Kostka functions associated to complex reflection groups

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March 25, 2017 Tokyo

Kostka polynomials $K_{\lambda,\mu}(t)$

$\lambda = (\lambda_1, \dots, \lambda_k)$: **partition of n**

$\lambda_i \in \mathbb{Z}_{\geq 0}, \quad \lambda_1 \geq \cdots \geq \lambda_k \geq 0, \quad \sum_i \lambda_i = n$

$\mathscr{P}_n$: the set of partitions of n

$s_\lambda(x) = s_\lambda(x_1, \dots, x_k) \in \mathbb{Z}[x_1, \dots, x_k]$: **Schur function**

$$s_\lambda(x) = \det(x_i^{\lambda_j + k - j}) / \det(x_i^{k-j})$$

$P_\lambda(x; t) = P_\lambda(x_1, \dots, x_k; t) \in \mathbb{Z}[x_1, \dots, x_k; t]$: **Hall-Littlewood function**

$$P_\lambda(x; t) = \sum_{w \in S_k / S_k^\lambda} w\left(x_1^{\lambda_1} \cdots x_k^{\lambda_k} \prod_{\lambda_i > \lambda_j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

$\{s_\lambda(x) \mid \lambda \in \mathscr{P}_n\}$, $\{P_\lambda(x;t) \mid \lambda \in \mathscr{P}_n\}$: bases of the space (free $\mathbb{Z}[t]$-module) of homog. symmetric functions of degee n

For $\lambda, \mu \in \mathscr{P}_n$, define **Kostka polynomials** $K_{\lambda,\mu}(t)$ by

$$s_\lambda(x) = \sum_{\mu \in \mathscr{P}_n} K_{\lambda,\mu}(t) P_\mu(x;t)$$

$(K_{\lambda,\mu}(t))_{\lambda,\mu\in\mathscr{P}_n}$: Transition matrix of two bases $\{s_\lambda(x)\}$, $\{P_\mu(x;t)\}$

$K_{\lambda,\mu}(t) \in \mathbb{Z}[t]$

Notation: $n(\lambda) = \sum_{i\ge 1}(i-1)\lambda_i$

$\widetilde{K}_{\lambda,\mu}(t) = t^{n(\mu)} K_{\lambda,\mu}(t^{-1})$: **modified Kostka polynomials**

Geometric realization of Kostka polynomials

Lusztig (1981) : geometric realization of Kostka polynomials in connection with unipotent classes.

$V = \mathbb{C}^n, \quad G = GL(V)$
$G_{\mathrm{uni}} = \{x \in G \mid x : \text{unipotent}\}$: unipotent variety

$$\mathscr{P}_n \simeq G_{\mathrm{uni}}/G$$

$$\lambda \longleftrightarrow \mathscr{O}_\lambda \ni x : \text{ Jordan type } \lambda$$

- **Dominance order of $\mathscr{P}_n$**

For partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k), \quad \mu = (\mu_1, \mu_2, \dots, \mu_k)$

$$\mu \le \lambda \Longleftrightarrow \sum_{i=1}^j \mu_i \le \sum_{i=1}^j \lambda_i \qquad \text{for each } j$$

- **Closure relations :**

$$\overline{\mathscr{O}}_\lambda = \coprod_{\mu \le \lambda} \mathscr{O}_\mu \quad (\overline{\mathscr{O}}_\lambda : \text{Zariski closure of } \mathscr{O}_\lambda)$$

$K = \mathrm{IC}(\overline{\mathscr{O}}_\lambda, \mathbb{C})$: intersection cohomology complex

$$K : \cdots \longrightarrow K_{i-1} \xrightarrow{d_{i-1}} K_i \xrightarrow{d_i} K_{i+1} \xrightarrow{d_{i+1}} \cdots$$

$K = (K_i)$: bounded complex of $\mathbb{C}$-sheaves on $\overline{\mathscr{O}}_\lambda$

$\mathscr{H}^i K = \mathrm{Ker}\, d_i / \mathrm{Im}\, d_{i-1}$: i-th cohomology sheaf

$\mathscr{H}^i_x K$: stalk of $\mathscr{H}^i K$ at $x \in \overline{\mathscr{O}}_\lambda$ (fin. dim. vector space ove $\mathbb{C}$)

Theorem (Lusztig 1981)

For any odd i, we have $\mathscr{H}^i K = 0$. Moreover, for $x \in \mathscr{O}_\mu \subset \overline{\mathscr{O}}_\lambda$,

$$\widetilde{K}_{\lambda,\mu}(t) = t^{n(\lambda)} \sum_{i \geq 0} (\dim_{\mathbb{C}} \mathscr{H}^{2i}_x K) t^i$$

In particular, $K_{\lambda,\mu}[t] \in \mathbb{Z}_{\geq 0}[t]$. (Theorem of Lascoux-Schützenberger)

Springer corresp. of GL_n and Kostka polynomials

$G = GL_n$, B : Borel subgroup of G, $T \subset B$: maximal torus,

$U \subset B$: maximal unipotent subgroup

$N_G(T)/T \simeq S_n$: Weyl group of GL_n

$$\pi_1 : \widetilde{G}_{\text{uni}} = \{(x, gB) \in G_{\text{uni}} \times G/B \mid g^{-1}xg \in U\} \to G_{\text{uni}}, \quad (x, gB) \mapsto x$$

