum_{j=1}^{n} \lambda_j e_j$ and $a_2 = \sum_{j=1}^{n} \mu_j e_j$.

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The “low-level" condition for $a_1 a_2 = 0$ is:
for every rank-two flat X and every $i \in X$, the matrix

$$\begin{bmatrix} \lambda_i & \sum_{j \in X} \lambda_j \\ \mu_i & \sum_{j \in X} \mu_j \end{bmatrix}$$

has rank 1. In particular, if $X = \{i, j\}$ is a flat, then

$$\begin{bmatrix} \lambda_i & \lambda_j \\ \mu_i & \mu_j \end{bmatrix}$$

has rank 1.

This leads to a dichotomy: either $|X| \geq 3$ and

$$\sum_{j \in X} \lambda_j = \sum_{j \in X} \mu_j = 0,$$

or the vectors

$$(\lambda_j, \mu_j)$$

are all parallel for all $j \in X$. The former condition gives an incidence matrix J whose kernel must contain λ and μ, and the second condition yields a partition of $\mathcal{A}$, which is shown to satisfy a certain condition: “neighborliness." (All this works over any field.)

One obtains a decompsition $\mathcal{R}^1(\mathcal{A}) = \bigcup_\Gamma V_\Gamma$ parameterized by neighborly partitions.

If the field has characteristic zero, one can use the matrix $Q = J^T J - E$, to assign multiplicities to the hyperplanes and refine the partition to form a “multinet" on $\mathcal{A}$, and then use that to show that there is a “Čeva pencil" associated with $\mathcal{A}$.

The case $p \geq 3$

Linear independence of $\{a_1, \ldots, a_p\}$ is not sufficient to conclude that $a_2 \cdots a_p \notin Aa_1$, so $a_1 \cdots a_p = 0$ does not imply $a_1 \in \mathcal{R}^{p-1}(\mathcal{A})$.
$a_2 \cdots a_p \in A^{p-1}$ is a decomposable cocycle for a_1, but may not represent a nontrivial element of $H^{p-1}(A, a_1)$.

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If $\mathcal{A}$ is a $(p-1)$ generic arrangement and $\{a_1, \ldots, a_p\}$ is linearly independent, then $a_2 \cdots a_p \notin Aa_1$. Then $a_1 \cdots a_p = 0$ implies $a_1 \in \mathcal{R}^{p-1}(\mathcal{A})$, the degree $(p-1)$ resonance variety of $\mathcal{A}$.

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$\mathcal{A}$ is *k-generic* if every set of k hyperplanes is independent. A $(p-1)$ generic arrangement of rank $(p+1)$ is also known as an INNC arrangement. Note, every arrangement of rank three is INNC.

The associated variety

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(Local components arise from the defining relations of A :

$$(e_{i_1} - e_{i_2}) \cdots (e_{i_p} - e_{i_{p+1}}) = 0$$

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Questions

Is $\mathcal{D}^{p-1}(\mathcal{A})$ linear for $p \geq 3$? Is $\exp(\mathcal{D}^{p-1}(\mathcal{A})) \subseteq (\mathbb{C}^*)^n$ topologically significant?

Master functions and 1-forms

Let $M = \mathbb{C}^\ell \setminus \bigcup_{j=1}^n H_j$, the complement of $\mathcal{A}$.

Recall, $A \cong H^*(M, \mathbb{C})$ under the identification

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The function $\Phi_\lambda := \prod_{j=1}^n f_j^{\lambda_j}$ is called the $\mathcal{A}$-master function corresponding to the weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. It is a multivalued function on M in general, single-valued if $\lambda \in \mathbb{Z}^n$.

Vanishing products and algebraic dependence

Let $\lambda^1, \ldots, \lambda^p$ be the weight vectors corresponding to $a_1, \ldots, a_p$, and write $\Phi_i = \Phi_{\lambda^i}$. Assume $\lambda^i \in \mathbb{Z}^n$, and let

$$\Phi = (\Phi_1, \ldots, \Phi_p) \colon M \to \mathbb{C}^p.$$

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Proposition

$a_1 \cdots a_p = 0$ if and only if $\overline{\mathrm{im}(\Phi)}$ is a proper subvariety of $\mathbb{C}^p$, equivalently, $\{\Phi_1, \ldots, \Phi_p\}$ is algebraically dependent.

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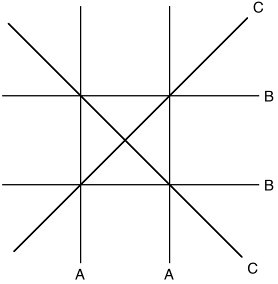
Proof.

