Class Field Theory
– Its Centenary and Prospect

ADVANCED STUDIES IN PURE MATHEMATICS 30

Chief Editor: Eiichi Bannai (Kyushu University)

Class Field Theory – Its Centenary and Prospect

Edited by

Katsuya Miyake (Tokyo Metropolitan Univ.)

Mathematical Society of Japan

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Advanced Studies in Pure Mathematics 30

Chief Editor

Eiichi Bannai (Kyushu University)

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Preface

The Seventh MSJ International Research Institute of the Mathematical Society of Japan was held in Tokyo for ten days from June 3rd to 12th, 1998. The theme was 'Class Field Theory – its Centenary and Prospect', which is taken as the title of the book. The program of this international conference is attached at the end of the preface. This volume is a collection of articles contributed by the speakers of the conference. All but a few of them are full scale papers. Some of them are expository on those subjects which are of central issues of algebraic number theory, and are prepared by the leading experts; they contains important and interesting problems with extensive references. Some of them are historical, and vividly explain how number theorists were motivated and exchanged their mathematical ideas.

In 1920 Takagi published the complete version of his class field theory as 'Ueber eine Theorie des relativ Abel'schen Zahlkörpers' in J. Coll. Sci. Tokyo, vol. 41. Chapter V of it is devoted to an affirmative solution to 'Kronecker's youth-dream'. This problem asks, roughly speaking, whether *all* abelian extensions of an imaginary quadratic number field could be obtained by the singular moduli and special values of elliptic functions which have complex multiplication by the elements of the quadratic field; and it was reformulated in a general frame-work by Hilbert as the twelfth problem of his celebrated 23 problems. There is another problem behind class field theory: that is, the principal ideal theorem which was finally proved by Furtwängler in 1930 based on Artin's general reciprocity law. Artin established this significant result in his short paper, 'Beweis des allgemeinen Reziprozitätsgesetzes', in Abh. Math. Sem. Univ. Hamburg, vol. 5 (1927). Like the former problem, the origin of the latter goes back to Kronecker. We may say, however, that the Takagi-Artin class field theory has its direct origins in several papers by Weber and by Hilbert published in 1897–1898:

H. Weber, Ueber Zahlengruppen in algebraischen Körpern I, Math. Ann. 48 (1897), 433–473; II, 49 (1897), 83–100; III, 50 (1898), 1–26;

D. Hilbert, Die Theorie der algebraischen Zahlkörper, Jber. Deutschen Math.-Ver.4 (1897), 175–546; Über die Theorie der relativ-Abelschen Zahlkörper, Nachr. Akad.Wiss. Göttingen 5 (1898), 377–399;

(cf. e.g. K. Miyake, The Establishment of the Takagi-Artin Class Field Theory, in *The Intersection of History and Mathematics* (ed. J. W.

Dauben et al), Birkhäuser Verlag, Basel-Boston-Berlin, 1994, pp.109–128). This is one of the reasons why the editor of this volume chose the above stated theme of the Seventh MSJ International Research Institute in 1998 when he was appointed as organizer.

After the marvelous results of Takagi and Artin, the next act of the drama of class field theory was played by H. Hasse. In this energetic performance, he established his reciprocity law in the form which beautifully embodied the *Local–Global Principle* in class field theory (cf. e.g. G. Frei's article in this volume). Under his influence F. K. Schmidt could demonstrate local class field theory in 1930 based on the global theory. Then Chevalley gave an 'arithmetic proof' to class field theory, and also introduced idèles (cf. e.g. C. Chevalley, La théorie du corps de classes, Ann. Math. 41 (1940), 394–418). And finally in 1950, K. Iwasawa and J. Tate independently developed functional analysis over idele groups of algebraic number fields to present zeta- and L-functions with functional equations in an elegant manner. Here ideles became a natural and basic concept of algebraic number theory. It should also be mentioned that A. Weil gave an important impetus by introducing the ring of adèles in his lecture notes, Adèles and algebraic groups, at Princeton in 1959.

In 1951 and 1952, G. Hochschild, T. Nakayama and J. Tate successfully gave a cohomological description to class field theory. In his lectures at Princeton in 1951–52, E. Artin treated class field theory with this method. It enabled him to give an axiomatic presentation, 'class formation'. Then extensive studies were carried out by Y. Kawada with the partial help of I. Satake.

