Floer theory for Lagrangian submanifolds *

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Noncommutativity, IHES
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moduli spaces of "geometric objects" (non-linear elliptic PDE)	suitable compactification transversality $\longrightarrow$ intersection theory Kuranishi structure	invariants with compatibility conditions algebraic structure associated to geometry

• **quantum cohomology, Gromov-Witten invariants** (Ruan-Tian, McDuff-Salamon; Fukaya- —, J. Li-G. Tian, Ruan, Siebert) Li-Tian, Behrend-Fantechi

• **Floer (co)homology for Hamiltonian systems** (Floer, Hofer-Salamon, — ; Fukaya- —, G. Liu-G. Tian) with pair-of-pants product (Schwarz, ...)

• **Floer (co)homology for Lagrangian intersections** (Floer, Oh; Fukaya-Oh-Ohta- —, filtered A_∞-algebras, bimodules, etc.)

• **SFT** (Eliashberg-Givental-Hofer), ...

(suitable compactification = stable map compactification due to Kontsevich)

Note. The first two cases: intersection theory on homology level

The thrid case: intersection theory on chain level

Symplectic manifolds

(M, ω)	symplectic manifold of dimension $2n$ (for simplicity, M is assumed to be closed.)
ω	$d\omega = 0, \omega^n \neq 0$ everywhere
J	almost complex structure compatible with ω, i.e., $g_J(u, v) = \omega(u, Jv)$ is a Riemannian metric

Locally modeled on symplectic vector space $(\mathbf{R}^{2n}, \omega_0)$ (Darboux).

Basic examples the total space of cotangent space of a manifold, Kähler manifolds, etc.

Fact. Existence, contractiblity of the space of such J's
$c_1(M) = c_1(TM, J) \in H^2(M; \mathbf{Z})$ is well-defined.

Basic Concepts

$\iota : L \to M$ Lagrangian submanifold (today: only embedding)

$$\iota^*\omega = 0, \quad \dim L = \frac{1}{2}\dim M.$$

Examples. • curves on a 2-dimensional symplectic manifold

• the graph of a closed 1-form on a C^∞-manifold X

Let G_η be the graph of $\eta \in \Omega^1(X)$ (a section of $T^*X \to X$).

$G_\eta \subset T^*X$ is Lagrangian $\Leftrightarrow d\eta = 0$

• the graph of a symplectomorphism $\phi \in \mathrm{Symp}(M,\omega)$

Let $\phi : M \to M$ be a diffeomorphism of M and Γ_ϕ its graph. Set $(P,\Omega) = (M \times M, -\mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega)$.

$\Gamma_\phi \subset P$ is Lagrangian $\Leftrightarrow \phi^*\omega = \omega$.

• the real part of projective algebraic manifold defined over $\mathbf{R}$

e.g., $\mathbf{R}P^n \subset \mathbf{C}P^n$ (the fixed point set of an anti-symplectic involution)

The Hamiltonian vector field X_h associated to $h \in C^\infty(M)$ is defined by

$$i(X_h)\omega = dh.$$

Note. $\mathcal{L}_{X_h}\omega = 0$. (Cf. $\mathcal{L}_X\omega = 0 \Leftrightarrow i(X)\omega$ a closed 1-form)
$\{X | \mathcal{L}_X\omega = 0\}/\{\text{Hamiltonian vector fields}\} \cong H^1(M;\mathbf{R})$.
$H = \{h_t\}$ a smooth family of functions, i.e., $H : \mathbf{R} \times M \to \mathbf{R}$, $h_t = H(t, \cdot)$,

$$H \overset{\text{Ham v. f.}}{\rightsquigarrow} \{X_{h_t}\} \overset{\text{integrate}}{\rightsquigarrow} \{\varphi_t^H\}$$

Definition. $\varphi \in \mathrm{Diff}(M)$ is called a Hamiltonian diffeomorphism iff $\varphi = \varphi_1^H$ for some $H = \{h_t\}$.

Arnold's conjecture for fixed points of Ham diffeo

For $\varphi \in \text{Ham}(M,\omega)$, $\#\text{Fix}(\varphi) \geq \min\{\#\text{Crit}(h) | h \in C^\infty(M)\}$.
RHS = LS category of $M \geq$ cup-length (M)
$\forall p \in \text{Fix}(\varphi)$ non-deg. $\Rightarrow \#\text{Fix}(\varphi) \geq \min\{\#\text{Crit}(h) | h$ Morse$\}$
Morse theory gives lower bound for RHS. (sum of Betti numbers, ...)

Note. Compare with the case of diffeomorphisms, homeomorphisms.

Analogous question for Lagrangian intersections

$L \subset M$ Lagrangian submanifold, $\varphi \in \text{Ham}(M,\omega)$
$\# L \cap \varphi(L) \geq \min\{\#\text{Crit}(f) | f \in C^\infty(L)\}$?
L transv. $\phi(L) \Rightarrow$ LHS $\geq \min\{\#\text{Crit}(f) | f$ Morse$\}$?

