Group actions on quasi-trees and quasi-morphisms

Koji Fujiwara

Kyoto University

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§0 Geometric group theory

dealing with infinite, discrete, (non-commutative) groups

Sample groups:

- $\mathbb{Z}^n$, $Aut(\mathbb{Z}^n) = GL(n, \mathbb{Z})$, $SL(n, \mathbb{Z})$.
- Lattices in Lie groups:
  - $SL(n, \mathbb{Z}) < SL(n, \mathbb{R})$ (arithmetic)
  - $G = \pi_1(M)$, M is a closed hyperbolic n-manifold. (geometric).
    $G < Isom(\mathbb{H}^n)$. $\mathbb{H}^n$ is the n-dim real hyperbolic space.
- Let S_g be the closed surface of genus g.
  $\pi_1(S_g)$, surface group.
  $Out(\pi_1(S_g)) = MCG(S_g)$, the mapping class group of S_g.
  $MCG(S_g) = Homeo_+(S_g)/\sim isotopy$.
  Example. $MCG(sphere) = 1, MCG(torus) = SL(2, \mathbb{Z})$.
- Free groups of rank n, F_n. $Aut(F_n), Out(F_n)$.
  $Out(F_2) = GL(2, \mathbb{Z})$.
- "Hyperbolic groups" (Gromov, 85)

§1 Abelianize

- The abelianization of $SL(3,\mathbb{Z})$ is trivial, ie, $H_1(SL(3,\mathbb{Z}),\mathbb{Z}) = 0$ and $[SL(3,\mathbb{Z}), SL(3,\mathbb{Z})] = SL(3,\mathbb{Z})$.
- It is generated by 6 elementary matrices, and each one (or its inverse) is the commutator of other two. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

  $[a,b] = aba^{-1}b^{-1}$ is the commutator of a and b.
- $SL(3,\mathbb{Z})$ is a lattice in the Lie group $SL(3,\mathbb{R})$, namely, a discrete subgroup such that the volume of $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ is finite.

Theorem (Matsushima 1964, Borel et al)

If Γ is an irreducible lattice in a semi-simple Lie group G of rank at least two, then $H^1(G,\mathbb{R}) = 0$, ie, $\beta_1(G) = 0$.

- $H^1(G,\mathbb{R}) = \{\textit{all homomorphisms}, f : G \to \mathbb{R}\}$
  $\beta_1 = \dim H^1(G,\mathbb{R})$, the 1st Betti number.
- The rank of $SL(n,\mathbb{R})$ is $(n-1)$. $\mathbb{R}^{n-1} < SL(n,\mathbb{R})$

$F_2 < SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$. Rank-1 case.

- $H^1(G,\mathbb{R}) = \{\textit{all homomorphisms},\ f : G \to \mathbb{R}\}$
- Matsushima's theorem says:
  Let $G < \Gamma$ be a lattice in a s.s. Lie group.
  The rank of Γ is at least 2 $\Rightarrow H^1(G,\mathbb{R}) = 0$.
- The converse is not true.
  Take $SL(2,\mathbb{Z})$ as a lattice in $SL(2,\mathbb{R})$, which is rank-1.
  $H^1(SL(2,\mathbb{Z}),\mathbb{R}) = 0$ since it is generated by
  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ s.t. $A^4 = B^6 = 1$.
  But there is a free group $F_2 < SL(2,\mathbb{Z})$ of finite index (=12).
  So, $F_2 < SL(2,\mathbb{R})$ is a lattice and $H^1(F_2,\mathbb{R}) = \mathbb{R}^2 \neq 0$.
- "rank is at least 2" is necessary in the theorem.

§2 Quasi-morphism

- A function on a group G, $f : G \to \mathbb{R}$, is a quasi-morphism if

$$D(f) = \sup_{g,h\in G} |f(gh) - f(g) - f(h)| < \infty$$

  $D(f)$ is the defect of f.
- f is a homomorphism iff $D(f) = 0$.
- Also, all bounded functions on G are quasi-morphisms.
- Define vector spaces:

$$QH(G) = \{all\ quasi\ morphisms\ on\ G\}$$

$$\widetilde{QH}(G) = QH(G)/\{homomorphisms + bounded\ functions\}$$

- Is quasi-morphism useful for anything at all?

