SURVEY OF RANDOM GROUPS

WHAT DOES A "GENERIC" GROUP LOOK LIKE?

GROUP PRESENTATION

$$G = \langle \underbrace{a_1, a_2, \dots a_k}_{\text{GENERATORS}} \mid \underbrace{r_1 = e, r_2 = e, \dots r_p = e}_{\text{RELATORS}} \rangle$$

EACH r_i IS A WORD IN THE $a_j^{\pm 1}$

EXAMPLE:

$$\mathbb{Z} \times \mathbb{Z} \simeq \langle a_1, a_2 \mid a_1 a_2 = a_2 a_1 \rangle$$

FORMALLY

$$G = F_k / \langle R \rangle$$

WHERE

F_k = FREE GROUP ON k LETTERS

$\langle R \rangle$ = NORMAL SUBGROUP OF F_k GENERATED BY THE RELATORS r_i

$\sim$

RANDOM GROUP: TAKE THE RELATORS r_i AT RANDOM AMONG ALL POSSIBLE WORDS.

DENSITY MODEL

- $2k(2k-1)^{\ell-1} \simeq (2k-1)^{\ell}$
  REDUCED WORDS OF LENGTH ℓ IN THE GENERATORS $a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_k^{\pm 1}$.

- CHOOSE $0 \leq d \leq 1$.
  TAKE $(2k-1)^{\ell d}$ REDUCED WORDS OF LENGTH ℓ AT RANDOM.
  USE THEM AS RELATORS

$\rightsquigarrow$ RANDOM GROUP AT DENSITY d

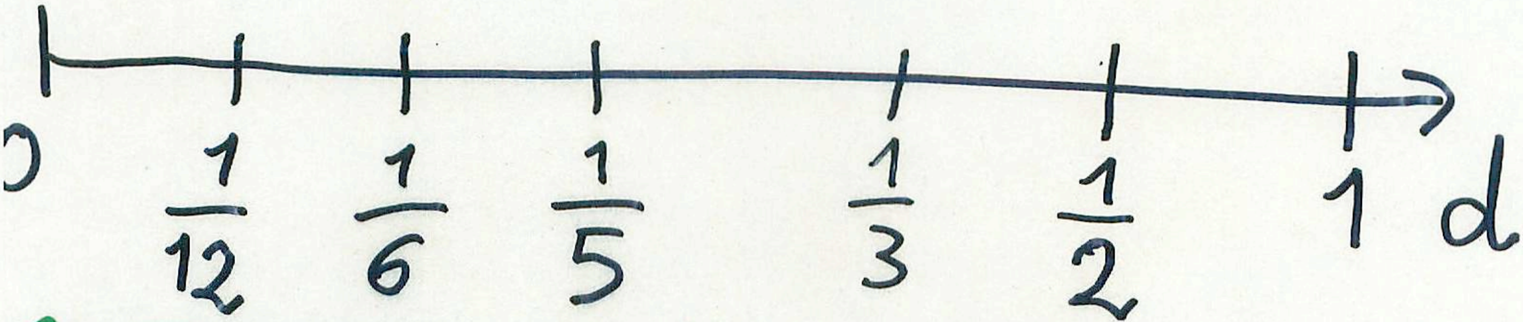
THM (GROMOV, 1992)

- IF $d > 1/2$ THEN
  $G = \{0\}$ OR $G = \{0, 1\}$
  WITH PROBA $\xrightarrow[\ell \to \infty]{} 1$

- IF $d < 1/2$ THEN
  G IS INFINITE AND HYPERBOLIC
  WITH PROBA $\xrightarrow[\ell \to \infty]{} 1$

PIGEON - HOLE PRINCIPLE

MORE THAN $\sqrt{N}$ PIGEONS
IN N PIGEON-HOLES
$\Rightarrow$ TWO PIGEONS IN
THE SAME HOLE
VERY PROBABLY



HYPERBOLIC

SMALL CANCELLATION

TRIVIAL

PROP. (T)

NO PROP (T)
DEHN ALGORITHM

ACTS ON CAT(0) CUBE COMPLEX
A-T-MENABLE

$d=0$

- FREE SUBGROUPS
- BOUNDARY AT INFINITY

TORSION-FREE
GROWTH EXPONENT
...

