Hypersphere Arrangement and Imaginary Cycles for Hypergeometric Integrals

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Abstract

We construct imaginary cycles for hypergeometric integrals associated with a hypersphere arrangement and discuss the relation between the twisted rational de Rham cohomology. We pose two geometric problems involved in it.

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Running title. Hypersphere Arrangement and Imaginary Cycles

1 Introduction

First we want to illustrate in one dimensional case the main objective discussed in this article. Let Q be the complex circle : $\{\xi = (\xi_1, \xi_2); \xi_1^2+\xi_2^2 = 1$ in the complex affine plane $\mathbf{C}^2$. Q is isomorphic to $\mathbf{C}^*$ by taking $\xi_1+\sqrt{-1}\xi_2 = \zeta$. Consider a family of m complex lines $H_j : f_j = 0$ where

$$f_j = u_{j,0} + u_{j,1}\xi_1 + u_{j,2}\xi_2$$

such that $u_{j,0}, u_{j,1}, u_{j,2} \in \mathbf{R}$ and that $u_{j,1}^2 + u_{j,2}^2 - u_{j,0}^2 = 1$. Denote

$$a_{i,j} = u_{i,1}u_{j,1} + u_{i,2}u_{j,2} - u_{i.0}u_{j,0},\ a_{i,0} = a_{0,i} = u_{i,0}$$

The intersection of Q and H_j consisits of 2 different points which we denote by ζ_j, ζ_j^* such that $|\zeta_j| = |\zeta_j^*| = 1$. Let R be the $\mathbf{C}[\xi_1, \xi_2]$ module

$$R = \sum_{\nu_1 \geq 0, \ldots, \nu_m \geq 0} \mathbf{C}[\xi_1, \xi_2] \prod_{j=1}^m f_j^{-\nu_j} = \sum_{\nu_1 \geq 0, \ldots, \nu_m \geq 0} \mathbf{C}[\zeta, \zeta^{-1}] \prod_{j=1}^m \{(\zeta - \zeta_j)(\zeta - \zeta_j^*)\}^{-\nu_j}$$

because f_j can be written as

$$f_j = \sqrt{-1} \frac{(\zeta - \zeta_j)(\zeta - \zeta_j^*)}{(\zeta_j^* - \zeta_j)\zeta}$$

Consider the multiplicative function

$$\Phi_0(\xi) = \prod_{j=1}^m f_j^{\lambda_j} \quad (\lambda_j \in \mathbf{R}_{>\mathbf{0}})$$

and the associated rational de Rham cohomology on $Y = Q - \cup_{j=1}^m H_j$

$$H^1(Y, \nabla_0) \cong R\tau_Q / \nabla_0(R)$$

defined by the covariant differential $\nabla_0(\psi) = d\psi + d \log \Phi_0 \psi$, where we denote

$$\tau_Q = -\xi_1 d\xi_2 + \xi_2 d\xi_1 = \sqrt{-1} \frac{d\zeta}{\zeta}$$

Suppose that $\zeta_1, \zeta_1^*, \ldots, \zeta_m, \zeta_m^*$ are different from each other. Then one can prove that for generic λ_j

$$H^1(Y, \nabla_0) \cong \mathbf{C}^{2m}$$

and it is spanned by

$$\varphi_Q(\emptyset) = \tau_Q, \varphi_Q(j) = \frac{\tau_Q}{f_j} = d \log \frac{\zeta - \zeta_j}{\zeta - \zeta_j^*} \, (1 \leq j \leq m),$$
$$\varphi_Q(j,k) = \frac{\tau_Q}{f_j f_k} \, (1 \leq j < k \leq m)$$

These 1 forms are not linearly independent on Y. For any different i, j, k there exists the fundamental linear relation

$$c_i \varphi_Q(i) + c_j \varphi_Q(j) + c_k \varphi_Q(k) + c_{j,k} \varphi_Q(j,k) + c_{k,i} \varphi_Q(k,i) + c_{i,j} \varphi_Q(i,j) = 0 \tag{1.1}$$

where $c_i, c_j, c_k, c_{j,k}, c_{k,i}, c_{i,j}$ can be written in terms of $a_{i,j}, a_{k,0}$ as

$$c_i = -\frac{A(0,i)}{A\begin{pmatrix} 0, & i, & k \\ 0, & i, & j \end{pmatrix}}, \quad c_{j,k} = \frac{A(j,k)}{A\begin{pmatrix} i, & j, & k \\ 0, & j, & k \end{pmatrix}}$$

$c_j, c_k, c_{k,i}, c_{i,j}$ being defined in the same way cyclically. Moreover $A(0,i) = -1 - a_{i,0}^2, A(j,k) = 1 - a_{j,k}^2$ and $A\begin{pmatrix} i, & j, & k \\ i', & j', & k' \end{pmatrix}$ denotes the determinant of the matrix whose components are $a_{p,q}\,(p = i,j,k; q = i',j',k')$.

The twisted homology $H_1(Y, \hat{\mathcal{L}}_0)$ dual to $H^1(Y, \nabla_0)$ is spanned by the linearly independent cycles which are expressed by the closures (arcs) of the connected components of $\Re Y$.

Suppose now that for a fixed pair i, j, one of ζ_i or ζ_i^* coincides with one of ζ_j or ζ_j^*. This occurs if and only if $A(i,j) = 0$ i.e.,$a_{i,j} = \pm 1$. If ζ_j tends to the point ζ_i^*, then the arc connecting the points ζ_i^*, ζ_j in $\Re Q$ reduces to the point. Hence if $\zeta_i^* = \zeta_j$, the dimension of $H_1(Y, \hat{\mathcal{L}}_0)$ decreses by one. On the other hand one can show that $\varphi_Q(i,j)$ can be described homologically as a linear combination of $\varphi_Q(k,i), \varphi_Q(k,j), \varphi_Q(k), \varphi_Q(\emptyset)$:

$$\begin{aligned} 2(\lambda_i + \lambda_j - 1)\varphi_Q(i,j) &\sim -\sum_{k \neq i,j} \lambda_k \left\{ \frac{A(k,i)}{a_{k,i} + a_{k,j}} \varphi_Q(k,i) \right. \\ &\left. + \frac{A(k,j)}{a_{k,i} + a_{k,j}} \varphi_Q(k,j) \right\} + \sum_{k=0}^{m} \lambda_k a_{k,0} \varphi_Q(k) - (\lambda_\infty - 1) \left\{ \frac{A(0,i)}{a_{i,0} + a_{j,0}} \varphi_Q(i) \right. \\ &\left. + \frac{A(0,j)}{a_{i,0} + a_{j,0}} \varphi_Q(j) \right\} - \lambda_\infty \varphi_Q(\emptyset) \end{aligned} \tag{1.2}$$

where we denote $\lambda_\infty = \sum_{j=1}^m \lambda_j$.

In particular consider the case where $m = 3$ and $A(1,2) = A(1,3) = A(2,3) = 0$, i.e., $\zeta_1^* = \zeta_2, \zeta_2^* = \zeta_3, \zeta_3^* = \zeta_1$. Then (1.1) reduces to the only one identity :

$$\varphi_Q(1) + \varphi_Q(2) + \varphi_Q(3) = 0$$

and there are three identities of type (1.2) :

$$\begin{aligned} 2(\lambda_i + \lambda_j - 1)\varphi_Q(i,j) &\sim \sum_{k=1}^{3} \lambda_k a_{k,0} \varphi_Q(k) \\ &+ (\lambda_\infty - 1) \left\{ \frac{1 + a_{i,0}^2}{a_{i,0} + a_{j,0}} \varphi_Q(i) + \frac{1 + a_{j,0}^2}{a_{i,0} + a_{j,0}} \varphi_Q(j) \right\} - \lambda_\infty \varphi_Q(\emptyset) \end{aligned}$$

Hence $H^1(Y, \nabla_0)$ is of dimension 3 and is spanned by a basis of representatives $\varphi_Q(\emptyset), \varphi_Q(1), \varphi_Q(2)$.

In this article we want to extend the above observation to $n \, (n \geq 1)$ dimensional cases. The twisted de Rham cohomology $H^n(Y, \nabla_0)$ can be formulated in the space Y, the complement of a union of $n-1$ dimensional hyperspheres in the n dimensional fundamental complex hypersphere Q. There arise two problems.

In the first place, based on my preceding articles (see [1],[4]), we formulate the twisted de Rham cohomology presenting explicitly its basis in terms of the invariants $a_{i,j}$ obtained by the Lorentz inner products of the coefficients of two linear functions defining hyperspheres, together with $a_{k,0}$ which are the constant terms of a linear function. The differential structures of the hypergeometric integrals are described by the invariants $a_{i,j}, a_{k,0}$ under Lorentz groups or orthogonal groups. In the final section we show how they should be modified in some degenerate cases.

In the second place the basis of the twisted homology $H_n(Y, \hat{\mathcal{L}}_0)$ cannot always be realized by real domains in $\Re Y$ except in special domain of parameters. We must construct some of them as imaginary cycles. We want to show that this can be done by deforming real cycles from a special domain of parameters involved where all the cycles can be realized by real domains (see Theorem 13).