Application 1. rank 1 vs rank at least 2

Theorem

Let Γ be an irreducible lattice in a semi-simple Lie group G. Then $\widetilde{QH}(\Gamma) = 0 \Leftrightarrow$ the rank of G is at least 2.

Proof.

($\Leftarrow$) Theorem (Burger-Monod, 02) If the rank of G is at least 2, then $\widetilde{QH}(\Gamma) = 0$.

For example $G = SL(n, \mathbb{R}), n \geq 3$.

($\Rightarrow$) If the rank of G is 1, then $\widetilde{QH}(\Gamma) \neq 0$ [F, 98].

For example $G = SL(2, \mathbb{R}), SL(2, \mathbb{C})$ etc. $\square$

- Remember that free groups $F_n, n \geq 2$ are lattices in $SL(2, \mathbb{R})$. It was the first example for $\widetilde{QH} \neq 0$ by a concrete construction.

Theorem (Brooks, 80)

$\widetilde{QH}(F_n) \neq 0$ if $n \geq 2$.

- He found a combinatorial way to construct many quasi morphisms. Key: a free group acts on a simplicial tree.

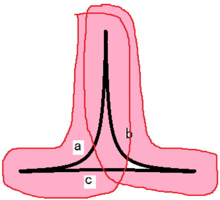
§3 Hyperbolic spaces

- Take a geodesic triangle $\Delta(a, b, c)$ in the hyperbolic plane $\mathbb{H}^2$. By Gauss-Bonnet theorem, $Area(\Delta) < \pi$.
- Then each side is contained in the 2-neighborhood of the union of the other two:

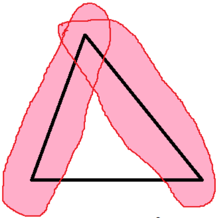
$$a \subset N_2(b \cup c), b \subset N_2(c \cup a), c \subset N_2(a \cup b)$$

  We say the triangle is 2-thin.

Definition. A geodesic space X is δ-hyperbolic if every geodesic triangle is δ-thin for a uniform constant δ.



δ -thin triangle



triangle is NOT δ -thin

Hyperbolicity and quasi-morphism

Examples of hyperbolic spaces.

- Hyperbolic spaces $\mathbb{H}^n$, complex hyperbolic spaces $\mathbb{CH}^n$, etc are all δ-hyperbolic for some δ.
- Trees are 0-hyperbolic. A geodesic triangle looks like "T".
- Euclidean spaces are not δ-hyperbolic for any δ.

Definition. A group G that acts on a δ-hyperbolic space X properly discontinuously, by isometries (ie, preserving the distance), with X/G compact, is called a hyperbolic group.

Examples. Free groups F_n, $\pi_1(S_g), (g \geq 2)$, but NOT $\mathbb{Z}^2$.

- (strangely enough) the notion of hyperbolicity is very useful.

Our guiding principle.

if G acts on a δ-hyperbolic space X by isometries, then G tends to have lots of quasi-morphisms.

- Actions do not have to be proper nor compact. This principle applies to hyperbolic groups, but also to MCG, $Out(F_n)$, which are not hyperbolic groups.
  For those groups, we will produce desired actions.

Application 2. Are $SL(n, \mathbb{Z})$ different from MCG?

Remember $MCG(torus) = SL(2, \mathbb{Z})$. Maybe $MCG(S_g)$ are all isomorphic to some $SL(n, \mathbb{Z})$?
$H_1(MCG(S_g)) = 0$ if $g \geq 2$ and $H_1(SL(n, \mathbb{Z})) = 0$ if $n \geq 3$.

Theorem (Kaimanovich-Masur, 96)

Let Γ be an irreducible lattice in a semi-simple Lie group of rank at least 2. Then $\Gamma \not< MCG(S)$, where S is a closed surface.

