HYPERPLANE ARRANGEMENTS, RANDOM WALKS AND EIGENVALUES

Christos A. Athanasiadis
University of Athens

and

Persi Diaconis
Stanford University

Let

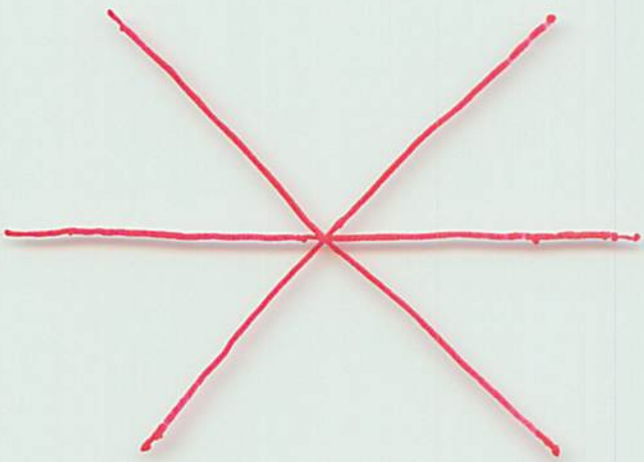
$\mathcal{A}$ = affine hyperplane arrangement in $V = \mathbb{R}^n$

$\mathcal{C}_{\mathcal{A}}$ = set of chambers (regions)

$\mathcal{F}_{\mathcal{A}}$ = set of faces

$\mathcal{L}$ = intersection poset of $\mathcal{A}$ with minimum element $\hat{0} = V$

