特殊関数と代数的線形常微分方程式

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§ Generalized Riemann Scheme and Universal Model

Euler

$$F(\alpha,\beta,\gamma;x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}$$

$$= 1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots$$

$$(\alpha)_k := \prod_{\nu=0}^{k-1}(\alpha+\nu) = \alpha(\alpha+1)\cdots(\alpha+k-1)$$

$$x(1-x)u'' + \big(\gamma-(\alpha+\beta+1)x\big)u' - \alpha\beta u = 0$$

Gauss summation formula: a connection coefficient

$$C_{\alpha,\beta,\gamma} := F(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)}$$

1. $\dfrac{C_{\alpha,\beta,\gamma+1}}{C_{\alpha,\beta,\gamma}} = \dfrac{\gamma(\gamma-\alpha-\beta)}{(\gamma-\beta)(\gamma-\alpha)}$ and $\displaystyle\lim_{n\to\infty} C_{\alpha,\beta,\gamma+n} = 1$

2. $F(\alpha,\beta,\gamma;x) = \dfrac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\displaystyle\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt \ \to x=1$

$\mathcal{F}\,:\,\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\} \supset U$ (simply connected domain) $\mapsto \mathcal{F}(U) = \mathbb{C}\phi_{U,1} + \mathbb{C}\phi_{U,2}$

- $\mathcal{F}(U_1)|_V = \mathcal{F}(U_2)|_V \quad (\forall V \subset U_1 \cap U_2$: connected open)

$x = 0, 1, \infty$: singularities

$V_{p,\theta,\epsilon} := \{x \in \mathbb{C}\,;\, \theta < \arg(x-p) < \theta + 2\pi,\ |x-p| < \epsilon\}$ ($x - p \mapsto \frac{1}{x}$ if $p = \infty$)

(N) $\phi \in \mathcal{F}(V_{p,\theta,\epsilon}) \Rightarrow \exists K,\ \exists m$ st. $|\phi(x)| < K|x-p|^{-m}$ (p : singular point)

$\mathcal{F}(V_{0,\theta,\frac{1}{2}}) = \varphi_1(x) + x^{1-\gamma}\varphi_2(x)$ with φ_1, φ_2 : analytic and not zero at $x = 0$

$$P\begin{Bmatrix} x=0 & 1 & \infty & \\ 1-\gamma & \gamma-\alpha-\beta & \alpha & ;\,x \\ 0 & 0 & \beta & \end{Bmatrix}$$

Riemann scheme

Characterized by the Riemann scheme

$u(x) \mapsto x^{\mu_1}(1-x)^{\mu_2}u(x)$:

$$P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} = 1-\gamma+\mu_1 & \lambda_{1,1} = \gamma-\alpha-\beta+\mu_2 & \lambda_{2,1} = \alpha-\mu_1-\mu_2 & ;\,x \\ \lambda_{0,2} = \mu_1 & \lambda_{1,2} = \mu_2 & \lambda_{2,2} = \beta-\mu_1-\mu_2 & \end{Bmatrix}$$

Fuchs relation : $\lambda_{0,1} + \lambda_{0,2} + \lambda_{1,1} + \lambda_{1,2} + \lambda_{2,1} + \lambda_{2,2} = 1$

Three regular singularities, 5 parameters, rank $= 2 \Leftrightarrow$ Gauss HG

$$ {}_3F_2(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2;x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{(\beta_1)_k(\beta_2)_k k!}x^k $$

$$ P\left\{\begin{matrix} x=0 & 1 & \infty & \\ 1-\beta_1 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \alpha_1 & \\ 1-\beta_2 & & \alpha_2 & ;x \\ 0 & -\beta_3 & \alpha_3 & \end{matrix}\right\} $$

$$ \Big(\prod_{j=1}^{n-1}(\vartheta-\beta_j)\cdot\frac{d}{dx}-\prod_{j=1}^{n}(\vartheta-\alpha_j)\Big)u=0 $$

$$ \vartheta := x\frac{d}{dx} \qquad (n=3) $$

Fuchs condition: $\alpha_1+\alpha_2+\alpha_3=\beta_1+\beta_2+\beta_3$

Characterized by the Riemann Scheme

with semisimple local monodromy (generic parameters)

$$ P\left\{\begin{matrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \begin{pmatrix} \lambda_{1,1} \\ \lambda_{1,1}+1 \end{pmatrix} & \lambda_{2,1} & \\ \lambda_{0,2} & & \lambda_{2,2} & ;x \\ \lambda_{0,3} & \lambda_{1,2} & \lambda_{2,3} & \end{matrix}\right\} $$

Functions $\Leftrightarrow$ Diff. Eq. (ODE)

$\mathcal{F} \quad \Leftrightarrow \quad P(x,\frac{d}{dx})u=0$

(FC) $\lambda_{0,1}+\lambda_{0,2}+\lambda_{0,3}+2\lambda_{1,1}+\lambda_{1,2}+\lambda_{2,1}+\lambda_{2,2}+\lambda_{2,3}=2$

Spectral Type $111,21,111=1^3,21,1^3$ (cf. Gauss HG: $11,11,11$)

Def. Generalized Riemann Scheme (GRS)

$$P\begin{Bmatrix} x = c_0 = \infty & c_1 & \cdots & c_p & \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} & \\ \vdots & \vdots & \vdots & \vdots & ;\ x \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} & \end{Bmatrix}$$

Semisimple local monodromies for generic $\lambda_{j,\nu}$ (without log terms)

$$[\lambda]_{(m)} := \begin{pmatrix} \lambda \\ \lambda+1 \\ \vdots \\ \lambda+m-1 \end{pmatrix}, \quad n = m_{j,1} + \cdots + m_{j,n_j}, \quad \lambda_{j,\nu} \in \mathbb{C}$$

$\mathbf{m} = (\mathbf{m}_0, \ldots, \mathbf{m}_p) = \big((m_{0,1}, \ldots, m_{0,n_0}), \ldots, (m_{p,1}, \ldots, m_{p,n_p})\big)$