$\widetilde{G}_{\text{uni}}$: smooth, π_1 : proper surjective (**Springer resolution** of G_{uni})

Theorem (Lusztig, Borho-MacPherson)

$(\pi_1)_*\mathbb{C}[\dim G_{\text{uni}}]$ is a semisimple perverse sheaf on G_{uni}, equipped with S_n-action, and is decomposed as

$$(\pi_1)_*\mathbb{C}[\dim G_{\text{uni}}] \simeq \bigoplus_{\lambda \in \mathscr{P}_n} V_\lambda \otimes \mathrm{IC}(\overline{\mathscr{O}}_\lambda, \mathbb{C})[\dim \mathscr{O}_\lambda],$$

where V_λ : irreducible S_n-module corresp. to $\lambda \in \mathscr{P}_n$.

For $x \in G_{\mathrm{uni}}$,

$$\mathscr{B}_x = \{gB \in G/B \mid g^{-1}xg \in B\} \simeq \pi_1^{-1}(x)$$

is called the **Springer fibre** of x.

$K = (\pi_1)_*\mathbb{C}$: complex with S_n-action

$$\mathscr{H}_x^i K \simeq H^i(\pi_1^{-1}(x), \mathbb{C}) \simeq H^i(\mathscr{B}_x, \mathbb{C}) \qquad \text{as } S_n\text{-modules}$$

S_n-module $H^i(\mathscr{B}_x, \mathbb{C})$: called the **Springer module** of S_n

Put $d_x = \dim \mathscr{B}_x$. $H^{2d_x}(\mathscr{B}_x, \mathbb{C})$: cohomology of highest degree

Corollary (Springer correps.)

For $x \in \mathscr{O}_\mu$, $H^{2d_x}(\mathscr{B}_x, \mathbb{C}) \simeq V_\lambda$ as S_n-modules. By the corresp. $x \mapsto H^{2d_x}(\mathscr{B}_x, \mathbb{C})$, obtain a natural bijection $G_{\mathrm{uni}}/G \to S_n^\wedge$.

Proposition

$$\widetilde{K}_{\lambda,\mu}(t) = \sum_{i \geq n(\lambda)} \langle V_\lambda, H^{2i}(\mathscr{B}_x, \mathbb{C})\rangle_{S_n} t^i \qquad (x \in \mathscr{O}_\mu)$$

Generalization of Kostka polynomials

$\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}), \quad \sum_{i=1}^{r} |\lambda^{(i)}| = n$: **r-partition of n**

$\mathscr{P}_{n,r}$: the set of r-partitions of n.

S (2004) Introduced Kostka functions $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbb{Q}(t)$ $(\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{n,r})$ assoc. to **complex reflection group** $W_{n,r} = S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$

- $(K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t))$:Transition matrix between basis $\{s_{\boldsymbol{\lambda}}(x)\}$ of Schur functions and basis $\{P^{\pm}_{\boldsymbol{\mu}}(x;t)\}$ of "Hall-Littlewood functions"
- If $r = 1, 2$, $K^{+}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = K^{-}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbb{Z}[t]$. (write as $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$)
- If $r = 2$, $W_{n,2}$: Weyl group of type B_n (C_n). But those Kostka functins have no relations with Sp_{2n} or SO_{2n+1}.

Characterization of $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$

For $\boldsymbol{\lambda} = \lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathscr{P}_{n,r}$, put

$$a(\boldsymbol{\lambda}) = r \cdot n(\boldsymbol{\lambda}) + |\lambda^{(2)}| + 2|\lambda^{(3)}| + \cdots + (r-1)|\lambda^{(r)}|$$

where $n(\boldsymbol{\lambda}) = n(\lambda^{(1)}) + \cdots + n(\lambda^{(r)})$.

$\widetilde{K}^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{a(\boldsymbol{\mu})} K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t^{-1})$: **modified Kostka function**

For $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathscr{P}_{n,r}$, choose $m \gg 0$ so that $\lambda^{(k)} = (\lambda^{(1)}_1, \dots, \lambda^{(k)}_m)$ for any k. Define a sequence $c(\boldsymbol{\lambda}) \in \mathbf{Z}^{rm}_{\geq 0}$ by

$$c(\boldsymbol{\lambda}) = (\lambda^{(1)}_1, \lambda^{(2)}_1, \dots, \lambda^{(r)}_1, \lambda^{(1)}_2, \lambda^{(2)}_2, \dots, \lambda^{(r)}_2, \dots, \lambda^{(1)}_m, \lambda^{(2)}_m, \dots, \lambda^{(r)}_m).$$

Define a partial order $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ in $\mathscr{P}_{n,r}$ (**dominance order on $\mathscr{P}_{n,r}$**) by $c(\boldsymbol{\mu}) \leq c(\boldsymbol{\lambda})$ under (gen. of) the dominance order on $\mathscr{P}_{mr}$.

For any character χ of $W_{n,r}$, define $R(\chi)$ by

$$R(\chi) = \frac{\prod_{1\le i\le r}(t^{ir}-1)}{|W_{n,r}|}\sum_{w\in W_{n,r}}\frac{\det_{\mathbf{V}}(w)\chi(w)}{\det_{\mathbf{V}}(t\cdot 1_{\mathbf{V}} - w)}$$

where $\mathbf{V}$: reflection representation of $W_{n,r}$.