There are cases when (a weaker version of) estimate hold: great circles on the round 2-sphere, the zero section of the cotangent bundle (Laudenbach-Sikorav, Hofer)
In general, no such estimates. (e.g., small circles on the round 2-sphere) It is not only a bad news. $L \cap \varphi(L) = \emptyset \Rightarrow$ Existence of non-constant pseudo-holomorphic discs with boundary on $L \Rightarrow$ non-existence of exact Lagrangian submanifold in $\mathbf{C}^n$ (Gromov), non-degeneracy of Hofer's distance (Chekanov, Oh, originally due to Hofer, Lalonde-McDuff), etc.

Set-up for Floer theory of Lagrangian intersections

$L_0, L_1 \subset M$ such that $L_0 \pitchfork L_2$

$\mathcal{P}(L_0, L_1):$ the space of paths from L_0 to L_1

Define the closed 1-form α, in a formal sense, on $\mathcal{P}(L_0, L_1)$ by

$$\alpha_{L_0,L_1}(\xi) = \int_0^1 \omega\Big(\xi(t), \dot\gamma(t)\Big)dt \text{ for } \xi \in \text{``}T_\gamma\mathcal{P}(L_0, L_1)''$$

Note that $\text{Zero}(\alpha_{L_0,L_1}) = L_0 \cap L_1$

Action functional

$$\begin{array}{ccc} \widetilde{\mathcal{P}}(L_0, L_1) & \overset{\mathcal{A}_{L_0,L_1}}{\longrightarrow} & \mathbf{R} \\ \downarrow & & \\ \mathcal{P}(L_0, L_1) & & \end{array} \qquad d\mathcal{A}_{L_0,L_1} = \pi^*\alpha_{L_0,L_1}$$

Maslov-Viterbo index

There are infinitely many positive, resp. negative eigenvalues of the Hessian operator of the action functional at critical points. Although the usual Morse index does not make sense, we can consider the relative index along paths joining critical points. If we take an appropriate covering space of $\widetilde{\mathcal{P}}(L_0, L_1)$, the absolute index can be defined:

$$\mu_{L_0,L_1} : \mathrm{Crit}(\mathcal{A}_{L_0,L_1}) \to \mathbf{Z}$$

$J = \{J_t\}$ a family of compatible alm cplx str $\Rightarrow$ (formal) L^2-metric
Formally, **gradient flow lines** are described as follows.

(Perturbed) Cauchy-Riemann equation

$$\frac{\partial u}{\partial \tau} + J(u)\frac{\partial u}{\partial t} = 0$$
$$\text{for } u : \mathbf{R} \times [0,1] \to M$$
$$u(\tau, i) \in L_i, i = 0, 1$$

with suitable asymptotic condition at $\tau \to \pm\infty$.

For solutions, the energy is given by $E(u) = \int |\frac{\partial u}{\partial \tau}|^2 \, d\tau dt$.
Condition for bounded flow lines

$$E(u) < \infty \Leftrightarrow \exists \gamma \pm = \lim_{\tau \to \pm\infty} u(\tau, t)$$

Floer complex $(\mathsf{CF}^\bullet, \delta)$ $\mathsf{CF}^\bullet$ is Floer-Novikov completion of the free module generated by $\mathrm{Crit}\mathcal{A}$. Grading is given by μ.

$$\delta(x) = \sum_{y:\mu(y)=\mu(x)+1} \#\mathcal{M}(x,y)y,$$

where $\mathcal{M}(x,y)$ is the moduli space of grad. flow lines from x to y.

Novikov ring Λ_{L_0,L_1}.

Floer-Novikov completion of the group ring of the covering transformation group of $\widetilde{\mathcal{P}}(L_0,L_1) \to \mathcal{P}(L_0,L_1)$.

$(\mathsf{CF}^\bullet, \delta)$ is a "complex" over the Novikov ring.

Orientation issue is non-trivial in the case of Lagrangian intersections.

Floer-Novikov completion.

R: ring (later consider R containing $\mathbf{Q}$)

Floer complex

$$\mathsf{CF}^q := \{\sum_{i=1}^{+\infty} a_i x_i | a_i \in R, x_i \in \mathsf{Crit}(\mathcal{A}) \text{ s.t. } (*)\}$$

$(*) \quad \mathcal{A}(x_i) \to +\infty \quad (i \to +\infty), \qquad \mu(x_i) = q$

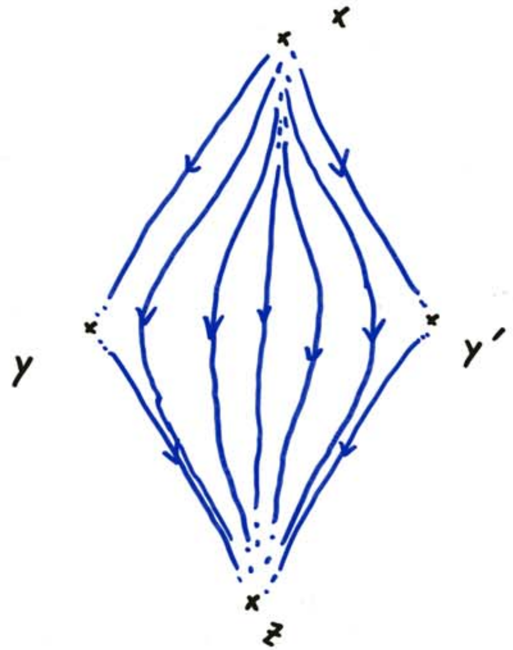
Novikov ring

Γ: covering transformation group of the Floer-Novikov covering

$$\Lambda := \{\sum_{i=0}^{+\infty} a_i g_i | a_i \in R, g_i \in \Gamma, \text{ s.t. } (**)\}$$