: $(p+1)$-tuples of partitions of $n = \operatorname{ord}\mathbf{m}$ (spectral type)

Fuchs Condition (FC):

$$\boxed{|\{\lambda_{\mathbf{m}}\}| := \sum m_{j,\nu}\lambda_{j,\nu} - \operatorname{ord}\mathbf{m} + \tfrac{1}{2}\operatorname{idx}\mathbf{m} = 0}$$

$\operatorname{idx}\mathbf{m} := \sum_{j,\nu} m_{j,\nu}^2 - (p-1)(\operatorname{ord}\mathbf{m})^2$ (index of rigidity (Katz))

$\mathbf{m}$: realizable $\overset{\text{def}}{\Leftrightarrow}$ $\exists\ P(x, \frac{d}{dx})$ with (GRS) for generic $\lambda_{j,\nu}$ under (FC)

$\mathbf{m}$: irreducibly realizable $\overset{\text{def}}{\Leftrightarrow}$ $\exists\ Pu = 0$ is irreducible for generic $\lambda_{j,\nu}$

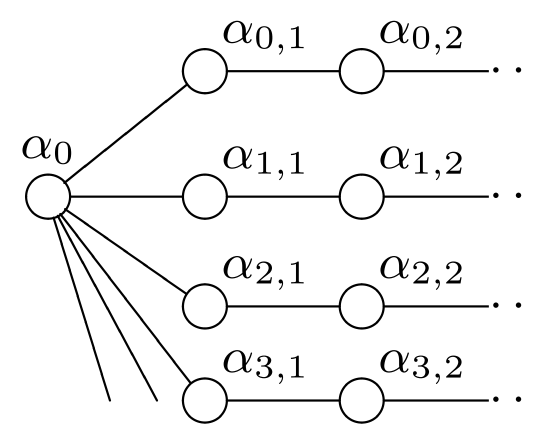
Problem. Classify such $\mathbf{m}$! (Deligne-Katz-Simpson problem)

$\mathbf{m}$: monotone $\overset{\text{def}}{\Leftrightarrow}$ $m_{j,1} \ge m_{j,2} \ge m_{j,3} \ge \cdots$

$\mathbf{m}$: basic $\overset{\text{def}}{\Leftrightarrow}$ monotone, $\gcd \mathbf{m} := \gcd\{m_{j,\nu}\} = 1$ and $d(\mathbf{m}) \le 0$

$$d(\mathbf{m}) = d_{1,\dots,1}(\mathbf{m}) := m_{0,1} + \cdots + m_{p,1} - (p-1)\operatorname{ord}\mathbf{m}$$

A Kac-Moody root system (Π, W)



$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi), \quad (\alpha_0|\alpha_{j,\nu}) = -\delta_{\nu,1},$$

$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \ne j \quad \text{or} \quad |\mu - \nu| > 1) \\ -1 & (i = j \quad \text{and} \quad |\mu - \nu| = 1) \end{cases}$$

$\Delta_+ = \Delta_+^{re} \cup \Delta_+^{im}$, Δ_+^{re}: positive real roots $(W\Delta_+^{re} = \Delta_+^{re} \cup \Delta_-^{re} = W\alpha_0)$

Δ_+^{im}: positive imaginary roots $(k\Delta_+^{im} \subset \Delta_+^{im} = W\Delta_+^{im},\ k = 2, 3, \dots)$

$\boxed{\mathbf{m} \leftrightarrow \alpha_{\mathbf{m}} = (\operatorname{ord}\mathbf{m})\alpha_0 + \sum_{j\ge 0,\, k\ge 1} \sum_{\nu > k} m_{j,\nu}\alpha_{j,k} \ \text{(Crawley-Boevey)}}$

Fact. $\operatorname{idx}\mathbf{m} = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}), \quad d(\mathbf{m}) = 2(\alpha_{\mathbf{m}}|\alpha_0), \quad W = \langle r_\alpha\,;\,\alpha\in\Pi\rangle$

$\Delta_+^{im} = \{kw\alpha_{\mathbf{m}}\,;\,k\in\mathbb{Z}_{>0},\ w\in W,\ \mathbf{m} : \text{basic}\}, \quad Q_+ := \sum_{\alpha\in\Pi}\mathbb{Z}_{\ge 0}\alpha$

$\widetilde{\Delta}_+ := \mathbb{Z}_{>0}\Delta_+ = \{\alpha\in Q_+\,;\,w\alpha\in Q_+\cup -Q_+\ (\forall w\in W)\}$

Thm. $\{\mathbf{m} : \text{realizable}\} \leftrightarrow \{\alpha\in\widetilde{\Delta}_+\,;\,\operatorname{supp}\alpha\ni\alpha_0\}$

Suppose $\mathbf{m}$ is realizable.

★ $\mathbf{m}$: irreducibly realizable $\Leftrightarrow$ $\gcd\mathbf{m} = 1$ or $\operatorname{idx}\mathbf{m} < 0$

★ $\exists\ P_{\mathbf{m}}$: a universal model with (GRS) $\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}$

$P_{\mathbf{m}} = \prod_j (x-c_j)^n\cdot\partial^n + \sum_{0\le\nu<n} a_\nu(x)\partial^\nu$ with $a_\nu(x)\in\mathbb{C}[x,\lambda_{j,\nu},r_i]$

★ $\forall\lambda_{j,\nu}$ under (FC), $\forall P$ with $\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}$ are $P_{\mathbf{m}}$

★ $r_1,\dots,r_N$: accessory parameters $\quad N = \begin{cases} 0 & (\operatorname{idx}\mathbf{m} > 0)\\ \gcd\mathbf{m} & (\operatorname{idx}\mathbf{m} = 0)\\ 1-\frac12\operatorname{idx}\mathbf{m} & (\operatorname{idx}\mathbf{m} < 0)\end{cases}$

