Matroids and Hyperplane Arrangements

Part Two

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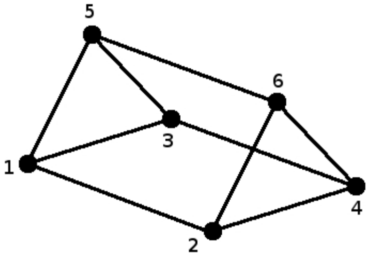
- X is a **flat** of $\mathcal{M}$ if $X = cl(X)$.

Example



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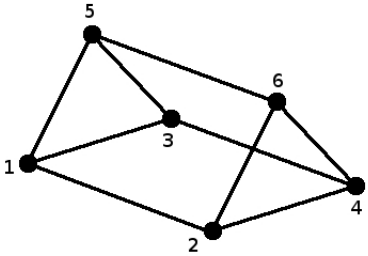

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- We do this by finding a matrix Λ with matroid $\mathcal{M}_{\Lambda}$ that satisfies certain properties so that the columns of Λ correspond to elements in $\mathcal{M}^2$.

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For the prism example, let $\mathfrak{X} = \{1234, 1256, 3456\}$. Then

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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- For the third property, we check that Γ, the set of maximal 2-cliques, satisfies the neighborly condition.

We say that Γ is **neighborly** if it satisfies the following properties for each rank-3 flat X in $\mathcal{M}_{\mathcal{A}}$.

(1) If X is an irreducible flat with $\sum_{j \in X} \lambda_{ji} \neq 0$ for some $i \leq 3$, then $X \subseteq S$ for some $S \in \Gamma$ (If X is irreducible and $X \notin \mathfrak{X}$, then X is contained in a 2-clique).

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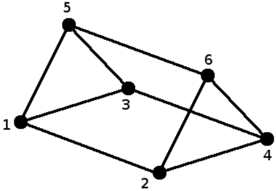
(2') If $X-\{i\}\subseteq S$ for some $i\in X$, $S\in\Gamma$, then $X\subseteq S$. (Generalization of condition (2). If $X-\{i\}$ is contained in a 2-clique for some $i\in X$, then so is X.)

Example



$$J_{\mathfrak{X}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

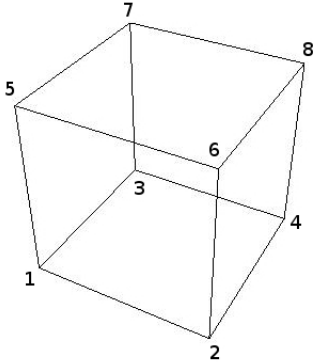
Consider the hyperplane arrangement

$$\mathcal{A} = \{x \pm y, y \pm z, z \pm w, w \pm x\}.$$

with matroid $\mathcal{M}_{\mathcal{A}}$ given by

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Cube



$$\mathfrak{X} = \{1357, 2358, 1458, 2457, 1368, 2367, 1467, 2468\}$$

Then:

$$J_{\mathfrak{X}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

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The matroid $\mathcal{M}_\Lambda$ is determined by the rows of Λ, and it consists of four double points.

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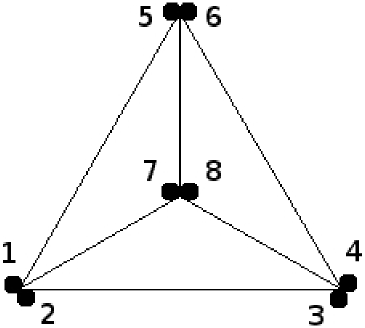

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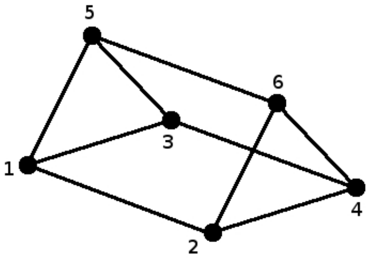
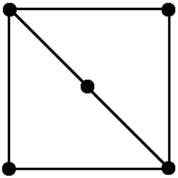
This tells us that in the OS Algebra for this arrangement,

$$[(e_1+e_2)-(e_7+e_8)]\wedge[(e_3+e_4)-(e_7+e_8)]\wedge[(e_5+e_6)-(e_7+e_8)] = 0$$

Graphic Matroids





Conclusion

Thank you.