HYPERBOLIC GROUPS

GROUP PRESENTATION

$$G = \langle a_1 \dots a_k \mid r_1 \dots r_p \rangle$$
$$= F_k / \langle R \rangle$$

w WORD

$w = e$ IN G

$\Longleftrightarrow$ $w \in \langle R \rangle$

$$\Longleftrightarrow \quad w = \prod_{j=1}^{N} u_j \, r_{i_j}^{\pm 1} \, u_j^{-1}$$

HOW LARGE IS N ?

G HYPERBOLIC

IF $\quad N(w) \leq C \cdot |w|$

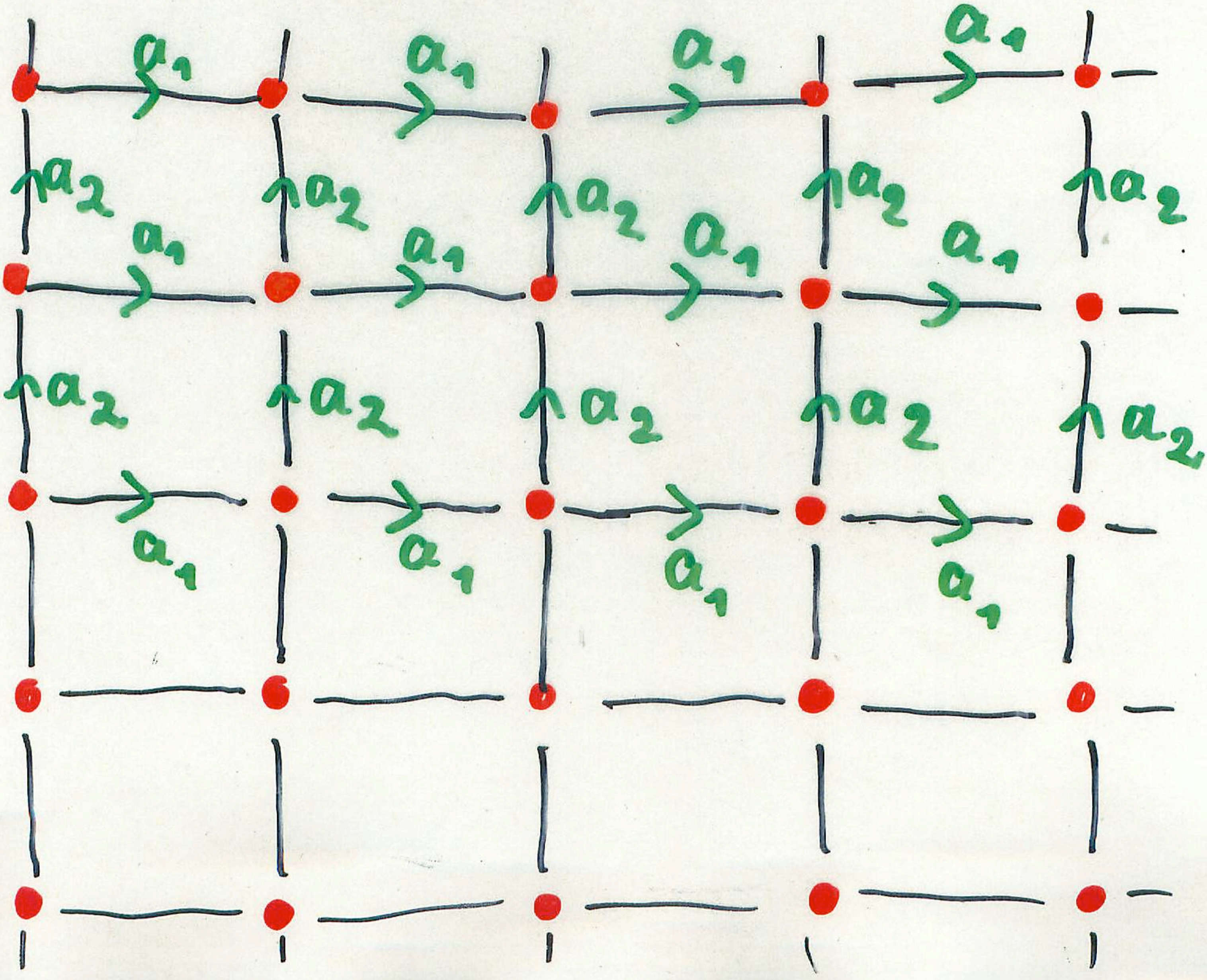
CAYLEY GRAPH OF G

VERTICES: ELEMENTS OF G