$\frac{\partial^2 P_{\mathbf{m}}}{\partial^2 r_i} = 0, \quad \operatorname{Top}(P_{\mathbf{m}}) = x^{L_i}\partial^{K_i}\operatorname{Top}(\frac{\partial P_{\mathbf{m}}}{\partial r_i})$

$\{(L_i,K_i)\,;\,i=1,\dots,N\}$ are explicitly given

$Q = (c_k x_k + \cdots + c_0)\partial^m + a_{m-1}(x)\partial^{m-1} + \cdots,\ c_k\neq 0 \quad\Rightarrow \operatorname{Top} Q = c_k x^k\partial^m$

Def. $\mathbf{m}$ is rigid $\overset{\text{def}}{\Leftrightarrow}$ irreducibly realizable and $\operatorname{idx}\mathbf{m} = 2$ $\quad(\Rightarrow N = 0)$

(corresponds to $\alpha \in \Delta_+^{re}$ with $\operatorname{supp}\alpha \ni \alpha_0$)

Rigid tuples : 9 (ord ≤ 4), 306 (ord $= 10$), 19286 (ord $= 20$)

$\mathrm{ord} = 2$ $\quad 11, 11, 11$ (${}_2F_1$; Gauss)

$\mathrm{ord} = 3$ $\quad 111, 111, 21$ (${}_3F_2$) $\qquad 21, 21, 21, 21$(Pochhammer)

$\mathrm{ord} = 4$ $\quad 1^4, 1^4, 31$ (${}_4F_3$) $\qquad 1^4, 211, 22$ (Even family) $\quad 211, 211, 211$

$31, 31, 31, 31, 31$ (Pochhammer) $\qquad 211, 22, 31, 31 \qquad 22, 22, 22, 31$

Remark. The existence of $P_{\mathbf{m}}$ for fixed rigid $\mathbf{m}$ and generic $\{\lambda_{j,\nu}\}$ was an open problem by N. Katz ("Rigid Local Systems", 1995).

Reduction by "fractional calculus" $\Leftarrow W$ (Katz's middle convolution)

$\mathbf{m} \rightsquigarrow$ trivial ($\Leftarrow$ $\mathbf{m}$: rigid) or

fundamental $:= \{k\alpha\,;\mathbf{m}$: basic, $k = 1, 2, \ldots$ $(k = 1 \Leftarrow \operatorname{idx}\mathbf{m} = 0)\}$

$\operatorname{idx}\mathbf{m} = 0 \;\rightarrow\; \tilde{D}_4$ ($\rightarrow$ Painlevé VI), $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (4 types)

$\operatorname{idx}\mathbf{m} = -2 \;\rightarrow$ 13 types, etc...

Reduction. $\mathbf{m}=(m_{j,\nu})$: monotone $\Rightarrow \partial\mathbf{m} : m_{j,1} \mapsto m_{j,1} - d(\mathbf{m})$

$$411, 411, 42, 33 \xrightarrow{15-2\cdot 6=3} 111, 111, 21 \xrightarrow{4-3=1} 11, 11, 11 \xrightarrow{3-2=1} 1, 1, 1$$

$$21, 21, 21, 111 \xrightarrow{7-2\cdot 3=1} 11, 11, 11, 11 \xrightarrow{4-2\cdot 2=0} 11, 11, 11, 11 \circlearrowright \text{ (Heun)}$$

$$22, 22, 1111 \xrightarrow{5-4=1} 21, 21, 111 \xrightarrow{5-3=2} \times$$

$$\begin{tikzcd}
1^2,1^2,1^2 \arrow[d] & 21,1^3,1^3 \arrow[l] & 31,1^4,1^4 \arrow[l] & 41,1^5,1^5 \arrow[l] & 51,1^6,1^6 \arrow[l] \\
1,1,1 & & 2^2,21^2,1^4 \arrow[ul] & 32,2^21,1^5 \arrow[l] & 3^2,321,1^6 \arrow[l] \\
 & & 21^2,21^2,21^2 \arrow[uul] & 32,21^3,21^3 \arrow[ul] \arrow[uu] & 42,2^3,1^6 \arrow[ul] \\
 & & & 31^2,2^21,21^3 \arrow[ul] \arrow[uul] & 321,31^3,2^3 \arrow[ull] \arrow[ul] \arrow[uul] \\
 & & & 2^21,2^21,2^21 \arrow[uul] & 321,321,2^21^2 \arrow[l] \arrow[ul]
\end{tikzcd}$$

$$r_\alpha(x) := x - 2\frac{(\alpha|x)}{(\alpha|\alpha)}\alpha$$

$$r_{\alpha_0}(\alpha_{\mathbf{m}}) = \alpha_{\partial\mathbf{m}} \ : \ \partial\mathbf{m}$$

$$r_{\alpha_{j,\nu}}(\alpha_{\mathbf{m}}) \ : \ m_{j,\nu} \leftrightarrow m_{j,\nu+1}$$

$$d(\mathbf{m}) = m_{0,1} + m_{1,1} + \cdots + m_{p,1} - (p-1)\operatorname{ord}\mathbf{m}$$

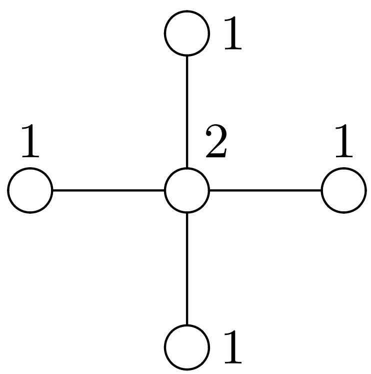
rigid tuples of partitions

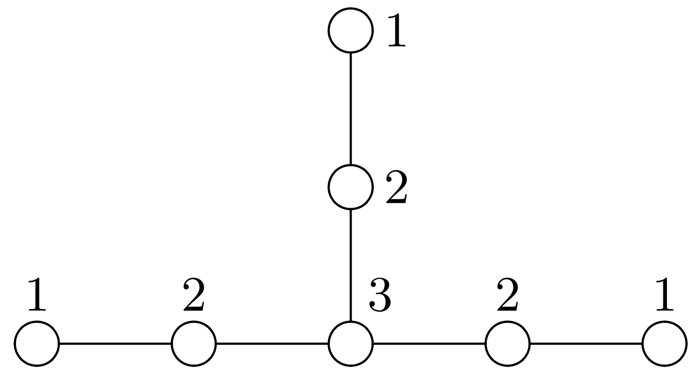
ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$
2	1	1	7	20	44	12	421	857	17	3276	6128
3	1	2	8	45	96	13	588	1177	18	5186	9790
4	3	6	9	74	157	14	1004	2032	19	6954	12595
5	5	11	10	142	306	15	1481	2841	20	10517	19269
6	13	28	11	212	441	16	2388	4644	21	14040	24748