w = probability measure on $\mathcal{F}_{\mathcal{A}}$

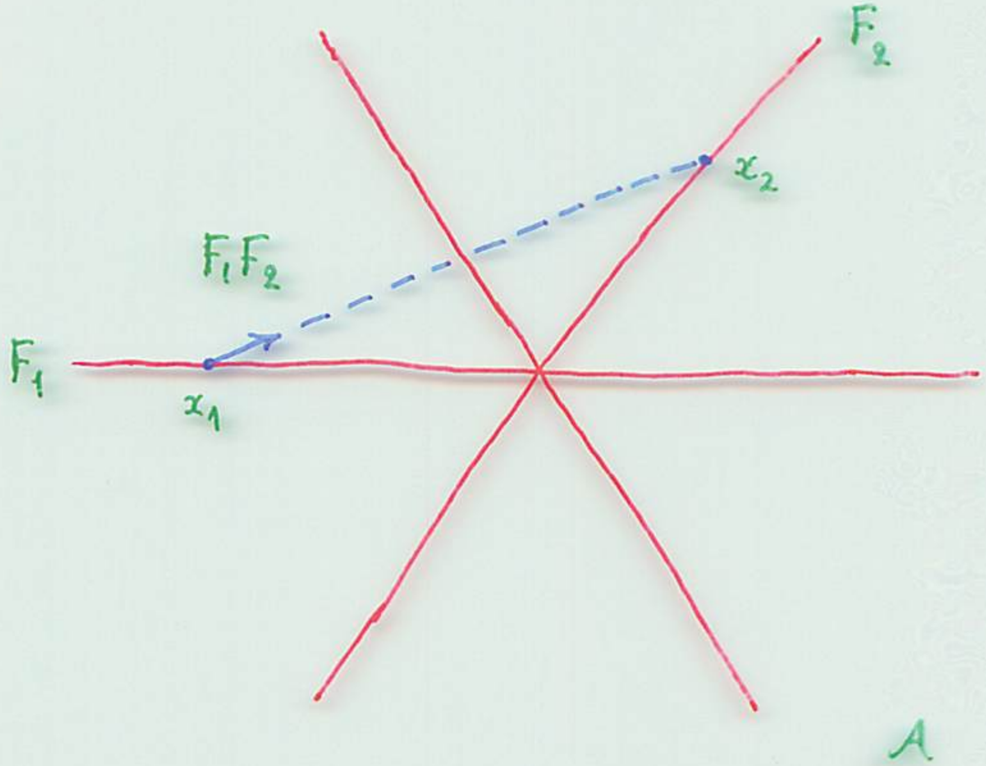


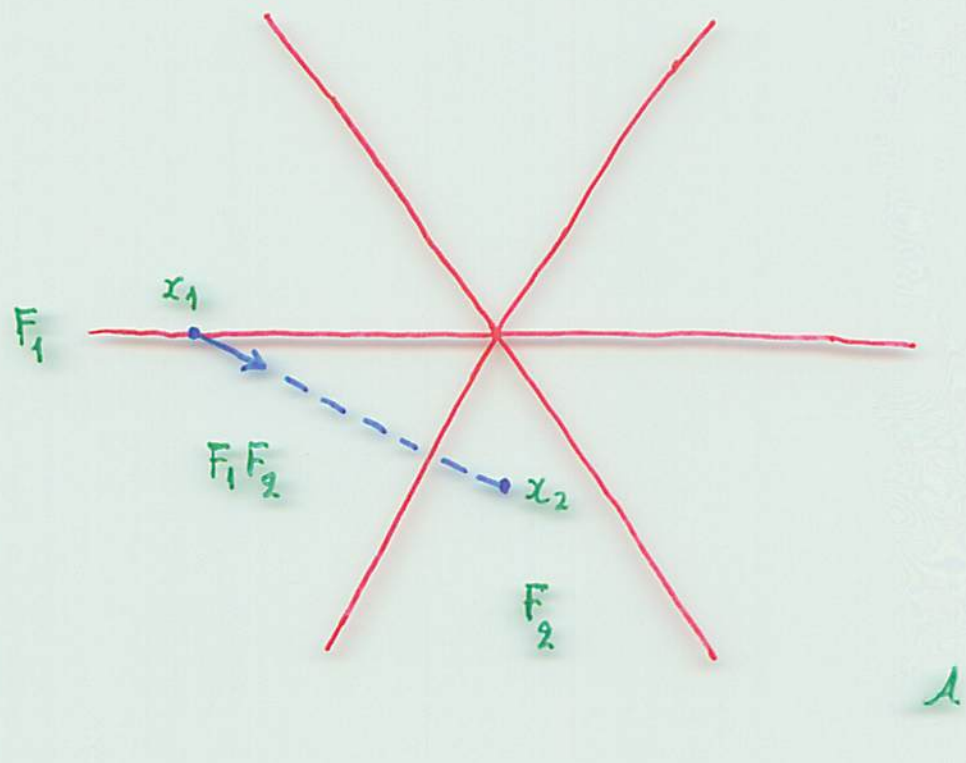
$\mathcal{A}$

$\hat{0} = V$

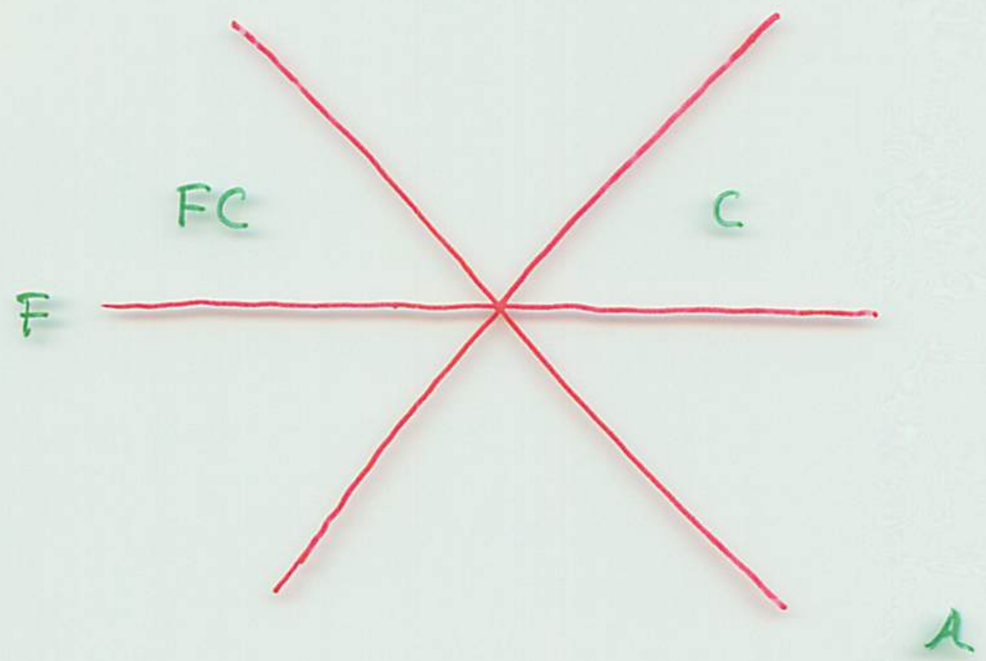
$\mathcal{L}$

There is a product on the set $\mathcal{F}_{\mathcal{A}}$ which turns it into a semigroup :





Recall that for all $F \in \mathcal{F}_{\mathcal{A}}$ and $C \in \mathcal{C}_{\mathcal{A}}$ we have $FC \in \mathcal{C}_{\mathcal{A}}$.



Bidigare, Hanlon and Rockmore defined a random walk on $\mathcal{C}_{\mathcal{A}}$ with transition matrix satisfying

$$K(C,C') = \sum_{\substack{FC = C' \\ F \in \mathcal{F}_{\mathcal{A}}}} w(F)$$

for $C, C' \in \mathcal{C}_{\mathcal{A}}$.