- They use Poisson boundary of groups.
- We present another proof using the following result on $\widetilde{QH}(G)$ to distinguish groups.

Theorem (Bestvina-F,02)

Let $G < MCG(S)$ be a finitely generated group. Then $\widetilde{QH}(G) \neq 0$ unless G contains $\mathbb{Z}^n$ as a subgroup of finite index.

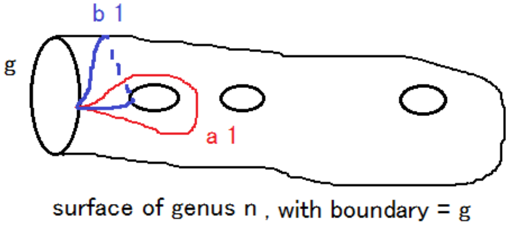
Proof of Kaimanovich-Masur thm. Remember $\widetilde{QH}(\Gamma) = 0$ for all lattices in s.s. Lie group of rank at least 2 (Burger-Monod). Also Γ does not contain $\mathbb{Z}^n$ as a subgroup of finite index. So, $\Gamma \not< MCG$. □

§4 Application 3. Stable commutator length (scl)

- $H_1(G) = 0$ is equivalent to $[G, G] = G$. Such group is called perfect. For example, $MCG(S_g), g \geq 2$ and $SL(n, \mathbb{Z}), n \geq 3$.
- By definition, $g \in [G, G]$ is written as a product of commutators

$$g = [a_1, b_1] \cdots [a_n, b_n], \ (a_i, b_i \in G)$$

The commutator length of g, $cl(g)$ is min n. If $g \notin [G, G]$, define $cl(g) = \infty$.



$cl(g)$ is the least genus of a surface that bounds g in G

§4 Application 3. Stable commutator length (scl)

- The sequence
$$cl(g), cl(g^2), cl(g^3), \cdots$$
is sub-additive ($cl(g^{n+m}) \leq cl(g^n) + cl(g^m)$), but does it grow linearly? Define the stable commutator length by
$$scl(g) = \liminf_{n\to\infty} \frac{cl(g^n)}{n} \leq \infty$$
$scl(g^n) = n \times scl(g), scl(1) = 0.$
- We want to know its image $scl(G) \subset \mathbb{R}_{\geq 0} \cup \infty$.
Contained in $\mathbb{Q}$? Discrete? Is 0 isolated? etc

Theorem (Burger-Monod, 02)

If G is an irr. lattices in a s.s. Lie group of rank at least 2, then $scl(g) = 0$ for every $g \in G$.

This immediately follows from $\widetilde{QH}(G) = 0$ and "Bavard duality".

scl on MCG, known cases

- We are curious about scl on MCG.
  Since [MCG, MCG]=MCG, $scl(g) < \infty$ for every g.
- Elements of MCG are classified into:
  (1) pseudo-Anosov elements, (2) reducible elements (Dehn twists etc), and (3) torsions.

Some known cases:

Theorem (For pseudo-Anosov and Dehn-twists)

1. $scl(g) > 0$ for every Dehn twist. [Endo-Kotschick, 01]
2. For a pseudo-Anosov element g, $scl(g) > 0$ iff there are no $h \in MCG(S)$ and $n > 0$ with $hg^n h^{-1} = g^{-n}$. [Calegari-F, 10]

- Endo-Kotschick uses Seiberg-Witten theory to show $scl(g) > 0$.
- Calegari-F uses

Proposition (cf. Milnor, 58)

If f is a "homogeneous" quasi-morphism on G such that $f(g) > 0$ on $g \in [G, G]$ then $scl(g) > 0$.

scl on MCG

Theorem (Bestvina-Bromberg-F, 14)

1. We can decide, in terms of Nielsen-Thurston theory, which elements $g \in MCG(S)$ have $scl(g) > 0$.
2. Moreover, there exists $C(S) > 0$ such that if $scl(g) > 0$ then $scl(g) \geq C(S)$.