$a_1 \cdots a_p = 0$ iff $\frac{d\Phi_1}{\Phi_1} \wedge \cdots \frac{d\Phi_p}{\Phi_p} = 0$ identically on M, iff $d\Phi_1 \wedge \cdots d\Phi_p = 0$ on M, iff the Jacobian of Φ has rank $< p$ at every point of M. □

Example



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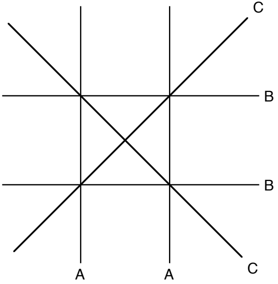

$$\lambda^1 = (-1,-1,0,0,1,1)$$

$$\lambda^2 = (-1,-1,1,1,0,0),$$

$$\Phi_1 = \frac{(x+y)(x-y)}{(x+z)(x-z)},$$

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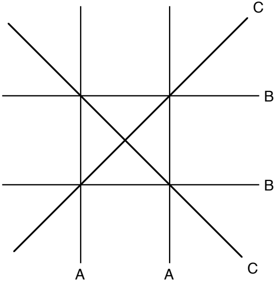

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$\Phi_1 + \Phi_2 = 1$, i.e. $\overline{\mathrm{im}(\Phi)}$ is a line in $\mathbb{C}^2$.

Critical loci

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Let $\Phi = (\Phi_1, \ldots, \Phi_p) \colon \mathbb{C}^\ell \to \mathbb{C}^p$ as above. Let $(y_1, \ldots, y_p)$ be coordinates on $\mathbb{C}^p$, $c_1, \ldots, c_p \in \mathbb{C}^p$ and $\lambda = \sum_{i=1}^{p} c_i \lambda^i$.

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Then

$$\Phi_\lambda = \prod_{i=1}^{p} \Phi_i^{c_i}$$

and

$$d\log(\Phi_\lambda) = \Phi^*(d\log \prod_{i=1}^{p} y_i^{c_i}) = \Phi^*(\sum_{i=1}^{p} c_i d\log y_i),$$

Now suppose $a_1 \cdots a_p = 0$, so $\overline{\mathrm{im}(\Phi)}$ is a proper subvariety of $\mathbb{C}^p$.

Then, for generic $c_1, \ldots, c_p$, the restriction of the one-form $\sum_{i=1}^{p} c_i d \log y_i$ to $\mathrm{im}(\Phi)$ vanishes on a nonempty, discrete set. Then the zero locus of $d \log(\Phi_\lambda)$ will be a union of fibers of Φ. We conclude

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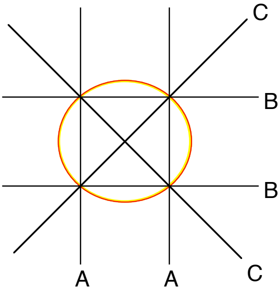
Theorem

If $a_1 \cdots a_p = 0$ then, for generic $c_1, \ldots c_p$, the critical locus of Φ_λ has codimension p in M.

Remark

In case $p = 2$, using the relation between resonance components, multinets and Čeva pencils (F.-Yuzvinsky), one can always choose λ^1 and λ^2 so that $\Phi_1 + \Phi_2 = 1$. Then $\sum_{i=1}^{p} c_i d \log y_i|_{\mathrm{im}(\Phi)}$ has a unique critical point iff $c_1 c_2 (c_1 + c_2) \neq 0$.

Example



$$\lambda = 2\lambda^1 - \lambda^2$$
$$= (-1,-1,-1,-1,2,2),$$

$$\Phi_\lambda = \frac{(x^2-y^2)^2}{(x^2-z^2)(y^2-z^2)},$$

$$\mathrm{crit}(\Phi_\lambda) = \Phi^{-1}(2,-1)$$

The tropicalization of $\overline{\mathrm{im}(\Phi)}$

The dimension of $\overline{\mathrm{im}(\Phi)}$ is equal to the dimension of its associated tropical variety. Using results of Dickenstein-Feichtner-Sturmfels and Feichtner-Sturmfels, we get necessary and sufficient conditions for the vanishing of $a_1 \cdots a_p$.

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Let $\Lambda = \left[\lambda^i_j\right]$. Then Φ factors as a linear map followed by a monomial map:

$$\mathbb{C}^\ell \xrightarrow{f} \mathbb{C}^n \xrightarrow{m_\Lambda} \mathbb{C}^p$$

where $f = (f_1, \ldots, f_n)$ and $m_\Lambda(x) = (x^{\lambda^1}, \ldots x^{\lambda^p})$, with the usual notation $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.

Tropical implicitization

Let us denote “tropicalization" by τ. Then, according to [DFS], $\tau(\overline{\text{im}(\Phi)})$ is the image of $\tau(\Phi)$, which is equal to $\tau(m_\Lambda) \circ \tau(f)$.