2 Basic Properties

Let $\mathcal{A}$ be an arrangement of m hyperplanes H_j $(1 \leq j \leq m)$ defined over the real field of coefficients in the $n+1$ dimensional complex affine space $\mathbf{C}^{n+1}$. Each hyperplane can be described as

$$H_j : u_{j,0} + \sum_{\nu=1}^{n} u_{j,\nu}\xi_\nu = 0$$

for $\xi = (\xi_1, \ldots, \xi_{n+1}) \in \mathbf{C}^{n+1}$

We denote by $N(\mathcal{A})$ the union of hyperplanes : $= \bigcup_{H_j \in \mathcal{A}} H_j$ and by $X = M(\mathcal{A})$ the compliment of $N(\mathcal{A}) := \mathbf{C}^{n+1} - N(\mathcal{A})$. Let Q be the complex hypersphere : $f_0 = 0$ defined by the quadratic polynomial$f_0 = 1 - \sum_{\nu=1}^{n+1} \xi_\nu^2$. Each intersection $H_j \cap Q$ defines a hypersphere in Q provided it does not reduce to a point i.e., $-u_{j,0}^2 + \sum_{\nu=1}^{n+1} u_{j,\nu}^2 \neq 0$. Throughout this article we shall assume this condition and so may assume that

$$-u_{j,0}^2 + \sum_{\nu=1}^{n+1} u_{j,\nu}^2 = 1$$

The family $\mathcal{A}' = \{H_j \cap Q\}_{1\le j\le m}$ defines a hypersphere arrangement in Q. We denote by Y the intersection of X and $Q : Y = Q - \cup_{j=1}^m H_j \cap Q$. We denote by $\Re Q$ the real part of Q which is identified with the n dimensional real hypersphere. The real part $S_j = H_j \cap \Re Q$ is a real $(n-1)$ dimensional hypersphere in $\Re Q$.

We define the $(m+1)\times(m+1)$ configuration matrix $A = (a_{i,j})_{0\le i,j\le m}$ associated with $\mathcal{A}'$, whose components are Lorentz inner products

$$a_{i,j} = -u_{i0}u_{j0} + \sum_{\nu=1}^{n+1} u_{i,\nu}u_{j,\nu}\ (1 \le i,j \le m);\ a_{i,0} = a_{0,i} = u_{i0};\ a_{0,0} = -1$$

so that $a_{i,i} = 1$ for $1 \le i \le m$.

For a set of indices $I = \{i_1,\ldots,i_p\} \subset \{0,1,2,\ldots,m\}$ the size p will be denoted by $|I|$. We say that I is admissible if $I \subset \{1,2,\ldots,m\}$. For two sets of indices $I = \{i_1,i_2,\ldots,i_p\}$ and $J = \{j_1,\ldots,j_p\}$ we define the subdeterminant

$$A\begin{pmatrix} I \\ J \end{pmatrix} = \begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \ldots & a_{i_1,j_p} \\ a_{i_2,j_1} & a_{i_2,j_2} & \ldots & a_{i_1,j_p} \\ \vdots & \vdots & & \vdots \\ a_{i_p,j_1} & a_{i_p,j_2} & \ldots & a_{i_p,j_p} \end{vmatrix}$$

We abbreviate $A\begin{pmatrix} I \\ I \end{pmatrix}$ by $A(I)$.

For an admissible set $I = \{i_1,i_2,\ldots,i_p\}$ and a set $J = \{j_1,j_2,\ldots,j_p\} \subset \{0,1,2,\ldots,n+1\}$ $(p \le n+2)$ we denote the subdeterminant

$$U\begin{pmatrix} i_1,i_2,\ldots,i_p \\ j_1,\ldots,j_p \end{pmatrix} = \begin{vmatrix} u_{i_1,j_1} & u_{i_1,j_2} & \ldots & u_{i_1,j_p} \\ u_{i_2,j_1} & u_{i_2,j_2} & \ldots & u_{i_2,j_p} \\ \vdots & \vdots & & \vdots \\ u_{i_p,j_1} & u_{i_p,j_2} & \ldots & u_{i_p,j_p} \end{vmatrix}$$

Remark The matrix A has at most rank $n+2$. Assume that I is admissible. Then $A(I) = 0$ for $|I| \geq n+3$, and

$$A(I) = -U^2 \begin{pmatrix} i_1, i_2, \dots, i_{n+2} \\ 0, 1, \dots, n+1 \end{pmatrix}$$

for $|I| = n+2$. On the other hand $A(0, I) = 0$ for $|I| \geq n+2$ and

$$A(0, I) = -U^2 \begin{pmatrix} i_1, i_2, \dots, i_{n+1} \\ 1, \dots, n+1 \end{pmatrix}$$

for $|I| = n+1$.

Lemma 1 *Fix an admissible set* I *and consider the intersection subspace* $V = \cap_{j \in I} H_j$ *in* $\mathbf{C}^{n+1}$.

(i) *In case where* $|I| \leq n$, $A(I) = 0$ *if and only if* V *has contact with* Q *at one point.*

(ii) *In case where* $|I| = n+1$, $A(I) = 0$ *if and only if* V *has a common point with* Q.

(iii) *In case where* $|I| = n+2$, $A(I) = 0$ *if and only if* V *is not empty.*

Assume that

($\mathcal{H}1$) : $A(0, I) < 0$ for an arbitrary admissible set I such that $|I| \leq n+1$, i.e., the homogeneous parts of $|I|$ linear functions f_j $(j \in I)$ are linearly independent.

Assume further that

($\mathcal{H}2$) : $A(I) \neq 0$ for an arbitrary admissible I $(2 \leq |I| \leq n+2)$.

Then for any $I \subset \{1, 2, \dots, m\}$ with $|I| = n+2$ the $(n+2) \times (n+2)$ symmetric submatrix $(a_{i,j})_{i,j \in I}$ has the signature of $n+1$ (+)sign and one (−)sign so that $A(I) < 0$ for $|I| = n+2$. This is equivalent to say that for any sequence of increasing admissible sets of indices

$$I_1 \subset I_2 \subset \cdots \subset I_{n+1} \subset I_{n+2}$$

such that $|I_r| = r$, the signs of $A(I_r)A(I_{r+1})$ $(1 \leq r \leq n+1)$ are positive except for one. Hence $A(I) < 0$ implies $A(J) < 0$ if $I \subset J$, $|J| \leq n+2$ (see [8]). In particular the following two cases are interesting :

($\mathcal{H}$2a): $A(I) > 0$ for all admissible I with $2 \leq |I| \leq n+1$ and $A(I) < 0$ for all admissible I with $|I| = n+2$.

($\mathcal{H}$2b): $A(I) < 0$ for all admissible I with $2 \leq |I| \leq n+2$.

Let τ be the $n+1$ form $d\xi_1 \wedge \cdots \wedge d\xi_{n+1.}$ on $\mathbf{C}^{n+1}$. We denote the n form $-\tau_Q$ on $\mathbf{C}^{n+1}$ such that its restriction to Q is the standard volume form on $\Re Q$:

$$-\tau_Q = \sum_{\nu=1}^{n+1} (-1)^{\nu-1} \xi_\nu d\xi_1 \wedge \cdots \wedge d\xi_{\nu-1} \wedge d\xi_{\nu+1} \wedge \cdots \wedge d\xi_{n+1}$$

such that $df_0 \wedge \tau_Q \equiv \tau \mod (f_0)$. We consider the multiplicative function on X

$$\Phi_0(\xi) = \prod_{j=1}^{m} f_j(\xi)^{\lambda_j}$$

where we assume that every $\lambda_j \in \mathbf{R}$ is positive and generic. We denote by $H^r(X - Y, \nabla_0)$ and $H^r(Y, \nabla_0)$ the r dimensional twisted rational de Rham cohomologies on $X - Y$ and Y associated with the covariant differentiation ∇_0 respectively :

$$\nabla_0(\psi) = d\psi + d \log \Phi_0 \wedge \psi$$

These cohomologies are defined in a standard way by using $\mathbf{C}[\xi_1, \ldots, \xi_{n+1}]$-module

$$R = \sum_{\nu_1 \geq 0, \ldots, \nu_m \geq 0} \mathbf{C}[\xi_1, \ldots, \xi_{n+1}] \prod_{k=1}^{m} f_k(\xi)^{-\nu_k}$$

$\mathcal{L}_0$ be the local systems on $X - Y$ and Y defined by $\Phi_0(\xi)$ respectively, and $\hat{\mathcal{L}}_0$ be their duals defined by $\Phi_0(\xi)^{-1}$. Then the $n+1$ and n dimensional homologies $H_{n+1}(X - Y, \hat{\mathcal{L}}_0)$ and $H_n(Y, \hat{\mathcal{L}}_0)$ represented by twisted cycles are dual to the twisted rational de Rham cohomologies $H^{n+1}(X - Y, \nabla_0)$ and $H^n(Y, \nabla_0)$ thorough the pairs of integrals respectively

$$H^{n+1}(X - Y, \nabla_0) \times H_{n+1}(X - Y, \hat{\mathcal{L}}_0) \ni (\varphi, \mathfrak{c}) \longrightarrow \langle \varphi, \mathfrak{c} \rangle = \int_{\mathfrak{c}} \Phi \varphi \tag{2.1}$$

$$H^n(Y, \nabla_0) \times H_n(Y, \hat{\mathcal{L}}_0) \ni (\varphi, \mathfrak{c}) \longrightarrow \langle \varphi, \mathfrak{c} \rangle = \int_{\mathfrak{c}} \Phi_0 \varphi \tag{2.2}$$

The following two Propositions have been proved in [2] and [3] (see Proposition 3.2_p, 3.3_p and Lemma 4.2 in [2]I, and also [5]).