Fix a total order $\preceq$ on $\mathscr{P}_{n,r}$ compatible with $\le$,
consider matrices indexed by $\mathscr{P}_{n,r}$ with respect to $\preceq$. .

Define a matrix $\Omega = (\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}})_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{n,r}}$ by

$$\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}} = t^{N^*}R(\rho^{\boldsymbol{\lambda}}\otimes\overline{\rho^{\boldsymbol{\mu}}}\otimes\overline{\det}_{\mathbf{V}})$$

where $\rho^{\boldsymbol{\lambda}}$: irred. character of $W_{n,r}$ corresp. to $\boldsymbol{\lambda}\in\mathscr{P}_{n,r}$,
N^* : number of reflections in $W_{n,r}$.

Theorem (S 2004)

There exist unique matrices $P^{\pm}, \varLambda$ on $\mathbb{Q}(t)$ satisfying the relation

$$P^{-}\varLambda\,{}^{t}P^{+} = \Omega,$$

where $\varLambda$: diagonal, $P^{\pm} = (p^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}})$: lower triangular with $p^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\lambda}} = t^{a(\boldsymbol{\lambda})}$.
Then $p^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \widetilde{K}^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$.

Remarks.

1. Construction of $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ depends on the choice of $\preceq$. Later we show independence of $\preceq$, and $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbb{Z}[t]$.
2. If $r = 1, 2$, $W_{n,r}$: Weyl group, Ω : symmetric, so $P^{+} = P^{-}$.
   If $r \geq 3$, Ω : not symmetric, so $P^{+} \neq P^{-}$.
3. $K^{-}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ have better properties than $K^{+}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ w.r.t geometry.

Enhanced variety $GL(V) \times V$

Achar-Henderson (2008) : geometric realization of $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ in the case where $r = 2$.

$G = GL(V)$, $\quad V = \mathbb{C}^n$: n-dim. vector space
$G_{\mathrm{uni}} \times V \subset G \times V$: **Enhanced variety** , G acts diagonally

(Achar-Henderson, Travkin) :

$$(G_{\mathrm{uni}} \times V)/G \simeq \mathscr{P}_{n,2}, \quad \mathscr{O}_{\boldsymbol{\lambda}} \leftrightarrow \boldsymbol{\lambda}$$

Theorem (Achar-Henderson 2008)

Put $K = \mathrm{IC}(\overline{\mathscr{O}}_{\boldsymbol{\lambda}}, \mathbb{C})$. If i is odd, then $\mathscr{H}^i K = 0$.
For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{n,2}$ and $(x, v) \in \mathscr{O}_{\boldsymbol{\mu}} \subseteq \overline{\mathscr{O}}_{\boldsymbol{\lambda}}$,

$$t^{a(\boldsymbol{\lambda})} \sum_{i \geq 0} (\dim_{\mathbb{C}} \mathscr{H}^{2i}_{(x,v)} K) t^{2i} = \widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t).$$

Exotic symmetric space $GL(V)/Sp(V) \times V$

$G = GL_{2n}(\mathbb{C}) \simeq GL(V)$, V : $2n$-dim. vector space

$\theta : G \to G$, $\theta(g) = J^{-1}({}^t g^{-1})J$: involution, $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

$H := \{g \in G \mid \theta(g) = g\} \simeq Sp_{2n}(\mathbb{C})$ G/H : symmetric space

Define $G^{\iota\theta} = \{g \in G \mid \theta(g) = g^{-1}\} = \{g\theta(g)^{-1} \mid g \in G\}$,

where $\iota : G \to G, g \mapsto g^{-1}$.

The map $G \to G$, $g \mapsto g\theta(g)^{-1}$ gives isom. $G/H \xrightarrow{\sim} G^{\iota\theta}$.

$$\mathscr{X} = G^{\iota\theta} \times V \quad : \textbf{exotic symmetric space}$$

$$\mathscr{X}_{\mathrm{uni}} = G^{\iota\theta}_{\mathrm{uni}} \times V \quad \simeq \textbf{exotic nilpotent cone} \text{ by Kato}$$

H acts diagonally on $\mathscr{X}$ and $\mathscr{X}_{\mathrm{uni}}$.

(Kato) $\mathscr{X}_{\text{uni}}/H \simeq \mathscr{P}_{n,2} \qquad (\mathscr{O}_{\boldsymbol{\lambda}} \leftrightarrow \boldsymbol{\lambda})$

$B = TU$: θ-stable Borel subgroup, maximal torus, and maximal unipotent subgroup of G
$T^\theta \subset B^\theta$: maximal torus and Borel subgroup of H
$\mathscr{B} = H/B^\theta$: flag variety of H, $\quad N_H(T^\theta)/T^\theta \simeq W_{n,2}$

Fix an isotropic flag in V stable by B^θ

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset V \qquad (\dim M_i = i)$$

Put $\widetilde{\mathscr{X}}_{\text{uni}} = \{(x, v, gB^\theta) \in G^{\iota\theta}_{\text{uni}} \times V \times \mathscr{B} \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}v \in M_n\}$
and $\pi_1 : \widetilde{\mathscr{X}}_{\text{uni}} \to \mathscr{X}_{\text{uni}} \quad (x, v, gB^\theta) \mapsto (x, v)$.

