Characterization of a Line Arrangement whose Fundamental Group of the Complement is a Direct Sum of Free Groups

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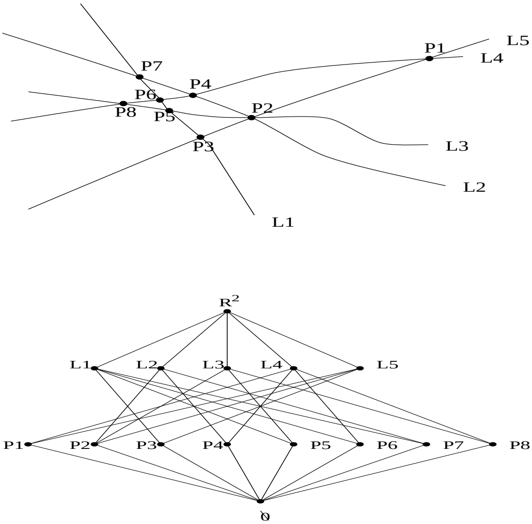
Definition

The incidence lattice of an arrangement

Let $\mathcal{L} = \{L_1, \cdots, L_n\}$ be an arrangement of lines. By $\mathrm{Lat}(\mathcal{L})$ we denote the partially-ordered set of non-empty intersections of the L_i, ordered by inclusion. We include the whole plane and the empty set in $\mathrm{Lat}(\mathcal{L})$, so that it becomes a lattice.

Example



L5
P1
L4
P7
P4
P6
P2
P8
P5
L3
P3
L2
L1
R^2
L1
L2
L3
L4
L5
P1
P2
P3
P4
P5
P6
P7
P8
0

The fundamental Group

Every line arrangement Σ is equipped with several invariants. The most important is the fundamental group of the complement $\pi_1(\mathbb{C}^2 - \Sigma)$.

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Computation

There are several algorithms to compute the fundamental group of the complement:

- Orlik-Terao
- Cohen-Suciu Algorithm.
- Moishezon-Teicher Algorithm + Van Kampen Theorem.

Combinatorics $\Rightarrow$ Fundamental Group ?

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Combinatorics ⇒ Fundamental Group?

Known Results

- Rybnikov (1994): two **complex** arrangements with the same lattice and different fundamental groups of the complement.

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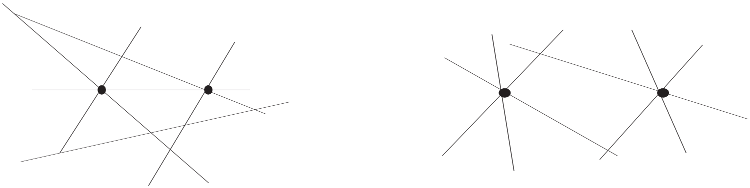
- Fan (1997): Up to 6 **complex** lines, the lattice determines the fundamental group of the complement.
- Garber Teicher Vishne (2002–3): Up to **real** 8 lines, the lattice determines the fundamental group of the complement.

Fundamental Group $\Rightarrow$ Combinatorics ?

Counterexample:

Figure: The arrangements Σ_1 and Σ_2



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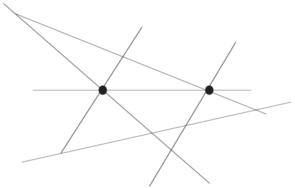

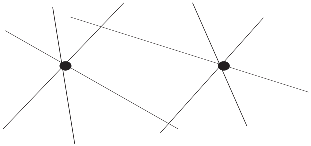

Figure: The arrangements Σ_1 and Σ_2

$$\pi_1(\mathbb{C}^2 - \Sigma_1) = \pi_1(\mathbb{C}^2 - \Sigma_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{Z}^2$$

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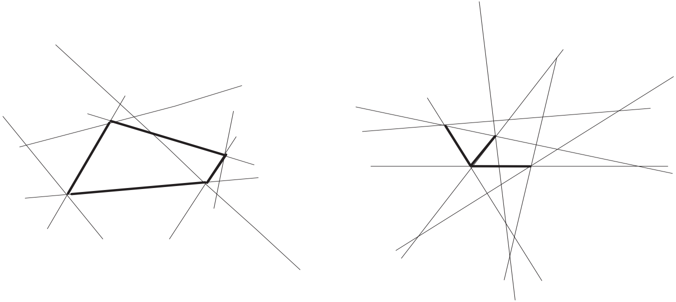

Figure: Graphs of line arrangements

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Theorem (Fan, 1997)

If $G(\mathcal{L})$ has no cycles, then:

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{F}_{m_1-1} \oplus \cdots \oplus \mathbb{F}_{m_k-1} \oplus \mathbb{Z}^{\ell-(\sum_{i=1}^{k}(m_i-1))-1},$$

where $m_1, \ldots, m_k$ are the multiplicities of the multiple intersection points in $\mathcal{L}$ and ℓ is the number of lines.

