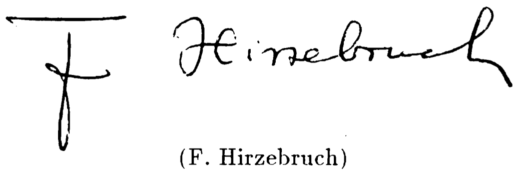
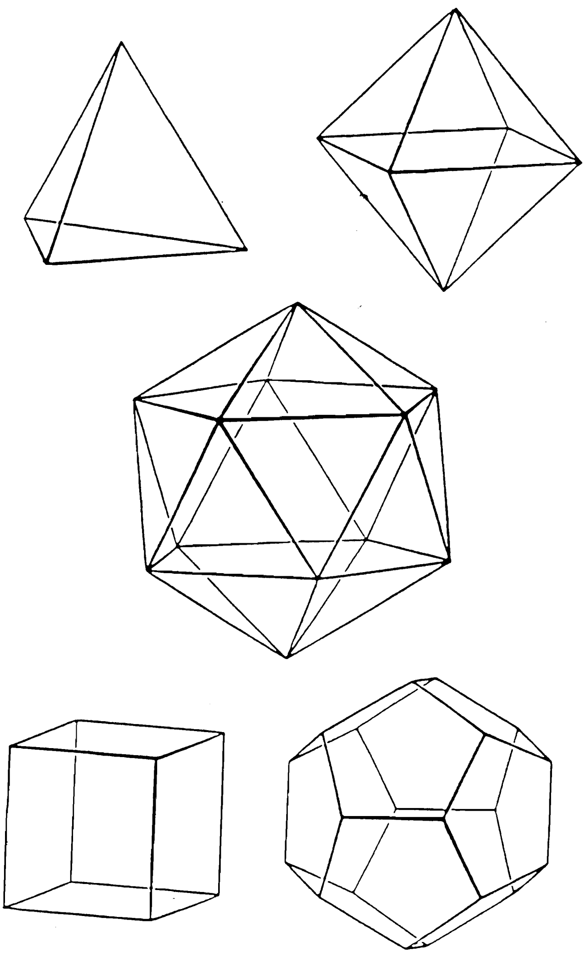
Regular Polyhedra and the Football



(F. Hirzebruch)

The regular polyhedra are discussed in Euclid's book (300 B.C.). I show them to you here.



They occur in nature and in art. Here are two examples for the dodecahedron.

Coccosphäre von Braarudosphaera bielowi
Vergrößerung etwa 5000 X

Aus dem Miozän (vor ca. 20 Mill. Jahren)
S. A. Jafar, Tübingen, 1975

Le dodécaèdre en argent trouvé à Saint-Pierre de Genève

Lors de la campagne archéologique de fouilles entreprise à l'occasion de la dernière restauration de la cathédrale de Saint-Pierre à Genève, un dé romain en forme de dodécaèdre, datant du 4e siècle après Jésus-Christ, a été mis au jour. Les 12 faces pentagonales en argent portent les 12 signes du zodiaque; il est rempli de plomb (poids 297 g). Le «dé» a probablement servi à la prédiction de l'avenir par le jeu, mais la provenance et l'utilisation restent inconnues pour l'instant.

I have this from Dr. Götze of Springer-Verlag.

The regular polyhedra have the following properties. There are natural numbers n, m such that: Each face is a regular n-gon. From each vertex (corner) m edges leave. Which n, m are possible? Here is a simple argument. The sum of the angles in a n-polygon equals

$$(n-2)\cdot 180^\circ.$$

Therefore

$$m\cdot\frac{n-2}{n}\cdot 180^\circ < 360^\circ$$

(the sum of the angles in each corner must be less than 360° because of convexity). This is equivalent to

$$\frac{1}{n}+\frac{1}{m}>\frac{1}{2}.$$

The solution of this inequality in natural numbers $\geqq 3$ are (3,3), (4,3), (3,4), (5,3),(3,5). I show you the following transparency (in German).

n = Valenz der Flächen

m = Valenz der Eckpunkte

Winkelsumme im n-Eck =

$(n-2)\cdot 180^0$, deshalb

$$m\,\frac{n-2}{n}\cdot 180^0 < 360^0$$

$$\frac{1}{n}+\frac{1}{m}>\frac{1}{2}$$

			e	k	f	
Polyeder	n	m	b_0	b_1	b_2	Plato
Tetraeder	3	3	4	6	4	Feuer
Hexaeder	4	3	8	12	6	Erde
Oktaeder	3	4	6	12	8	Luft
Dodekaeder	5	3	20	30	12	Kosmos
Ikosaeder	3	5	12	30	20	Wasser

The Greek names for the polyhedra are similar in English and I suppose in Japanese. Platon associates the basic elements of our existence to the five regular polyhedra. b_0, b_1, b_2 are the numbers of corners (vertices), edges, faces of the polyhedron. The Greek name comes from the number of faces. In German often e, k, f are used (Eeken, Kanten, Flächen). Can we calculate b_0, b_1, b_2 from n and m. We have

$$\begin{aligned} b_0 m &= 2b_1 \\ b_2 n &= 2b_1 \end{aligned}$$

and we add courageously Euler's formula

$$b_0 - b_1 + b_2 = 2$$

and obtain

$$\frac{1}{n} + \frac{1}{m} - \frac{1}{2} = \frac{1}{b_1}.$$

Now we have formulas for b_0, b_1, b_2 in terms of n, m . Interchanging n, m leads to interchanging b_0, b_2 .