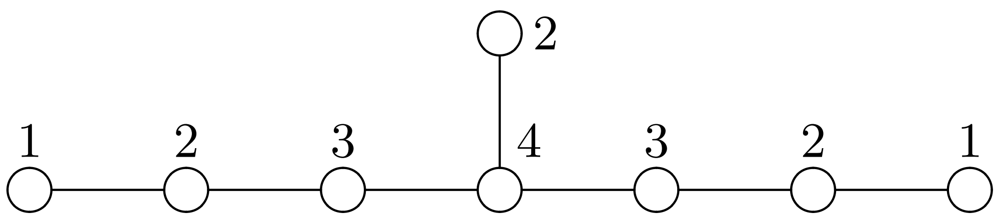
2:11,11,11
3:111,111,21
3:21,21,21,21
4:211,211,211
4:1111,211,22
4:1111,1111,31
4:211,22,31,31
4:22,22,22,31
4:31,31,31,31,31
5:2111,221,311
5:2111,2111,32
5:221,221,221
5:11111,221,32
5:11111,11111,41
5:221,221,41,41
5:221,32,32,41
5:311,311,32,41
5:32,32,32,32
5:32,32,41,41,41
5:41,41,41,41,41,41
6:3111,3111,321
6:2211,2211,411
6:2211,321,321
6:222,3111,321
6:21111,222,411
6:21111,2211,42
6:21111,3111,33
6:2211,2211,33
6:222,222,321
6:21111,222,33
6:111111,321,33
6:111111,222,42
6:111111,111111,51
6:2211,222,51,51
6:2211,33,42,51
6:222,33,33,51
6:222,33,411,51
6:3111,33,411,51
6:321,321,42,51
6:321,42,42,42
6:33,33,33,42
6:33,33,411,42
6:33,411,411,42
6:411,411,411,42
6:33,42,42,51,51
6:321,33,51,51,51
6:411,42,42,51,51
6:51,51,51,51,51,51,51

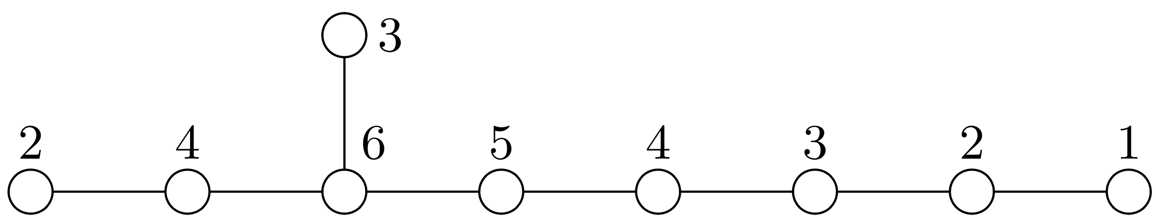
Fudamental tuples: $\mathrm{idx} = 0$ ($\Rightarrow$ affine)

Heun :