Proposition 2 *Under the conditions* $(\mathcal{H}1), (\mathcal{H}2)$ *we have the isomorphism*

$$H^n(Y, \nabla_0) \cong \mathbf{C}^{\kappa_n}$$

where $\kappa_n = \sum_{\nu=0}^{n} \binom{m}{\nu} + \binom{m-1}{n}$. $H^n(Y, \nabla_0)$ *has a basis represented by the differential* n *forms*

$$\varphi_Q(I) = \frac{\tau_Q}{f_{i_1} \dots f_{i_p}}$$

where I *moves over the admissible sets* I *of indices such that* $0 \le |I| \le n+1$. *We denote* $\varphi_Q(\emptyset)$ *for* $|I| = 0$. *There exist the fundamental relations among them of the following type. For an arbitrary admissible set of indices* J *with* $|J| = n+2$ *there exists the identity :*

$$\frac{1}{2}\sum_{\mu \neq \nu}(-1)^{\mu+\nu}\varphi_Q(\partial_\mu\partial_\nu J)\frac{A(0, \partial_\mu\partial_\nu J)}{A\begin{pmatrix} 0, \partial_\mu J \\ 0, \partial_\nu J \end{pmatrix}} + \sum_{\mu=1}^{n+2}(-1)^{\mu-1}\varphi_Q(\partial_\mu J)\frac{A(\partial_\mu J)}{A\begin{pmatrix} 0, \partial_\mu J \\ J \end{pmatrix}} = 0 \tag{2.3}$$

where $\partial_\mu J$ *denotes the subset of* J *deleted by the* μ*th index* j_μ. *Further for* $|I| = n+2$ *a partial fraction gives*

$$U\begin{pmatrix} I \\ 0, 1, \dots, n+1 \end{pmatrix}\varphi_Q(I) = \sum_{\mu=1}^{n+2}(-1)^{\mu-1}U\begin{pmatrix} \partial_\mu I \\ 1, \dots, n+1 \end{pmatrix}\varphi_Q(\partial_\mu I) \tag{2.4}$$

We denote by $\mathcal{B}$ a linear space spanned by the representatives $\varphi_Q(I), 0 \le |I| \le n+1$.

Proposition 3 *Under the condition* $(\mathcal{H}1), (\mathcal{H}2a)$ $H_n(Y, \hat{\mathcal{L}}_0)$ *has a basis represented by the closures of all the connected components of* $\Re Y = \Re Q \cap Y$. *Their number is equal to* κ_n. *In other words,* $H_n(Y, \hat{\mathcal{L}}_0)$ *is spanned by only real twisted cycles defined by connected components of* $\Re Y$.

For example we have $\kappa_1 = 2m$, $\kappa_2 = m^2 - m + 2$, $\kappa_3 = \frac{1}{3}m^3 - m^2 + \frac{8}{3}m$.

Remark The number κ_n is also equal to the number of non-compact connected components of $\mathbf{R}^{n+1} - N(\mathcal{A})$.

3 Twisted imaginary cycles

We may assume without losing generality

$$u_{j,0} \leq 0 \quad \text{for all } j, 1 \leq j \leq m \tag{3.1}$$

Define the set

$$S_{j,+} : \{\xi \in \Re Q; f_j(\xi) > 0\}$$

as the inside of the real hypersphere $S_j = \Re Q \cap H_j$. We denote by $\boldsymbol{\nu}_j$ the unit normal of $\Re H_j$:

$$\boldsymbol{\nu}_j = \frac{(u_{j,1}, u_{j,2}, \ldots, u_{j,n+1})}{\sqrt{\sum_{\nu=1}^{n+1} u_{\mathrm{j},\nu}^2}} \tag{3.2}$$

Remark that $\boldsymbol{\nu}_j \in S_{j,+}$.

First notice the following:

Lemma 4 *Suppose I is admissible. The real affine subspace $\bigcap_{j\in I} \Re H_j$ is disjoint with $\Re Q$ if and only if $A(I) < 0$.*

Proof. In fact the square of the distance between the subspace $\bigcap_{j\in I} \Re H_j$ and the origin is equal to $\{A(I) + A(0, I)\}/A(0, I)$. It is bigger than 1 if and only if $A(I) < 0$ because $A(0, I) < 0$.

Corollary 5 *Suppose that $A(i, j) < 0$, i.e., $a_{i,j}^2 > 1$ for every pair $i, j \in \{1, 2, \ldots, m\}, i \neq j$ then every S_j is disjoint with each other. In this case, $S_{i,+}, S_{j,+}$ are disjoint, or the one is included in the other, according as $a_{i,j} < -1$ or $a_{i,j} > 1$.*

Proposition 6 *Under the condition $(\mathcal{H}2)$ consider an admissible set I such that $2 \leq |I| \leq n + 1$. Suppose further $A(I)$ is a positive number near 0 and that $A(J) < 0$ for any admissible $J \supset I$, $|J| > |I|$. Then the compact domain*

$$\mathfrak{l}(I) := \{\xi \in \Re Q; f_j(\xi) \geq 0 \quad (j \in I)\}$$

gives a twisted real cycle representing an element $H_n(Y, \hat{\mathcal{L}}_0)$. This cycle vanishes if it is deformed in an isotopic way by the matrix A as $A(I)$ tend to 0, any other $A(K)$ being never equal to 0.

Proof. Since $A(I) > 0$ and near 0, $\mathfrak{l}(I)$ is one of the compact components of $\Re Y$. This reduces to a point for $A(I) \to 0$ as is seen from Lemma 4.

Definition 7 The cycle $\mathfrak{l}(I)$ mentioned in Proposition 6 is called the twisted vanishing cycle (Lefschetz cycle) at the singularity $A(I) = 0$.

Assume now the conditions $(\mathcal{H}2b)$ together with $(\mathcal{H}1)$. Then each $n-1$ dimensional hypersphere $\Re Q \cap H_j$ is disjoint with each other. This means that $\Re Y$ has only $m+1$ connected components which make only a part of the basis of $H_n(Y, \hat{\mathcal{L}}_0)$. We want to construct a basis of $H_n(Y, \hat{\mathcal{L}}_0)$ represented by imaginary cycles in addition to real ones.

Definition 8 We start from an admissible $I = \{i, j\}, |I| = 2$. By hypothesis we have $A(I) < 0$ i.e., $a_{i,j} < -1$. By an orthogonal transformation we may choose the new coordinates $\xi = (\xi_1, \dots, \xi_{n+1})$ such that $\boldsymbol{\nu}_i$ coincides with the positive ξ_1-axis and $\boldsymbol{\nu}_j$ lies in the (ξ_1, ξ_2)-plane, i.e.,

$$f_i(\xi) = f_i(\xi_1) = u_{i,0} + u_{i,1}\xi_1, \quad (u_{i,1} > 0)$$
$$f_j(\xi) = f_j(\xi_1, \xi_2) = u_{j,0} + u_{j,1}\xi_1 + u_{j,2}\xi_2 \quad (u_{j,2} > 0)$$

The set of all points $\xi = (\xi_1, \xi_2, \sqrt{-1}\xi_3^*, \dots, \sqrt{-1}\xi_{n+1}^*) \in (\mathbf{R}^2 \times (\sqrt{-1}\mathbf{R})^{n-1}) \cap Q$ which is a piece of an ultra hyperboloid

$$\xi_1^2 + \xi_2^2 - \xi_3^{*2} - \cdots - {\xi_{n+1}^*}^2 = 1; f_i(\xi_1) \le 0,\ f_j(\xi_1, \xi_2) \le 0\}$$

is denoted by $\mathfrak{l}^*(I)$. More generally let I be an admissible set such that $|I| = p, 2 \le p \le n$. We have $A(I) < 0$. We may assume without losing generality $I = \{1, 2, \dots, p\}$ and choose the new coordinates $\xi = (\xi_1, \dots, \xi_{n+1})$ of $\mathbf{R}^{n+1}$ such that $\boldsymbol{\nu}_1$ coincides with the positive ξ_1-axis and that $\boldsymbol{\nu}_r$ lies in the r dimensional $(\xi_1, \dots, \xi_r)$-subspace $(1 \le r \le p)$ such that

$$f_r(\xi) = f_r(\xi_1, \dots, \xi_r) = u_{r,0} + \sum_{\nu=1}^{r} u_{r,\nu}\xi_\nu,\ (u_{r,r} > 0) \quad (1 \le r \le p) \tag{3.3}$$

We denote by $\Delta^*(I)$ the set of all points $\xi = (\xi_1, \dots, \xi_p, \sqrt{-1}\xi_{p+1}^*, \dots, \sqrt{-1}\xi_{n+1}^*)$ $\in (\mathbf{R}^p \times (\sqrt{-1}\mathbf{R})^{n+1-p}) \cap Q$ which is a piece of an ultra hyperboloid

$$\xi_1^2 + \cdots + \xi_p^2 - {\xi_{p+1}^*}^2 - \cdots - {\xi_{n+1}^*}^2 = 1 \tag{3.4}$$
$$f_r(\xi_1, \dots, \xi_r) \le 0 \quad (1 \le r \le p) \tag{3.5}$$

and by $\partial_{p,+}\Delta^*(I)$ the set of all points $\xi = (\xi_1, \ldots, \xi_{p-1}, \sqrt{-1}\xi_p^*, \ldots, \sqrt{-1}\xi_{n+1}^*)$ $\in (\mathbf{R}^{p-1} \times (\sqrt{-1}\mathbf{R})^{n+2-p}) \cap Q$ which is a piece of an ultra hyperboloid :

$$\xi_1^2 + \cdots + \xi_{p-1}^2 - \xi_p^{*2} - \cdots - {\xi_{n+1}^*}^2 = 1 \tag{3.6}$$

$$f_r(\xi_1, \ldots, \xi_r) \leq 0 \quad (1 \leq r \leq p-1), \xi_p^* \geq 0 \tag{3.7}$$