It would be too much for me to give even a brief sketch of the development of the theory of automorphic functions and forms. I would like, however, to point out just two significant works. Mainly motivated by Hilbert's twelfth problem, E. Hecke developed his works on 'Hilbert modular functions' of two variables in his two papers published in Math. Ann. 71 (1912) and *ibid.* 74 (1913); the title of the second paper is 'Über die Konstruktionen relativ-Abelscher Zahlkörper durch Modulfunktionen von zwei Variabeln'. Here he deals with SL_2 over the ring of integers of a real quadratic number field which discontinuously acts on a direct product of two copies of the complex upper half plane. The other work I point out here is one of C. L. Siegel's works, Symplectic Geometry, Amer. J. Math. 65 (1943). In this paper he worked on Sp_n over the rational number field and established the analytic theory of moduli of (polarized) Abelian varieties. There followed extensive works on automorphic functions and forms in many variables with respect to semi-simple and reductive algebraic groups over algebraic number fields. Weil's lecture notes referred above should also be mentioned here.

Towards the end of the 1960's there appeared three important works by K. Iwasawa, by G. Shimura and by R. P. Langlands. Concerning the first one and the third, we have splendid articles by R. Greenberg and by B. Casselman, respectively, in this book. Let me just point out one of Shimura's works here:

G. Shimura, On canonical models of arithmetic quotients of bounded symmetric domains, Ann. Math. 91 (1970), 144–222; II, *ibid.* 92 (1970), 528–558.

This paper represented a modern summit in the language of adele geometry of which Kronecker once dreamed and then Hilbert anticipated with his twelfth problem.

On the history of Hilbert's twelfth problem, N. Schappacher gave two interesting lectures in our conference as you will see from the program at the end of the preface. However, we could not include his article in this book. For those who are interested in it, I refer to his article, On the history of Hilbert's twelfth problem: a comedy of errors, Matériaux pour l'histoire des mathématiques au XX^{e} siècle (Nice, 1996), 243–273, Sémin. Congr., 3, Soc. Math. France, Paris, 1998. In our conference John Coates gave two inspiring lectures on Iwasawa theory of elliptic curves for graduate students and young mathematicians at the request of the organizers. Those who are interested in the subject are recommended to see his article, Fragments of the GL_2 Iwasawa theory of elliptic curves without complex multiplication, Arithmetic theory of elliptic curves (Cetraro, 1997), 1–50, Lecture Notes in Math., 1716, Springer, Berlin, 1999.

I have mentioned only a strictly limited number of important works on those topics which are closely related with class field theory. It is, of course, apparent that this list is far from sufficient. I certainly failed to mention such important topics as class field theory in positive characteristic, higher dimensional class field theory, and so on. (As for the first theme, we are happy to include an article of P. Roquette in this volume.) I just hope this humble review of mine may help younger number theorists who might be interested in this volume.

Acknowledgement

The chief organizer of the international conference and the editor of this volume would like to express his gratitude to those who financially supported these scientific activities:

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December 25, 2000
Katsuya Miyake,
Editor

All papers in this volume have been refereed and are in final form. No version of any of them will be submitted for publication elsewhere.

The Seventh International Research Institute of the Mathematical Society of Japan

– Class Field Theory – its Centenary and Prospect

DATE: ARRIVAL DAY **Tuesday, June 2, 1998**
TALKS BEGIN **Wednesday, June 3, 1998**
TALKS END **Friday, June 12, 1998**
(There will be no talks in the afternoon of Saturday, June 6, and on Sunday, June 7.)

PLACE: Building 55, School of Science and Engineering, Waseda University Okubo, Shinjuku, Tokyo 169, Japan

PROGRAM:

June 3

10:10 Opening Address
10:20 - 10:50 **Shokichi Iyanaga** (Japan Acad.)
Memories of Professor Teiji TAKAGI
11:00 - 12:00 **Günther Frei** (Univ. Laval)
How H. Hasse was led to the Local-Global Principle, the Reciprocity Laws and Local Class Field Theory.
13:30 - 14:20 **Noriyuki Otsubo** (Univ. Tokyo)
Recent Progress on the Finiteness of Torsion Algebraic Cycles
14:30 - 15:20 **Kanetomo Sato** (Tokyo Inst. Tech.)
Finiteness of a certain Motivic Cohomology of Varieties over Local and Global Fields
15:50 - 16:40 **Moulay Chrif Ismaili** (Univ. Oujda)
The Capitulation Problem for Certain Number Fields
16:50 - 17:50 **Hiroshi Suzuki** (Nagoya Univ.)
On the Capitulation Problem

June 4

11:15 - 12:15 **Peter Stevenhagen** (Univ. Amsterdam)
Generating Class Fields using Shimura Reciprocity
13:45 - 14:45 **Takeshi Saito** (Univ. Tokyo)
Modular Forms and p-adic Hodge Theory
15:40 - 16:40 **Spencer Bloch** (Univ. Chicago)
Higher Theta Functions
16:50 - 17:50 **Kazuya Kato** (Univ. Tokyo)
Log Hodge Structures and Classifying Spaces

June 5

10:00 - 11:00 **Shuji Saito** (Tokyo Inst. Tech.)
Generalization of the Theorem of Brauer-Hasse-Noether