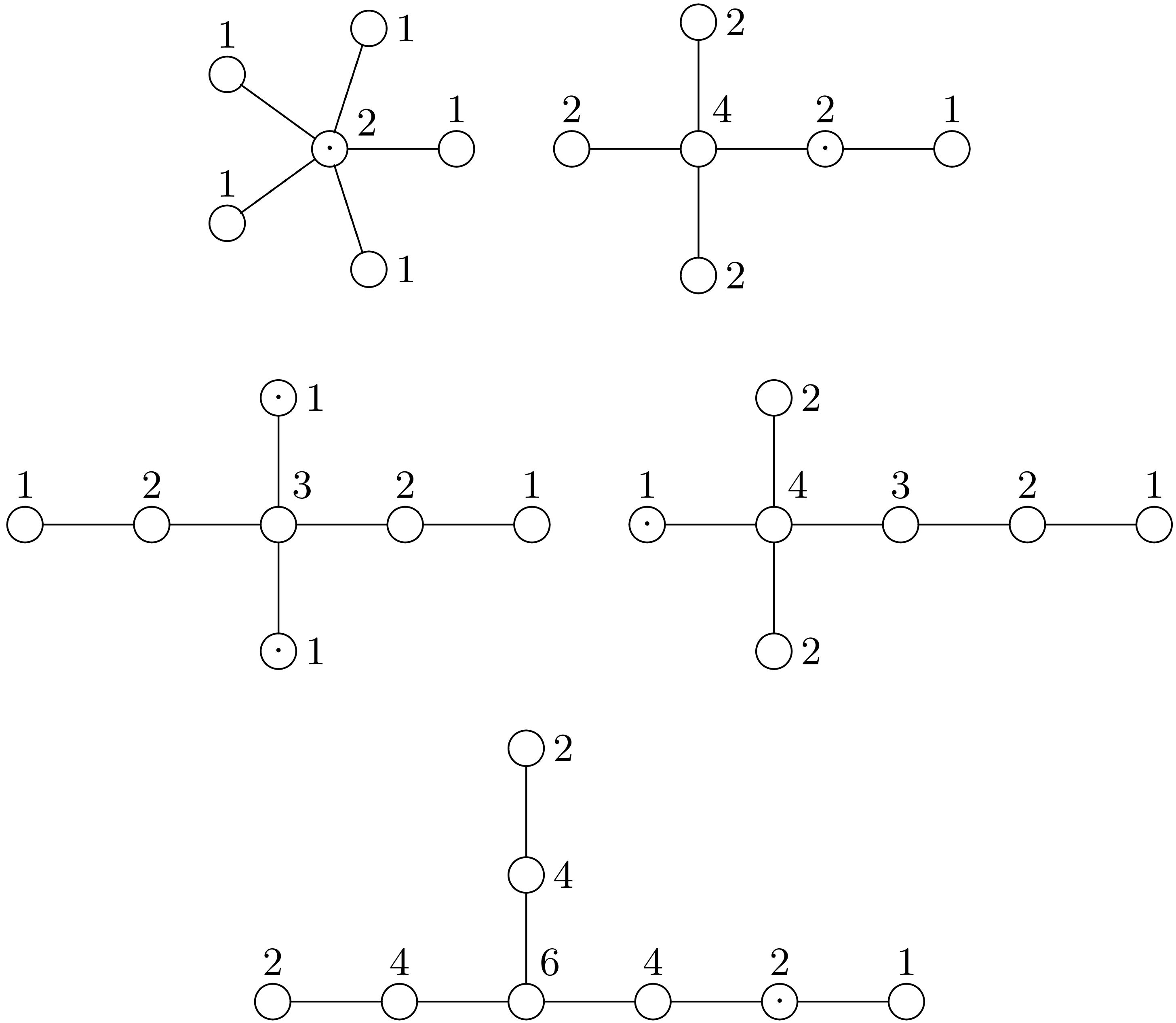


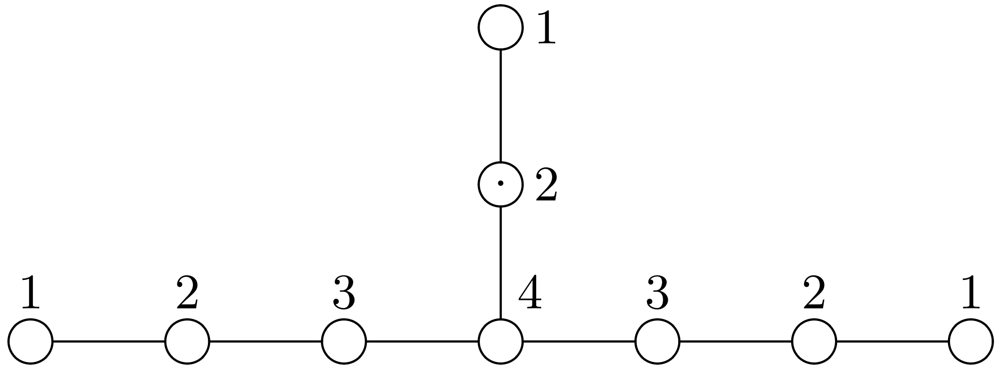


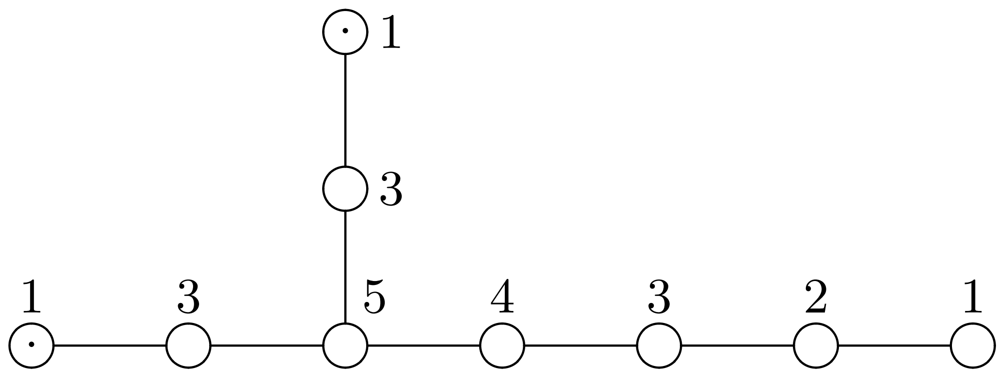


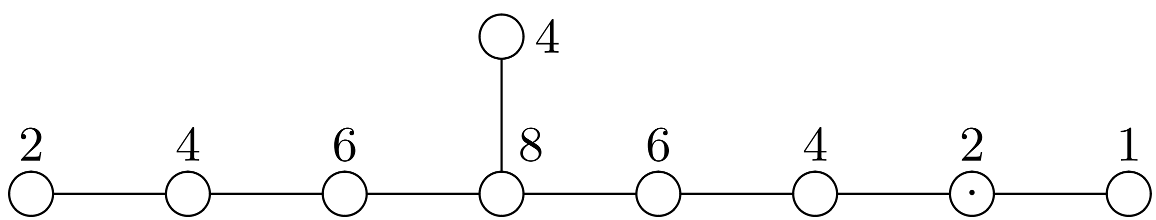
Prop. $\mathbf{m}:$ fund.$\Rightarrow \mathrm{ord}\,\mathbf{m} \le 6 - 3\cdot\mathrm{idx}\,\mathbf{m}$ $(\le 2 - \mathrm{idx}\,\mathbf{m} \Leftarrow \mathbf{m} \notin \mathcal{P}_3)$

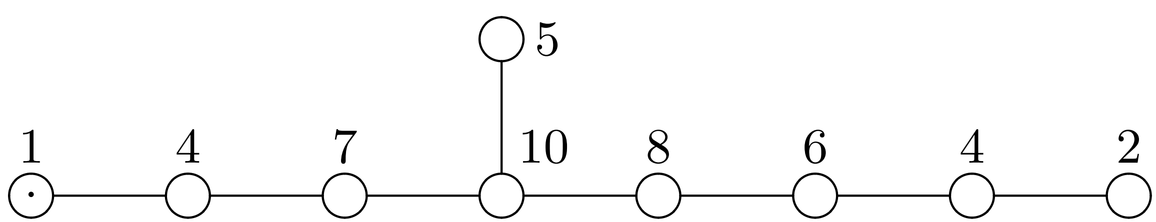
Fundamental tuples: $\mathrm{idx} = -2$ ($\Rightarrow$ Lorentzian)