EDGES: MULTIPLICATION BY A GENERATOR

EXAMPLE:

$$\mathbb{Z} \times \mathbb{Z} = \langle a_1, a_2 \mid a_1 a_2 = a_2 a_1 \rangle$$



HYPERBOLICITY, 2

WORD $w = e$ IN G

$\Longleftrightarrow$ w CLOSED PATH IN THE CAYLEY GRAPH

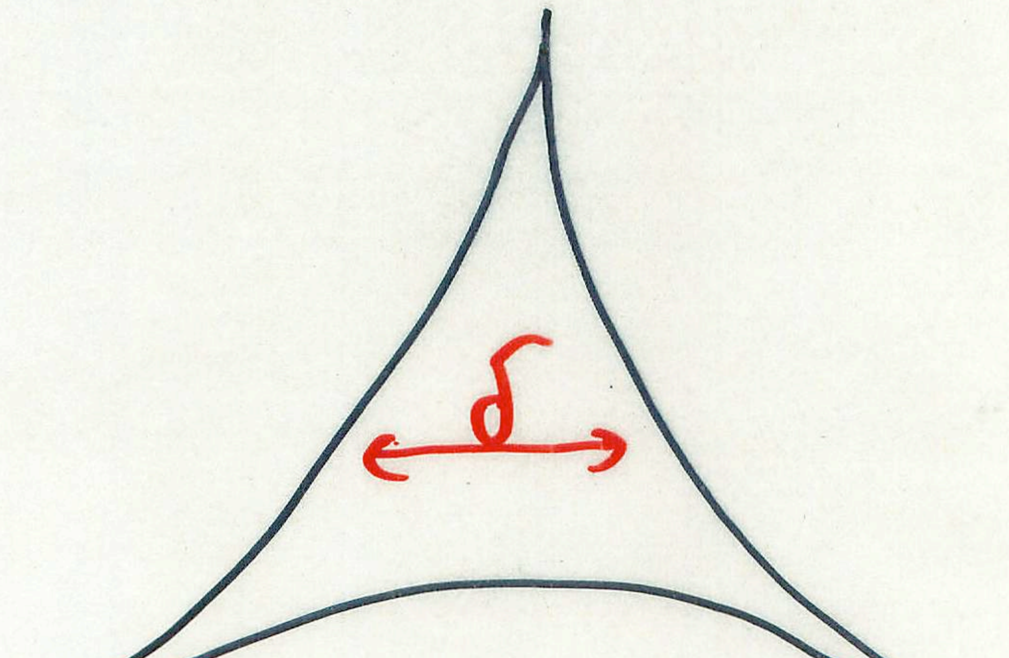
HYPERBOLICITY $\Longleftrightarrow$ EVERY CLOSED PATH IN THE CAYLEY GRAPH CAN BE FILLED USING AT MOST LINEARLY MANY RELATORS.

THIS IS THE LINEAR ISOPERIMETRIC INEQUALITY RELATED TO NEGATIVE CURVATURE.