Some definitions

A probability distribution π on $\mathcal{C}_A$ is stationary for K if

$$K^{\ell}(C,C') \longrightarrow \pi(C') \quad \text{as} \quad \ell \to \infty$$

for all $C, C' \in \mathcal{C}_A$. When it exists, it is characterized by

$$\sum_{C \in \mathcal{C}_A} \pi(C)\, K(C,C') = \pi(C')$$

for all $C' \in \mathcal{C}_A$.

For probability distributions P, Q on a finite set X we define the total variation distance

$$\| P - Q \|_{TV} = \max_{A \subseteq X} |P(A) - Q(A)| .$$

We are interested in the rate of convergence of $\| K_C^{\ell} - \pi \| \to 0$ as $\ell \to \infty$.

Theorem (BHR, 1999) The characteristic polynomial of the matrix K is equal to

$$\det(xI - K) = \prod_{W \in \mathcal{L}} (x - \lambda_W)^{m_W},$$

where

$$\lambda_W = \sum_{\substack{F \in \mathcal{F}_A \\ F \subseteq W}} \omega(F)$$

is an eigenvalue,

$$m_W = |\mu_{\mathcal{L}}(\hat{0}, W)| = (-1)^{\operatorname{codim}(W,V)} \mu_{\mathcal{L}}(V, W)$$

and $\mu_{\mathcal{L}}$ is the Möbius function on $\mathcal{L}$.

Corollary (Las Vergnas, Zaslavsky, 1975)

$$\sum_{W \in \mathcal{L}} |\mu_{\mathcal{L}}(\hat{0}, W)| = \# C_A .$$

Theorem (Brown – Diaconis, 1998)

(a) The matrix K is diagonalizable over $\mathbb{R}$.

(b) K has a unique stationary distribution π if and only if for each $H \in \mathcal{A}$ there exists a face $F \in \mathcal{F}_{\mathcal{A}}$ such that $F \notin H$ and $\omega(F) > 0$.

Moreover, assuming π exists:

(c) Sample without replacement from ω, thus getting an ordering $F_1, \dots, F_m$ of the set $\{F \in \mathcal{F}_{\mathcal{A}} : \omega(F) > 0\}$. Then $C = F_1 F_2 \cdots F_m$ is a chamber distributed from π.

(d) We have

$$\| K_C^{\ell} - \pi \|_{TV} \leq \sum_{H \in \mathcal{A}} \lambda_H^{\ell},$$

where K_C^{ℓ} is the distribution of the chain started from $C \in \mathcal{C}_{\mathcal{A}}$ after ℓ steps, i.e. $K_C^{\ell}(C') = K^{\ell}(C, C')$.

Example Suppose $\mathcal{A}$ is an essential, central arrangement of hyperplanes in $V = \mathbb{R}^3$.

Five hyperplanes in $\mathbb{R}^3$

Theorem (Billera - Brown - Diaconis, 1999) If w is supported and is uniform on the one-dimensional faces of $\mathcal{A}$, then

$$\pi(C) = \frac{i-2}{2(f_0-2)},$$

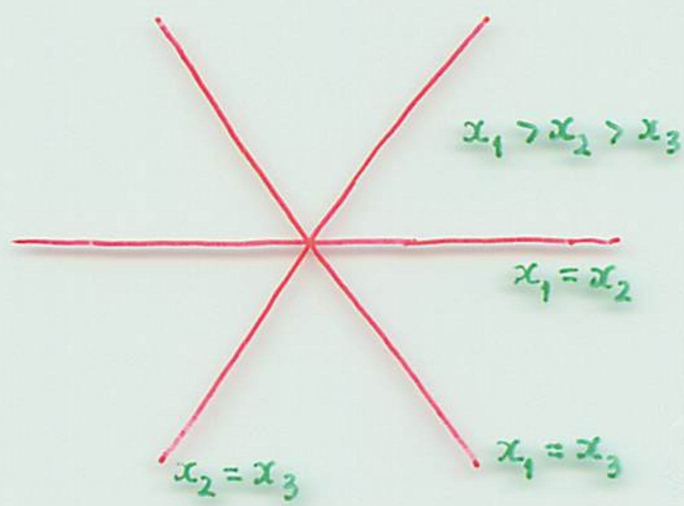
where i is the number of facets (sides) of $C \in \mathcal{C}_{\mathcal{A}}$ and f_0 is the number of one-dimensional faces of $\mathcal{A}$.

The braid arrangement

Let

$$\mathcal{A} = \{ x_i - x_j = 0 \; : \; 1 \le i < j \le n \}$$

be the braid arrangement in $\mathbb{R}^n$.