Tetrahedron	---	*Tetrahedron*
Hexahedron(cube)	---	*Octahedron*
Dodecahedron	---	*Icosahedron*

We shall see in a moment the close relationship between the left and the right side.

Let's first talk about Euler's formula. Euler was born in Basel (Switzerland) in 1707, he worked in the Prussian Academy from 1741 to 1766 under Frederick the Great in Berlin who treated him badly, though Euler was a genius also in "Applied Mathematics" and helped the King in many technical problems. From 1766 until his death in 1783 he was in the Russian Academy in Sankt Petersburg under Katherina II who treated him better. Euler's picture is on a Swiss bank note

I show you also Carl Friedrich Gauß(1777–1855) on a German bank note whose work on curvature is closely related to Euler's formula. How to prove Euler's formula?

We show that it is true for an arbitrary convex polyhedron. The surface of the polyhedron can be mapped onto the plane. The faces become countries. One country being a big ocean. We imagine a point at infinity (in the ocean) to make the plane to a sphere. The idea is explained in the following picture.

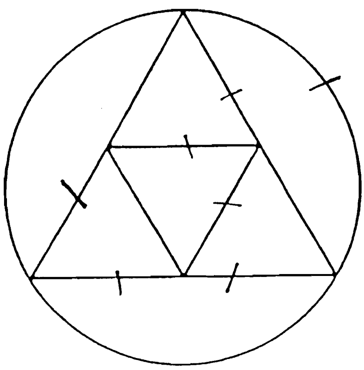


Fig. 48

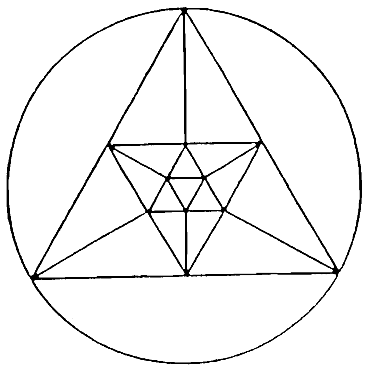
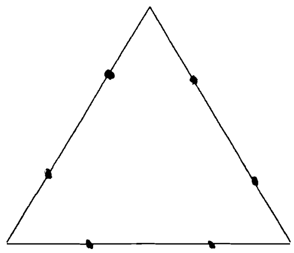


Fig. 49

(From the famous book by Rademacher and Toeplitz "Von Zahlen und Figuren"). It shows the diagram of an icosahedron and an octahedron. Open a dam (remove an edge) to let water in. An edge disappears and two countries are united. Open again if there is water only on one side. Each time b_1 is reduced by 1 and also b_2. Then $b_0 - b_1 + b_2$ stays the same until you reach a connected string of edges (without a cycle) and one country (the ocean has taken over). Obviously, at the end of this process, $b_0 - b_1 = 1$ and $b_2 = 1$. Thus $b_0 - b_1 + b_2 = 2$ at the end and at the beginning of the process.

The relation between the icosahedron and the dodecahedron becomes clear by studying the group of symmetries. A symmetry is a rotation around an axis by a certain angle which carries the polyhedron to itself. Icosahedron and dodecahedron have six symmetry axis (through a vertex and its opposite for the icosahedron, through the center of a face and its opposite for the dodecahedron). We can rotate by $k \cdot 72°$ ($k = 1, 2, 3, 4$, five fold symmetry). Similarly we have 10 axis with threefold and 15 with twofold symmetry. The number of symmetries is $6 \cdot 4 + 10 \cdot 2 + 15 \cdot 1 = 59$. The group of symmetries has $N = 60$ elements because we have to include the identity. For all regular polyhedra $N = 2b_1$. From the point of view of the symmetry group icosahedron and dodecahedron, cube and octahedron cannot be distinguished. The face centers of one correspond to the vertices of the other.

Let us consider the icosahedron. Take a point on it which is not special (not a vertex, not the center of an edge, not the center of a face). Apply all symmetries to it. Then we get 60 points. This is the orbit of the given point. For example, take a face (triangle) of the icosahedron and a point on the boundary, but not a vertex and not the center of an edge. Then the orbit has 60 points, two on each edge.



Now study a football. It usually has 12 black pentagons with 60 vertices. This is such an orbit. The usual football corresponds to the case when the edge of the triangle is divided into three equal parts. Inside a face we obtain a regular hexagon. We can cut off the corners of the icosahedron by a planar cut through the 5 points of the orbit near the corner. Then we obtain the truncated icosahedron with 12 regular pentagons and 20 regular hexagons. Therefore $b_2 = 12 + 20 = 32$. We have $b_1 = 30 + 60 = 90$ and $b_0 = 60$. Euler's equation $60 - 90 + 32 = 2$ checks.

Next we come to the application of the truncated icosahedron in chemistry.

