First MSJ Seasonal Institute – Kyoto 2008
Probabilistic Approach to Geometry
Heat kernel estimates

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Probabilistic Approach to Geometry

Heat kernel estimates, III

$$\lim_{t\to 0}\left(-t\log \mathbf{P}_\mu(X_0\in A\ \&\ X_t\in B)\right)=\frac{d(A,B)^2}{4}$$

Manifolds with ends

We will consider manifolds with ends:

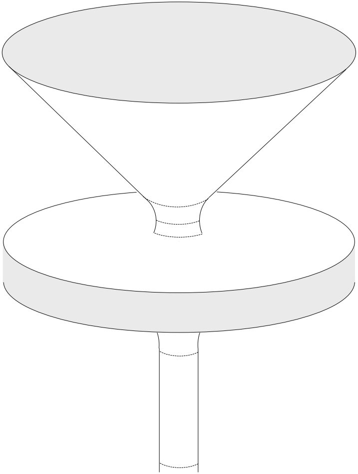
$$M = M_1 \# M_2 \# \dots \# M_k$$

where the ends M_i, $1 \leq i \leq k$ are of Harnack type.

$$M = K \cup E_1 \cup \dots \cup E_k \text{ (disjoint union)}$$

with K compact with smooth boundary and E_i isometric to an open set in M_i (we can allow $\overline{E_i} = M_i$).
Curvature conditions that yield such manifolds:
(c1) Asymptotically non-negative sectional curvature,
(c2) Non-negative Ricci curvature outside a compact set with ends satisfying (RCA)

Euclidean domains



The heat kernel on manifolds with ends

Consider $M = M_1 \# \cdots \# M_k$ and assume that each M_k is of Harnack type, transient. Then the heat kernel is bounded above and below by expressions of the type

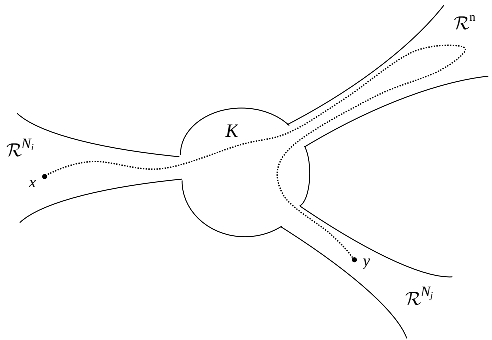
$$\frac{1}{\sqrt{V_{i_x}(x,\sqrt{t})V_{i_y}(y,\sqrt{t})}} \exp\left(-\frac{d_\emptyset(x,y)^2}{t}\right) +$$

$$\left(\frac{H(x,t)H(y,t)}{V_0(\sqrt{t})} + \frac{H(x,t)}{V_{i_y}(\sqrt{t})} + \frac{H(y,t)}{V_{i_x}(\sqrt{t})}\right) \exp\left(-\frac{d_+(x,y)^2}{t}\right)$$

$|x| = d(o,x)$, $V_0(r) = \min_i V_i(r)$ and

$$H(x,t) = \min\left\{1, \frac{|x|^2}{V_{i_x}(|x|)} + \left(\int_{|x|^2}^{t} \frac{ds}{V_{i_x}(\sqrt{s})}\right)_+\right\}, \; x \in M_{i_x}.$$

Brownian motion on M



Gluing 1

For any closed set $\Gamma \subset M$, define the first hitting time by

$$\tau_\Gamma = \inf\{t \geq 0 : X_t \in \Gamma\}.$$

Let us set

$$\psi_\Gamma(t,x) := \mathbb{P}_x(\tau_\Gamma \leq t).$$

This is the probability to hit Γ before time t, starting from x. It is increasing with t.

Let Ω be open with smooth boundary Γ. Then for all $x \in \Omega$, $y \in M$, and $t > 0$

$$p(t,x,y) \leq p_\Omega(t,x,y) + \sup_{\substack{0 \leq s \leq t \\ z \in \Gamma}} p(s,z,y)\, \psi_\Gamma(t,x).$$

Gluing 1

Furthermore

$$
\begin{aligned}
p(t,x,y) &\leq p_\Omega(t,x,y) + \sup_{\substack{t/2\leq s\leq t\\ z\in\Gamma}} p(s,z,y)\,\psi_\Gamma(\frac{t}{2},x) \\
&+ \sup_{t/2\leq s\leq t} \psi'_\Gamma(s,x) \int_0^{t/2} \sup_{z\in\Gamma} p(\theta,z,y)\,d\theta
\end{aligned}
$$

Gluing 1

and

$$
\begin{aligned}
p(t,x,y) \quad &\geq \quad p_{\Omega}(t,x,y) + \inf_{\substack{t/2\leq s\leq t \\ z\in\Gamma}} p(s,z,y)\ \psi_{\Gamma}(\frac{t}{2},x) \\
&+ \quad \inf_{t/2\leq s\leq t} \psi'_{\Gamma}(s,x) \int_0^{t/2} \inf_{z\in\Gamma} p(\theta,z,y)d\theta
\end{aligned}
$$