$(**) \quad I_{\mathcal{A}}(g_i) \to +\infty \quad (i \to +\infty),$

where $I_{\mathcal{A}}(g_i) = \mathcal{A}(g_i(x)) - \mathcal{A}(x)$, $x \in \widetilde{\mathcal{L}}M$ or $\widetilde{P}(L_0, L_1)$.



limits of grad. flow lines

||

broken grad. flow lines

$$\delta \circ \delta = 0$$

Crucial Problem. (1) δ well defined? (2) $\delta \circ \delta = 0$?

Ham systems Floer: monotone case ($c_1(M) = \lambda[\omega]$ for $\exists \lambda > 0$), Hofer-Salamon, —: weakly monotone (semi-positive) case (no J-holomorphic spheres with negative first Chern # for generic J)
In these cases, $\mathsf{HF}^\bullet(\{\varphi_t^H\}, J) \cong \mathsf{H}^{\bullet+n}(M)$.

Lagrangian intersection Floer: $\pi_2(M, L) = 0 \Rightarrow$ (1) and (2) holds. Moreover, $\mathsf{HF}^\bullet(L, \varphi(L)) \cong \mathsf{H}^{\bullet+c}(L)$ with $\mathbf{Z}/2\mathbf{Z}$-coeff. Oh: monotone Lagrangian submanifolds L_0, L_1 with min Maslov $\# \geq 3$, monotone Lagrangian submanifold L and $\varphi_1^H(L)$ with min Maslov $\# \geq 2 \Rightarrow$ (1) and (2) hold.
$\exists$ spectral sequence converging to $\mathsf{HF}^\bullet(L, \varphi(L))$. (There are cases when it degenerates at E_2-level.)

Hamiltonian systems Work with $\mathbf{Q}$-coefficients.

Theorem.(Fukaya- —, Liu-Tian) For any closed symplectic manifold, (1) and (2) holds. Moreover, $\mathsf{HF}^\bullet(\{\varphi_t^H\}, J) \cong \mathsf{H}^{\bullet+n}(M;\mathbf{Q}) \otimes \Lambda_\omega$.

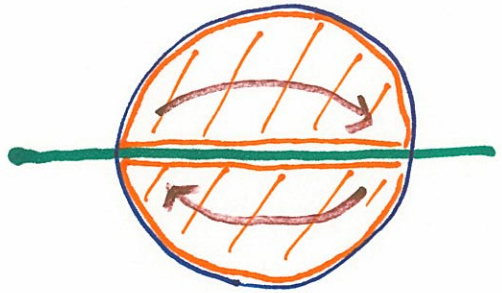
Corollary For $\varphi \in \mathsf{Ham}(M,\omega)$ only with non-deg fixed points,

$$\#\mathsf{Fix}(\varphi) \geq \sum_p b_p(M).$$

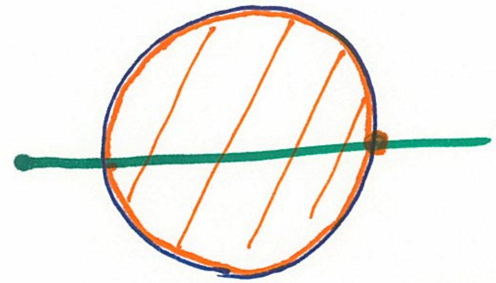
Lagrangian case
In general, $\delta \circ \delta$ may not be 0. E.g., small circles on the round sphere.

Obstruction theory (Fukaya-Oh-Ohta- —)
By Kuranishi structure, multi-valued perturbation technique, compatible system of orientations (cf. weak spin structure) $\Rightarrow$ virtual fundamental chain $\mathcal{M}(p,q)$. In contrast to case of Ham systems, not only bubbling-off of J-hol spheres, but also J-hol discs (real codimension-one phenomenon), which may cause $\delta \circ \delta \neq 0$

Systematic study of all hol discs $\Rightarrow$ obstruction classes to define HF, filtered A_∞-algebra associated to L.



split to
a broken trajectory



bubbling-off
of hol. disc

$\bullet\forall$ obstruction vanishes ($\Leftrightarrow$ $\exists$ sol. of Maurer-Cartan eq. for filtered A_∞-algebra) $\Rightarrow$ HF defined by revising δ, Hamiltoinan invariance (under suitable choice of sol. M-C eq.), $\exists$ spectral sequence converging to $\mathrm{HF}^\bullet(L, L)$.

Filtered A_∞-algebra associated to $L \subset M$

A_∞-algebra

$C^\bullet$: graded module. Set $C[1]^q := C^{q+1}$.

$\overline{\mathfrak{m}}_k : (C[1]^\bullet)^{\otimes^k} \to C[1]^\bullet$ of degree 1 (after shifting degrees)

$BC[1] := \bigoplus_k (C[1]^\bullet)^{\otimes^k}$, bar complex

Extend $\overline{\mathfrak{m}}_k$ to coderivation $\widehat{\overline{\mathfrak{m}}}_k$ on $BC[1]$ and write $\widehat{d} = \sum_k \widehat{\overline{\mathfrak{m}}}_k$.