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Main Theorem (E-Liberman-Schaps-Teicher, 2009)

Fan's conjecture is true.

Presentation of The Fundamental Group

Orlik-Terao (1988), Arvola (1992), Cohen-Suciu (1997)

Let $\Sigma = \{L_1, \ldots, L_n\} \subseteq \mathbb{C}^2$ be a line arrangement.
We associate a generator Γ_i to each line L_i such that

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Every intersection point of $L_{i_1}, \ldots, L_{i_m}$ creates a set of relations

$$\Gamma_{i_1}^{x_1}\Gamma_{i_2}^{x_2}\cdots\Gamma_{i_m}^{x_m} = \Gamma_{i_m}^{x_m}\Gamma_{i_1}^{x_1}\cdots\Gamma_{i_{m-1}}^{x_{m-1}} = \Gamma_{i_2}^{x_2}\cdots\Gamma_{i_m}^{x_m}\Gamma_{i_1}^{x_1}$$

where $x_i \in G$ and $\Gamma_i^{x_i} = {x_i}^{-1}\Gamma_i x_i$.
It is equivalent to:

$$[\Gamma_{i_j}^{x_j}, \Gamma_{i_1}^{x_1}\cdots\Gamma_{i_m}^{x_m}] = e, 1 \leq j \leq m.$$

Algebraic Background

Let G be a group. $G_2 = [G, G], G_3 = [G, [G, G]]$
$G_2/G_3 = [G, G]/[G, [G, G]]$.

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Lemma

Let G be a group and let $\{x_1, \ldots, x_k\}$ be the generators of G. Then :

$$G_2/G_3 = \left\langle [x_i, x_j] \,\middle|\, \begin{array}{l} i \neq j,\ 1 \leq i, j \leq k \\ \textit{induced relations from relations of } G. \end{array} \right\rangle$$

In the case $G = \pi_1(\mathbb{C}^2 - \Sigma)$, it is easy to calculate G_2/G_3:

Lemma

An implementation for line arrangements. *Let Σ be a line arrangement and $G = \pi_1(\mathbb{C}^2 - \Sigma)$. Then the abelian group G_2/G_3 can be written as*

$$G_2/G_3 = \left\langle [\Gamma_i, \Gamma_j] \;\middle|\; \begin{array}{l} [\Gamma_i, \Gamma_j] = [\Gamma_j, \Gamma_i]^{-1}, \\ [\Gamma_i, \Gamma_j][\Gamma_k, \Gamma_l] = [\Gamma_k, \Gamma_l][\Gamma_i, \Gamma_j], \\ \prod_{\Gamma_x \in \Gamma(p)} [\Gamma_x, \Gamma_y], p \in \mathcal{P}, \Gamma_y \in \Gamma(p) \end{array} \right\rangle .$$

where $\Gamma(p)$ are the generators related to lines intersect in p.

Remark

We can see that if Γ_1 and Γ_2 are related to lines meeting in one point and Γ_3 and Γ_4 are related to lines meeting in a different point, there is no relation combining $[\Gamma_1, \Gamma_2]$ and $[\Gamma_3, \Gamma_4]$.
Therefore,

$$G_2/G_3 = \bigoplus_{p \in \mathcal{P}} C_p$$

where

$$C_p = \left\langle [\Gamma_i, \Gamma_j], \Gamma_i, \Gamma_j \in \Gamma(p) \;\middle|\; \begin{array}{c} [\Gamma_i, \Gamma_j] = [\Gamma_j, \Gamma_i]^{-1} \\ [\Gamma_i, \Gamma_j][\Gamma_k, \Gamma_l] = [\Gamma_k, \Gamma_l][\Gamma_i, \Gamma_j] \\ \Gamma_i, \Gamma_j, \Gamma_k, \Gamma_l \in \Gamma(p) \\ \displaystyle\prod_{\Gamma_x \in \Gamma(p)} [\Gamma_x, \Gamma_y], \Gamma_y \in \Gamma(p) \end{array} \right\rangle .$$