- Nielsen-Thurston theory is a refined classification of elements of MCG (like Jordan normal forms for matrices).
- Two sufficient algebraic conditions for $scl(g) = 0$ were known (one is in the Calegari-F thm). We showed that they are necessary, and also decide which $g \in MCG$ satisfies them.
- For the necessary part, for each candidate element $g \in MCG$, we find a homogeneous quasi-morphism f with $f(g) > 0$, which verifies $scl(g) > 0$ using the proposition by Milnor. To produce f, we construct a suitable action on a hyperbolic space.
- In particular, we recover Endo-Kotschick theorem without using Seiberg-Witten theory.

§5 Bass-Serre theory. Group actions on trees.

Theorem (Ihara, 66)

Every torsion-free discrete subgroup G in $SL(2, \mathbb{Q}_p)$ is a free group.

- Serre interpreted Ihara's combinatorial argument as follows: construct a simplicial tree T on which $SL(2, \mathbb{Q}_p)$ acts by automorphism. Then prove G acts on T freely. It follows $T \to T/G$ is a covering, and $G \simeq \pi_1(T/G)$ is a free group.

Serre established a theory of groups acting on trees. It's called Bass-Serre theory.
For example,
$SL(2, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$
acts on a tree s.t. the quotient is one edge with a blue vertex and a red vertex.

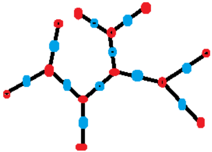


Figure: $SL(2, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ acts on a tree.

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The theory only applies when G acts on some tree T without a fixed point (non-trivial action). For example, $SL(3, \mathbb{Z})$ does not have any non-trivial actions on trees, which is equivalent to that $SL(3, \mathbb{Z})$ is not decomposed as an amalgamation or an HNN-extension (property FA).

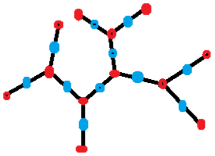


Figure: $SL(2, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ acts on a tree.

Quasi-isometry and quasi-tree

Unfortunately, MCG(S) does not act on any trees non-trivially. We will make it act on something else, but similar, that is quasi-trees.

- Let X, Y be metric spaces, and $f : X \to Y$ a map.
  1. f is a quasi-isometric (QI) embedding if $\exists K, L$ such that

$$\forall x, y \in X, \ \frac{|x-y|}{K} - L \leq |f(x) - f(y)| \leq K|x-y| + L$$

  2. Moreover, X and Y are quasi-isometric if $\forall y \in Y, \exists x \in X, \ |y - f(x)| \leq L$

Definition. A graph is a quasi-tree if it is quasi-isometric to a simplicial tree.

- A quasi-tree is δ-hyperbolic. It turns out group actions on quasi-trees give lots of information on G.

Example of a quasi-tree

The hyperbolic plane $\mathbb{H}^2$ is tessellated by ideal triangles.
Make it into a planer graph, that is the Farey graph, $\mathcal{F}$, s.t.
each edge has length 1 and $\mathcal{F}$ is a geodesic space. $\mathcal{F}$ is a quasi-tree.
If you remove any edge from $\mathcal{F}$, then $\mathcal{F}$ is disconnected.
(cf, if you remove a point from a tree, then the tree is disconnected.)

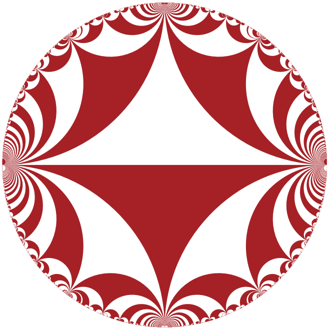
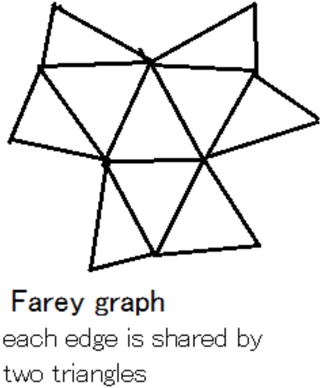


Figure: Farey tesselation of $\mathbb{H}^2$



§6 Group actions on quasi-trees

We now discuss a group action on a quasi-tree by isometries (ie, preserving the distance).