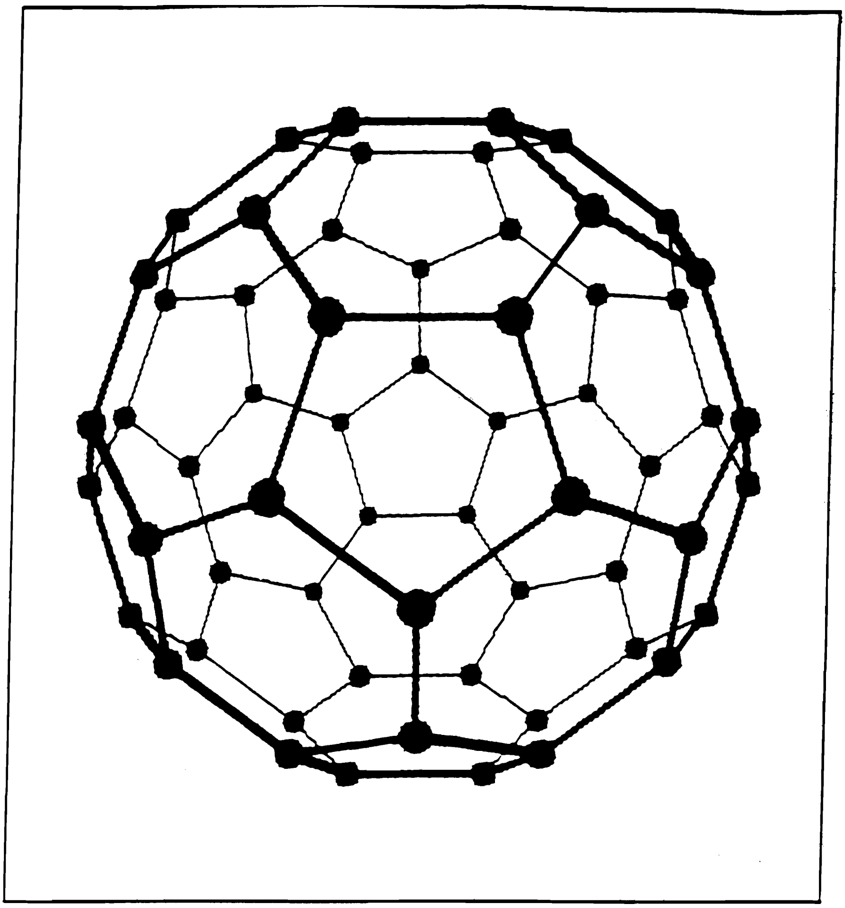


Abb. 1. Perspektivische Darstellung des fußballförmigen C_{60} Moleküls. Die Kohlenstoffatome befinden sich an den Ecken eines Polyeders, das Fünf- und Sechsecke als begrenzende Flächen hat (sog. gekapptes Ikosaeder). Im Unterschied zum idealen Polyeder mit gleichen Kantenlängen sind im realen C_{60} Molekül die Abstände benachbarter Kohlenstoffatome nicht alle identisch. Einschließlich seiner Elektronenwolke hat das Molekül etwa 1 nm Durchmesser.

Dr. Wolfgang Krätschmer, Max-Planck-Institut für Kernphysik, Heidelberg

Konstinos Fostiropoulos
Donald R. Huffman, Tucson, Arizona

There is the carbon molecule C_{60} with 60 atoms sitting in the vertices of the truncated icosahedron. As explained before this is an orbit of the icosahedral group. The Frankfurter Allgemeine Zeitung of October 10, 1996, has an article "Forscher im Fussballfieber" and reports that the Nobel Prize for chemistry was given to Harold W. Croto (Great Britain), Robert F. Curl, Jr. and Richard E. Smalley (USA) for the discovery of C_{60}. Congratulations! Maybe, Wolfgang Krätschmer of the Max-Planck-Institut für Kernphysik in Heidelberg, who produced C_{60} in macroscopic quantities, could also have been a candidate. I understand that a Japanese scientist is also involved. But, in any case, Euclid and Archimedes also deserve part of this prize.

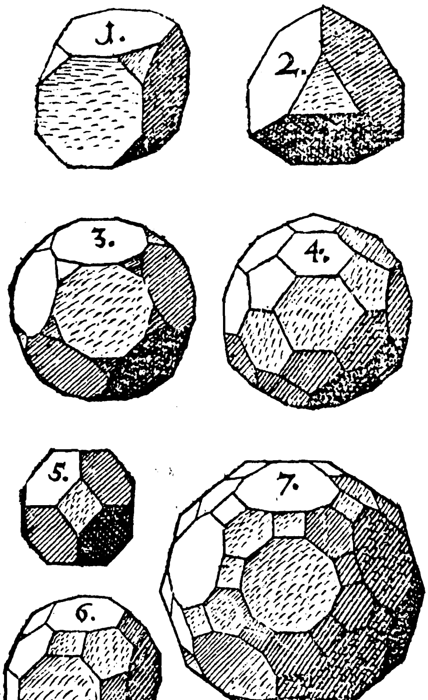
A polyhedron is called an archimedean polyhedron if all its faces are regular n-gons (but "n" may vary, for example $n = 5, 6$ for the truncated icosahedron), if all edges have equal length, and if every corner can be moved by a symmetry to any other corner. There are 13 interesting archimedean polyhedra. Kepler listed and investigated them in his "Harmonices mundi" in 1619.

que imparilaterarum rejicitur , per XXIII. cum duobus Octogonicis , planum locum implet: cum majoribus etiam superat 4 rectos; nec assurgit ad solidum angulum formandum. Ita transactum est cum Tetragono , cum duæ solæ debent esse planorum species.