There are bijections

- $\mathcal{C}_{\mathcal{A}} \cong$ symmetric group $\mathfrak{S}_n$ of permutations of $[n] := \{1, 2, \ldots, n\}$
- $\mathcal{F}_{\mathcal{A}} \cong$ set of ordered partitions $(B_1, B_2, \ldots, B_k)$ of the set $[n]$

mapping e.g. $\tau \in \mathfrak{S}_n$ to the chamber

$$x_{\tau(1)} > x_{\tau(2)} > \cdots > x_{\tau(n)} .$$

Moreover, if

- $\mathcal{F}_A \ni F \longleftrightarrow (B_1, B_2, \dots, B_k) = B$
- $\mathcal{C}_A \ni C \longleftrightarrow \tau \in S_n$,

then $FC \in \mathcal{C}_A$ corresponds to the permutation obtained from B by linearly ordering each block B_i according to τ.

Example: If $n = 9$ and

$$\tau = (8, 1, 4, 7, 9, 2, 6, 3, 5)$$

$$B = \big(\{6, 9\}, \{1, 3, 7\}, \{4\}, \{2, 5, 8\} \big),$$

then the action of B on τ results in the permutation

$$\tau' = (9, 6, 1, 7, 3, 4, 8, 2, 5) \in S_9.$$

Two interesting measures on $\mathcal{F}_{\mathcal{A}}$:

RANDOM TO TOP: Let $w_1, w_2, \ldots, w_n$ be positive numbers summing to 1 and let

$$\omega(\{i\}, [n] \setminus \{i\}) = w_i \quad \text{for} \quad 1 \le i \le n$$

and $\omega(B) = 0$ for other ordered partitions B.

prob $= w_i$

$$(\tau(1), \tau(2), \ldots, i, \ldots, \tau(n))$$

Thus the BHR walk proceeds by selecting the coordinate of a permutation $\tau \in S_n$ equal to i with probability w_i and moving it in front.

Corollary (i) (Phatarfod, 1991) For each subset $S \subseteq [n]$ (other than those with $n-1$ elements) there exists an eigenvalue

$$\lambda_S = \sum_{i \in S} w_i$$

of K of multiplicity equal to the number of permutations $\tau \in S_n$ with set of fixed points equal to S.

(ii) The stationary distribution π is given by

$$\pi(\tau) = \frac{w_{\tau(1)} \, w_{\tau(2)} \cdots w_{\tau(n)}}{(1-w_{\tau(1)})(1-w_{\tau(1)}-w_{\tau(2)}) \cdots (1-w_{\tau(1)}-\cdots-w_{\tau(n-1)})}.$$

(iii) If $w_1 = w_2 = \cdots = w_n = 1/n$ then

$$\| K_\tau^\ell - \pi \|_{TV} \leq \binom{n}{2} \left(1 - \frac{2}{n}\right)^\ell .$$

It gets $n \log n$ steps for the walk to reach stationarity.

INVERSE α - SHUFFLES : Let $\alpha \geq 2$ be an integer. Assign weight (measure) $1/\alpha^n$ to each of the α^n weak ordered set partitions

$$(B_1, B_2, \ldots, B_\alpha)$$

of $[n]$ (and zero otherwise).

Example: For $\alpha = 2$ the BHR walk proceeds by selecting some of the coordinates of a permutation $\tau \in S_n$, uniformly at random, and moving them in front, keeping their relative order as in τ.

$$\begin{array}{l}
(8, 1, 4, 7, 9, 2, 6, 3, 5) \\
\downarrow \\
({}^{8}, 1, {}^{4}, 7, 9, {}^{2}, {}^{6}, {}^{3}, 5) \\
\downarrow \\
(8, 4, 2, 6, 3, 1, 7, 9, 5)
\end{array}$$

Corollary

(i) (Hanlon, 1990; Bayer-Diaconis, 1992)

The eigenvalues of K are $1, 1/a, 1/a^2, \ldots, 1/a^{n-1}$.

The multiplicity of $1/a^i$ is equal to the number of permutations $\tau \in S_n$ with $n-i$ cycles.

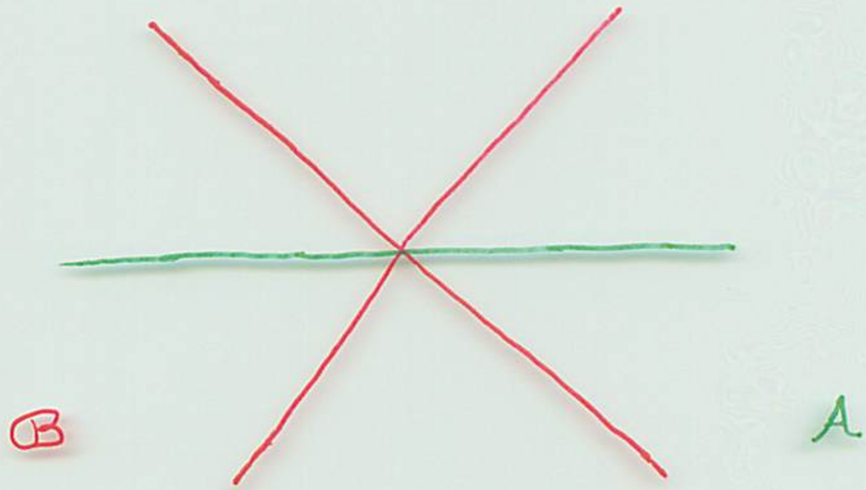
(ii) The stationary distribution π is uniform on S_n and