For $z = (x, v) \in \mathscr{O}_{\boldsymbol{\lambda}}$, consider $\pi^{-1}(z) \subset \widetilde{\mathscr{X}}_{\text{uni}}$ **(exotic Springer fibre)**

$$\pi^{-1}(z) \simeq \mathscr{B}_z = \{gB^\theta \in \mathscr{B} \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}v \in M_n\}$$

Put $d_{\boldsymbol{\lambda}} = (\dim \mathscr{X}_{\text{uni}} - \dim \mathscr{O}_{\boldsymbol{\lambda}})/2$

Theorem (Kato, S.-Sorlin)

$(\pi_1)_*\mathbb{C}[\dim \mathscr{X}_{\mathrm{uni}}]$ is a semisimple perverse sheaf on $\mathscr{X}_{\mathrm{uni}}$, equipped with $W_{n,2}$-action, and is decomposed as

$$(\pi_1)_*\mathbb{C}[\dim \mathscr{X}_{\mathrm{uni}}] \simeq \bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}_{n,2}} \widetilde{V}_{\boldsymbol{\lambda}} \otimes \mathrm{IC}(\overline{\mathscr{O}}_{\boldsymbol{\lambda}}, \mathbb{C})[\dim \mathscr{O}_{\boldsymbol{\lambda}}],$$

where $\widetilde{V}_{\boldsymbol{\lambda}}$; irred. representation of $W_{n,2}$ corresp. to $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$.

Corollary (Springer corresp.) (Kato, S.-Sorlin)

1. $H^i(\mathscr{B}_z, \mathbb{C})$ has a structure of $W_{n,2}$-module **(Springer module)**.
2. $\dim \mathscr{B}_z = d_{\boldsymbol{\lambda}}$ for $z \in \mathscr{O}_{\boldsymbol{\lambda}}$.
3. $H^{2d_{\boldsymbol{\lambda}}}(\mathscr{B}_z, \mathbb{C}) \simeq \widetilde{V}_{\boldsymbol{\lambda}}$ as $W_{n,2}$-modules for $z \in \mathscr{O}_{\boldsymbol{\lambda}}$.
   The corresp. $\mathscr{O}_{\boldsymbol{\lambda}} \mapsto H^{2d_{\boldsymbol{\lambda}}}(\mathscr{B}_z, \mathbb{C})$ gives a natural bijection

$$\mathscr{X}_{\mathrm{uni}}/H \simeq W_{n,2}^{\wedge}$$

Geometric realization of Kostka functions

Theorem (Kato, S. -Sorlin)

Put $K = \mathrm{IC}(\overline{\mathscr{O}}_{\boldsymbol{\lambda}}, \mathbb{C})$. Then for $z \in \mathscr{O}_{\boldsymbol{\mu}} \subset \overline{\mathscr{O}}_{\boldsymbol{\lambda}}$, we have $\mathscr{H}^i K = 0$ unless $i \equiv 0 \pmod 4$, and

$$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{a(\boldsymbol{\lambda})} \sum_{i \geq 0} (\dim \mathscr{H}_z^{4i} K) t^{2i}$$

Corollary

For $z \in \mathscr{O}_{\boldsymbol{\mu}}$, $\quad \widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = \sum_{i \geq a(\boldsymbol{\lambda})} \langle H^{2i}(\mathscr{B}_z, \mathbb{C}), \widetilde{V}_{\boldsymbol{\lambda}} \rangle_{W_{n,2}}\, t^i.$

Remark. Springer correspondence can be formulated also for the enhanced variety. But in that case, $W_{n,2}$ does not appear. Only the produc of symmetric groups $S_m \times S_{n-m}$ $(0 \leq m \leq n)$ appears.

Exotic symmeric space of higher level

For an integer $r \geq 2$, consider the varieties

$$\mathscr{X} = G^{\iota\theta} \times V^{r-1} \supset \mathscr{X}_{\text{uni}} = G^{\iota\theta}_{\text{uni}} \times V^{r-1}$$

with diagonal action of H on $\mathscr{X}, \mathscr{X}_{\text{uni}}$ **(exotic sym. space of level r)**.

Remark. If $r = 2$, $\mathscr{X} = G^{\iota\theta} \times V$: exotic symmetric space.
If $r \geq 3$, $\mathscr{X}_{\text{uni}}$ has **infinitely many** H-orbits.

In fact, since $\dim G^{\iota\theta}_{\text{uni}} = 2n^2 - 2n$,

$$\dim \mathscr{X}_{\text{uni}} = 2n^2 - 2n + (r-1)2n > \dim H = 2n^2 + n$$

Put

$$\widetilde{\mathscr{X}}_{\text{uni}} = \{(x, \mathbf{v}, gB^\theta) \in G^{\iota\theta}_{\text{uni}} \times V^{r-1} \times \mathscr{B} \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}\mathbf{v} \in M_n^{r-1}\},$$

$$\pi_1 : \widetilde{\mathscr{X}}_{\text{uni}} \to \mathscr{X}_{\text{uni}}, \quad (x, \mathbf{v}, gB^\theta) \mapsto (x, \mathbf{v}) \quad \text{: } \textbf{not surjective}.$$

Fix $\mathbf{m} = (m_1, \dots, m_{r-1}) \in \mathbb{Z}_{\geq 0}^{r-1}$ s.t. $\sum m_i = n$.
Put $p_i = m_1 + \cdots + m_i$ for each i.