Remark that

$$\Delta^*(I) \cap \partial_{p,+}\Delta^*(I) = \{\xi_p^* = 0\} \cap \partial_{p,+}\Delta^*(I) = \Delta^*(I) \cap \Delta^*(\partial_p I)$$

One can define similarly the chains $\partial_{j,+}\Delta^*(I)$ for $1 \leq j \leq p-1$ by exchange of coordinates and can have the identities

$$\Delta^*(I) \cap \partial_{j,+}\Delta^*(I) = \Delta^*(I) \cap \Delta^*(\partial_j I)$$

Then the n-chain

$$\mathfrak{l}^*(I) = \Delta^*(I) + \sum_{j=1}^{p} (-1)^{j-1} \partial_{j,+}\Delta^*(I) \tag{3.8}$$

defines an n-cycle in $H_n(Y, \hat{L}_0)$. In fact

$$\partial\mathfrak{l}^*(I) = \sum_{k=1}^{p} (-1)^{k-1} \Delta^*(\partial_k I) \cap \Delta^*(I) - \sum_{j=1}^{p} (-1)^{j-1} \Delta^*(\partial_j I) \cap \Delta^*(I)$$
$$+ \sum_{k=1, k \neq j}^{p} (-1)^{k-1} \sum_{j=1}^{p} (-1)^{j-1} \partial_{k,+}\partial_{j,+}\Delta^*(I) = 0$$

since $\partial_{k,+}\partial_{j,+} + \partial_{j,+}\partial_{k,+} = 0$.

Note that $\mathfrak{l}^*(I)$ coincides with $\Delta^*(I)$ if $|I| = 2$. $\mathfrak{l}^*(I)$ is called twisted Lefschetz cycle associated with I. If $A(I)$ is near 0, this is a deformation of $\mathfrak{l}(I)$ as the matrix A moves from the part $A(I) > 0$ to the one $A(I) < 0$ being detoured from the singularity $A(I) = 0$.

The following lemma immediately follows from the above Definition.

Lemma 9 *The number of Lefschetz cycles is equal to* $\sum_{\nu=2}^{m} \binom{m}{\nu}$.

Now we construct the $\binom{m-1}{n}$ remaining imaginary cycles. Since this number vanishes unless $m-1 \geq n$, we may assume $m \geq n+1$.

First consider the case where $n=1$, $m \geq 2$. By hypothesis every $a_{i,j}^2 > 1$. The $2m$ points $\cup_{j=1}^m \Re Q \cap H_j$ are different from each other. Therefore $\Re Y$ consists of $2m$ connected components which make a basis of $H_1(Y, \hat{\mathcal{L}}_0)$. None of imaginary cycles occur. Next consider the case where $n=2$, $m \geq 3$. Suppose I is admissible with $|I|=3$. Let $\Delta(I)$ be the geodesic triangle in $\Re Q$ with the vertices $\boldsymbol{\nu}_i, \boldsymbol{\nu}_j, \boldsymbol{\nu}_k$. Define the chain $\Delta^*(I)$ as

$$\Delta^*(I) = \overline{\Delta(I) - S_{i,+} - S_{j,+} - S_{k,+}}$$

where the overline $\overline{\Delta - \cdots}$ denotes the closure.

To each geodesic $\widehat{\boldsymbol{\nu}_i, \boldsymbol{\nu}_j}$ going through $\boldsymbol{\nu}_i, \boldsymbol{\nu}_j$, by an orthogonal transformation there exist the new coordinates ξ_1, ξ_2, ξ_3 such that $\boldsymbol{\nu}_i$ coincides with the ξ_1-axis, the geodesic lies in the ξ_1, ξ_2-plane and the inner normal of the geodesic in $\Delta(I)$ coincides with the positive ξ_3-axis. Then the Lefschetz cycle $\mathfrak{l}^*(\{i,j\})$ is defined as the chain

$$\{\xi = (\xi_1, \xi_2, \sqrt{-1}\xi_3^*) \in Q \cap (\mathbf{R}^2 \times \sqrt{-1}\mathbf{R}), f_i(\xi_1) \leq 0, f_j(\xi_1, \xi_2) \leq 0\}$$

The 2 dimensional cell

$$\begin{aligned}&\partial_{k,+}\Delta^*(\{i,j,k\}) = \{\xi = (\xi_1, \xi_2, \sqrt{-1}\xi_3^*) \in Q \cap (\mathbf{R}^2 \times \sqrt{-1}\mathbf{R}),\\ &\xi_3^* \geq 0,\ f_i(\xi_1) \leq 0, f_j(\xi_1, \xi_2) \leq 0\}\end{aligned}$$

has the intersection $\overline{\Delta(\{i,j\}) - S_{i,+} - S_{j,+}}$ with $\Delta^*(I)$. $\partial_{i,+}\Delta^*(\{i,j,k\})$, $\partial_{j,+}\Delta^*(\{i,j,k\})$ can be constructed similarly by exchange of coordinates. We can choose the unique orientations of $\partial_{i,+}\Delta^*(\{i,j,k\}), \partial_{j,+}\Delta^*(\{i,j,k\})$, $\partial_{k,+}\Delta^*(\{i,j,k\})$ such that the boundaries satisfy

$$\begin{aligned}&\partial\{\Delta^*(I) + \partial_{i,+}\Delta^*(\{i,j,k\}) - \partial_{j,+}\Delta^*(\{i,j,k\}) + \partial_{k,+}\Delta^*(\{i,j,k\})\}\\ &\equiv 0 \mod S_i \cup S_j \cup S_k\end{aligned}$$

Hence we have constructed the new 2 dimensional cycle

$$\mathfrak{a}^*(\Delta) = \Delta^*(I) + \partial_{i,+}\Delta^*(\{i,j,k\}) - \partial_{j,+}\Delta^*(\{i,j,k\}) + \partial_{k,+}\Delta^*(\{i,j,k\})$$

The above construction can be extended to higher dimensional cases by induction on dimensions as follows.

Consider now the case where $n \geq 3$. I be an admissible set such that $|I| = n+1$. We may assume for simplicity that $I = \{1, 2, \ldots, n+1\}$. We have the n dimensional geodesic simplex $\Delta(I)$ with the vertices $\boldsymbol{\nu}_j$ $(1 \leq j \leq n+1)$. in $\Re Q$ such that its edges are geodesic segments and its higher dimensional faces are all totally geodesic. Define the subdomain $\Delta^*(I)$ as

$$\Delta^*(I) = \overline{\Delta(I) - \sum_{j=1}^{n+1} S_{j,+}}$$

Remark that

$$\partial\Delta^*(I) = \sum_{j=1}^{n+1} (-1)^{j-1} \{\Delta(\partial_j I) - \bigcup_{k \neq j} S_{k,+}\}$$

For each j, $(1 \leq j \leq n+1)$, by an orthogonal transformation there exists the new coordinates system $\xi = (\xi_1, \ldots, \xi_{n+1})$ such that the j th face $\Delta(\partial_j I)$ spanned by the normals $\boldsymbol{\nu}_k$, $(k \neq j)$ included in the $\xi_1, \ldots, \xi_n$-coordinate subspace and that its inner normal to $\Delta(\partial_j I)$ coincides with the positive ξ_{n+1}-axis. Suppose for simplicity that $j = n+1$ and the subspace spanned by $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_r$ coincides with the $\xi_1, \ldots, \xi_r$-subspace, i.e.,

$$f_r(\xi) = u_{r,0} + \sum_{k=1}^{r} u_{r,k}\xi_k, \ (u_{r,r} > 0) \quad (1 \leq r \leq n) \tag{3.9}$$

The n dimensional cell

$$\partial_{n+1,+}\Delta^*(I) = \{\xi = (\xi_1, \ldots, \xi_n, \sqrt{-1}\xi^*_{n+1}) \in Q \cap (\mathbf{R}^n \times \sqrt{-1}\mathbf{R}), \xi^*_{n+1} \geq 0,$$
$$f_r(\xi_1, \ldots, \xi_r) \leq 0 \, (1 \leq r \leq n)\} \tag{3.10}$$

whose intersection with $\Delta^*(I)$ coincides with $\overline{\Delta(\partial_{n+1} I) - \bigcup_{k=1}^{n} S_{k,+}}$. Similarly by exchange of coordinates we can construct the cell $\partial_{j,+}\Delta^*(I))$ $(1 \leq j \leq n)$ whose intersection with $\Delta^*(I)$ coincides with $\overline{\Delta(\partial_j I) - \bigcup_{k \neq j} S_{k,+}}$. Hence there exist a suitable orientation for each $\partial_{j,+}\Delta^*(I)$ such that the boundary vanishes:

$$\partial\{\Delta^*(I) + \sum_{j=1}^{n+1} (-1)^{j-1} \partial_{j,+}\Delta^*(I)\} \equiv 0 \mod \bigcup_{j=1}^{n+1} S_j$$

In the same way as above one can prove that the following chain

$$\mathfrak{a}^*(I) = \Delta^*(I) + \sum_{j=1}^{n+1} (-1)^{j-1} \partial_{j,+} \Delta^*(I) \in H_n(Y, \hat{\mathcal{L}}_0)$$

is an n dimensional cycle.

Definition 10 The $\mathfrak{a}^*(I)$ ($|I| = n+1$) will be called <u>twisted adjacent cycle</u> associated with I.

