First MSJ Seasonal Institute – Kyoto 2008
Probabilistic Approach to Geometry
Heat kernel estimates

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Probabilistic Approach to Geometry

Heat kernel estimates, II

$$\lim_{t \to 0} \left(-t \log \mathbf{P}_\mu(X_0 \in A \;\&\; X_t \in B)\right) = \frac{d(A, B)^2}{4}$$

Spaces of Harnack type

On, say, a weighted complete Riemannian manifold:

Theorem (Grigor'yan, 91; LSC, 92)

The following properties are equivalent:

(a) *The conjunction of*

- *The doubling property:* $V(x, 2r) \leq DV(x, r)$*, for all* x, r*.*
- *The Poincaré inequality: For all* $B = B(x, r)$*,*

$$\forall\, f \in Lip(B), \quad \int_B |f - f_B|^2 d\mu \leq Pr^2 \int_B |\nabla f|^2 d\mu.$$

(b) *The two-sided Gaussian bound: for all* $x, y,\ t > 0$*,*

$$p(t, x, y) \simeq \frac{1}{V(x, \sqrt{t})} \exp\left(-\frac{d(x, y)^2}{t}\right).$$

Call this a space of Harnack type.

Sobolev and elliptic Harnack

Fix $R \in (0,\infty]$. Assume that (for some $\alpha > 2$)

$$\left(\int_B |f|^{2\alpha/(\alpha-2)} d\lambda\right)^{(\alpha-2)/\alpha} \le \frac{Cr^2}{V(x,r)^{2/\alpha}} \int_B (|\nabla f|^2 + r^{-2}|f|^2) d\lambda$$

$B = B(x,r)$, $f \in \mathcal{C}_c(B)$, $x \in M$, $r \in (0,R)$.

Theorem (W. Hebish, LSC, 2001)

Under this hypothesis, the following properties are equivalent:

- *The two-sided Gaussian bound: for all x, y, $t \in (0, \sqrt{R})$,*

$$p(t,x,y) \simeq \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{t}\right).$$

- *The Elliptic Harnack inequality up to scale R:*
  $\exists C$, $\forall u \ge 0$ harmonic in $B(x,2r)$,

$$u(y) \le Cu(z), \quad y,z \in B(x,r).$$

Elliptic Harnack on $M \times \mathbb{R}$

Theorem (W. Hebish, LSC, 2001)

For any fixed $R \in (0, \infty]$, the following properties are equivalent:

- *The two-sided Gaussian bound: for all $x, y \in M$, $t \in (0, \sqrt{R})$,*

$$p(t, x, y) \simeq \frac{1}{V(x, \sqrt{t})} \exp\left(-\frac{d(x, y)^2}{t}\right).$$

- *The elliptic Harnack inequality up to scale R on $M \times \mathbb{R}$.*

A Riemannian manifold of the product form $M \times \mathbb{R}$ is of Harnack type if and only if it satisfies the elliptic Harnack inequality.

Examples of spaces of Harnack type

- Convex domains in Euclidean space.
- Complete Riemannian manifolds with $\mathrm{Ric} \geq 0$.
  Bishop-Gromov (Cheeger-Gromov-Taylor) and P. Buser, 1982.
  Li-Yau, 1986.
- Lie groups with polynomial volume growth.
  Gromov 1981, Varopoulos 1987.
- Quotients of any space of Harnack type by an isometric group action.
- Spaces that are (measure) quasi-isometric to a space of Harnack type. (Kanai, Coulhon, LSC)
- Coverings of compact manifolds which have polynomial volume growth
- and more ...

Manifolds with ends

We will consider manifolds with ends:

$$M = M_1 \# M_2 \# \dots \# M_k$$

where the ends M_i, $1 \le i \le k$ are of Harnack type.

$$M = K \cup E_1 \cup \cdots \cup E_k \text{ (disjoint union)}$$

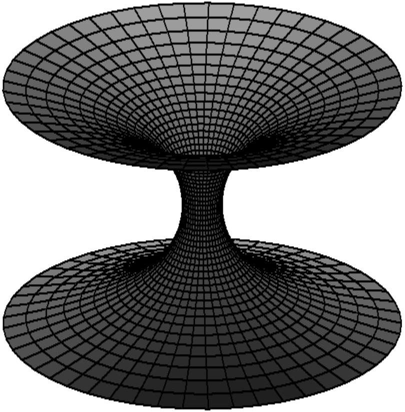
with K compact with smooth boundary and E_i isometric to an open set in M_i (we can allow $\overline{E_i} = M_i$).