$(C^\bullet, \{\overline{\mathfrak{m}}_k\}_{k=1}^\infty)$ an A_∞-algebra iff $\widehat{d} \circ \widehat{d} = 0$.

Note. If $\overline{\mathfrak{m}}_k = 0$ for $\forall k > 2$, DGA (before shifting the degree).

intersection of chains:

transversality fails for e.g. self-intersection.

$\overline{\mathfrak{m}}_1$ usual boundary operation (up to sign)

To define $\overline{\mathfrak{m}}_2$, take intersection of chains after perturbation $\Rightarrow$ associativity fails in strict sense, but holds up to homotopy.

A few formulae

$\overline{\mathfrak{m}}_1 \circ \overline{\mathfrak{m}}_1 = 0$

$\overline{\mathfrak{m}}_1 \circ \overline{\mathfrak{m}}_2 + \overline{\mathfrak{m}}_2 \circ (\overline{\mathfrak{m}}_1 \otimes id. \pm id. \otimes \overline{\mathfrak{m}}_1) = 0$

$\overline{\mathfrak{m}}_1 \circ \overline{\mathfrak{m}}_3 + \overline{\mathfrak{m}}_2 \circ (\overline{\mathfrak{m}}_2 \otimes id. \pm id. \otimes \overline{\mathfrak{m}}_2) + \overline{\mathfrak{m}}_3 \circ (\overline{\mathfrak{m}}_1 \otimes id. \otimes id. \pm id. \otimes \overline{\mathfrak{m}}_1 \otimes id. \pm id. \otimes id. \otimes \overline{\mathfrak{m}}_1) = 0$

$\cdots$

filtered case: $1 \in \Lambda_{0,nov} \subset B(C[1] \otimes \Lambda_{nov})$

$\mathfrak{m}_1 \circ \mathfrak{m}_0 = 0$

$\mathfrak{m}_1 \circ \mathfrak{m}_1 + \mathfrak{m}_2 \circ (\mathfrak{m}_0(1) \otimes id. \pm id. \otimes \mathfrak{m}_0(1)) = 0$

$\mathfrak{m}_1 \circ \mathfrak{m}_2 + \mathfrak{m}_2 \circ (\mathfrak{m}_1 \otimes id. \pm id. \otimes \mathfrak{m}_1) + \mathfrak{m}_3 \circ (\mathfrak{m}_0(1) \otimes id. \otimes id. \pm id. \otimes \mathfrak{m}_0(1) \otimes id. \pm id. \otimes id. \otimes \mathfrak{m}_0(1)) = 0$

$\cdots .$

$C^\bullet(L)$: a suitable countably generated subcomplex of singular chain module with the grading given by codimension of chain (We adopt cohomological convention.)
More precisely, we deal with currents represented by singular chains.
$\overline{\mathfrak{m}}_1(P) = \pm\partial P$, $\overline{\mathfrak{m}}_2(P_1, P_2)$ perturbed intersection of P_1 and P_2.
$\overline{\mathfrak{m}}_k(P_1, \ldots, P_k)$ using parametrized family of perturbations of the diagonal depending on $P_1, \ldots, P_k$.
Obtain A_∞-algebra, which is "homotopy equivalent" to degree shift of de Rham DGA of L.

filtered A_∞-algebras

universal Novikov ring e, T free generators, $\deg e = 2, \deg T = 0$

$$\Lambda_{nov} = \{\sum a_i e^{n_i} T^{\lambda_i} | a_i \in \mathbf{Q}, n_i \in \mathbf{Z}, \lambda_i \to +\infty\}$$
$$\Lambda_{0,nov} = \{\sum a_i e^{n_i} T^{\lambda_i} \in \Lambda_{nov} | \lambda_i \geq 0\}.$$

Consider free module over $\Lambda_{0,nov}$ generated by **energy zero** elements, its bar complex and take its completion.

Fix $\{\beta_i = (\lambda_i, n_i)\}$ such that $\lambda_i \to +\infty$.

$\mathfrak{m}_{k,\beta_i} : (C[1]^\bullet)^{\otimes k} \to C[1]^\bullet$, $\widehat{\mathfrak{m}}_{k,\beta_i}$ its extension as coderivation.

(When $\beta = 0$, $\mathfrak{m}_{k,0} = \overline{\mathfrak{m}}_k$. The contribution from $\beta = 0$ is "quantum effect" by holomorphic discs. Our filtered A_∞-algebra is considered as "quantum deformation" of (classical) A_∞-algebra.)

$\widehat{d} = \sum_{k,\beta_i} \widehat{\mathfrak{m}}_{k,\beta_i} \otimes e^{n_i} T^{\lambda_i}$ degree 1

Here $\mathfrak{m}_{0,\beta} \neq 0$ only when $\beta = (\lambda, n)$ with $\lambda > 0$.

$(C^\bullet \otimes \Lambda_{0,nov}, \{\mathfrak{m}_{k,\beta}\})$ filtered A_∞-algebra iff $\widehat{d} \circ \widehat{d} = 0$.