All the cycles thus constructed are not necessarily linearly independent in $H_n(Y, \hat{\mathcal{L}}_0)$. In fact we have

Lemma 11 *For an admissible I with $|I| = n+2$, the following identity holds:*

$$\sum_{j=1}^{n+2} (-1)^{j-1} \mathfrak{a}^*(\partial_j I) = 0 \tag{3.11}$$

Proof. We may assume that $I = \{1, 2, \ldots, n+2\}$. Then

$$\mathfrak{a}^*(\partial_j I) = \Delta^*(\partial_j I) + \sum_{k=1}^{j-1} (-1)^{k-1} \partial_{k,+} \Delta^*(\partial_j I) + \sum_{k=j+1}^{n+2} (-1)^{k} \partial_{k,+} \Delta^*(\partial_j I)$$

On the other hand by definition for $j \neq k$

$$\partial_{j,+} \Delta^*(\partial_k I) = \partial_{k,+} \Delta^*(\partial_j I)$$

also

$$\sum_{j=1}^{n+2} (-1)^{j-1} \Delta^*(\partial_j I) = 0$$

Hence the LHS of (3.11) equals

$$\sum_{j=1}^{n+2} (-1)^{j-1} \{ \sum_{k=1}^{j-1} (-1)^{k-1} \partial_{k,+} \Delta^*(\partial_j I) + \sum_{k=j+1}^{n+2} (-1)^{k} \partial_{k,+} \Delta^*(\partial_j I) \} = 0$$

which implies Lemma 11. Q.E.D.

As an immediate consequence we have

Corollary 12 *Among* $\mathfrak{a}^*(I)$ $(|I| = n+1)$*, there exist* $\binom{m-1}{n}$ *linearly independent ones, say* $\mathfrak{a}^*(I), 1 \in I$ *such that all the others are linear combination of the latter.*

Summing up the above we have proved the following :

Theorem 13 *As a basis of* $H_n(Y, \hat{\mathcal{L}}_0)$*, one can choose the representatives of twisted cycles of the following kinds:*

(i) *Real cycles. This can be realized by the real chambers which are the closures of the connected components of* $\Re Y$*. Their numbers are* $1 + m$*.*

(ii) *Imaginary Lefschetz cycles* $\mathfrak{l}^*(I)$*. Their numbers are equal to* $\sum_{\nu=2}^{n} \binom{m}{\nu}$*.*

(iii) *Adjacent cycles* $\mathfrak{a}^*(I)$ *such that* $1 \in I$*. Their numbers are equal to* $\binom{m-1}{n}$*.*

4 Stereographic projection

The cycles defined in the previous section can also be described in the n dimensional Euclidean space as below. The compliment of the south pole, $Q - \{(-1, 0, \ldots, 0)\}$, is isomorphic to $\mathbf{R}^n$ through the stereopgraphic projection

$$\eta_1 = \frac{\xi_2}{1+\xi_1}, \ldots, \eta_n = \frac{\xi_{n+1}}{1+\xi_1} \tag{4.1}$$

which is a conformal transformation. Then a hypersphere S in $\Re Q$ corresponds to a hypersphere or a hyperplane $\tilde{S}$ in $\mathbf{R}^n$:

$$\sum_{\nu=1}^{n} (\eta_\nu - v_\nu)^2 = r^2 \quad (r > 0)$$

where a hyperplane can be regarded as a limiting case for $r = \infty$. Denote the center of $\tilde{S}$ by $\mathbf{v} = (v_1, \ldots, v_n)$ and its length by $||\mathbf{v}|| = \sqrt{\sum_{\nu=1}^{n} v_\nu^2}$. Then we have

$$S : u_0 + \sum_{\nu=0}^{n+1} u_\nu \xi_\nu = 0, \ (u_0 \leq 0)$$

$$u_0 = \frac{r^2 - 1 - ||\mathbf{v}||^2}{2r}, u_1 = \frac{r^2 + 1 - ||\mathbf{v}||^2}{2r}, u_{\nu+1} = \frac{v_\nu}{r} \ (1 \leq \nu \leq n)$$

or

$$-u_0 = \frac{r^2 - 1 - ||\mathbf{v}||^2}{2r}, -u_1 = \frac{r^2 + 1 - ||\mathbf{v}||^2}{2r}, -u_{\nu+1} = \frac{v_\nu}{r} \ (1 \le \nu \le n)$$

according as $r^2 - 1 - ||\mathbf{v}||^2 \le 0$ or > 0, namely

Lemma 14 *S_+ corresponds to the inside or the outside of $\tilde{S}$ according as $r^2 - 1 - ||\mathbf{v}||^2 < 0$ or > 0.*

As for $a_{i,j}$

$$a_{i,j} = \frac{r_i^2 + r_j^2 - ||\mathbf{v}^{(i)} - \mathbf{v}^{(j)}||^2}{2r_i r_j}$$

where $r_i, r_j, \mathbf{v}^{(i)}, \mathbf{v}^{(j)}$ denote the radii and the centers of $\tilde{S}_i, \tilde{S}_j$ respectively. Hence

Lemma 15 *We have*

$$A(i,j) = \frac{(r_i - r_j + a)(-r_i + r_j + a)(r_i + r_j + a)(r_i + r_j - a)}{4r_i^2 r_j^2}$$

where we put $a = ||\mathbf{v}^{(i)} - \mathbf{v}^{(j)}||$. This implies $a_{i,j} > 1$ if and only if $|r_i - r_j| > a$. $a_{i,j} < -1$ if and only if $r_i + r_j < a$.

In the same way

Lemma 16 *Suppose that $|a_{i,j}| < 1$ for an admissible $I = \{i,j,k\}$ and put $-\cos\alpha_{i,j} = a_{i,j}$ such that $0 < \alpha_{i,j} < \pi$. Then*

$$A(i,j,k) = -4\cos\frac{\alpha_{i,j} + \alpha_{j,k} + \alpha_{i,k}}{2} \cdot \cos\frac{-\alpha_{i,j} + \alpha_{j,k} + \alpha_{i,k}}{2} \cdot \cos\frac{\alpha_{i,j} - \alpha_{j,k} + \alpha_{i,k}}{2} \cdot \cos\frac{\alpha_{i,j} + \alpha_{j,k} - \alpha_{i,k}}{2}$$

The three hyperspheres $\tilde{S}_i, \tilde{S}_j, \tilde{S}_k$ intersect each other. $\pi - \alpha_{i,j}$ is equal to the angle subtended by the tangents of $\tilde{S}_i, \tilde{S}_j$ at an intersection point of $\tilde{S}_i \cap \tilde{S}_j$. $A(1,2,3) = 0$ if and only if $\alpha_{i,j} + \alpha_{j,k} + \alpha_{i,k} = \pi$, or $-\alpha_{i,j} + \alpha_{j,k} + \alpha_{i,k} = \pi$, or $\alpha_{i,j} - \alpha_{j,k} + \alpha_{i,k} = \pi$, or $\alpha_{i,j} + \alpha_{j,k} - \alpha_{i,k} = \pi$.

Lemma 17 *For an arbitrary admissible* $I, |I| \leq n+1$ *there exist the new coordinates* $\eta = (\eta_1, \dots, \eta_n)$ *of* $\mathbf{R}^n$ *such that* $\mathbf{v}^{(i_1)} = 0$ *and* $\mathbf{v}^{(i_{r+1})}$ *lies in the* $\eta_1, \dots, \eta_r$*-subspace* $2 \leq r \leq |I| - 1$.

Proof. We may assume that $I = \{1, 2, \dots, p\}$. By the change of coordinates (4.1), there exist the coordinates $\xi_1, \dots, \xi_{n+1}$ such that $\boldsymbol{\nu}_{r+1}$ lies in the $\xi_1, \dots, \xi_{r+1}$-subspace $(1 \leq r \leq p)$. Since $u_{1,\nu} = 0$ for $\nu \geq 2$, $\mathbf{v}^{(1)} = 0$. And $u_{r,\nu} = 0$ for $\nu \geq r+1$, $\mathbf{v}^{(r+1)}$ lies in the $\eta_1, \dots, \eta_r$-subspace.

The cycles equivalent to the one constructed in section 3 are described as follows:

Consider the case where $m = 2$. Let O_1, O_2 be the centers of $\tilde{S}_1, \tilde{S}_2$ and the insides of $\tilde{S}_1, \tilde{S}_2$ be denoted by $\tilde{S}_{1,+}\tilde{S}_{2,+}$ respectively. Suppose first that $|a_{1,2}| < 1$. Then $\tilde{S}_{1,+} \cap \tilde{S}_{2,+}$ is a non-empty domain so that $\mathbf{R}^n - \tilde{S}_1 \cup \tilde{S}_2$ consists of 4 connected components:

$$\mathbf{R}^n - \tilde{S}_{1+} \cup \tilde{S}_{2,+}, \tilde{S}_{1,+} - \tilde{S}_{2,+}, \tilde{S}_{2,+} - \tilde{S}_{1,+}, \tilde{S}_{1,+} \cap \tilde{S}_{2,+}$$

Their closures make the representatives of a basis of $H_n(Y, \hat{\mathcal{L}}_0)$.

Suppose that $a_{1,2} < -1$. Then $\tilde{S}_{1,+}$ is disjoint with $\tilde{S}_{2,+}$. We have three real domains $\tilde{S}_{1,+}, \tilde{S}_{2,+}, \mathbf{R}^n - \tilde{S}_{1+} \cup \tilde{S}_{2,+}$. On the other hand suppose that $a_{1,2} > 1$. Then $\tilde{S}_{1,+}$ includes or is included in $\tilde{S}_{2,+}$. Assume for example that $\tilde{S}_{1,+} \supset \tilde{S}_{2,+}$. Then there are three real domains $\mathbf{R}^n - \tilde{S}_{1,+}, \tilde{S}_{1,+} - \tilde{S}_{2+}, \tilde{S}_{2,+}$.