Duo Pentagonici cum uno Hexagonico aut quocunq; alio unico rejectitium quid inchoant, per XXIII, quod suprà etiam de Trigonico & Tetragonico cum binis Pentagonicis usurpavimus. Insuper cum uno Decagonico planitiem sternunt , nec cum illo aut majoribus assurgunt in soliditatem.

X. Truncum Icosiedron.

Unus ergò Pentagonicus cum duobus Hexagonicis minus facit 4 rectis; & congruunt duodecim Pentagoni cum viginti Hexagonis in unum Triacontakadyhedron , quod appello Truncum Icosihedron. Formam habes signatam numero 4. Nec plura expectanda à Pentagono. Nam unus Pentagonus cum duobus Heptagonicis jam superat 4. rectos.



Hexagonicus cum duobus alijs implet planitiem , cum majoribus superat 4 rectos. Itaq; hic finis est mixtorum ex duabus speciebus.

Quod si trium specierum Plana concurrere possunt ad unum angulum solidum: Primum anguli duo plani, unus Tetragoni, alter Pentagoni superant 2 rectos; majores his, multò magis: tres verò Trigonorum trium , æquant 2 rectos : nequeunt igr tres Trigonici admitti, ne summa omniū superet 4 rectos. Duo verò Trigonici cum uno Tetragonico & uno Pentagonico vel pro eo Hexagonico , aut quocunque majori, rejiciuntur, per pr. XXIII. quia Trigonus imparilatera figura cingi deberet Tetragono & Pentagono , vel pro eo Hexagono &c.

XI. Rhomb Icosidodecaedron.

Unus igitur Trigonicus cum duobus Tetragonicis & uno Pentagonico, minus efficiunt 4 rectis , & congruunt 20 Trigoni cum 30 Tetragonis & 12 Pentagonis, in unum Hexacontadyhedron , quod appello Rhombicosidodecaëdron, seu sectum Rhombum Icosidodecaëdricum. Pingitur num. 11. fol. antecedentis

Unus Trigonicus , duo Tetragonici , cum uno Hexagonico, æquant rectos quatuor ; cum uno majori , superant ; nec ad solidum assurgunt. Mittamus igitur duos Tetragonicos.

Unus Trigonicus, unus Tetragonicus, & duo Pentagonici superant 4 rectos; multoq; magis si bini majores plani anguli admiscerentur. Desinunt igitur misceri anguli plani quaterni ad formandum unum solidum; desinit ergò & Trigonus ingredi mixturam triplicem. Nam unus Trigonicus, unus Tetragonicus &

62 DE FIGURARUM HARMON:

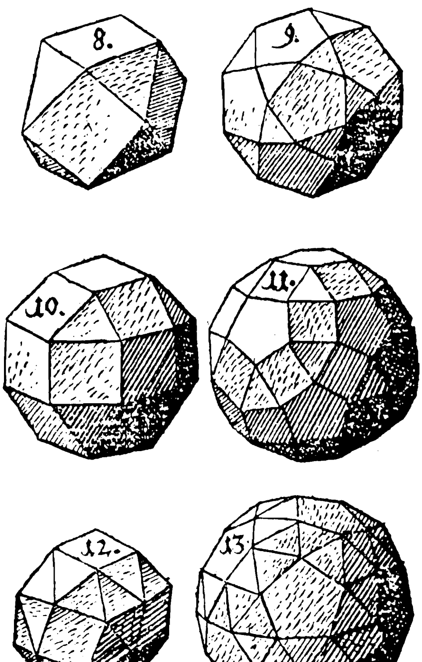
Cum enim misceantur in hoc gradu figuræ diversæ, quare per propos. XXI. miscebuntur aut duarum aut trium specierum figuræ. Quod si duarum, tunc inter eas vel sunt Trigoni vel non sunt.

Igr ex Trigonis & Tetragonis fiunt solida tria, quibus quidem def. IX. competat. Nam illa rejicit formas hasce tres, in quibus solidum angulum claudunt, cum uno Tetragonico plano angulo, tam duo, quàm tres plani Trigonici; aut cum

Oct *duobus Tetragonicis, unus Trigonicus; quia in primo casu unus solus Tetragonus est, fitq; dimidium Octaëdri, & anguli solidi sunt diversiformes: in secundo duo soli Tetragoni, in tertio duo soli Trigoni: quæ p X, sunt imperfectæ congruentiæ. Restant ergò modi hi, in quibus angulum solidum claudunt plani, Primum,*

I. Cubus simus. *quatuor Trigonici & unus Tetragonicus. Sunt enim minores 4 rectis. Congruunt igitur sex Tetragoni & Triginta duo (id est 20 & 12.) Trigoni, & fit figura Triacontaoctohedrica, quod appello Cubum simum. Hic in schemate sequenti pictus est Numero 12.*

Quinq; enim Trigonici plani & unus Tetragonicus superant quatuor rectos, cùm debeant ad solidum claudendum esse minores quatuor rectis, per XVI. Sic etiam quatuor Trigonici & duo Tetragonici. Tres verò Trigonici & duo Tetragonici faciunt quatuor rectos.