$$\| K_\tau^\ell - \pi \|_{TV} \leq \binom{n}{2} (1/a)^\ell$$

for every $\ell = 1, 2, 3, \ldots$

It takes $\frac{3}{2} \log_2 n$ steps for the walk to reach stationarity if $a = 2$ (Aldous-Diaconis, 1986; Bayer-Diaconis, 1992); the answer is $\frac{3}{2} \log_a n$ in general.

Let $\mathcal{A}$ be any arrangement in $V = \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathcal{A}$ be a subarrangement.



Recall that

$$K_{\mathcal{A}}(C, C') = \sum_{FC = C'} w(F), \qquad C, C' \in \mathcal{C}_{\mathcal{A}}$$

is the transition matrix of the BHR walk on $\mathcal{C}_{\mathcal{A}}$. At each step of the walk, record the chamber $D \in \mathcal{C}_{\mathcal{B}}$ of $\mathcal{B}$ in which the current chamber $C \in \mathcal{C}_{\mathcal{A}}$ lies.

This process defines a Markov subchain of the Markov chain

$$C_0, \quad F_1 C_0, \quad F_2 F_1 C_0, \quad F_3 F_2 F_1 C_0, \ldots$$

Indeed, given $D, D' \in \mathcal{C}_{\mathcal{B}}$ and choosing $C \in \mathcal{C}_{\mathcal{A}}$ with $C \subseteq D$, the probability

$$K_{\mathcal{B}}(D, D') = \sum_{\substack{C' \in \mathcal{C}_{\mathcal{A}} \\ C' \subseteq D'}} K_{\mathcal{A}}(C, C')$$

of moving from D into D' is independent of the chamber $C \subseteq D$ of $\mathcal{A}$ chosen.

<u>Question:</u> What can one say about the analysis of these subchains on $\mathcal{C}_{\mathcal{B}}$?

<u>Note:</u> In a special case, this was considered by J.-C. Uyemura Reyes, 2002.

Motivation: Let $\mathcal{A}$ be the braid arrangement in $V = \mathbb{R}^n$, so that $\mathcal{B} \subseteq \mathcal{A}$ corresponds to a simple graph $G \subseteq \binom{[n]}{2}$ on the node set $[n] = \{1, 2, \dots, n\}$.

- If $G = \{\{1,2\}, \{1,3\}, \dots, \{1,k\}\}$ for some k, $2 \le k \le n$, then the subchain records the position of 1 relative to each of $2, \dots, k$ at each step of the walk.
- If $G = \binom{[k]}{2}$ for some $2 \le k \le n$, then the subchain records the position of $1, 2, \dots, k$ relative to each other.
- If $G = \{\{1,2\}, \{2,3\}, \dots, \{n-1,n\}\}$, then the subchain records the descent set

$$\mathrm{Des}(\tau^{-1})$$

of the permutation τ^{-1}.

Back to the general case of an arrangement $\mathcal{A}$ in $V = \mathbb{R}^n$, subarrangement $\mathcal{B} \subseteq \mathcal{A}$ and probability measure w on $\mathcal{F}_{\mathcal{A}}$.

Lemma: We have

$$K_{\mathcal{B}}(D, D') = \sum_{\substack{F \in \mathcal{F}_{\mathcal{B}} \\ FD = D'}} w^*(F)$$

for $D, D' \in \mathcal{C}_{\mathcal{B}}$, where

$$w^*(F) = \sum_{\substack{E \in \mathcal{F}_{\mathcal{A}} \\ E \subseteq F}} w(E)$$

for $F \in \mathcal{F}_{\mathcal{B}}$.

Note: For W in the intersection poset $\mathcal{L}_{\mathcal{B}}$ of $\mathcal{B}$ we have

$$\sum_{\substack{F \in \mathcal{F}_{\mathcal{B}} \\ F \subseteq W}} w^*(F) = \lambda_W = \sum_{\substack{E \in \mathcal{F}_{\mathcal{A}} \\ E \subseteq W}} w(E).$$

Corollary: (A – Diaconis, 2008)

Let $\mathcal{L}_{\mathcal{B}}$ denote the intersection poset of $\mathcal{B}$ and $\mu_{\mathcal{B}}$ denote the Möbius function of $\mathcal{L}_{\mathcal{B}}$.