Using $b \in (C[1] \otimes \Lambda_{0,nov})^0$ with strictly positive energy, i.e., $b = \sum b_i e^{n_i} T^{\lambda_i}$, $\lambda_i > 0$, sdeg $b_i + 2n_i = 0$ for $\forall i$, we can deform the operations
$\mathfrak{m}^b_{k,\beta}(P_1, \ldots, P_k) = \sum \mathfrak{m}_{k+\ell}(b, \ldots, b, P_1, b, \ldots, b, P_i, b, \ldots, b, P_k, b, \ldots, b)$,
where ℓ the number of inserted b's in arbitrary positions.

- $\{\mathfrak{m}^b_{k,\beta}\}$ is also filtered A_∞-structure.

Write $e^b := \sum b \otimes \cdots \otimes b$.
b sol. of Maurer-Cartan equation iff $\widehat{d}(e^b) = 0$.

Note. This is equivalent to that $\mathfrak{m}^b_{0,\beta} = 0$ for $\forall \beta$. Such a b is called bounding (co)chain.
It implies that $\mathfrak{m}^b_1 \circ \mathfrak{m}^b_1 = 0$, where $\mathfrak{m}^b_1 = \sum \mathfrak{m}^b_{1,\beta}$. Thus $(C[1]^\bullet \otimes \Lambda_{0,nov}, \mathfrak{m}^b_1)$ is a cochain complex.

Filtered A_∞-algebra associated to L

$\beta \in \pi_2(M, L)$ By abuse of notation, we write $\beta = (\int_\beta \omega, \mu_L(\beta))$, where μ_L is the Maslov index.

$\mathcal{M}_{k+1}(\beta; J)$: moduli space of bordered stable maps of genus 0 with $k+1$-marked points on boundary

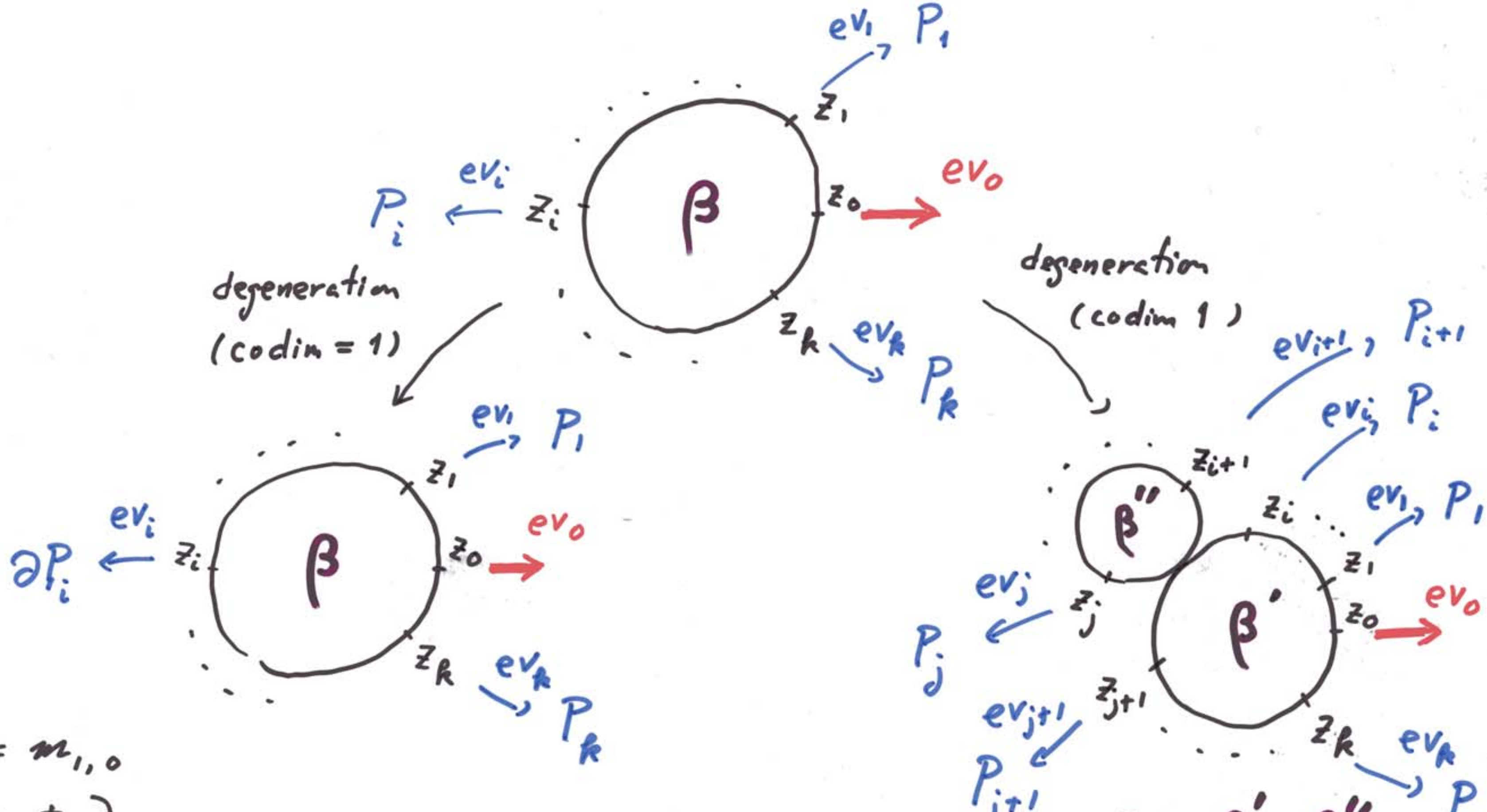
Take the fiber product by the evaluation maps $ev_1, \ldots, ev_k$:

$\mathcal{M}_{k+1}(\beta : P_1, \ldots, P_k; J) = \mathcal{M}_{k+1}(\beta; J) \times_{L\times\cdots\times L} (P_1 \times \cdots \times P_k)$

Orientation issue, multi-valued perturbation, suitable countable generated complex ...