11:15 - 12:15 **Ivan Fesenko** (Univ. Nottingham)
Neukirch Construction of the Reciprocity Map

13:45 - 14:45 **Masakazu Yamagishi** (Nagoya Inst. Tech.)
A Survey of p-Extensions

15:15 - 16:15 **Helmut Koch** (Humboldt-Univ., Berlin)
History of the Theorem of Shafarevich in the Theory of Class Formations

16:30 - 17:30 **Thong Nguyen Quang Do** (Univ. Franche-Comté)
Galois Module Structure of Class-Formations and free pro-p-Extensions

18:30 - 20:30 RECEPTION at Avaco

June 6

10:00 - 11:00 **Peter Roquette** (Univ. Heidelberg)
Class Field Theory in Characteristic p: its Origin and History

11:15 - 12:15 **Tsuneo Tamagawa** (Yale Univ.)
Class field theory is easy: Hazewinkel-Iwasawa Theory of Class Fields of Function Fields over Finite Fields

June 8

10:00 - 10:50 **Winfried Kohnen** (Univ. Heidelberg)
Class Numbers of Imaginary Quadratic Fields

11:00 - 11:50 **Ryotaro Okazaki** (Doshisha Univ.)
On Parities of Relative Class Numbers of Certain CM-Extensions

13:20 - 14:10 **Akito Nomura** (Kanazawa Univ.)
On Embedding Problems with restricted Ramifications

14:20 - 15:10 **Sang Geun Hahn** (KAIST)
Some Congruences for Binomial Coefficients

15:40 - 16:40 **Florian Pop** (Univ. Bonn)
On the Absolute Galois Group of the Rationals

June 9

10:00 - 11:00 **Ralph Greenberg** (Univ. Washington)
Iwasawa Theory – Past and Present

11:10 - 12:00 **Manabu Ozaki** (Waseda Univ.)
Iwasawa Invariants of $\mathbf{Z}_p$-Extensions over an imaginary Quadratic Field

13:30 - 14:20 **Masato Kurihara** (Tokyo Metropolitan Univ.)
Iwasawa Theory and Ideal Class Groups of large Number Fields

14:30 - 15:20 **Hiroaki Nakamura** (Tokyo Metropolitan Univ.)
Artin Braid Groups and Families of Elliptic Curves

15:50 - 16:40 **Hisao Taya** (Tohoku Univ.)
On p-adic Zeta Functions and Class Groups of $\mathbf{Z}_p$-Extensions of certain totally real Fields

16:50 - 17:50 **John H. Coates** (Univ. Cambridge)
Iwasawa Theory of Elliptic Curves without Complex Multiplication I

June 10

10:00 - 11:00 **Norbert Schappacher** (Univ. Strasbourg)
On the History of Hilbert's 12th Problem, 1900–1932

11:15 - 12:15 **William Casselman** (Univ. British Columbia)
Early Years of the L-group

13:45 - 14:45 **Norbert Schappacher** (Univ. Strasbourg)
Complex Multiplication in Arithmetic Algebraic Geometry: Hecke Characters, Abelian Varieties, and Motives

15:15 - 16:15 **Roland Gillard** (Institut Fourier)
Obstructions in Deformation Problems

16:30 - 17:30 **Thomas Zink** (Univ. Bielefeld)
P-adic Periods and Displays

June 11

10:00 - 11:00 **René Schoof** (Univ. Roma II)
Abelian Varieties over Number Fields with good Reduction everywhere

11:15 - 12:15 **Hiromichi Yanai** (Aichi Inst. Tech.)
Exceptional Hodge Cycles and unramified Class Fields

13:45 - 14:45 **David R. Kohel** (Univ. Singapore)
Hecke Module Structure of Quaternions

15:15 - 16:15 **Ram Murty** (Queen's Univ. at Kingston)
What is known about Artin L-Functions ?

16:30 - 17:30 **Ichiro Satake** (Univ. Chuo)
Classification Theory of Semisimple Algebraic Groups

June 12

10:00 - 11:00 **John H. Coates** (Univ. Cambridge)
Iwasawa Theory of Elliptic Curves without Complex Multiplication II

11:15 - 12:15 **Jean Cougnard** (Univ. Caen)
On Rings of Integers which are stably free and non free $H_8 \times C_2$ Modules

13:45 - 14:45 **M. Morishita** (Kanazawa Univ.) and **T. Watanabe** (Osaka Univ.)
Adele Geometry of Numbers

15:05 - 16:05 **Takashi Ono** (Johns Hopkins Univ.)
On Shafarevich-Tate Sets

16:15 - 17:15 **Yoshihiko Yamamoto** (Osaka Univ.)
The Canonical Power Series associated with Elliptic Curves over **Q**

17:30 - 18:30 Farewell Party

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