There is the Lefschetz cycle enclosed by two pieces of hyperboloids

$$\tilde{\mathfrak{l}}(\{1,2\}) : \{\eta = (\eta_1, \sqrt{-1}\eta_2^*, \dots, \sqrt{-1}\eta_n^*) \in \mathbf{R} \times (\sqrt{-1}\mathbf{R})^{n-1};$$
$$\eta_1^2 - \sum_{\nu=2}^{n} \eta_\nu^{*2} \geq r_1^2, (\eta_1 - v_1^{(2)})^2 - \sum_{\nu=2}^{n} \eta_\nu^{*2} \geq r_2^2\}$$

More generally suppose that $A(I) < 0$ for $|I| = p\,(2 \leq p \leq n)$. We may assume that $I = \{1, 2, \dots, p\}$. There exist the new coordinates $\eta = (\eta_1, \dots, \eta_n)$ such that $\tilde{S}_j\,(1 \leq j \leq p)$ are defined by

$$\sum_{\nu=1}^{j-1} (\eta_\nu - v_\nu^{(j)})^2 + \sum_{\nu=j}^{n} \eta_\nu^2 = r_j^2$$

We define the chain enclosed by p pieces of ultra hyperboloids

$$\tilde{\Delta}^*(I) = \{\eta = (\eta_1, \dots, \eta_{p-1}, \sqrt{-1}\eta_p^*, \dots, \sqrt{-1}\eta_n^*) \in \mathbf{R}^{p-1} \times (\sqrt{-1}\mathbf{R})^{n-p+1};$$
$$\sum_{\nu=1}^{j-1} (\eta_\nu - v_\nu^{(j)})^2 + \sum_{\nu=j}^{p-1} \eta_\nu^2 - \sum_{\nu=p}^{n} \eta_\nu^{*2} \geq r_j^2\,(1 \leq j \leq p)\}$$

Further we put

$$\partial_{p,+}\tilde{\Delta}^*(I)=\{\eta=(\eta_1,\ldots,\eta_{p-2},\sqrt{-1}\eta^*_{p-1},\ldots,\sqrt{-1}\eta^*_n)\in\mathbf{R}^{p-2}$$
$$\times(\sqrt{-1}\mathbf{R})^{n-p+2};\sum_{\nu=1}^{j-1}(\eta_\nu-v^{(j)}_\nu)^2+\sum_{\nu=j}^{p-2}\eta_\nu^2-\sum_{\nu=p-1}^{n}\eta^{*2}_\nu\geq r_j^2\,(1\leq j\leq p-1),$$
$$\eta^*_{p-1}\geq 0\}$$

which is the chain enclosed by $p-1$ pieces of ultra hyperboloids and the hyperplane $\eta^*_{p-1}=0$. By exchange of coordinates one can similarly define the chains $\partial_{k,+}\tilde{\Delta}^*(I)\,(1\leq k\leq p-1)$. Then the Lefschetz cycle $\tilde{\mathfrak{l}}^*(I)$ is defined to be

$$\tilde{\mathfrak{l}}^*(I)=\tilde{\Delta}^*(I)+\sum_{j=1}^{p}(-1)^{j-1}\partial_{j,+}\tilde{\Delta}^*(I)$$

Finally suppose $A(I)<0$ for $|I|=n+1$. We may assume $I=\{1,2,\ldots,n+1.\}$ Denote by $\tilde{\Delta}$ the Euclidean n-simplex with the vertices $\mathbf{v}^{(j)}$. We want to construct a series of chains $\partial_{j,+}\tilde{\Delta}^*(I)$ associated with each face $\tilde{\Delta}(\partial_j I)$ as follows. For simplicity we may assume $j=n+1$. There exist the coordinates $\eta=(\eta_1,\ldots,\eta_n)$ of $\mathbf{R}^n$ such that $\mathbf{v}^{(1)}=0$, and $\mathbf{v}^{(j)}$ lies in the $\eta_1,\ldots,\eta_{j-1}\,(2\leq j\leq n.)$ so that the face $\partial_{n+1}\tilde{\Delta}$ lie in the $\eta_1,\ldots,\eta_{n-1}$-subspace. Define

$$\partial_{n+1,+}\tilde{\Delta}^*(I)=\{\eta=(\eta_1,\ldots,\eta_{n-1}),\sqrt{-1}\eta^*_n)\in\mathbf{R}^{n-1}\times\sqrt{-1}\mathbf{R};$$
$$\sum_{\nu=1}^{j-1}(\eta_\nu-v^{(j)}_\nu)^2+\sum_{\nu=j}^{n-1}\eta_\nu^2-{\eta^*_n}^2\geq r_j^2\,(1\leq j\leq n),\eta^*_n\geq 0\}$$

In the same way one can construct the chains $\partial_{j,+}\tilde{\Delta}^*(I)\,(1\leq j\leq n)$ and finally put

$$\tilde{\mathfrak{a}}^*(I)=\tilde{\Delta}^*(I)+\sum_{j=1}^{n+1}(-1)^{j-1}\partial_{j,+}\tilde{\Delta}^*(I)$$

Then we have

$$\partial(\tilde{\mathfrak{a}}^*(I))\equiv 0\mod\bigcup_{j=1}^{n+1}\tilde{S}_j$$

i.e., $\tilde{\mathfrak{a}}^*(I)\in H_n(Y,\hat{\mathcal{L}}_0)$. Furthermore we have as Lemma 11

Lemma 18 *For any admissible I such that $|I| = n+2$, the identity holds*

$$\sum_{j=1}^{n+2}(-1)^{j-1}\partial(\tilde{\mathfrak{a}}^*(\partial_j I)) = 0$$

In conclusion we may choose admissible I with $|I| = n+1$, such that $1 \in I$ so that any other can be a linear combination of them. We have the same conclusion as Theorem 13.

5 Degenerate cases

In Section 3, and 4 we have assumed ($\mathcal{H}1$) and ($\mathcal{H}2$). In this section we discuss the cases where these conditions are not necessarily satisfied.

First note the following (for example, see [2]I, Lemma 4.2) :

Lemma 19 *We have the commutative diagram :*

$$\begin{array}{ccc} H^{n+1}(X-Y;\nabla_0) & \xrightarrow{\text{Res}} & H^n(Y;\nabla_0) \\ \updownarrow & & \updownarrow \\ H_{n+1}(X-Y;\hat{L}_0) & \xleftarrow{\delta} & H_n(Y;\hat{L}_0) \end{array}$$

where Res *denotes the Residue along Y, and δ means the boundary operation (Leray map) into a tubular neighborhood of Y in $X-Y$.*

For an arbitrary $\varphi(\xi)\tau \in R\tau$ such that its representative $\in H^{n+1}(X-Y,\nabla_0)$, denote

$$\varphi^{(1)}\tau_Q = \text{Res}\left(\frac{\varphi}{f_0}\tau\right) = \left[\frac{\varphi\tau}{df_0}\right]_Y, \ \varphi^{(2)}\tau_Q = \text{Res}\left(\frac{\varphi}{f_0^2}\tau\right)$$

Then $\varphi^{(1)}(I)$ is equal to the restriction of

$$\varphi(I) = \frac{1}{f_{i_1}\cdots f_{i_p}}$$

to Q. As for $\varphi^{(2)}(I)$ the following two recurrence relations play an important role in the sequel:

Lemma 20 *For an admissible I with $|I| = p\,(0 \le p \le n+1)$*

$$A(I)\varphi^{(2)}(I) \sim \sum_{k\notin I} \lambda_k A\begin{pmatrix} 0, & I \\ k, & I \end{pmatrix} \varphi_Q(k, I) + (\lambda_\infty + n - p - 1)A(0, I)\varphi_Q(I)$$

$$-\sum_{\nu=1}^{p}(-1)^{\nu-1} A\begin{pmatrix} I \\ 0, \partial_\nu I \end{pmatrix} \varphi^{(2)}(\partial_\nu I) \tag{5.1}$$

In particular

$$\varphi^{(2)}(\emptyset) \sim \sum_{k=1}^{m} \lambda_k a_{k,0}\varphi_Q(k) - (\lambda_\infty + n - 1)\varphi(\emptyset),$$

$$\varphi^{(2)}(j) \sim \sum_{k\neq j} \lambda_k A\begin{pmatrix} 0, & j \\ k, & j \end{pmatrix} \varphi_Q(k, j) - \sum_{k=1}^{m} \lambda_k a_{j,0}a_{k,0}\varphi_Q(k)$$

$$+(\lambda_\infty + n - 2)A(0, j)\varphi_Q(j) + (\lambda_\infty + n - 1)a_{j,0}\varphi_Q(\emptyset)$$

Therefore $\varphi^{(2)}(I)$ can be described as a linear combination of $\varphi_Q(J)$ such that $|J - J \cap I| \le 1$ with the coefficients of rational functions of $a_{i,j}$, $a_{k,0}$ whose denominators are products of $A(K)$ for $K \subset I$.

For the proof see [2]I,Proposition 4.2.