Gluing 2

Let Ω_1 and Ω_2 be two open sets in M with boundaries Γ_1 and Γ_2 respectively. Assume that Γ_2 separates Ω_2 from Γ_1.
Set

$$\overline{G}(t) := \int_0^t \sup_{v\in\Gamma_1, w\in\Gamma_2} p(s,v,w)ds, \quad \underline{G}(t) := \int_0^t \inf_{v\in\Gamma_1, w\in\Gamma_2} p(s,v,w)ds.$$

Then, for all $x \in \Omega_1$, $y \in \Omega_2$, and $t > 0$,

Gluing 2

$$
\begin{aligned}
p(t,x,y) &\le p_{\Omega_1}(t,x,y) \\
&+ 2\left(\sup_{s\in[t/4,t]}\ \sup_{v\in\Gamma_1, w\in\Gamma_2} p(s,v,w)\right)\psi_1(t,x)\psi_2(t,y) \\
&+ \overline{G}(t)\left[\sup_{s\in[t/4,t]} \psi_1'(s,x)\right]\psi_2(t,y) \\
&+ \overline{G}(t)\left[\sup_{s\in[t/4,t]} \psi_2'(s,y)\right]\psi_1(t,x)
\end{aligned}
$$

Gluing 2

$$
\begin{aligned}
p(t,x,y) \quad &\geq \quad (1/2)p_{\Omega_1}(t,x,y) \\
&+ \quad \left[\inf_{s\in[t/4,t]} \inf_{v\in\Gamma_1, w\in\Gamma_2} p(s,v,w)\right] \psi_1(\frac{t}{4},x)\psi_2(\frac{t}{4},y) \\
&+ \quad \underline{G}(\frac{t}{4}) \left[\inf_{s\in[t/4,t]} \psi_1'(s,x)\right] \psi_2(\frac{t}{4},y) \\
&+ \quad \underline{G}(\frac{t}{4}) \left[\inf_{s\in[t/4,t]} \psi_2'(s,y)\right] \psi_1(\frac{t}{4},x).
\end{aligned}
$$

Hitting probability ψ_K

Let M be of Harnack type. Fix a compact set K with non-empty interior and a reference interior point $o \in K$. set $|x| = d(o, y)$ and $H_*(x,t) := \min\left\{1,\ \frac{|x|^2}{V(o,|x|)} + \left(\int_{|x|^2}^{t} \frac{ds}{V(o,\sqrt{s})}\right)_+\right\}$.
Then, for all $x \in M \setminus K_\delta$ and $t > 0$ (Grigor'yan and LSC, 2002),

$$\psi_K(t,x) \simeq H_*(x,t) \exp\left(-\frac{|x|^2}{t}\right)$$

$$\partial_t \psi_K(t,x) \le \frac{C}{V(o,\sqrt{t})} \exp\left(-c\frac{|x|^2}{t}\right).$$

Dirichlet heat kernel p_Ω^D

Let M be of Harnack type and transient. Fix a compact set K with non-empty interior and $\Omega = M \setminus K$. Then, at bounded distance of K,

$$p_\Omega^D(t, x, y) \simeq p(t, x, y).$$

The central estimate

Set $V_0(r) = \min_i V_i(r)$. Then, for all x, y at bounded distance from o,

$$p(t,x,y) \le \frac{C}{V_0(\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{t}\right).$$

Also, if all ends are transient, for x, y at bounded distance of the central point o,

$$p(t,x,y) \ge \frac{c}{V_0(\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{t}\right).$$

The heat kernel on manifolds with ends

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$$\frac{1}{\sqrt{V_{i_x}(x,\sqrt{t})V_{i_y}(y,\sqrt{t})}} \exp\left(-\frac{d_\emptyset(x,y)^2}{t}\right) +$$

$$\left(\frac{H(x,t)H(y,t)}{V_0(\sqrt{t})} + \frac{H(x,t)}{V_{i_y}(\sqrt{t})} + \frac{H(y,t)}{V_{i_x}(\sqrt{t})}\right) \exp\left(-\frac{d_+(x,y)^2}{t}\right)$$

The harmonic function h

(RCA) with respect to o: There exists $A > 1$ such that any two points x_1, x_2 with $d(o, x_i) = r$, $R > a$, are connecetd in $B(o, Ar) \setminus B(o, A^{-1}R)$.