$p(t,x,y) \simeq ?$, $\sup_y p(t,x,y) \simeq ?$, $\sup_{x,y} p(t,x,y) \simeq ?$

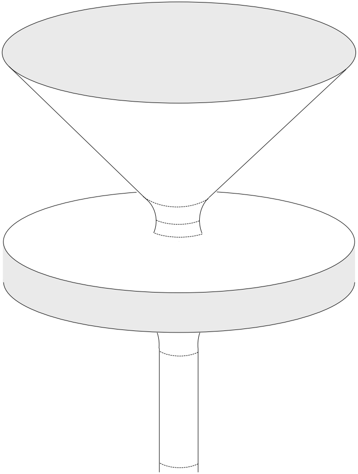
For a fixed x, describe roughly the set

$$\{y : p(t,x,y) \ge \epsilon \sup_z p(t,x,z)\}$$

Catenoid like surfaces



Euclidean domains



Curvature conditions

Consider the following two curvature conditions:

(c1) Asymptotically non-negative sectional curvature:
$\exists\ k : (0, \infty) \to (0, \infty)$, continuous decreasing,
$\int^{\infty} sk(s)ds < \infty$ such that $\mathrm{Sect}(x) \geq -k(d(o, x))$.

(c2) Non-negative Ricci curvature outside a compact set whose ends satisfy condition (RCA) below.

Under any one of these two conditions:

- M has finitely many ends (Cai, Kasue, Liu, Li-Tam)
- these ends are Harnack type (Grigor'yan, LSC).
- these ends also have relatively connected annuli, i.e., (RCA):
  For any $o \in M$, any two points x, y at distance $r > A^2$ from o are connected in $B(o.Ar) \setminus B(o, A^{-1}r)$.

Can such manifolds be Harnack?

Consider $M = M_1 \# \cdots \# M_k$ and assume that each M_k is of Harnack type and satisfies (RCA). Fix o and set $V(r) = \mu(B(o,r))$, $V_i(r) = \mu(B(o,r) \cap M_i)$.

Theorem (Grigor'yan, LSC 2005)

M is of Harnack type if and only if M has only one end or:

(v1) $V_i(r) \simeq V_j(r)$, $1 \le i < j \le k$;

(v2) $\int_1^r \frac{sds}{V(s)} \simeq \frac{r^2}{V(r)}$ *(when $V(r) \simeq r^\alpha$, this means $\alpha \in (0,2)$).*

Sketch of proof: All ends satisfy the same good localized Sobolev inequality (or Faber-Krahn), Hence M also. Conditions (v1)-v(2) can be used to prove the elliptic Harnack inequality (they are in fact necessary for it).
The result of Hebisch-LSC then gives that M is Harnack type.

The heat kernel on manifolds with ends

Consider $M = M_1 \# \cdots \# M_k$ and assume that each M_k is of Harnack type, transient. Then the heat kernel is bounded above and below by expressions of the type

$$\frac{1}{\sqrt{V_{i_x}(x,\sqrt{t})V_{i_y}(y,\sqrt{t})}} \exp\left(-\frac{d_\emptyset(x,y)^2}{t}\right) +$$

$$\left(\frac{H(x,t)H(y,t)}{V_0(\sqrt{t})} + \frac{H(x,t)}{V_{i_y}(\sqrt{t})} + \frac{H(y,t)}{V_{i_x}(\sqrt{t})}\right) \exp\left(-\frac{d_+(x,y)^2}{t}\right)$$

$|x| = d(o,x)$, $V_0(r) = \min_i V_i(r)$ and

$$H(x,t) = \min\left\{1, \frac{|x|^2}{V_{i_x}(|x|)} + \left(\int_{|x|^2}^{t} \frac{ds}{V_{i_x}(\sqrt{s})}\right)_+\right\}, \ x \in M_{i_x}.$$

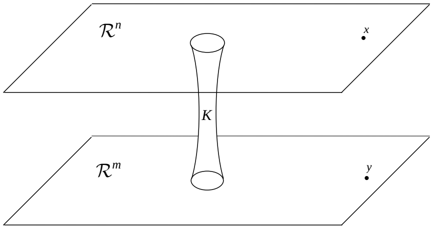
$\mathcal{R}^n \# \mathcal{R}^m$, $n \leq m$

For $x \in \mathcal{R}^n$ and $y \in \mathcal{R}^m$,

$$p(t,x,y) \simeq \left(\frac{1}{t^{n/2}|y|^{m-2}} + \frac{1}{t^{m/2}|x|^{n-2}} \right) \exp\left(-\frac{d(x,y)^2}{t} \right).$$

For fixed x, y and $t \to \infty$, $p(t,x,y) \simeq t^{-n/2}$.

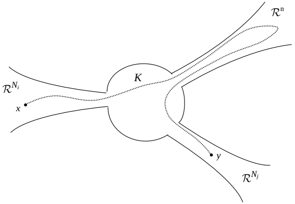
For $|x|, |y|, t \to \infty$, $|x| \simeq |y| \simeq \sqrt{t}$, $p(t,x,y) \simeq t^{-(n+m)/2+1}$.



$\mathcal{R}^{n_1}\#\mathcal{R}^{n_2}\#\mathcal{R}^{n_3}$, $n=\min\{n_i\}$

For $x \in E_i$ and $y \in E_j$, $i \neq j$ and all $t \geq 1$ the heat kernel is estimated by:

$$\left(\frac{1}{t^{n/2}\left|x\right|^{n_i-2}\left|y\right|^{n_j-2}}+\frac{1}{t^{n_j/2}\left|x\right|^{n_i-2}}+\frac{1}{t^{n_i/2}\left|y\right|^{n_j-2}}\right)\exp\left(-\frac{d(x,y)^2}{t}\right).$$



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