Lemma 21 *Fix an admissible I with $p = |I| \le n + 1$. Then an arbitrary $\mu, 1 \le \mu \le p$*

$$(-1)^{\mu-1}(\lambda_{i_\mu} - 1)A(I)\frac{\varphi_Q(I)}{f_{i_\mu}} \sim -\sum_{k\notin I} \lambda_k A\begin{pmatrix} I \\ k, \partial_\mu I \end{pmatrix} \varphi_Q(k, I)$$

$$-(\lambda_\infty + n - p - 1)A\begin{pmatrix} I \\ 0, \partial_\mu I \end{pmatrix} \varphi_Q(I) + \sum_{\nu=1}^{p}(-1)^{\nu-1} A\begin{pmatrix} \partial_\mu I \\ \partial_\nu I \end{pmatrix} \varphi^{(2)}(\partial_\nu I) \tag{5.2}$$

For the proof see [1]I, Proposition 4.2.

Owing to Lemma 20 and 21 an arbitray form $\frac{\tau_Q}{\prod_{k=1}^{m} f_k^{\nu_k}}$ $(\nu_k \ge 0)$ can be described explicitly as a linear combination of the representatives of admissible forms $\varphi_Q(I)\tau_Q$.

Proposition 22 *In addition to $(\mathcal{H}1)$ suppose the following condition :*

($\mathcal{H}IV(p)$) *For a fixed admissible* I *with* $p = |I| \le n$, $A(I) = 0$. *But for any other admissible* J *such that* $|J| \le n+2$ $A(J) \neq 0$.

Then $\mathfrak{l}^*(I)$ *vanishes. The dimension of* $H_n(Y, \hat{\mathcal{L}}_0)$ *decreases by one and is equal to* $\kappa_n - 1$. *On the other hand the representatives* $\varphi_Q(I)$ *in* Proposition 2 *does not make a basis of* $H^n(Y, \nabla_0)$. *We have a linear relation*

$$(\sum_{j\in I}\lambda_j + \lambda_\infty + n - p - 1)A(0,I)\varphi_Q(I) + \sum_{k\notin I}\lambda_k A\begin{pmatrix} 0, & I \\ k, & I \end{pmatrix}\varphi_Q(k,I)$$
$$-\sum_{k\notin I}\lambda_k\sum_{\nu=1}^{p}(-1)^{\nu-1}\frac{A\begin{pmatrix} I & \\ 0, & \partial_\nu I\end{pmatrix}A\begin{pmatrix} k, & \partial_\nu I \\ 0, & \partial_\nu I\end{pmatrix}}{A(\partial_\nu I)}\varphi_Q(k,\partial_\nu I) \equiv 0$$
$$\bmod \mathcal{B}(\mathcal{I}) \tag{5.3}$$

where $\mathcal{B}(I)$ *denotes a linear space spanned by* $\varphi_Q(J)$ *such that* $|J - J \cap I| \le 1$ *and* $|J| < |I|$. $H^n(Y, \nabla_0)$ *is of dimension* $\kappa_n - 1$ *and is spanned by* $\varphi_Q(J)$ *such that* $J \neq I$ *and* $|J| \le n+1$ *with the fundamental relations* (2.3), (2.4).

Proof. In fact since $A(I) = 0$, the LHS of (5.1) vanishes. A repeated application of (5.1) to $\varphi^{(2)}(\partial_\nu I)$ shows the RHS of (5.1) equals the RHS of (5.3) in view of the Jacobi identities

$$A^2\begin{pmatrix} I & \\ 0, & \partial_\nu I\end{pmatrix} = -A(0,I)A(\partial_\nu I)$$

Proposition 23 *In addition to* ($\mathcal{H}1$) *suppose the following condition :*

$\mathcal{H}IV(n+1)$ *For a fixed admissible* I *with* $|I| = n+1$, $A(I) = 0$. *But for any other admissible* J *such that* $|J| \le n+2$ $A(J) \neq 0$. *Then* $\mathfrak{l}^*(I)$ *vanishes and* $\dim H_n(Y, \hat{L}_0) = \kappa_n - 1$. *We have a linear relation*

$$2(\sum_{j\in I}\lambda_j - 1)A(0,I)\varphi_Q(I) \equiv \sum_{k\notin I}\sum_{\nu=1}^{n+1}(-1)^{\nu-1}\lambda_k$$
$$\cdot\frac{A(0,I)A\begin{pmatrix} I \\ k, \partial_\nu I\end{pmatrix}A(k,\partial_\nu I)}{A(k,I)A(\partial_\nu I)}\varphi_Q(k,\partial_\nu I) \quad \bmod \mathcal{B}(\mathcal{I}) \tag{5.4}$$

$H^n(Y, \nabla_0)$ *is of dimension* $\kappa_n - 1$ *and is spanned by* $\varphi_Q(J)$ *such that* $J \neq I$ *and* $|J| \le n+1$ *with the fundamental relations* (2.3), (2.4).

Proof. Since $A(I) = 0$ the LHS of (5.1) vanishes. Applying repeatedly (2.4) to $\varphi_Q(k, I)$ and (5.1) to $\varphi^{(2)}(\partial_\nu I)$ one sees that the RHS of (5.1) equals

$$-2(\sum_{j\in I} \lambda_j - 1)A(0, I)\varphi_Q(I)$$

$$+\sum_{k\notin I}\sum_{\nu=1}^{n+1}(-1)^{\nu-1}\lambda_k \frac{A\begin{pmatrix} 0, I \\ k, I \end{pmatrix} A\begin{pmatrix} I \\ 0, \partial_\nu I \end{pmatrix} A(k, \partial_\nu I)}{A(k, I)A(\partial_\nu I)}\varphi_Q(k, \partial_\nu) \mod \mathcal{B}(\mathcal{I})$$

Hence (5.4) follows owing to the identities

$$A(k, I) = -U^2\begin{pmatrix} k, I \\ 0, 1, \ldots, n+1 \end{pmatrix},$$
$$A\begin{pmatrix} 0, I \\ k, I \end{pmatrix} A\begin{pmatrix} 0, \partial_\nu I \\ I \end{pmatrix} = A(0, I)A\begin{pmatrix} I \\ k, \partial_\nu I \end{pmatrix}$$

Corollary 24 *Suppose that $m = n + 2, n \geq 1$ and that $A(I) = 0$ for all admissible I with $|I| = n+1$. Then $H^n(Y, \nabla_0)$ is of dimension $\kappa_n - (n+2) = 2^{n+2} - n - 4$ and is spanned by the representatives $\varphi_Q(I)$ with $|I| \leq n$ with the one fundamental relation: For $J = \{1, 2, \ldots, n+2\}$*

$$\sum_{\mu\neq\nu}(-1)^{\mu+\nu}\varphi_Q(\partial_\nu\partial_\nu J)\frac{A(0, \partial_\mu\partial_\nu J)}{A\begin{pmatrix} 0, & \partial_\mu J \\ 0, & \partial_\nu J \end{pmatrix}} = 0 \tag{5.5}$$

$\varphi_Q(I)$ $(|I| = n+1)$ can be expressed as

$$2(\sum_{j\in I}\lambda_j - 1)A(0, I)\varphi_Q(I) \equiv 0 \mod \mathcal{B}(\mathcal{I}) \tag{5.6}$$

over the coefficients of rational functions of $a_{i,j}, a_{k,0}$ with the denominators $A(K)$ $(|K| \leq n)$. This identity is just an n dimensional version of (1.2).

Proof. (5.5) is a special case of (2.3) since $A(\partial_\mu J) = 0$. On the other hand (5.6) is a special case of (5.4) since $A(k, \partial_\nu I) = 0$.

Proposition 25 *In addition to $(\mathcal{H}1)$ suppose the following condition :*
$(\mathcal{H}IV(n+2))$ *For a fixed admissible I with $|I| = n+2$, $A(I) = 0$. But for any other admissible J such that $|J| \leq n+2$ $A(J) \neq 0$. Then there is*

no vanishing of Lefschetz cycles and $\dim H_n(Y, \hat{\mathcal{L}}_0) = \kappa_n$*. On the other hand, for any fixed* μ

$$(\sum_{\nu \in I} \lambda_\nu - 1) U \begin{pmatrix} \partial_\mu I \\ 1, \ldots, n+1 \end{pmatrix} \varphi_Q(I) \equiv 0 \mod \mathcal{B}(\mathcal{I}) \tag{5.7}$$

$H^n(Y, \nabla_0)$ *is of dimension* κ_n *and is spanned by* $\varphi_Q(J)$ *such that* $|J| \leq n+1$ *with the fundamental relations* (2.3), (2.4).

Proof. By hypothesis the LHS of (2.4) vanishes. By a multiplication by f_{i_μ} of both sides of (2.4) one sees that $\varphi_Q(I)$ is linearly dependent on $\frac{\varphi_Q(\partial_\nu I)}{f_{i_\mu}}$ $(\nu \neq \mu)$. On the other hand due to (5.2) each $\frac{\varphi_Q(\partial_\nu I)}{f_{i_\mu}}$ is linearly dependent on admissible $\varphi_Q(J)$ with $|J| \leq n+1$. Hence the Proposition follows.

Finally we consider the special case where $n \geq 2, m = n+2$ and $A(i,j) = 0$, i.e., $a_{i,j} = \pm 1$ for all $i, j \in \{1, 2, \ldots, n+2\}$ $(i \neq j)$. Since the signature of A is of type $(n+1, 1)$, we have $A(1,2) = 0, A(1, 2, \ldots, p) < 0$ if $3 \leq p \leq n+2$.

By a suitable Lorentz transformation we may assume that $a_{i,j} = -1$ for all i, j $(i \neq j)$. In fact

Lemma 26 *There exist a diagonal matrix* P *with diagonal elements equal to* ± 1 *such that* $B = P \cdot A \cdot {}^t P$ *is the matrix with diagonal elements* 1 *and off-diagonal elements* -1.