Define

$$\mathscr{X}_{\mathbf{m},\mathrm{uni}} = \bigcup_{g \in H} g(U^{\iota\theta} \times \prod_{i=1}^{r-1} M_{p_i}) \subset \mathscr{X}_{\mathrm{uni}},$$

$$\widetilde{\mathscr{X}}_{\mathbf{m},\mathrm{uni}} = \pi_1^{-1}(\mathscr{X}_{\mathbf{m},\mathrm{uni}}),$$

let $\pi_{\mathbf{m},1} : \widetilde{\mathscr{X}}_{\mathbf{m},\mathrm{uni}} \to \mathscr{X}_{\mathbf{m},\mathrm{uni}}$: restriction of π_1,

$$\widetilde{\mathscr{P}}(\mathbf{m}) = \{\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathscr{P}_{n,r} \mid |\lambda^{(i)}| = m_i \text{ for } 1 \leq i \leq r-2\}.$$

Theorem (S) (Springer correspondence for $W_{n,r}$)

Let $d'_{\mathbf{m}} = \dim \mathscr{X}_{\mathbf{m},\mathrm{uni}}$. Then $(\pi_{\mathbf{m},1})_*\mathbb{C}[d'_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathscr{X}_{\mathbf{m},\mathrm{uni}}$ equipped with $W_{n,r}$-action, and is decomposed as

$$(\pi_{\mathbf{m},1})_*\mathbb{C}[d'_{\mathbf{m}}] \simeq \bigoplus_{\boldsymbol{\lambda} \in \widetilde{\mathscr{P}}(\mathbf{m})} \widetilde{V}_{\boldsymbol{\lambda}} \otimes \mathrm{IC}(\overline{X}_{\boldsymbol{\lambda}}, \mathbb{C})[\dim X_{\boldsymbol{\lambda}}],$$

where $\widetilde{V}_{\boldsymbol{\lambda}}$: irred. rep. of $W_{n,r}$ corresp. to $\boldsymbol{\lambda} \in \widetilde{\mathscr{P}}(\mathbf{m}) \subset \mathscr{P}_{n,r}$.

Varieties $X_{\boldsymbol{\lambda}}$

Note: $\coprod_{\mathbf{m}} \widetilde{\mathscr{P}}(\mathbf{m}) = \mathscr{P}_{n,r}$

For each $\boldsymbol{\lambda} \in \mathscr{P}_{n,r}$, one can construct a locally closed, smooth, irreducible, H-stable subvariety $X_{\boldsymbol{\lambda}}$ of $\mathscr{X}_{\text{uni}}$ satisfying the following properties;

1. $\pi_1(\widetilde{\mathscr{X}}_{\text{uni}}) = \bigcup_{\boldsymbol{\lambda} \in \mathscr{P}_{n,r}} \overline{X}_{\boldsymbol{\lambda}}$.
2. $\overline{X}_{\boldsymbol{\lambda}(\mathbf{m})} = \mathscr{X}_{\mathbf{m},\text{uni}}$ for $\boldsymbol{\lambda}(\mathbf{m}) = ((m_1), (m_2), \dots, (m_{r-1}), \emptyset)$.
3. Assume that $\boldsymbol{\mu} \in \widetilde{\mathscr{P}}(\mathbf{m})$. Then $X_{\boldsymbol{\mu}} \subset \mathscr{X}_{\mathbf{m},\text{uni}}$.
4. If $r = 2$, $X_{\boldsymbol{\lambda}}$ coincides with H-orbit $\mathscr{O}_{\boldsymbol{\lambda}}$.

Remark. $X_{\boldsymbol{\lambda}}$: analogue of H-orbit $\mathscr{O}_{\boldsymbol{\lambda}}$ in the case $r = 1, 2$. However, if $r \geq 3$, $\pi_1(\widetilde{\mathscr{X}}_{\text{uni}}) \neq \bigcup_{\boldsymbol{\lambda} \in \mathscr{P}_{n,r}} X_{\boldsymbol{\lambda}}$, and $X_{\boldsymbol{\lambda}}$ are not mutually disjoint.

Springer fibres

For $z = (x, \mathbf{v}) \in \mathscr{X}_{\mathbf{m},\mathrm{uni}}$, put

$$\mathscr{B}_z = \pi_1^{-1}(z) \simeq \{gB^\theta \in \mathscr{B} \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}\mathbf{v} \in M_n^{r-1}\},$$

$$\mathscr{B}_z^{(\mathbf{m})} = \{gB^\theta \in \mathscr{B}_z \mid g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i}\}..$$

Hence $\mathscr{B}_z^{(\mathbf{m})} \subset \mathscr{B}_z$.

$\mathscr{B}_z$: called **Springer fibre**

$\mathscr{B}_z^{(\mathbf{m})}$: called **small Springer fibre**

Remark. If $r \geq 3$, $\dim \mathscr{B}_z, \dim \mathscr{B}_z^{(\mathbf{m})}$ are **not necessarily constant** for $z \in X_{\boldsymbol{\lambda}}$.