HYPERBOLICITY, 3

THM (GROMOV)

G IS HYPERBOLIC $\iff$
$\exists\ \delta > 0$ S.T. THE CAYLEY GRAPH IS **δ-HYPERBOLIC**
i.e. EVERY GEODESIC TRIANGLE IS δ-THIN.



ITERATED QUOTIENTS

THM (YO, 2002)

LET G_0 BE A NON-ELEMENTARY TORSION-FREE HYPERBOLIC GROUP. LET B_ℓ BE THE BALL OF RADIUS ℓ IN G_0.

CHOOSE $0 \leq d \leq 1$. CHOOSE AT RANDOM A SUBSET $R \subset B_\ell$ WITH $\# R = (\# B_\ell)^d$.

LET $G = G_0 / \langle R \rangle$.

THEN:

- IF $d > 1/2$ THEN $G = \{0\}$
- IF $d < 1/2$ THEN G IS NON-ELEMENTARY HYPERBOLIC.

(WITH PROBA $\xrightarrow[\ell \to \infty]{} 1$)

APPLICATION:
1-RELATOR GROUPS

THM (I. KAPOVICH, SCHUPP, SHPILRAIN)
LET $I_\ell(m)$ BE THE NUMBER OF GROUPS ON m GENERATORS WITH A SINGLE RELATOR OF LENGTH $\leq \ell$, **UP TO ISOMORPHISM**. THEN

$$I_\ell(m) \sim \frac{1}{m!\,2^{m+1}} \frac{(2m-1)^\ell}{\ell}$$

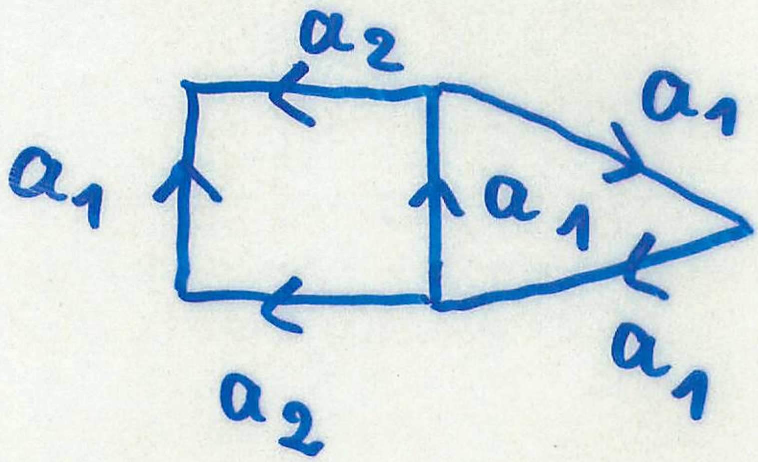
IDEA OF PROOF:

- MOST GROUPS ARE TYPICAL
- TYPICAL GROUPS HAVE FEW ISOMORPHISMS

GRAPHICAL PRESENTATIONS

GOAL: PUT AN ARBITRARY GRAPH IN SOME CAYLEY GRAPH

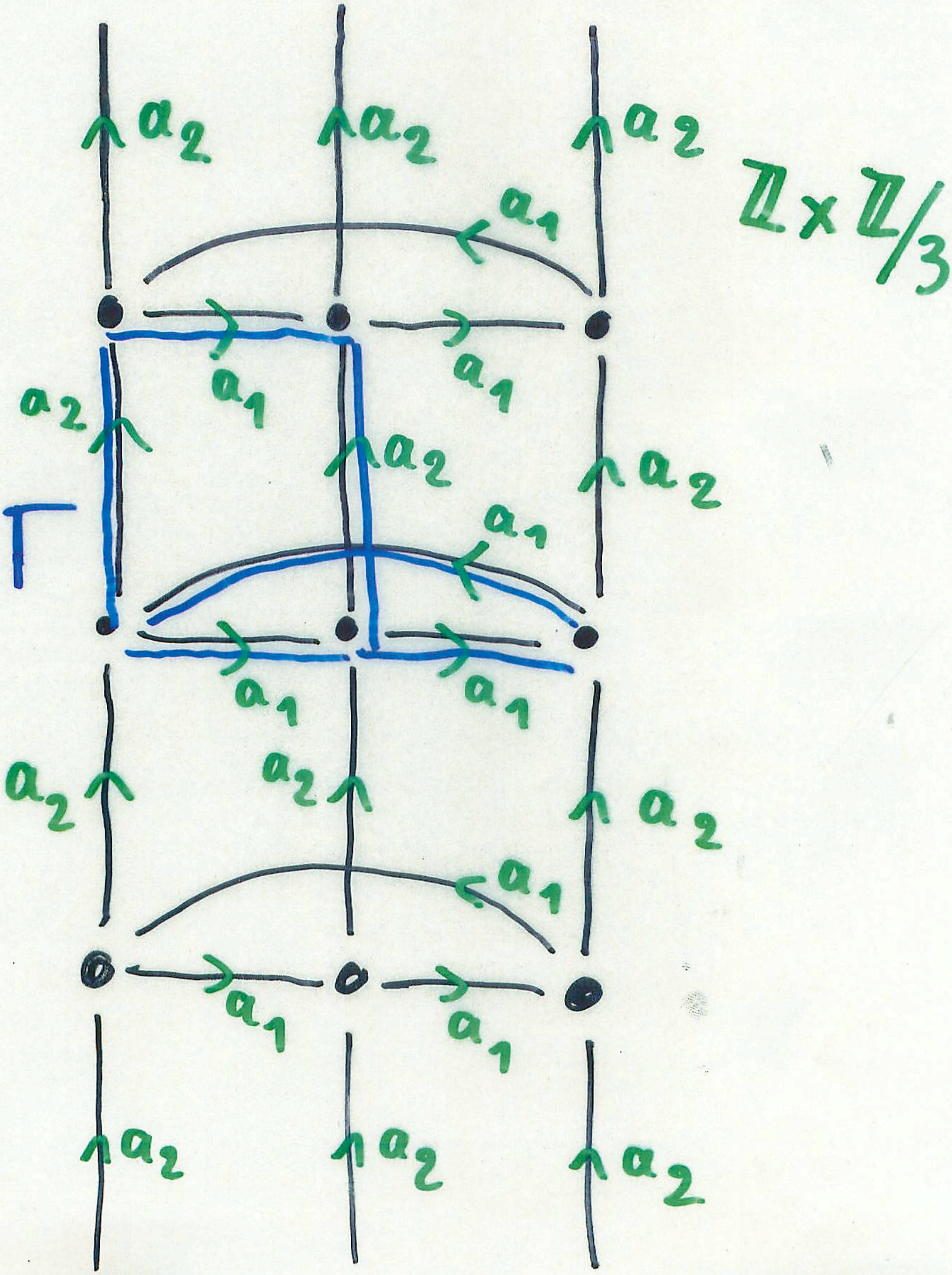
LET Γ BE A GRAPH SUCH THAT EACH EDGE OF Γ IS LABELLED BY A LETTER IN $a_1^{\pm 1}, \dots, a_k^{\pm 1}$.



DEFINE A GROUP $G(\Gamma)$:

- GENERATORS = $a_1, \dots a_k$
- RELATORS = ALL WORDS READ ON CYCLES OF Γ.

PROP: $\Gamma \longrightarrow$ CAYLEY GRAPH OF $G(\Gamma)$



PROP: (GROMOV)

LET Γ BE A "REASONABLE" GRAPH (BOUNDED DIAMETER, DEGREE, DIAMETER/GIRTH RATIO), TAKE A RANDOM LABELLING OF Γ. THEN WITH HIGH PROBA, THE MAP $\Gamma \longrightarrow \mathrm{CAYLEY}(G)$ IS (ALMOST) INJECTIVE.

ITERATION $\leadsto$ A GROUP WHOSE CAYLEY GRAPH CONTAINS EXPANDERS.

REMARK: NOT TYPICAL