Let $M = M_1 \# \ldots \# M_k$. Assume that M is transient and that, for each $i = 1, \ldots, k$, M_i is Harnack type and satisfies *(RCA)*. Then there exists a positive harmonic function h on M such that, for all $x \in M$,

$$h(x) \asymp 1 + \left(\int_1^{|x|^2} \frac{ds}{V_{i_x}(\sqrt{s})} \right)_+ .$$

(Sung,Tam and Wang — Grigor'yan and LSC)

The general transient case

Consider $M = M_1 \# \cdots \# M_k$. Assume that each M_k is of Harnack type, and satisfies (RCA). Assume that M is transient. Then the heat kernel is bounded above and below by expressions of the type

$$h(x)h(y)\left(\frac{\widetilde{H}(x,t)\widetilde{H}(y,t)}{\widetilde{V}_{\min}(\sqrt{t})} + \frac{\widetilde{H}(x,t)}{\widetilde{V_{i_y}}(\sqrt{t})}\right.$$

$$\left. + \quad \frac{\widetilde{H}(y,t)}{\widetilde{V_{i_x}}(\sqrt{t})}\right) \exp\left(-\frac{d_+^2(x,y)}{t}\right)$$

$$+ \quad \frac{h(x)h(y)}{\sqrt{\widetilde{V}_{i_x}(x,\sqrt{t})\widetilde{V}_{i_y}(y,\sqrt{t})}} \exp\left(-\frac{d_\emptyset^2(x,y)}{t}\right).$$

The tilde means relative to $(M, h^2 d\mu)$!

The general transient case

Set

$$\eta_i(r) := 1 + \left(\int_1^{r^2} \frac{ds}{V_i(\sqrt{s})} \right)_+$$

Then

$$\widetilde{V}_i(r) \asymp \eta_i^2(r) V_i(r). \tag{1}$$

and

$$\widetilde{H}(x,t) \simeq \frac{|x|^2}{\eta_{i_x}^2(|x|) V_{i_x}(|x|)} + \frac{1}{\eta_{i_x}(|x|)\, \eta_{i_x}\left(\sqrt{t}\right)} \left(\int_{|x|^2}^{t} \frac{ds}{V_{i_x}(\sqrt{s})} \right)_+ .$$

The general transient case

Corollary

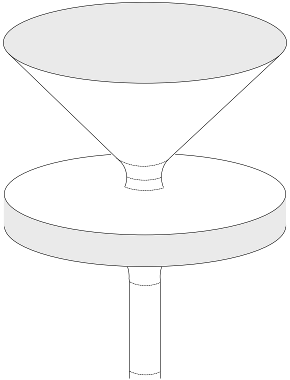
Assume that $M = M_1 \# \cdots \# M_k$ is transient with each M_k of Harnack type and satisfying (RCA). Then

- $\sup_{x,y}\{p(t,x,y)\} \simeq max_i\{V_i(\sqrt{t})^{-1}\}$
- $\sup_y\{p(t,x,y)\} \simeq max_i\{[\eta_i(\sqrt{t})V_i(\sqrt{t})]^{-1}\}$
- $p(t,x,y) \simeq \max_i\{[\eta_i(\sqrt{t})^2 V_i(\sqrt{t})]^{-1}\}$

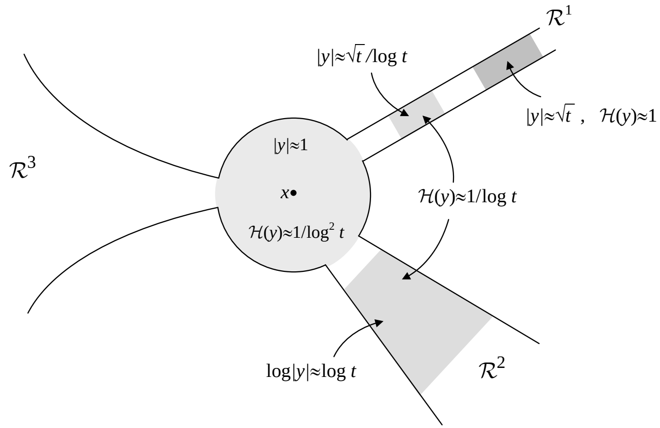
$$\eta_i(r) := 1 + \left(\int_1^{r^2} \frac{ds}{V_i(\sqrt{s})}\right)_+ .$$

$\mathcal{R}^1 \# \mathcal{R}^2 \# \mathcal{R}^3$

$$\mathcal{H}(y) = \frac{p(t,x,y)}{\sup_z p(t,x,z)}.$$



Example $\mathcal{R}^1 \# \mathcal{R}^2 \# \mathcal{R}^3$



$\mathcal{R}^2 \# \mathcal{R}^2$

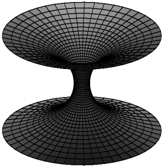
Set

$$Q(x,t) \asymp \frac{1}{\log(2+|x|)} + \left(\frac{1}{2} - \frac{\log(2+|x|)}{\log(2+t)}\right)_+$$

$$D(x,t) \asymp \frac{\log(2+|x|)}{\log(2+|x|)+\log(2+t)}.$$

For $t \geq 1$, $x \in E_1$, $|y| \in E_2$ and $|x|, |y| \leq C\sqrt{t}$, we have

$$p(t,x,y) \asymp \frac{C}{t}\left(Q(x,t)D(y,t) + D(x,t)Q(y,t) + Q(x,t)Q(y,t)\right).$$



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