Basic idea: Use $ev_0 : \mathcal{M}_{k+1}(\beta : P_1, \ldots, P_k; J) \to L$ to define $\mathfrak{m}_{k.\beta}$.

The equation $\widehat{d} \circ \widehat{d} = 0$ follows from the study of stable compactification of moduli spaces of bordered stable maps of genus 0. $(C(L; \Lambda_{0,nov})^\bullet, \{\mathfrak{m}_{k,\beta}\})$ filtered A_∞-algebra associated to L. ("homotopy type" is well-defined)



$$\overline{m}_1 = m_{1,0} = \pm \partial$$

$$\beta = \beta' + \beta''$$

$$\sum_{\beta' + \beta'' = \beta} \sum_{1 \leq i \leq j \leq k} m_{k-j+i-1,\beta'}(P_1 \otimes \cdots \otimes m_{j-i,\beta''}(P_{i+1} \otimes \cdots \otimes P_j) \otimes P_{j+1} \otimes \cdots \otimes P_k) = 0$$

$$[\ \beta'' = 0 \Rightarrow i < j\ ]$$

$\phi \in \mathrm{Symp}(M, \omega)$ induces a filtered A_∞-morphism $\widehat{\phi}_*$, which is a morphism of **coalgebra** $BC(L, \Lambda_{0,nov}))[1] \to BC(\phi(L), \Lambda_{0,nov}))[1]$ s.t.

$$\widehat{d} \circ \widehat{\phi}_* = \widehat{\phi}_* \circ \widehat{d}.$$

The filtered A_∞-algebra depends on choice of J, perturbation, $C^\bullet(L)$, etc. Introduce the notion of **homotopy** between filtered A_∞-morphisms, then **homotopy equivalence**. The homotopy type of the filtered A_∞-algebra is uniquely determined by $L \subset M$.

If $\exists b \in C(L; \Lambda_{0,nov})^0$ sol. of Maurer-Cartan equation, i.e., L is **unobstructed**, $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0 \Rightarrow$ Floer complex for (L, L) of **Bott-Morse** type

For $\phi \in \text{Symp}(M, \omega)$, define $\phi_*(b)$ by $\widehat{\phi}_*(e^b) = e^{\phi_*(b)}$. Then b: bounding cochain $\Rightarrow$ $\phi_*(b)$: bounding cochain

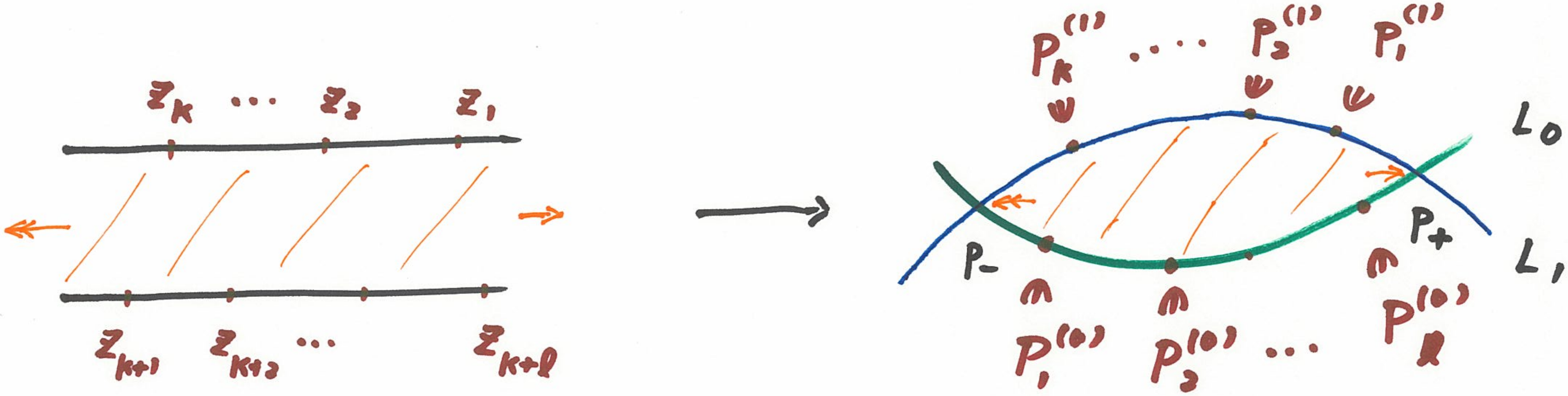
We can also deform $\mathfrak{m}_{k,\beta}$ using cycles in M (infinitesimal deformation or bulk/boundary deformation). If the Maurer-Cartan equation has solutions in the filtered A_∞-algebra after infinitesimal (or bulk/boundary) deformation, L is called unobstructed after infinitesimal deformation.