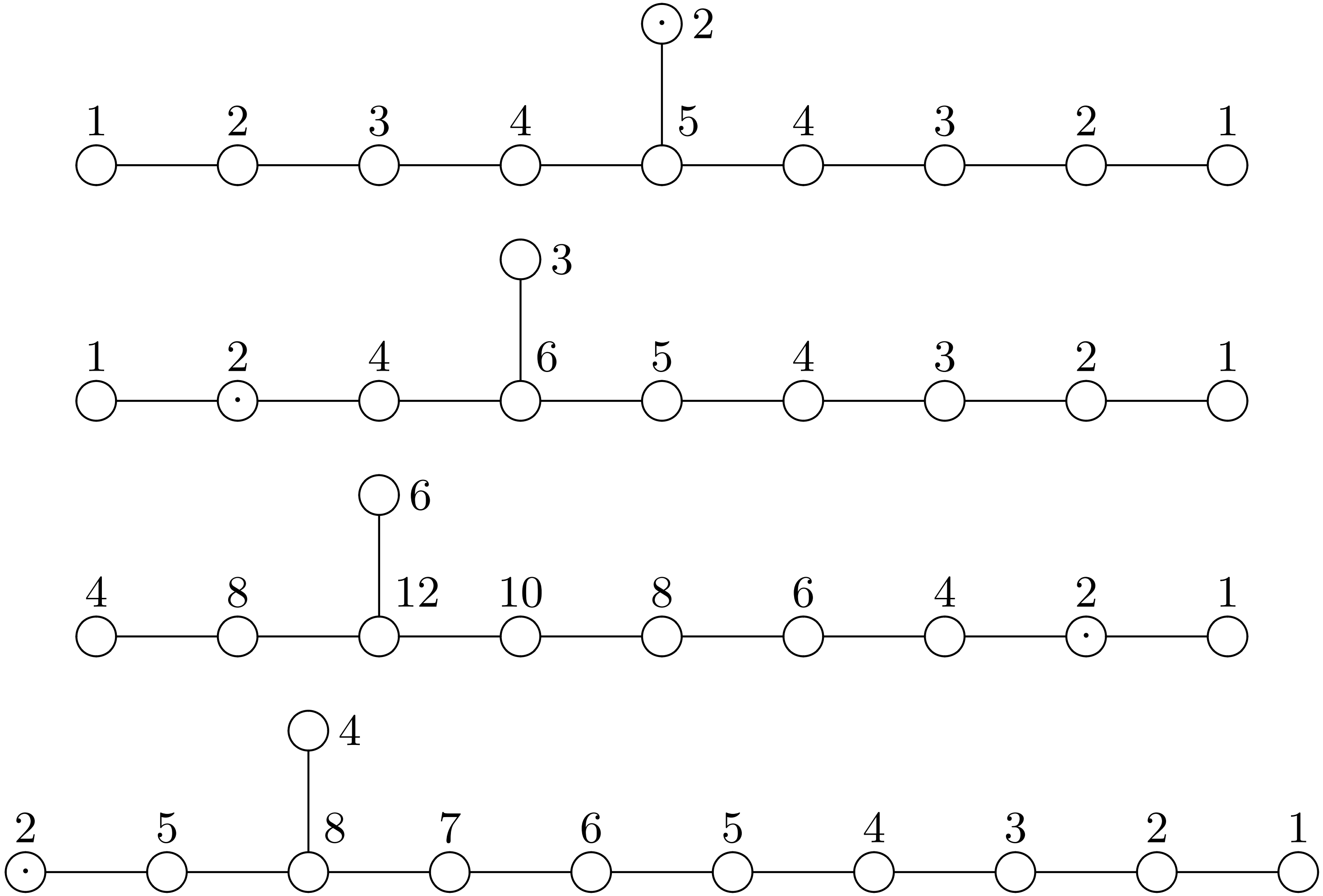



1
2
1 2 3 4 3 2 1


1
3
1 3 5 4 3 2 1


4
2 4 6 8 6 4 2 1


5
1 4 7 10 8 6 4 2



idx of rigidity	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20
#fundamental	4	13	36	67	94	162	243	305	420	582	720
# triplets	3	9	24	44	60	97	144	163	223	303	342
# tuples of 4	1	3	9	17	25	45	68	95	128	173	239

§ Fractional Calculus of Weyl algebra

Unified and computable interpretation ($\Rightarrow$ a computer program) of

Construction of equations

Integral representation of solutions

Series expansion of solutions

Reducibility, Monodromy

Connection problem

Contiguity relations

Congruences

Several variables (PDE)

$$W[x] := \langle x, \partial, \xi \rangle \otimes \mathbb{C}(\xi) \subset \overline{W}[x] := W[x] \otimes \mathbb{C}(x, \xi)$$

$$\simeq \overline{W}_L[x] := W[x] \otimes \mathbb{C}(\partial, \xi)$$

$\mathrm{R} : \overline{W}[x],\ \overline{W}_L[x] \to W[x]$ (reduced representative)

$\mathrm{L} : \partial_j \mapsto x_j,\ x_j \mapsto -\partial_j$ (Laplace transf.)