Assume that $\boldsymbol{\lambda} \in \widetilde{\mathscr{P}}(\mathbf{m})$ (then $X_{\boldsymbol{\lambda}} \subset \mathscr{X}_{\mathbf{m},\mathrm{uni}}$). Put

$$d_{\boldsymbol{\lambda}} = (\dim \mathscr{X}_{\mathbf{m},\mathrm{uni}} - \dim X_{\boldsymbol{\lambda}})/2.$$

Proposition (Springer correspondence)

Assume that $\boldsymbol{\lambda} \in \widetilde{\mathscr{P}}(\mathbf{m})$.

1. $X^0_{\boldsymbol{\lambda}} = \{z \in X_{\boldsymbol{\lambda}} \mid \dim \mathscr{B}_z^{(\mathbf{m})} = d_{\boldsymbol{\lambda}}\}$ forms open dense subset of $X_{\boldsymbol{\lambda}}$.
2. Take $z \in X^0_{\boldsymbol{\lambda}}$. Then $H^{2d_{\boldsymbol{\lambda}}}(\mathscr{B}_z, \mathbb{C}) \simeq \widetilde{V}_{\boldsymbol{\lambda}}$ as $W_{n,r}$-modules.

The map $X_{\boldsymbol{\lambda}} \mapsto H^{2d_{\boldsymbol{\lambda}}}(\mathscr{B}_z, \mathbb{C})$ gives a bijective correspondence

$$\{X_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,r}\} \simeq W^{\wedge}_{n,r}$$

Remarks.

1. In the case of $GL(V) \times V^{r-1}$ (enhanced variety of higher level), Springer correspondence can be proved. In that case, subgroups $S_{m_1} \times \cdots \times S_{m_r} \subset S_n$ with $\sum_i m_i = n$ appear.
2. Geometric realization of Kostka functions for $r \geq 3$ is not yet known. Only exist partial results in case of enhanced vareity of higher level.

Kostka functions with multi-variables

$y = (y_1, y_2, \dots)$: infinitely many variables
$\varLambda = \varLambda(y) = \bigoplus_{n \ge 0} \varLambda^n$: ring of symmetric functions w.r.t. y over $\mathbb{Z}$.
$\{s_\lambda(y) \mid \lambda \in \mathscr{P}_n\}$: free $\mathbb{Z}$-basis of $\varLambda^n$.

Prepare r-types variables $x = (x^{(1)}, \dots, x^{(r)})$
with $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dots)$: infinitely many variables

$$\Xi = \Xi(x) = \varLambda(x^{(1)}) \otimes \cdots \otimes \varLambda(x^{(r)})$$

$\Xi = \bigoplus_{n \ge 0} \Xi^n$: ring of symmetric functions w.r.t. $x^{(1)}, \dots, x^{(r)}$.

For $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathscr{P}_{n,r}$, define **Schur function** $s_{\boldsymbol{\lambda}}(x)$ by

$$s_{\boldsymbol{\lambda}}(x) = \prod_{k=1}^{r} s_{\lambda^{(k)}}(x^{(k)}).$$

Then $\{s_{\boldsymbol{\lambda}}(x) \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,r}\}$: free $\mathbb{Z}$-basis of Ξ^n.

$\mathbb{Z}[\mathbf{t}] = \mathbb{Z}[t_1, \dots, t_r]$: polynomial ring with r-variables $\mathbf{t} = (t_1, \dots, t_r)$.

$\Xi[\mathbf{t}] = \mathbb{Z}[\mathbf{t}] \otimes_{\mathbb{Z}} \Xi = \bigoplus_{n \geq 0} \Xi^n[\mathbf{t}]$, $\quad \Xi(\mathbf{t}) = \mathbb{Q}(\mathbf{t}) \otimes_{\mathbf{Z}} \Xi = \bigoplus_{n \geq 0} \Xi^n_{\mathbb{Q}}(\mathbf{t})$

Fixing $m \geq 1$, consider finitely many variables $x_1^{(k)}, \dots, x_m^{(k)}$ for each k.

For a parameter t, define a function $q_{s,\pm}^{(k)}(x;t)$ by

$$q_{s,\pm}^{(k)}(x;t) = \sum_{i=1}^{m} (x_i^{(k)})^{s-1} \frac{\prod_{j \geq 1}(x_i^{(k)} - t x_j^{(k \mp 1)})}{\prod_{j \neq i}(x_i^{(k)} - x_j^{(k)})},$$

if $s \geq 1$, and by $q_{s,\pm}^{(k)}(x;t) = 1$ if $s = 0$. (**Here we regard** $k \in \mathbb{Z}/r\mathbb{Z}$)

Note : $q_{s,\pm}^{(k)}(x;t) \in \mathbb{Z}[x;t]$, symmetric w.r.t. $x^{(k)}$ and $x^{(k \mp 1)}$.

For $\boldsymbol{\lambda} \in \mathscr{P}_{n,r}$, choose $m \gg 0$ s.t. $\lambda^{(k)} = (\lambda_1^{(k)}, \dots, \lambda_m^{(k)})$ for any k.