II. Rh. Cuboctaedron.

Secundò duo Trigonici & duo Tetragonici minus habent quatuor rectis; Hic igitur congruunt octo Trigoni & sex Tetragoni ad formandum unum Tessareskædecaëdron, quod cuboctaëdron appello. Pictum est hic num: octavo. Duo verò Trigonici cum tribus Tetragonicis superant 4 rectos.

III. Rhombici Cuboctaedron.

Tertiò unus Trigonicus & tres Tetragonici minus habent 4. rectis Hic ergò congruunt octo Triangula & octodecim (id est 12 & 6) quadrangula, ad unum Icosihexaëdron, quod appello sectum Rhombū Cuboctaëdricum: vel Rhombicuboctaëdron. Pictus est hic numero 10.

In his igr tribus sunt Tetragoni juxta Trigonos: sequitur ut & Pentagonicos ijs seorsim associemus.

Quinq; plani Trigonici juxta unum Pentagonicum non stant, quia neq; juxta minorem eo, Tetragonicum, stare poterant. Quatuor ergò Trigonici, cum uno Pentagonico, minus efficiunt 4 rectis, & congruunt octoginta (id est 20. &

IV. Dodecaëdron simū *60) Trigoni, cum duodecim Pentagonis, ad formandum Ennenecontakædyhedron, quod appello Dodecaëdron simum. Pingitur hic numero 13. Et in hoc ordine simorum, Icosaëdron posset esse tertium, quod est quasi Tetraëdron simum.*

Tres

Number 4 is the truncated icosahedron (football):

> "... et congruunt duodecim Pentagoni cum viginti Hexagonis in unum Triacontakedyhedron, quod appello Truncum Icosihedron. Formam habes signatam numero 4."

The chemists are interested in polyhedra such that three edges leave every vertex and that we have pentagons and hexagons only. They try to find carbon molecules whose atoms sit in the vertices. We learn in classical chemistry that carbon has valence 4. Now using such a polyhedron, each atom is connected with 3 others by edges. Therefore, to one edge should be given multiplicity 2. Such a Kekulé structure can be given to the football: The 30 edges between 2 hexagons get multiplicity 2. In chemistry polyhedra with the above properties are called Fullerenes after the famous architect Buckminster Fuller (1895–1983).

It follows from Euler's formula that for a Fullerene the number of pentagons is always equal to 12. To show this we first prove same general formulas valid for any convex polyhedron. Let $b_0(r)$ be the number of vertices from which r edges leave and $b_2(r)$ the number of faces with r edges in their boundary ($r \geq 3$). Then

$$\sum b_0(r) = b_0 \quad , \quad \sum b_2(r) = b_2$$
$$\sum r b_0(r) = 2b_1 \quad , \quad \sum r b_2(r) = 2b_1.$$

Then the following equations are consequences of Euler's formula

$$(*) \quad \begin{array}{l} 12 + \sum(r-6)b_0(r) + \sum(2r-6)b_2(r) = 0 \\ 12 + \sum(r-6)b_2(r) + \sum(2r-6)b_0(r) = 0 \end{array}$$

The first equation is equivalent to

$$12 + 2b_1 - 6b_0 + 4b_1 - 6b_2 = 0$$

and this is Euler's formula. The same proof works for the second equation (interchange 0 and 2). The second equation gives for a

Fullerene $b_2(5) = 12$. The first equation implies

$$\begin{aligned} b_0 &= 20 + 2b_2(6) \\ b_1 &= 30 + 3b_2(6) \\ b_2 &= 12 + b_2(6). \end{aligned}$$

For $b_2(6) = 0$ we have the regular dodecahedron, for $b_2(6) = 20$ the football. One can show that the number $b_2(6)$ of hexagons can have any value ≥ 2. In the chemical literature (I forgot the precise reference) there are computer calculations to find the combinatorial types of all Fullerenes with given b_0. The number of types for $b_0 = 60$ equals 1760. But there is only one type with disjoint pentagons, namely the football. There are 21822 types for $b_0 = 78$. But there are only 5 types with disjoint pentagons.

The second equation (*) implies for any convex polyhedron

$$3b_2(3) + 2b_2(4) + b_2(5) \geqq 12.$$

Equality holds if and only if $b_0(r) = 0$ for $r \neq 3$ and $b_2(r) = 0$ for $r \geqq 7$. This is satisfied for the cube, the tetrahedron, the dodecahedron and all Fullerenes.

Kepler mentions that in a corner one regular pentagon and two regular heptagons (7-gons) are impossible because the sum of the angle's is greater than 360°.

$$108° + 2 \cdot \frac{5 \cdot 180°}{7} > 360°.$$

> "Nam unus Pentagonus cum duobus Heptagonicis jam superat 4 rectos."

We introduce for each corner the deficit δ as 360° minus the sum of all angles coming together in this corner. It is positive. A result of Descartes is equivalent to

$$(**) \quad \sum \delta_i = 720° (\text{sum over all corners})$$

Leibniz (the great contemporary of Seki Takakasu) copied Descartes' result in a Paris library. Descartes' manuscript is lost, but the copy of Leibniz exists.