(a) The matrix $K_{\mathcal{B}}$ is diagonalizable over $\mathbb{R}$ and its eigenvalues are included among those of $K_{\mathcal{A}}$. Specifically,

$$\det(xI - K_{\mathcal{B}}) = \prod_{W \in \mathcal{L}_{\mathcal{B}}} (x - \lambda_W)^{n_W},$$

where

$$n_W = |\mu_{\mathcal{B}}(\hat{0}, W)| = (-1)^{\operatorname{codim}(W,V)} \mu_{\mathcal{B}}(V, W).$$

(b) $K_{\mathcal{B}}$ has a unique stationary distribution $\pi_{\mathcal{B}}$ if and only if for each $H \in \mathcal{B}$ there exists a face $E \in \mathcal{F}_{\mathcal{A}}$ such that $E \not\subseteq H$ and $w(E) > 0$.

Moreover, assuming $\pi_{\mathcal{B}}$ exists:

(c) Sample without replacement from ω, thus getting an ordering $E_1, E_2, \dots, E_m$ of $\{E \in \mathcal{F}_{\mathcal{A}} : \omega(E) > 0\}$. Let $C = E_1 E_2 \cdots E_m \in \mathcal{C}_{\mathcal{A}}$ and assume that $\pi_{\mathcal{A}}$ exists. Then the unique $D \in \mathcal{C}_{\mathcal{B}}$ which contains C is a chamber distributed from $\pi_{\mathcal{B}}$.

Equivalently, we have

$$\pi_{\mathcal{B}}(D) = \sum_{\substack{C \in \mathcal{C}_{\mathcal{A}} \\ C \subseteq D}} \pi_{\mathcal{A}}(C)$$

for each $D \in \mathcal{C}_{\mathcal{B}}$.

(d) We have

$$\| K_{\mathcal{B}}^{\ell} - \pi_{\mathcal{B}} \|_{TV} \leq \sum_{H \in \mathcal{B}} \lambda_H^{\ell},$$

where $K_{\mathcal{B}}^{\ell}(D') = K_{\mathcal{B}}^{\ell}(D, D')$ for some arbitrary but fixed initial chamber $D \in \mathcal{C}_{\mathcal{B}}$ and every $D' \in \mathcal{C}_{\mathcal{B}}$.

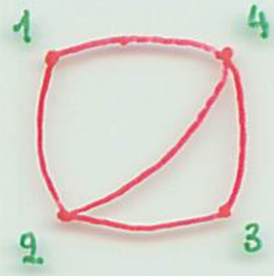
Subarrangements of the braid arrangement

Let $\mathcal{A}$ be the braid arrangement in $\mathbb{R}^n$. Then $\mathcal{B} \subseteq \mathcal{A}$ is the graphical arrangement corresponding to a simple graph

$$G \subseteq \binom{[n]}{2}$$

on the node set $[n]$ and $\mathcal{C}_{\mathcal{B}}$ bijects to the set of acyclic orientations of G.

Example:

$G =$ 

$\mathcal{B}$ consists of $x_1 = x_2$, $x_1 = x_4$, $x_2 = x_3$, $x_2 = x_4$, $x_3 = x_4$.

acyclic

not acyclic

Lets focus on our two standard measures on $\mathcal{F}_A$.

<u>RANDOM</u> <u>TO</u> <u>TOP</u>: We are given positive numbers $w_1, w_2, \dots, w_n$ summing to 1, assigned to the nodes $1, 2, \dots, n$ of G.

The walk proceeds from the current acyclic orientation of G by picking node i with probability w_i and redirecting towards i every edge of G incident to i.

Corollary: (i) The corresponding transition matrix K on the set of acyclic orientations of G (is diagonalizable over $\mathbb{R}$ and) satisfies

$$\det(xI - K) = \prod_{s \subseteq [n]} (x - \lambda_s)^{m_s}$$

where

$$\lambda_s = \sum_{i \in s} w_i$$

and

$$m_s = \sum_{t \supseteq s} (-1)^{t \setminus s} \, n_t ,$$

where n_t is the number of acyclic orientations of $G \setminus t$.

(ii) The stationary distribution is given by

$$\pi(o) = \sum_{\tau \in \mathcal{L}(o)} \frac{w_{\tau(1)} w_{\tau(2)} \cdots w_{\tau(n)}}{(1 - w_{\tau(1)})(1 - w_{\tau(1)} - w_{\tau(2)}) \cdots (1 - w_{\tau(1)} - \cdots - w_{\tau(n-1)})}$$

where $\mathcal{L}(o)$ is the set of linear extensions of the acyclic orientation o of G, i.e. permutations $\tau \in S_n$ such that $\tau^{-1}(i) < \tau^{-1}(j)$ whenever there is an edge $j \to i$ in o.