If $\text{H}^\bullet(M; \mathbf{Q}) \to \text{H}^\bullet(L; \mathbf{Q})$ is surjective, L is unobstructed after infinitesimal deformation.

Filtered A_∞-bimodule associated to (L_0, L_1).
(L_0, L_1): Lagrangian submanifolds with clean intersection
Construct the filtered A_∞-bimodule. (For simiplicity, assume transversal intersection.) Let $P_i^{(0)}$ and $P_j^{(1)}$ be chains in L_0 and L_1, respectively. Put marked points $z_i^{(0)}$ and $z_j^{(1)}$ on $\mathbf{R} \times \{0,1\}$ and consider Floer grad. flow lines u such that $u(z_i^{(0)}) \in P_i^{(0)}$ and $u(z_j^{(1)}) \in P_j^{(1)}$. (After taking stable compactification, multi-valued perturbation in the sense of Kuranishi structure,) count such objects to define

$$BC(L_1, \Lambda_{0,nov})^\bullet \otimes CF^\bullet(L_1, L_0) \otimes BC(L_0, \Lambda_{0,nov})^\bullet \stackrel{\mathfrak{n}_{k_1,k_0}}{\to} CF^\bullet(L_1, L_0).$$

Use $\widehat{d}$'s on $BC(L_i, \Lambda_{0,nov})^\bullet$, $i = 1, 0$, and $\mathfrak{n}_{k_1,k_0}$
$\Rightarrow$ $\widehat{d}$ on $BC(L_1, \Lambda_{0,nov})^\bullet \otimes CF^\bullet(L_1, L_0) \otimes BC(L_0, \Lambda_{0,nov})^\bullet$, which satisfies $\widehat{d} \circ \widehat{d} = 0$ filtered A_∞-bimodule structure.



$\{ z \in \mathbb{C} \mid \operatorname{Im} z \in [0, 1] \}$

Extend the coefficient ring from $\Lambda_{0,nov}$ to Λ_{nov}. $\varphi_i \in \mathrm{Ham}(M, \omega)$ induces a filtered A_∞-bimodule morphism $(\varphi_1, \varphi_0)_*$, which is homotopy equivalence of filtered A_∞-bimodules.

If $\exists b_i$ boundings cochains for L_i, $i = 0, 1$, deform δ to $\delta^{b_1,b_0}(\cdot) = \widehat{d}(e^{b_1} \otimes \cdot \otimes e^{b_0}) \Rightarrow$ Floer complex $\mathrm{CF}^\bullet(L_1, L_0), \delta^{b_1,b_0})$. Denote by $\mathrm{HF}^\bullet((L_1, b_1), (L_0, b_0)$. After extending the coefficient ring to Λ_{nov}, $(\varphi_1, \varphi_0)_*$ induces

$$\mathrm{HF}^\bullet((L_1, b_1), (L_0, b_0) \cong \mathrm{HF}^\bullet((\varphi_1(L_1), \varphi_{1*}(b_1)), (\varphi_0(L_0), \varphi_{0*}(b_0))$$

Spectral sequence.

$\exists$ spectral sequence $E_r^{\bullet,\bullet}$ with $E_1^{\bullet,\bullet} = \mathsf{CF}^\bullet(L,L)$ converging to $\mathsf{HF}^\bullet((L,b)(L,b))$, s.t. $E_2^{\bullet,\bullet} \cong \mathsf{H}^\bullet(L;\mathbf{Q}) \otimes \Lambda_{0,nov}$.

Some examples of applications.

Theorem. L: Lagrangian submanifold such that $\mathsf{H}^\bullet(M;\mathbf{Q}) \to \mathsf{H}^\bullet(L;\mathbf{Q})$ is surjective. For $\varphi \in \mathsf{Ham}(M,\omega)$ such that L and $\varphi(L)$ are transversal, $\#(L \cap \varphi(L)) \geq \sum b_p(L)$.

Theorem. L: spin Lagrangian submanifold in $(\mathbf{R}^{2n}, \omega_{can})$ such that $\mathsf{H}^2(L;\mathbf{Q}) = 0$. Then the Maslov class $\mu_L \in \mathsf{H}^1(L;\mathbf{Z})$ is non-zero.

partial affirmative answer to Arnold-Givenal's conjecture, ...

An application to Audin's conjecture.

Audin's conjecture Let L be an embedded Lagrangian torus in the symplectic vector space. Then its minimal Maslov number is 2.

There are some results due to Polterovich, Viterbo, Oh concerning minimal Maslov number. Also recent works by Fukaya, and Cieliebak-Mohnke. Based on our theory, we give a partial result, which is independently obtained by Lev Buhovsky.

Theorem. Let L be a monotone Lagrangian torus. Then its minimal Maslov number is 2.