Descartes 1596 – 1650

Progymnasmata
de solidorum elementis
excerpta ex manuscripto Cartesii
Leibniz 1676
(vgl. P. Costabel 1987)

Si quatuor angular plani recti
ducantur per numerum angulorum
solidorum & ex producto tollantur
8 anguli recti plani, remanet
aggregatum ex omnibus angulis
planis qui in superficie talis
corporis solidi existunt

$$360^\circ \cdot e - 720^\circ = 180^\circ \sum_{r \geq 3} (r-2) f_r$$

$$e - 2 = k - f$$

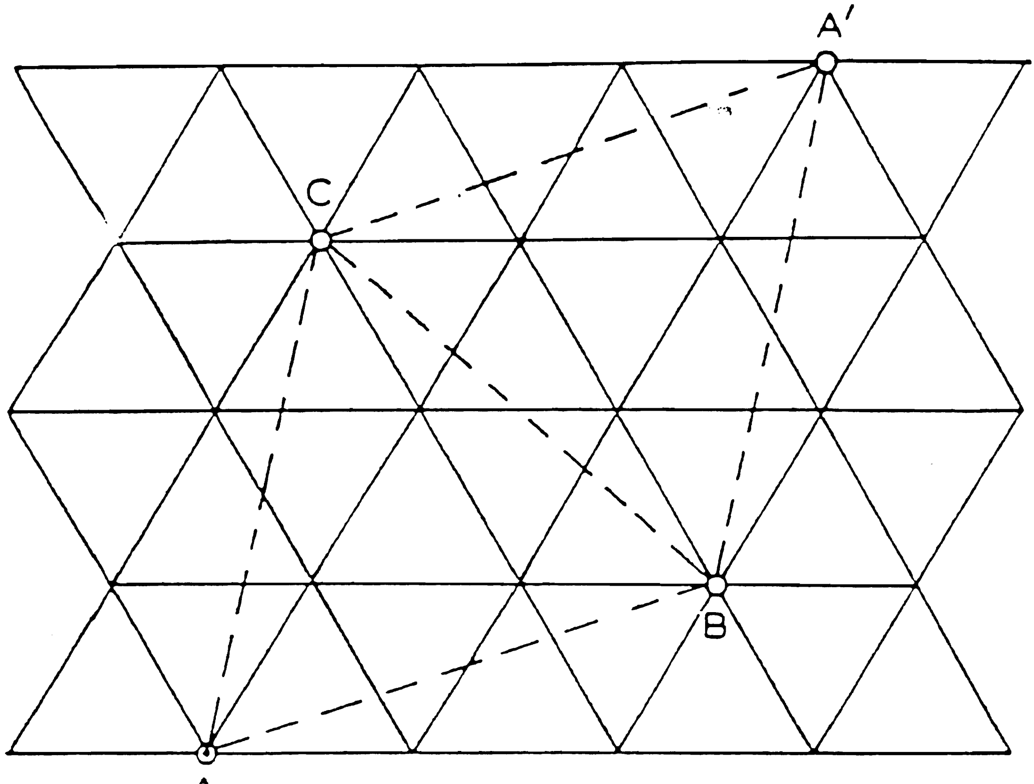
$$\sum_i \delta_i = 720^\circ$$

δ_i (für einen Eckpunkt i) =
360° – Summe der Winkel in i

The transparency is in Latin and German ($k = b_1, f = b_2, f_r = b_2(r)$). We see that Descartes' result is equivalent to Euler's formula. The formula (**) is a discrete version of the famous formula of Gauß that the integral $\int \kappa dF$ over the curvature of a convex surface equals 4π ($= 720°$). There should be a new French bank note with Descartes. Proposal:

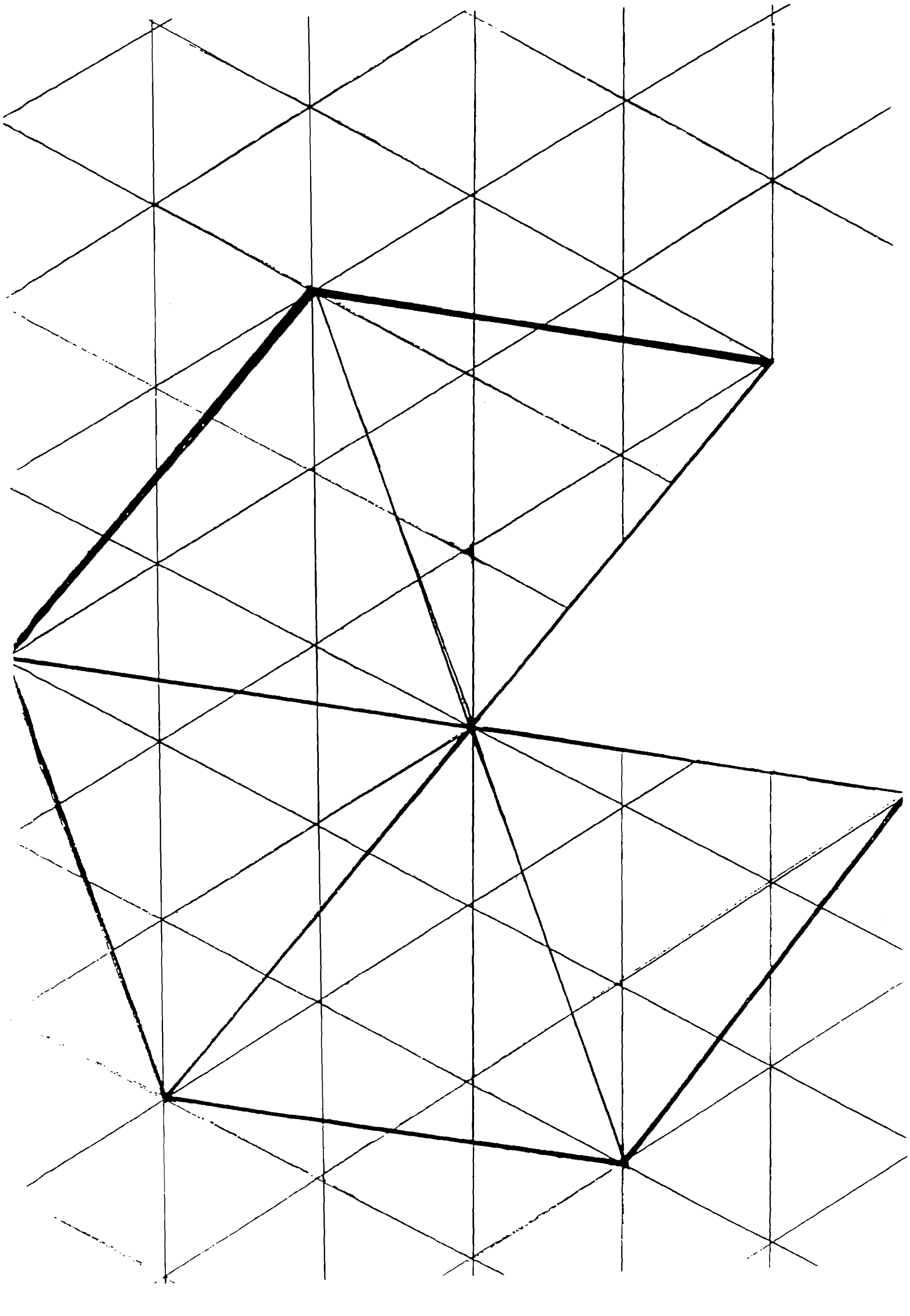
How to construct Fullerenes? There are many methods. I do not have a complete survey. Recently, in connection with my work on Hilbert modular surfaces, I studied an article by Bertram Kostant "The Graph of the Truncated Icosahedron and the Last Letter of Galois" (Notices of the American Mathematical Society, September 1995). I talked about this in the Kyoto colloquium. Kostant mentions P. W. Fowler and D. E. Manolopoulos, An Atlas of Fullerenes, Oxford, 1995. I did not see this book yet. I shall explain here a construction given by M. Goldberg in 1936. It is possible to triangulate an icosahedron in $20(a^2 + ab + b^2)$ triangles as explained by the following diagrams:

M. Goldberg 1936



$a = 2$, $b = 1$

Δ zerfällt in $a^2 + ab + b^2$ kleine Dreiecke.



This gives the combinatorial structure of a convex polyhedron which has only triangles:

$$b_2(r) = 0 \text{ for } r \neq 3, \quad b_2 = 20(a^2 + ab + b^2).$$

We have $b_0(5) = 12$ and $b_0(r) = 0$ for $r \neq 5, 6$ and

$$\begin{aligned} b_0 &= 10(a^2 + ab + b^2) + 2 \\ b_1 &= 30(a^2 + ab + b^2) \\ b_2 &= 20(a^2 + ab + b^2) \end{aligned}$$