Define $q^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}) \in \mathbb{Z}[x;\mathbf{t}]$ by

$$q^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}) = \prod_{k\in\mathbb{Z}/r\mathbb{Z}}\prod_{i=1}^{m} q^{(k)}_{\lambda^{(k)}_i,\pm}(x;t_{k-c}),$$

where $c = 1$ in the "+"-case, and $c = 0$ in the "-"-case.

Bt taking $m \mapsto \infty$, obtain symmetric function $q^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}) \in \Xi^n[\mathbf{t}]$.
$\{q^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}) \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,r}\}$: $\mathbb{Q}(\mathbf{t})$-basis of $\Xi^n_{\mathbb{Q}}(\mathbf{t})$.

We fix the total order $\preceq$ on $\mathscr{P}_{n,r}$ compatible with $\le$.

Theorem (S, 2004, 2017)

For any $\boldsymbol{\lambda} \in \mathscr{P}_{n,r}$, there exists a unique function $P^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}) \in \Xi^n_{\mathbb{Q}}(\mathbf{t})$ satisfying the following properties;

1. $P^{\pm}_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu}\succeq\boldsymbol{\lambda}} c_{\boldsymbol{\lambda},\boldsymbol{\mu}} q^{\pm}_{\boldsymbol{\mu}}$ with $c_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{Q}(\mathbf{t})$, and $c_{\boldsymbol{\lambda}\boldsymbol{\lambda}} \ne 0$.
2. $P^{\pm}_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu}\preceq\boldsymbol{\lambda}} u_{\boldsymbol{\lambda}\boldsymbol{\mu}} s_{\boldsymbol{\mu}}$ with $u_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{Q}(\mathbf{t})$ and $u_{\boldsymbol{\lambda}\boldsymbol{\lambda}} = 1$,

Similarly, there exists a unique function $Q^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}) \in \Xi^n_{\mathbb{Q}}(\mathbf{t})$, defined by the condition $c_{\boldsymbol{\lambda}\boldsymbol{\lambda}} = 1$ in (i), and $u_{\lambda\lambda} \ne 0$ in (ii).

Bt taking $m \mapsto \infty$, we obtain $P^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}), Q^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}) \in \Xi^n_{\mathbb{Q}}(\mathbf{t})$, called **Hall-Littlewood functions with multi-parameter**.

- By definition, $P^{\pm}_{\boldsymbol{\lambda}}$ coincides with $Q^{\pm}_{\boldsymbol{\lambda}}$, up to constant $\in \mathbb{Q}(\mathbf{t})$.
- $\{P^{\pm}_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,r}\}$, $\{Q^{\pm}_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,r}\}$ give $\mathbb{Q}(\mathbf{t})$-bases of $\Xi^n_{\mathbb{Q}}(\mathbf{t})$.

For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{n,r}$, define $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathbf{t})$ by

$$s_{\boldsymbol{\lambda}}(x) = \sum_{\boldsymbol{\mu} \in \mathscr{P}_{n,r}} K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathbf{t}) P^{\pm}_{\boldsymbol{\mu}}(x;\mathbf{t}).$$

$K^{\pm}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\mathbf{t}) = K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t_1, \dots, t_r)$: **Kostka function with multi-variable**

Remarks. (i) When $t_1 = \cdots = t_r = t$, $P^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t})$ coincides with $P^{\pm}_{\boldsymbol{\lambda}}(x;t)$ introduced in (S, 2004). Hence in this case, $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t,\dots,t)$ coincides with one variable $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ introduced there.

(ii) In multi-variable case, no formula such as $P^- \Lambda\, {}^tP^+ = \Omega$. So the relation with complex reflection groups is not clear.

Hall-Littlewood functions

From the definition, $P^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}), Q^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t})$ are functions of x with coefficients in $\mathbb{Q}(\mathbf{t})$, and depend on the choice of $\preceq$ on $\mathscr{P}_{n,r}$.
Hence $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathbf{t}) \in \mathbb{Q}(\mathbf{t})$**, and may depend on the choice of** $\preceq$.

Theorem

There exists a closed formula for $P^{\pm}_{\boldsymbol{\lambda}}$ and $Q^{\pm}_{\boldsymbol{\lambda}}$. In particular we have

1. $P^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}), Q^{\pm}_{\boldsymbol{\lambda}}(x;\mathbf{t}) \in \Xi^n[\mathbf{t}]$.
2. $\{P^{\pm}_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,r}\}$ gives a $\mathbb{Z}[\mathbf{t}]$-basis of $\Xi^n[\mathbf{t}]$.
3. $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathbf{t}) \in \mathbb{Z}[\mathbf{t}]$.
4. In the defining formula of $P^{\pm}_{\boldsymbol{\lambda}}$ in terms of $q^{\pm}_{\boldsymbol{\mu}}$ or $s_{\boldsymbol{\mu}}$, possible to replace $\preceq$ by $\leq$. Hence, $P^{\pm}_{\boldsymbol{\lambda}}$, $K^{\pm}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ are independent of the choice of $\preceq$.

Remark. For the proof of the closed formula, we use, in an essential way, the argument based on the multi-parameter case.