Here monotonicity for L means that $\mu_L : \pi_2(M, L) \to \mathbf{R}$ and $\omega : \pi_2(M, L) \to \mathbf{R}$ are positively proportional.

moduli spaces of "geometric objects" (non-linear elliptic PDE)	suitable compactification transversality $\longrightarrow$ intersection theory Kuranishi structure	invariants with compatibility conditions algebraic structure associated to geometry

• **quantum cohomology, Gromov-Witten invariants** (Ruan-Tian, McDuff-Salamon; Fukaya- —, J. Li-G. Tian, Ruan, Siebert)

• **Floer (co)homology for Hamiltonian systems** (Floer, Hofer-Salamon, —; Fukaya- —, G. Liu-G. Tian) with pair-of-pants product (Schwarz, ...)

• **Floer (co)homology for Lagrangian intersections** (Floer, Oh; Fukaya-Oh-Ohta- —, filtered A_∞-algebras, bimodules, etc.)

• **SFT** (Eliashberg-Givental-Hofer), ...

(suitable compactification = stable map compactification due to Kontsevich)

Note. The first two cases: intersection theory on homology level

The thrid case: intersection theory on chain level

Appendix

Stable maps and compactification of moduli spaces
stable maps (Kontsevich)
Σ pre-stable curve (at worst nodes) with possibly marked points on non-sing part, $p : \widetilde{\Sigma} = \bigcup \widetilde{\Sigma}_i \to \Sigma$ normalization
$f : \Sigma \to M$ stable map iff f is continuous, $f \circ p$ is J-hol on each $\widetilde{\Sigma}_i$ and $\#\mathrm{Aut}(f) = \#\{\phi|$ automorphism of Σ s.t. $f \circ \phi = f\} < \infty$.

Fundamental Theorem The moduli space $\mathcal{M}_{g,k}(\alpha, J)$ of stable maps is a compact Hausdorff space. Here $g =$ genus of Σ, $k = \#$marked points, $\alpha \in \mathrm{H}_2(M; \mathbf{Z})$.

We have analogous notions and results for grad. flow lines in Floer theory and for bordered J-holomorphic map with Lagrangian condition.

In general, transversality fails, especially at **infinity** of (uncompactified) moduli spaces. (e.g., bubbling-off of multiple cover of J-hol sphere with negative $c_1(M)$- number) In order to overcome this trouble, use Kuranishi structure machinery.

Kuranishi structure

A Kuranishi structure on a compact metrizable space X is a compatible system of local models.

(U, E, Γ, s, ψ): Kuranishi neighborhood of $x \in X$

U open subset of $\mathbf{R}^m$, Γ finite group, $E = U \times \mathbf{R}^k \to U$ Γ-equiv. vector bundle, s Γ-invariant **single-valued** section of $E \to U$, $\psi : s^{-1}(0) \to X$ homeo onto its image s.t. $x \in \mathrm{Im}\psi$.

coordinate change from (U, E, Γ, s, ψ) to $(U', E', \Gamma', s', \psi')$

$\Gamma \to \Gamma'$ injective homomorphism

$$\begin{array}{ccccl} E & \subset & E' & & (\Gamma, \Gamma')\text{-equivariant embedding} \\ \downarrow & & \downarrow & & (TU'|_U)/TU \cong (E'|_U)/E \\ U & \subset & U' & & (\Gamma, \Gamma')\text{-equivariant embedding} \end{array}$$

s is the restriction of s'. $\mathrm{rank}E - \dim U = \mathrm{rank}E' - \dim U'$ **Require certain compatiblity condition for coordinate changes**

Taking compatible system of **multi-valued** perturbation of s, obtain **virtual fundamental chain** $[X]^{\mathrm{vir}}$ over $\mathbf{Q}$ from X with oriented Kuranishi structure. (X with Kuranishi structure "without boundary" $\Rightarrow$ **virtual fundamental cycle**.)

Stable maps and compactification of moduli spaces
stable maps (Kontsevich)
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Before compactification:
$\overline{\partial}_J u = 0$ (non-linear elliptic PDE) $\Rightarrow$ Fredholm set-up (approximation by finite dimensional models) $\Rightarrow$ Kuranishi neighborhood

stable map compactification (#Aut finite):
gluing theorem with index computation, etc $\Rightarrow$ Kuranishi neighborhood

Obtain a compatible system of Kuranishi neighborhood, i.e., Kuranishi structure.

Σ without boundary $\Rightarrow$ bubbling-off of hol sphere, degeneration are real codimension-two or more $\Rightarrow$ $\mathcal{M}_{g,k}(\alpha, J)$ carries **virtual fundamental cycle**
$\Rightarrow$ G-W invariants with Kontsevich-Manin's axiom (except motivic axiom), quantum cohomology and its associativity.

Floer complex for Hamiltonian systems $\Rightarrow$ $[\mathcal{M}(p,q)]^{\text{vir}}$ enjoy expected properties (essentially, lack of compactness in codimension-one:broken grad. flow lines) $\Rightarrow$ $\mathrm{HF}^\bullet(H, J)$, invariance under Ham deformations, etc
For computation, pick $h_t \equiv h$ C^2-small Morse function. S^1-symmetry in Kuranishi structure $\Rightarrow$ $\mathrm{HF}^\bullet(H, J) \cong \mathrm{H}^{\bullet+n}(M; \mathbf{Q}) \otimes \Lambda_\omega$(Similar argument works for $\{\phi_t\}$ with small flux.)