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The image of $\tau(f)$ is the Bergman fan of $\mathcal{A}$, and $\tau(m_\Lambda)$ is the linear map with matrix Λ. Thus we have

Proposition

The dimension of $\mathrm{im}(\Phi)$ is the maximum dimension of $\Lambda(C)$, as C ranges over the cones in the Bergman fan of $\mathcal{A}$.

Nested sets

According to [FS], the Bergman fan is subdivided by the nested set fan. Recall, a nested set is a set of irreducible lattice elements $\{X_1, \ldots, X_r\}$ for which the join of any pairwise incomparable subset is reducible. The cones in the nested set fan are generated by the sets of characteristic vectors of nested sets.

Example

In our example, $\{1, 2\}$ and $\{1, 135\}$ are nested, but $\{1, 3\}$ is not. The corresponding cones are generated by $\{e_1, e_2\}$ and $\{e_1, e_1 + e_3 + e_5\}$.

The rank condition

We obtain

Theorem

The dimension of $\mathrm{im}(\Phi)$ *is strictly less than p if and only if, for every nested set* $\{X_1, \ldots, X_r\}$, *the rank of the matrix*

$$\begin{bmatrix} \lambda^1_{X_1} & \cdots & \lambda^1_{X_r} \\ \vdots & & \vdots \\ \lambda^p_{X_1} & \cdots & \lambda^p_{X_r} \end{bmatrix}$$

is less than p.

Here $\lambda_X = \sum_{i \in X} \lambda_i$.

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- Nested sets describe the incidence structure among proper transforms of hyperplanes and exceptional divisors in a wonderful compactification of M: is there some geometric content in the rank condition?
- We have seen that much more can be said in case $\overline{\mathrm{im}(\Phi)}$ is linear. In general the degree of $\overline{\mathrm{im}(\Phi)}$ is encoded in its tropicalization. Can one derive general conditions for linearity?

k-generic arrangements

We now return to the case $\mathcal{A}$ is a $(p-1)$ generic arrangement, and apply the rank condition. For simplicity let us assume $p = 3$, and $\mathcal{A}$ has rank 4. The irreducible lattice elements are the hyperplanes and the irreducible rank-three flats (the INNC's).
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The maximal nested sets have two forms:

- $\{i, j, X\}$, with $i, j \in X$, and
- $\{i, j, k\}$, with $X = \{i, j, k\}$ a reducible rank-three flat.

The rank condition implies:

Theorem

$\operatorname{im}(\Phi)$ *has dimension* < 3 *if and only if,*

- *for every irreducible rank-three flat* $X = \{i_1, \ldots, i_q\}$, *the matrix*

$$\begin{bmatrix} \lambda^1_{i_1} & \cdots & \lambda^1_{i_q} & \lambda^1_X \\ \lambda^2_{i_1} & \cdots & \lambda^2_{i_q} & \lambda^2_X \\ \lambda^3_{i_1} & \cdots & \lambda^3_{i_q} & \lambda^3_X \end{bmatrix}$$

  has rank < 3, *and*

- *for every reducible rank-three flat* $\{i, j, k\}$,

$$\begin{bmatrix} \lambda^1_i & \lambda^1_j & \lambda^1_k \\ \lambda^2_i & \lambda^2_j & \lambda^2_k \\ \lambda^3_i & \lambda^3_j & \lambda^3_k \end{bmatrix}$$

  has rank < 3.

A decomposition of $\mathcal{D}^3(\mathcal{A})$

One is led to a similar dichotomy, but no parallel classes as in the $p = 2$ case. The key idea is to think of the partition in the $p = 2$ case as being a loop-free matroid of rank two on $\mathcal{A}$, with the blocks of the partition corresponding to multiple points. Similarly, one decomposes $\mathcal{D}^3(\mathcal{A})$ according to the matroid realized by Λ.

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Fix a set of irreducible rank-three flats $\mathcal{X}$, and a loop-free rank 3 matroid (also called Λ) on the ground set $\mathcal{A}$. One gets a component of $\mathcal{D}^3(\mathcal{A})$ precisely when Λ has a realization in the kernel of the incidence matrix determined by $\mathcal{X}$. Moreover, Λ is required to satisfy the following "neighborliness conditions":

- if X is a rank-three flat of $\mathcal{A}$ and $X \notin \mathcal{X}$, then X has rank two in Λ.

- if X is a rank-three flat of $\mathcal{A}$ and $X \notin \mathcal{X}$, then X has rank two in Λ.
- if X is a rank-three flat of $\mathcal{A}$, and $X - \{i\}$ has rank two in Λ, then so does X.