$$\mathrm{Ad}(f) \in \mathrm{Aut}(\overline{W}[x]),\ \partial_i \mapsto f(x,\xi) \circ \partial_i \circ f(x,\xi)^{-1} = \partial_i - \tfrac{f_i}{f},\quad h_i = \tfrac{f_i}{f} \in \mathbb{C}(x,\xi)$$

$$\widetilde{\Delta}_+ := \{k\alpha\,;\ k = 1, 2, \ldots,\ \alpha \in \Delta_+,\ \operatorname{supp}\alpha \ni \alpha_0\}$$

$$\begin{array}{ccc}
\{P_{\mathbf{m}} : \text{ Fuchsian differential operators}\} & \leftrightarrow & \widetilde{\Delta}_+ = \{\alpha_{\mathbf{m}}\} \\
\Big\downarrow {\scriptstyle \text{Fractional operations} \atop \mathrm{RAd}(\partial^{-\mu})\circ\mathrm{RAd}(\prod_j (x-c_j)^{\lambda_j})} & & \Big\downarrow W\text{-action} \\
\{P_{\mathbf{m}} : \text{ Fuchsian differential operators}\} & \leftrightarrow & \widetilde{\Delta}_+ = \{\alpha_{\mathbf{m}}\}
\end{array}$$

$$\mathrm{RAd}(\partial^{-\mu}) := \mathrm{L} \circ \mathrm{R} \circ \mathrm{Ad}(x^{\mu}) \circ \mathrm{L}^{-1}, \quad \mathrm{RAd}(f(x)) := \mathrm{R} \circ \mathrm{Ad}(f(x))$$

"W-action" for operators, series expansions and integral representations of solutions, contiguity relations, connection coefficients, monodromies,... are concretely determined.

Remark. On Fuchsian systems of Schlesinger canonical form

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{A_j}{x - c_j} u$$

the W-action is given by Katz + Dettweiler-Reiter + Crawley-Boevey.

Example: Jordan-Pochhammer Eq. ($p = 2 \Rightarrow$ Gauss)

$p-11, p-11, \ldots, p-11$: $(p+1)$-tuple of partitions of p

$$P := \mathrm{RAd}\big(\partial^{-\mu}\big) \circ \mathrm{RAd}\Big(x^{\lambda_0}\prod_{j=1}^{p-1}(1-c_jx)^{\lambda_j}\Big)\partial$$

$$= \mathrm{RAd}(\partial^{-\mu}) \circ \mathrm{R}\Big(\partial - \frac{\lambda_0}{x} + \sum_{j=2}^{p-1}\frac{c_j\lambda_j}{1-c_jx}\Big)$$

$$= \partial^{-\mu+p-1}\Big(p_0(x)\partial + q(x)\Big)\partial^{\mu} = \sum_{k=0}^{p} p_k(x)\partial^{p-k}$$

$$p_0(x) = x\prod_{j=2}^{p-1}(1-c_jx) \qquad q(x) = p_0(x)\Big(-\frac{\lambda_0}{x} + \sum_{j=2}^{p-1}\frac{c_j\lambda_j}{1-c_jx}\Big)$$

$$p_k(x) = \binom{-\mu+p-1}{k}p_0^{(k)}(x) + \binom{-\mu+p-1}{k-1}q^{(k-1)}(x)$$

$$u(x) = \frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\lambda_0+1)\Gamma(\mu)} \int_0^x \Bigl(t^{\lambda_0} \prod_{j=2}^{p-1} (1-c_j t)^{\lambda_j}\Bigr)(x-t)^{\mu-1} dt$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_{p-1}=0}^{\infty} \frac{(\lambda_0+1)_{m_1+\cdots+m_{p-1}}(-\lambda_1)_{m_1}\cdots(-\lambda_{p-1})_{m_{p-1}}}{(\lambda_0+\mu+1)_{m_1+\cdots+m_{p-1}} m_1!\cdots m_{p-1}!}$$

$$c_2^{m_2}\cdots c_{p-1}^{m_{p-1}} x^{\lambda_0+\mu+m_1+\cdots+m_{p-1}}$$

$$P\left\{\begin{matrix} x=0 & 1=\frac{1}{c_1} & \cdots & \frac{1}{c_{p-1}} & \infty \\ [0]_{(p-1)} & {\color{blue}[0]}_{(p-1)} & \cdots & [0]_{(p-1)} & [1-\mu]_{(p-1)} \\ {\color{red}\lambda_0+\mu} & {\color{blue}\lambda_1+\mu} & \cdots & \lambda_{p-1}+\mu & -\lambda_1-\cdots-\lambda_{p-1}-\mu \end{matrix}\right\}$$

$$c({\color{red}\lambda_0+\mu} \rightsquigarrow {\color{blue}\lambda_1+\mu}) = \frac{\Gamma(\lambda_0+\mu+1)\Gamma(-\lambda_1-\mu)}{\Gamma(\lambda_0+1)\Gamma(-\lambda_1)} \prod_{j=2}^{p-1}(1-c_j)^{\lambda_j}$$

$$c({\color{red}\lambda_0+\mu} \rightsquigarrow {\color{blue}0}) = \frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\mu)\Gamma(\lambda_0+1)} \int_0^1 t^{\lambda_0}(1-t)^{\lambda_1+\mu-1} \prod_{j=2}^{p-1}(1-c_j t)^{\lambda_j} dt$$

Example: ${}_nF_{n-1} \xrightarrow{\mathrm{RAd}(\partial^{\mu_n-1})\circ\mathrm{RAd}(x^{\lambda_n})} {}_{n+1}F_n : \sum_{k\ge 0} \frac{(\alpha_1)_k\cdots(\alpha_{n+1})_k}{(\beta_1)_k\cdots(\beta_n)_k}\frac{x^k}{k!}$

Versal Pochhammer operator

$$p_0(x)=\prod_{j=1}^{p}(1-c_jx),\quad q(x)=\sum_{k=1}^{p}\lambda_k x^{k-1}\prod_{j=k+1}^{p}(1-c_jx)$$

$$P\left\{\begin{matrix} x=\frac{1}{c_j}\ (j=1,\dots,p) & \infty \\ [0]_{(p-1)} & [1-\mu]_{(p-1)} \\ \displaystyle\sum_{k=j}^{p}\frac{\lambda_k}{c_j\prod_{\substack{1\le\nu\le k\\ \nu\ne j}}(c_j-c_\nu)}+\mu & \displaystyle\sum_{k=1}^{p}\frac{(-1)^k\lambda_k}{c_1\dots c_k}-\mu \end{matrix}\right\}$$

$$u_C(x)=\int_C\left(\exp\int_0^t\sum_{k=1}^{p}\frac{-\lambda_k s^{k-1}}{\prod_{1\le\nu\le k}(1-c_\nu s)}ds\right)(x-t)^{\mu-1}dt$$