(iii) We have

$$\| K^{\ell} - \pi \|_{TV} \leq \sum_{\{i,j\} \in G} (1 - w_i - w_j)^{\ell}.$$

In particular

$$\| K^{\ell} - \pi \|_{TV} \leq m \left(1 - \frac{2}{n}\right)^{\ell},$$

where m is the number of edges of G, if $w_1 = w_2 = \cdots = w_n = 1/n$.

INVERSE a- SHUFFLES: The walk proceeds from a given acyclic orientation of G by picking uniformly at random a weak ordered set partition

$$(B_1, B_2, \dots, B_a)$$

of the node set $[n]$, with a parts, and redirecting from v to u each edge $\{u,v\}$ of G with $u \in B_i$, $v \in B_j$ and $i < j$.

Corollary: (i) The corresponding transition matrix K (is diagonalizable over $\mathbb{R}$ and) satisfies

$$\det(xI - K) = \prod_{i=0}^{n-1} \left(x - \frac{1}{a^i}\right)^{m_i}$$

where

$$\chi_G(q) = \sum_{i=0}^{n-1} (-1)^i \, m_i \, q^{n-i}$$

is the chromatic polynomial of G.

(4i) The stationary distribution is given by

$$\pi(o) = \frac{e(o)}{n!},$$

where $e(o)$ is the number of linear extensions of the acyclic orientation o of G.

(4ii) We have

$$\| K^{\ell} - \pi \|_{TV} \leq \frac{m}{a^{\ell}},$$

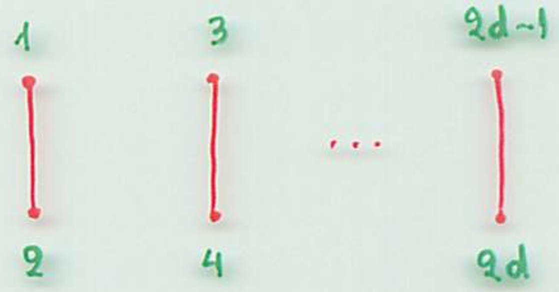
where m is the number of edges of G.

Back to the random to top measure. Assume that $w_1 = w_2 = \cdots = w_n = 1/n$, so that

$$\pi(o) = \frac{e(o)}{n!},$$

as before.

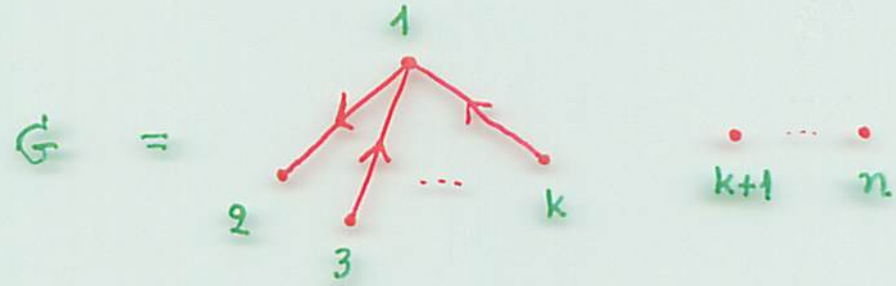
<u>Example:</u> Let $G = \{\{1,2\}, \{3,4\}, \dots, \{2d-1, 2d\}\}$, so that $n = 2d$.



$= G$

The acyclic orientations of G are in one-to-one correspondence with the vertices of the d-dimensional cube $[0,1]^d$ and our random walk is the familiar nearest neighbor random walk on $[0,1]^d$.

Example: Let $G = \{\{1,2\}, \{1,3\}, \dots, \{1,k\}\}$, so the set of acyclic orientations of G bijects to the set of subsets of $\{2, 3, \dots, k\}$.



Corollary (i) The eigenvalues of the transition matrix K are the following:

$$\begin{cases} 1, & \text{with multiplicity one} \\ \dfrac{n-i-1}{n}, & \text{with multiplicity } \binom{k-1}{i} \text{ for } 1 \le i \le k-1. \end{cases}$$

(ii) In the stationary distribution, our subset has $0, 1, \dots$ or $k-1$ elements with equal probability $1/k$ and each subset of $\{2, 3, \dots, k\}$ with i elements has chance

$$\frac{1}{k \cdot \binom{k-1}{i}}.$$

(iii) We have

$$\| K^{\ell} - \pi \|_{TV} \leq (k-1)\left(1 - \frac{q}{n}\right)^{\ell}.$$

We can also prove

$$\| K^{\ell} - \pi \|_{TV} \leq \left(1 - \frac{1}{n}\right)^{\ell}$$

using the following:

Lemma: (Brown - Diaconis, 1998) For any BHR walk on an arrangement $\mathcal{A}$ we have

$$\| K^{\ell} - \pi \|_{TV} \leq \text{Prob}\{ F_1 \cdots F_{\ell} \notin \mathcal{C}_{\mathcal{A}} \}$$

$$\leq \sum_{H \in \mathcal{A}} \lambda_H^{\ell}.$$

New Proof: By a coupling argument. ■

Corollary: In the above situation, $\| K^{\ell} - \pi \|_{TV}$ is bounded above by the probability that ℓ vertices of G, chosen uniformly at random (with repetition allowed) do not cover all edges of G.