Proof. Denote by B_r the matrix of size $r+2$ with diagonal elements 1 and off-diagonal elements -1. Let A_r be the matrix with the (i,j) elements $a_{i,j}$ $(1 \leq i, j \leq r+2)$. For $r = 0$ the Lemma is trivial. Suppose that the Lemma is true for A_{r-1}. There exists a diagonal matrix P_{r-1} with diagonal elements ± 1 such that $B_{r-1} = P_{r-1} \cdot A_{r-1} \cdot {}^t P_{r-1}$. Let $\tilde{P}_r$ be the diagonal matrix of size $r+2$ such that the first $r+1$ diagonal elements coincides with the ones of P_{r-1} and the last one equal to 1. Then $\tilde{P}_r \cdot A_r \cdot {}^t \tilde{P}_r$ has the same components as B_r except for the off-diagonal components in the last column or row. Denote these components by $\varepsilon_1, \ldots, \varepsilon_{r+1.}$ Then we have

$$\det(\tilde{P}_r \cdot A_r \cdot^t \tilde{P}_r) = (1-r)2^r + (r-2)2^{r-1} \sum_{k=1}^{r+1} \varepsilon_k^2 - 2^r \sum_{1 \leq i < j \leq r+1} \varepsilon_i \varepsilon_j < 0$$

But this inequality goes to a contradiction except for the case where all ε_j equal 1 or all ε_j equal -1. One sees that the first case is equivalent to B_r, while the last one coincides with B_r. Q.E.D.

Lemma 27 *Suppose $I = \{1, 2, \ldots, n+2.\}$ The matrix A for all off diagonal elements $a_{i,j} = -1$ defines the hypersphere arrangement $\mathcal{A}'$ if and only if $\{a_{j,0}\}_j$ satisfy the quadratic relation*

$$(n-1)\sum_{j=1}^{n+2} a_{j,0}^2 - 2\sum_{1\le j\le k\le n+2} a_{j,0}a_{k,0} + 2n = 0$$

Proof. First remark that if $A(0, I) \le 0$ then $A(0, J) < 0$ for $J \subset I, J \neq I$. In fact it is sufficient to show this in case $J = \{1, \ldots, r\}$ $(3 \le r \le n+1.)$ This follows by lowering induction from the identity

$$A(0,1,\ldots,r)A(1,\ldots,r+1) - A^2\begin{pmatrix} 0,1,\ldots,r \\ r+1,1,\ldots,r \end{pmatrix}$$
$$= A(0,1,\ldots,r+1)A(1,\ldots,r)$$

because $A(1, \ldots, r), A(1, \ldots, r+1)$ are both negative. On the other hand

$$A(0,I) = 2^n\{(n-1)\sum_{j=1}^{n+2} a_{j,0}^2 - 2\sum_{1\le j\le k\le n+2} a_{j,0}a_{k,0} + 2n\}$$

Hence the Lemma.

We now apply to it the formula (5.3) for $p = 2$.

For $I = \{i, j\}$ we have

$$A(0,i,j) = -(a_{i,0} + a_{j,0})^2, A\begin{pmatrix} 0,i,j \\ k,i,j \end{pmatrix} = 2(a_{i,0} + a_{j,0})$$

Hence (5.3) and Lemma 20 give

$$(\lambda_\infty + n - 3 + \lambda_i + \lambda_j)(a_{i,0} + a_{j,0})\varphi_Q(i,j) + \sum_{k\neq i,j} \lambda_k(a_{k,0} + a_{i,0})\varphi_Q(k,i)$$
$$+\sum_{k\neq i,j} \lambda_k(a_{k,0} + a_{j,0})\varphi_Q(k,j) \sim w_{i,j} \tag{5.8}$$

where we put

$$w_{i,j} = 2\sum_{k\neq i,j} \lambda_k\varphi_Q(k,i,j) + (a_{i,0} + a_{j,0})\sum_{k=1}^{n+2} \lambda_k a_{k,0}\varphi_Q(k) - (\lambda_\infty + n - 1)$$
$$\cdot(a_{i,0} + a_{j,0})\varphi_Q(\emptyset) - (\lambda_\infty + n - 2)\{A(0,i)\varphi_Q(i) + A(0,j)\varphi_Q(j)\}$$

To solve (5.4) with respect to $\varphi_Q(i,j)$ we put $v_{i,j} = (a_{i,0} + a_{j,0})\varphi_Q(i,j)$ and

$$V_i = \sum_{k \neq i} \lambda_k v_{k,i},\ V_\infty = \sum_{i \neq j} \lambda_i \lambda_j v_{i,j},\ W_i = \sum_{k \neq i} \lambda_k w_{k,i},\ W_\infty = \sum_{i \neq j} \lambda_i \lambda_j w_{i,j}$$

Then (5.8) is equivalent to

$$(\lambda_\infty + n - 3)v_{i,j} + V_i + V_j \sim w_{i,j} \tag{5.9}$$

(5.9) can be uniquely solved for $v_{i,j}$:

$$\begin{aligned}(\lambda_\infty + n - 3)v_{i,j} &\sim w_{i,j} + V_\infty \left(\frac{1}{2\lambda_\infty + n - 3 - 2\lambda_i} + \frac{1}{2\lambda_\infty + n - 3 - 2\lambda_j} \right) \\ &- \left(\frac{W_i}{2\lambda_\infty + n - 3 - 2\lambda_i} + \frac{W_j}{2\lambda_\infty + n - 3 - 2\lambda_j} \right)\end{aligned} \tag{5.10}$$

where V_i and V_∞ are uniquely determined by

$$\begin{aligned}&(2\lambda_\infty + n - 3 - 2\lambda_i)V_i \sim W_i - V_\infty, \\ &\left(1 + \sum_{k=1}^{n+2} \frac{\lambda_k}{2\lambda_\infty + n - 3 - 2\lambda_k}\right) V_\infty \sim \sum_{k=1}^{n+2} \frac{\lambda_k W_k}{2\lambda_\infty + n - 3 - 2\lambda_k}\end{aligned}$$

provided none of $2\lambda_\infty + n - 3 - 2\lambda_k$ or the symmetric polynomial

$$G(\lambda) = \prod_{k=1}^{n+2}(2\lambda_\infty + n - 3 - 2\lambda_k) + \sum_{k=1}^{n+2} \lambda_k \prod_{j \neq k}(2\lambda_\infty + n - 3 - 2\lambda_j)$$

vanishes. In this way we can conclude

Proposition 28 *For $m = n + 2, n \geq 2$, suppose that in addition to $(\mathcal{H}1)$, $a_{i,j} = -1$ for all i, j $(i \neq j)$, and $A(I) < 0$ for all admissible I with $3 \leq |I| \leq n + 2$. Suppose further that neither of $2\lambda_\infty + n - 3 - 2\lambda_k$ or $\lambda_\infty + n - 3$ or $G(\lambda)$ vanish. Then all the Lefschetz cycles $\mathfrak{l}^*(I)$ $(|I| = 2)$ vanish. $H^n(Y, \nabla_0)$ is of dimension $\kappa_n - \binom{n+2}{2}$ and has a basis of representatives $\varphi_Q(I)$ with $0 \leq |I| \leq n + 1, |I| \neq 2$ satisfying the fundamental relations* (2.3),(2.4). *$\varphi_Q(i,j)$ can be described as a linear combination of these representatives as in* (5.10).

6 2 problems

As is seen from Propositon 2, $H^n(Y, \nabla_0)$ is spanned by the representatives $\varphi_Q(I)$. The result due to Orlik-Terao (see [9]) suggests that this fact still holds in general in the following sense :

Conjecture 1 . Without any of the hypotheses $(\mathcal{H}1)$ or $(\mathcal{H}2)$, $H^n(Y, \nabla_0)$ is spanned by the representatives $\varphi_Q(I)$, $I \subset \{1, 2, \ldots, m\}$ including $\varphi_Q(\emptyset)$.

The complex hypersphere Q has the Kähler metric

$$ds^2 = \sum_{\nu=1}^{n+1} |d\xi_\nu|^2 = \sum_{\mu,\nu=1}^{n} g_{\mu,\overline{\nu}} d\zeta^\mu d\overline{\zeta^\nu}$$

with respect to local coordinates $\zeta = (\zeta^\nu)_{1\le\nu\le n}$. We put

$$\lambda_j = N l_j + \lambda'_j \quad (N \in \mathbf{Z}_{>0})$$

for fixed $l = (l_j)_j \in (\mathbf{Z}_{>0})^m$, $\lambda' = (\lambda'_j)_j \in \mathbf{R}^m$. For a large N the asymptotic behavior of the integral (2.2) can be explicitly evaluated if the cycle $\mathbf{c}$ is a stable cycle defined by the gradient vector field on Q :

$$d\zeta^\mu = \sum_{\nu=1}^{n} \frac{\partial}{\partial\overline{\zeta_\nu}} (\sum_{j=1}^{m} l_j \log |f_j|) g^{\mu,\overline{\nu}} dt$$

where $g^{\mu,\overline{\nu}}$ denotes the inverse matrix of the metric tensor $g_{\mu,\overline{\nu}}$. The critical points are determined by the equations on Q

$$\sum_{j=1}^{m} l_j \, d \log f_j = 0$$

Every cycle mentioned in Theorem 13 seems to have one-to-one relation with a stable cycle corresponding to these critical points. This fact suggests :

Conjecture 2 All the critical points of the gradient vector field lie in $\Re Q$.

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