$p=2\Rightarrow$ Unifying Gauss $+$ Kummer $+$ Hermite-Weber

$$c_1=\cdots=c_p=0\Rightarrow u_C(x)=\int_\infty^x\exp\Big(-\sum_{k=1}^{p}\frac{\lambda_k t^k}{k!}\Big)(x-t)^{\mu-1}dt$$

Thm. $\mathbf{m}$: rigid monotone with $m_{0,n_0} = m_{1,n_1} = 1$, $\frac{1}{c_0} = 0$, $\frac{1}{c_1} = 1$

$$c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) = \frac{\displaystyle\prod_{\nu=1}^{n_0-1} \Gamma\bigl(\lambda_{0,n_0} - \lambda_{0,\nu} + 1\bigr) \cdot \prod_{\nu=1}^{n_1-1} \Gamma\bigl(\lambda_{1,\nu} - \lambda_{1,n_1}\bigr)}{\displaystyle\prod_{\substack{\mathbf{m}' \oplus \mathbf{m}'' = \mathbf{m} \\ m'_{0,n_0} = m''_{1,n_1} = 1}} \Gamma\bigl(|\{\lambda_{\mathbf{m}'}\}|\bigr) \cdot \prod_{j=2}^{p-1} (1 - c_j)^{L_j}}$$

$$|\{\lambda_{\mathbf{m}'}\}| = \sum m'_{j,\nu} \lambda_{j,\nu} - \operatorname{ord} \mathbf{m}' + 1$$

$\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}'' \overset{\text{def}}{\Longleftrightarrow} \mathbf{m},\ \mathbf{m}'$ realizable and $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$

$$\text{Gauss}: \left\{\begin{matrix} x = \frac{1}{c_0} = 0 & \frac{1}{c_1} = 1 & \infty \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} \\ \overline{\lambda_{0,2}} & \underline{\lambda_{1,2}} & \lambda_{2,2} \end{matrix}\right\} = \left\{\begin{matrix} x = 0 & 1 & \infty \\ 1 - \gamma & \gamma - \alpha - \beta & \alpha \\ 0 & 0 & \beta \end{matrix}\right\} \qquad \begin{matrix} 1\bar{1}, 1\underline{1}, 11 \\ = 0\bar{1}, 10, 10 \\ \oplus 10, 0\bar{1}, 01 \end{matrix}$$

$$c(\lambda_{0,2} \rightsquigarrow \lambda_{1,2}) = \frac{\Gamma(\lambda_{0,2} - \lambda_{0,1} + 1)\,\Gamma(\lambda_{1,1} - \lambda_{1,2})}{\Gamma(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,1})\,\Gamma(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,2})} \qquad \begin{matrix} \updownarrow \\ \leftrightarrow \end{matrix}$$

$$P\left\{\begin{matrix} x=\frac{1}{c_0}=0 & \frac{1}{c_1}=1 & \cdots & \frac{1}{c_p}=\infty \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix}\right\}$$

$\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$: rigid $\iff$ $\alpha_{\mathbf{m}}=\alpha_{\mathbf{m}'}+\alpha_{\mathbf{m}''}$: positive real roots

$\mathrm{ord}\le 40,\ p=2\Rightarrow$ 4,111,704 independent cases by a computer

non-rigid case : $c(\lambda_{0,n_0}\rightsquigarrow\lambda_{1,n_1})=$

a gamma factor $\times$ a connection coefficient of a fundamental case

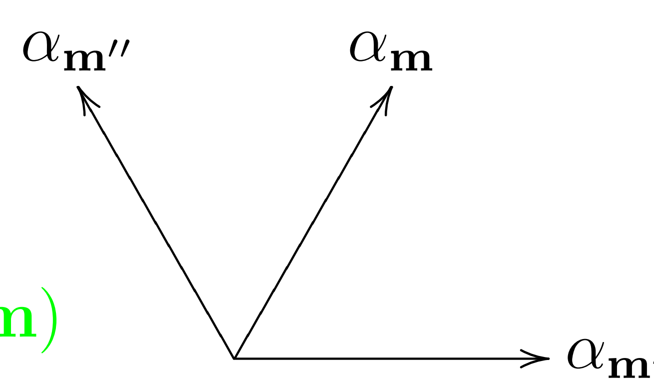
Thm (Irreducibility): Suppose $\mathbf{m}$ is monotone

$\mu:=\lambda_{0,1}+\cdots+\lambda_{j_o,1}+\cdots+\lambda_{p,1}-1\notin\mathbb{Z}$

$\lambda_{j_o,1}\mapsto\lambda_{j_o,\nu_o}\ \uparrow$ if $m_{j_o,\nu_o}>m_{j_o,1}-d(\mathbf{m})$

$\lambda_{0,0}\mapsto\lambda_{0,0}-2\mu,\ \lambda_{0,\nu}\mapsto\lambda_{0,\nu}-\mu\ (\nu>1)$

$\lambda_{k,\nu}=\lambda_{k,\nu}+\mu\ (k>0),\ \ m_{j,1}\mapsto m_{j,1}-d(\mathbf{m})$



Thank you! End!

· *Classification of Fuchsian systems and their connection problem*, arXiv:0811.2916, 29 pages, 2008

· *Fractional calculus of Weyl algebra and Fuchsian differential equations*, preprint, 95 pages, 2009

· `ftp://akagi.ms.u-tokyo.ac.jp/ftp/pub/math` `okubo.exe, muldif.rr`

reduction, integral representation, series expansion, connection formula,. . .