Conjecture of Finkelberg-Ionov

$\varepsilon_1, \dots, \varepsilon_m$: standard basis of $\mathbb{Z}^m$,
$R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le m\}$: positive roots of type A_{m-1}

For any $\xi = (\xi_i) \in \mathbb{Z}^m$, define a polynomial $L(\xi; t) \in \mathbb{Z}[t]$ by

$$L(\xi; t) = \sum_{(m_\alpha)} t^{\sum m_\alpha},$$

where (m_α) runs over all non-negative integers such that $\xi = \sum_{\alpha \in R^+} m_\alpha \alpha$. $L(\xi; t)$: called **Lusztig's partition function**.

- **Kostka polynomials and Lusztig's partition function**

For $\lambda, \mu \in \mathscr{P}_n$, take $m \gg 0$ s.t. $\lambda = (\lambda_1, \dots, \lambda_m)$, $\mu = (\mu_1, \dots, \mu_m)$.
Put $\delta_0 = (m-1, m-2, \dots, 1, 0)$. Then $\lambda + \delta_0, \mu + \delta_0 \in \mathbb{Z}^m$.

$$K_{\lambda,\mu}(t) = \sum_{w \in S_m} \varepsilon(w) L(w^{-1}(\lambda + \delta_0) - (\mu + \delta_0); t)$$

Remark. Lusztig : gave q-analogue of weight multiplicity by affine Kazhdan-Lusztig polynomials

Lusztig's conjecture : q-analogue of Kostant's weight multiplicity formula by Lusztig's partition function $\Longrightarrow$ proved by **S.-I. Kato**.

• Generalization to multi-variable Kostka functions

Fix $m \gg 0$, corresp. to $\boldsymbol{\lambda} \in \mathscr{P}_{n,r}$, consider

$$\{x_i^{(k)}\} \longleftrightarrow \{\lambda_i^{(k)}\} \quad (1 \le k \le r, 1 \le i \le m).$$

Put $\mathscr{M} = \{(k, i) \mid 1 \le k \le r, 1 \le i \le m\}$. Put $M = |\mathscr{M}| = rm$.

Give a total order on $\mathscr{M}$ by (similarly as $\boldsymbol{\lambda} \leftrightarrow c(\boldsymbol{\lambda}) \in \mathbb{Z}^M$)

$$(1,1) < (2,1) < \cdots < (r,1) < (1,2) < (2,2) < \cdots < (r,m).$$

By using this order, identify $\mathbb{Z}^M \simeq \mathbb{Z}^{\mathscr{M}}$

Let $\varepsilon_1, \dots, \varepsilon_M$: standard basis of $\mathbb{Z}^M$, $R^+ \subset \mathbb{Z}^M$: positive roots

For $\xi = (\xi_i) \in \mathbb{Z}^M$, define $L_-(\xi; \mathbf{t})$ by

$$L_-(\xi; \mathbf{t}) = \sum_{(m_\alpha)} \prod_\alpha (t_{b(\nu)})^{m_\alpha}, \quad \textbf{(Lusztig's partition function)}$$

where (m_α) runs over all non-negative integers satisfying (*):

(*) $\xi = \sum_{\alpha \in R^+} m_\alpha \alpha$ with $\alpha = \varepsilon_\nu - \varepsilon_{\nu'}$ s.t. $b(\nu') = b(\nu) + 1$.
(Here $b(\nu) = k \in \mathbb{Z}/r\mathbb{Z}$ for $\nu = (k, i) \in \mathscr{M}$.)

Theorem (conjecture of F-I)

Put $\delta = (M-1, M-2, \dots, 1, 0)$. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{n,r}$, we have

$$K^-_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathbf{t}) = \sum_{w \in (S_m)^r} \varepsilon(w) L_-(w^{-1}(c(\boldsymbol{\lambda}) + \delta) - (c(\boldsymbol{\mu}) + \delta); \mathbf{t}).$$

In particular, $K^-_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathbf{t})$ is a monic of degree $a(\boldsymbol{\mu}) - a(\boldsymbol{\lambda})$.

Stability of Kostka functions

Finkelberg-Ionov : geometric realization of $K^-_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathbf{t})$ by coherent sheaf on a certain vector bundle on $(GL_m)^r$ (under condition below).

Corollary

Assume that $\mu^{(k)}_1 > \cdots > \mu^{(k)}_m > 0$ for any k. Then $K^-_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathbf{t}) \in \mathbb{Z}_{\geq 0}[\mathbf{t}]$.

Let $\boldsymbol{\theta} = (\theta^{(1)}, \dots, \theta^{(r)})$ be r-partitions s.t. $\theta^{(k)} = (\theta_1, \dots, \theta_m)$ (indep. of k). For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{n,r}$, we have $\boldsymbol{\lambda} + \boldsymbol{\theta}, \boldsymbol{\mu} + \boldsymbol{\theta} \in \mathscr{P}_{n',r}$ for some n'.

Proposition (stability)

Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{n,r}$, and assume that $\theta_1 \gg \theta_2 \gg \cdots \gg \theta_m > 0$. Then $K^-_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathbf{t})$ takes a value independent of the choice of $\boldsymbol{\theta}$. We have

$$K^-_{\boldsymbol{\lambda}+\boldsymbol{\theta},\boldsymbol{\mu}+\boldsymbol{\theta}}(\mathbf{t}) = L_-(c(\boldsymbol{\lambda}) - c(\boldsymbol{\mu}); \mathbf{t}).$$