Definition

$f : G/G_2 \times G/G_2 \to G_2/G_3$
$f(\overline{a}, \overline{b}) = [a, b]/G_3.$

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Lemma

Let $a, b, c \in G/G_2$. *Then:*

1. $f(a \cdot b, c) = f(a, c) \cdot f(b, c)$.
2. $f(a, b \cdot c) = f(a, b) \cdot f(a, c)$.
3. *if* $n, m \in \mathbb{Z}$ *then* $f(a^n, b^m) = f(a, b)^{nm}$ *for* $m, n \in \mathbb{Z}$.
4. $f(b, a) = (f(a, b))^{-1}$.

For any $\overline{x} \in G/G_2$, we define:

$$S(\overline{x}) = \{\overline{y} \in G/G_2 | f(\overline{y}, \overline{x}) = e\} \leq G/G_2.$$

Meaning: $S(\overline{x}) \leq G/G_2$ contains elements whose quotient commutes with $\overline{x}$ in G_2/G_3.

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Theorem

Let $Q \in \mathcal{P}$ be an intersection point of $\{L_{i_1}, \dots, L_{i_m}\}$.
Let $\Gamma(Q) = \{\Gamma_{i_1}, \dots, \Gamma_{i_m}\}$ and the induced relations of the point Q is $\Gamma_{i_1}^{x_1}\Gamma_{i_2}^{x_2}\cdots\Gamma_{i_m}^{x_m} = \Gamma_{i_m}^{x_m}\Gamma_{i_1}^{x_1}\cdots\Gamma_{i_{m-1}}^{x_{m-1}} = \Gamma_{i_2}^{x_2}\cdots\Gamma_{i_m}^{x_m}\Gamma_{i_1}^{x_1}$.
Let $M = \Gamma_{i_1}^{x_1}\cdots\Gamma_{i_m}^{x_m}$.
Then

$$S(\overline{M}) = \left\langle \overline{\Gamma(Q)} \cup \left(\bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma}) \right) \right\rangle.$$

We call $S(\overline{M})$ the stabilizer *of the intersection point Q.*

Theorem

Assume:

$$G = \pi_1(\mathbb{C}^2 - \Sigma) \simeq (\bigoplus_{i=1}^{n} A_i) \oplus \mathbb{Z}^l$$

where A_i is a free group. Then for any multiple point Q of k lines $\{l_1, \ldots, l_k\}$, there exists $r, 1 \leq r \leq n$, and a projection onto A_r, $\varphi_Q : G \to G$ such that $A_r = \langle \varphi_Q(\Gamma_1), \ldots, \varphi_Q(\Gamma_k) \rangle \cong \mathbb{F}_{k-1}$.
If l_j is a line do not pass through the point, then $\varphi_Q(\Gamma_j) = e$.

Moreover, if $\{p_1, \ldots, p_m\}$ are the multiple points of Σ and n_i is the number of lines pass through the point p_i, then $G \cong (\bigoplus_{i=1}^{m} C_i) \oplus B$, where $C_i \cong \mathbb{F}_{n_i - 1}$. If l is a line which does not pass through p_i and let Γ be its corresponding generator, then $\mathrm{pr}_i(\Gamma) = e$ (where pr_i is the projection onto C_i).

Proof of main theorem

Theorem

Let $\Sigma \subseteq \mathbb{C}^2$ be a line arrangement which has no pair of parallel lines. Then if

$$\pi_1(\mathbb{C}^2 - \Sigma) = \bigoplus_{i=1}^{r} A_i \oplus \mathbb{Z}^l,$$

where A_i are free groups. Then $G(\Sigma)$ has no cycles.

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Proof:

Assume by negation there is at least one cycle in the graph. We choose the minimal one.

By the previous Theorem we can write : $G \cong (\bigoplus_{i=1}^{m} C_i) \oplus B$, where $C_i \cong \mathbb{F}_{n_i - 1}$.

Define:

$\{\Gamma_1, \ldots, \Gamma_n\}$- generators related to the lines of the arrangement.
$\{\Gamma_{x_1}, \ldots, \Gamma_{x_t}\}$ - generators related to the lines of the cycle.
$Z := \Gamma_1 \cdots \Gamma_n$
$N := \langle \Gamma_{x_1}, \ldots, \Gamma_{x_t}, Z \rangle$
$H := G/N$

There is a contradiction related to the rank of H.

The End!