Def. A group action on a quasi-tree is non-trivial if it has an unbounded orbit of a point.

Theorem (BBF, 14)

The following groups act on some quasi-trees, non-trivially.

1. $MCG(S_g), g \geq 1$.
2. $Out(F_n), n \geq 2$
3. *every infinite hyperbolic groups.*

- MCG, $Out(F_n)$ do not act trees non-trivially.
  For hyperbolic groups, some do, and some don't.
- Every group actions by $SL(n, \mathbb{Z}), n \geq 3$, on a quasi-tree is trivial.[Manning, 06]
- Unknown for general lattices in $SL(n, \mathbb{R})$.

Embedding of MCG

- Our method for the construction of quasi-trees and group actions is systematic and produces many actions for a given group.
- Bass-Serre theory relies on the algebraic structure of the group while our method relies on geometry using nearest point projections.
- The following is the main theorem. We first produce many actions of MCG on quasi-trees. Then we put them together and produce one nice action.

Main Theorem (BBF, 14)

MCG(S) properly acts on a finite product of hyperbolic graphs, $X = X_1 \times \cdots \times X_n$, such that embedding $MCG(S) \to X$ by an orbit is a quasi-isometric embedding: fix a base point $x_0 \in X$,

$$g \in MCG(S) \mapsto g(x_0) \in X$$

We put a "word metric" on MCG in the theorem.

Consequences of Main thm

Using the embedding of MCG into a finite product of hyperbolic graphs, we can prove with some extra efforts,

Theorem (BBF, 2014)

The asymptotic dimension of MCG is finite.

Remark. Asymptotic dimension (Gromov, 93) is defined for a metric space. It is a quasi-isometric invariant. It is defined for a finitely generated group using its "Cayley graph" with a word metric. Sometimes dimension is infinite.

Example. $asdim(\mathbb{E}^n) = n, asdim(\mathbb{H}^n) = n, asdim(tree) \leq 1, asdim(\mathbb{Z}^n) = n, asdim(F_n) = 1$. Exact number is unknown for asdim(MCG).

In many case, it coincides with its (virtual) cohomological dimension.

Consequences of Main thm

Using the embedding of MCG into a finite product of hyperbolic graphs, we can prove with some extra efforts,

Theorem (BBF, 2014)

The asymptotic dimension of MCG is finite.

One motivation to define the asym. dim is Novikov conjecture. It follows from the theorem, combined with a theorem by Yu,

Theorem (Kida 06, Hamenstaedt 09)

Novikov conjecture holds for MCG.

Another immediate corollary of Main thm is

Theorem (Farb-Lubotzky-Minsky 01)

Every element $g \in MCG$ of infinite order is not distorted.

ie, $\liminf_{n\to\infty} \frac{||g^n||}{n} > 0$, where $||g||$ is a word norm.

§7 Quasi-homomorphism into non-commutative groups

- Let's change the target group $\mathbb{R}$ to a non-commutative, discrete group, H. For example, a free group.

  $f : G \to H$ is a quasi-homomorphism if

  $$\{(f(gh))^{-1} f(g)f(h) \mid g, h \in G\} \subset H \textit{ is finite}.$$

- It turns out that we do not gain any new information on G.

Theorem (Kapovich-F, 2015)

Let G be any group and H a torsion-free hyperbolic group. Then any quasi-homomorphism $f : G \to H$ is either a homomorphism or the image is cyclic (ie, a quasi-morphism into $\mathbb{Z}$).

- In fact, the target group H can be any discrete group, and every quasi-homomorphism is "essentially" either
  1. homomorphism, or
  2. quasi-morphism into $\mathbb{Z}^n \subset H$.
- Ozawa (2011) showed the result when G is an irr. lattice in a s.s. Lie group of rank at least two, and H is a hyperbolic group.