Example: Let $G = \{\{1,2\},\{2,3\},\dots,\{n-1,n\}\}$.

$\vec{G} =$

The acyclic orientations of $\vec{G}$ are again in bijection with the subsets of $[n-1]$.

Corollary (i) The eigenvalues of the transition matrix K are $0, 1/n, \dots, \frac{n-2}{n}$ and 1. For $0 \le k \le n$, the multiplicity of the eigenvalue k/n is equal to the number of compositions of n with exactly k parts equal to 1.

(ii) The stationary distribution π satisfies

$$\pi(S) = \frac{1}{n!} \# \{ \tau \in S_n : \mathrm{Des}(\tau) = S \}$$

for each $S \subseteq [n-1]$.

(iii) We have

$$\| K^{\ell} - \pi \|_{TV} \le (n-1) \left(1 - \frac{2}{n}\right)^{\ell}.$$

Enumeration of walks and eigenvalues

Let $A = (a_{ij}) \in \mathbb{C}^{r\times r}$ be a matrix and

let

$$f(A,\ell) = \mathrm{tr}(A^\ell) = \sum_i a_{ii_1} a_{i_1 i_2} \cdots a_{i_{\ell-1} i}$$

for $\ell \in \{1, 2, 3, \ldots\}$.

Lemma: If $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{C}$ are numbers such that

$$f(A,\ell) = \lambda_1^\ell + \lambda_2^\ell + \cdots + \lambda_r^\ell$$

for all $\ell \in \{1, 2, 3, \ldots\}$, then $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the eigenvalues of the matrix A.

Example: If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$, then

$$f(A,\ell) = \begin{cases} 2, & \text{if } \ell \text{ even}, \\ 0, & \text{if } \ell \text{ odd} \end{cases}$$

$$= 1^\ell + (-1)^\ell$$

and hence 1 and -1 are the eigenvalues of A.

Proof of BHR: In view of the Lemma, it suffices to show that

$$f(K,\ell) = \sum_{W \in \mathcal{L}} m_W \lambda_W^{\ell}$$

for $\ell = 1, 2, 3, \ldots$, where $m_W = |\mu_{\mathcal{L}}(\hat{0}, W)|$. By definition, we have

$$f(K,\ell) = \sum_{C \in \mathcal{C}_{\mathcal{A}}} \sum_{F_1 F_2 \cdots F_\ell C = C} w(F_1)\, w(F_2) \cdots w(F_\ell).$$

Also, by Las Vergnas - Zaslavsky we have

$$\#\{C \in \mathcal{C}_{\mathcal{A}} : FC = C\} = \sum_{\substack{\mathrm{aff}(F) \subseteq W \\ W \in \mathcal{L}}} m_W$$

for every $F \in \mathcal{F}_{\mathcal{A}}$. Hence

$$f(K,\ell) =$$

$$\sum_{C \in \mathcal{C}_{\mathcal{A}}} \sum_{\substack{F \in \mathcal{F}_{\mathcal{A}} \\ FC = C}} \sum_{F_1 \cdots F_\ell = F} \omega(F_1)\,\omega(F_2) \cdots \omega(F_\ell) =$$

$$\sum_{F \in \mathcal{F}_{\mathcal{A}}} \#\{C \in \mathcal{C}_{\mathcal{A}} : FC = C\} \sum_{F_1 F_2 \cdots F_\ell = F} \omega(F_1)\,\omega(F_2) \cdots \omega(F_\ell) =$$

$$\sum_{F \in \mathcal{F}_{\mathcal{A}}} \sum_{\substack{\mathrm{aff}(F) \subseteq W \\ W \in \mathcal{L}}} m_W \sum_{F_1 F_2 \cdots F_\ell = F} \omega(F_1)\,\omega(F_2) \cdots \omega(F_\ell) =$$

$$\sum_{W \in \mathcal{L}} m_W \sum_{\mathrm{aff}(F_1 \cdots F_\ell) \subseteq W} \omega(F_1)\,\omega(F_2) \cdots \omega(F_\ell) =$$

$$\sum_{W \in \mathcal{L}} m_W \sum_{F_1, \ldots, F_\ell \subseteq W} \omega(F_1)\,\omega(F_2) \cdots \omega(F_\ell) =$$

$$\sum_{W \in \mathcal{L}} m_W \left(\sum_{F \subseteq W} \omega(F) \right)^\ell = \sum_{W \in \mathcal{L}} m_W (\lambda_W)^\ell .$$

$\blacksquare$