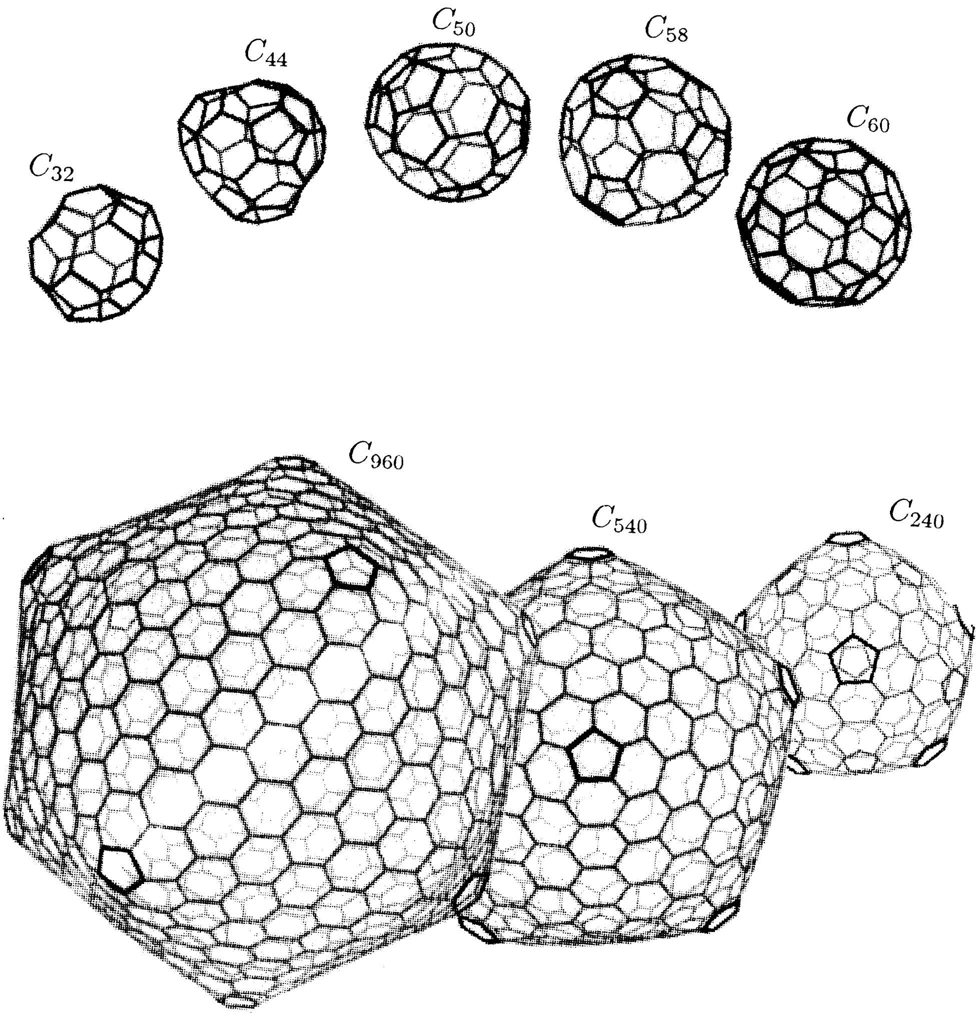
According to D. L. D. Caspar and A. Klug 1962 (Nobel Prize) some virusses have protective ball like structures (capsides) consisting of capsomeres sitting in the corners of such a polyhedron.

We can pass to the dual polyhedra. The centers of the faces correspond to the vertices (corners) of the dual polyhedron. Then b_0 and b_2 are interchanged. We obtain Fullerenes with

$$\begin{aligned} b_0 &= 20(a^2 + ab + b^2) \\ b_1 &= 30(a^2 + ab + b^2) \\ b_2 &= 10(a^2 + ab + b^2) + 2 \\ b_2(6) &= 10(a^2 + ab + b^2 - 1) \end{aligned}$$

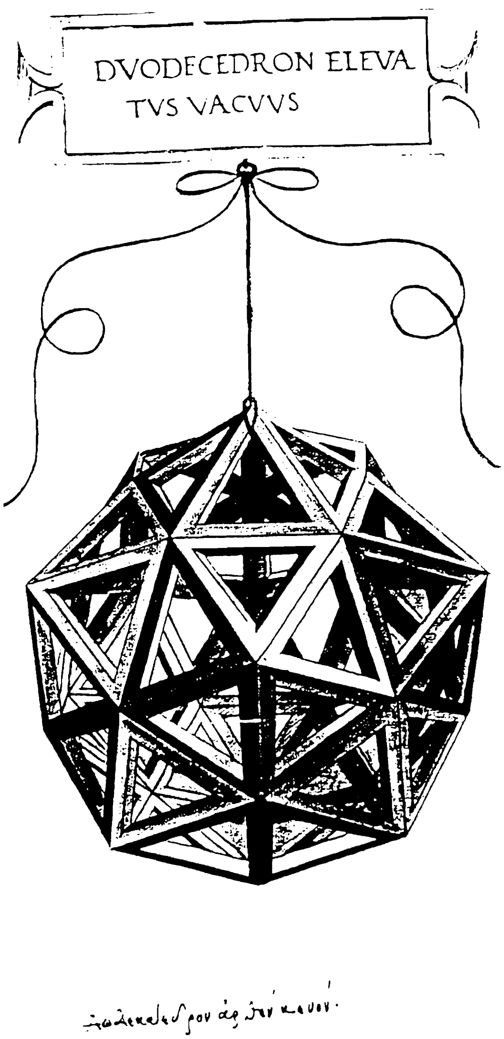
For $a = b = 1$ we get the football. For $a = b$ we have $b_0 = 60a^2$. Here are pictures of C_{60a^2}.

5 0周年記念講演会講演



Fullerenes
by Robert F. Curl and Richard E. Smalley,
Scientific American,
October, 1991.

The dual football ($a = b = 1,\ b_0 = 32$) was drawn by Leonardo da Vinci.



Leonardo da Vinci
in Luca Pacioli
De Divina Proportione
Milano , Bibliotheca Ambrosiana
1509

It has icosahedral symmetry and is the protective cover of the Picorna virus with 32 capsomere. The centers of the 60 triangles give the vertices of the football. I refer to H. S. M. Coxeter "Virus macromolecules and geodesic domes" 1971.

[This lecture is a very shortened and much modified version of lectures in the ETH Zurich and the Siemens Foundation in Munich.]

（フリードリッヒ ヒルツェブルッフ）