Knot concordance and exotica

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Since the work of Freedman and Donaldson in the early 1980's, there have been techniques available for producing closed *exotic* 4-manifolds, i.e. 4-manifolds which admit more than one smooth structure. However, these (gauge theoretic) tools are difficult to wield on 4-manifolds with very little algebraic topology, and cannot be used to demonstrate that S^4 is exotic.

There is an alternate, well-known strategy which a priori *can* be used to distinguish smooth structures on very simple closed 4-manifolds; produce a knot K in S^3 which is (smoothly) slice in one smooth filling W of S^3, but not slice in some homeomorphic smooth filling W'. However, this strategy had never actually been used in practice, even to reproduce known complicated exotica. In this manuscript I will discuss joint work with Manolescu and Marengon [MMP20] which gives the first application of this strategy. I will also discuss joint work with Manolescu [MP21] which gives a systematic approach towards using this strategy to produce candidates for exotic homotopy spheres.

This manuscript only aims to provide an overview, for more detailed exposition and references, see the full papers [MMP20, MP21]. All manifolds and embedding are taken to be smooth unless otherwise specified.

1 Introduction

Definition 1.1 A smooth 4-manifold X is *exotic* if there exists a smooth 4-manifold Y such that Y is homeomorphic but not diffeomorphic to X.

The following is a brief survey of the simplest 4-dimensional exotica known: for non-compact manifolds it can be shown as a consequence of Donaldson's theorem [Don83] and work of Freedman [Fre82] that $\mathbb{R}^4$ is exotic. In the setting of compact 4-manifolds with boundary, it is known that there are contractible exotic 4-manifolds, [AR16]. In the closed case, the smallest (in terms if b_2) known exotic manifold is $\mathbb{C}P^2\#_2\overline{\mathbb{C}P^2}$, [AP10].

In the results above, the diffeomorphism obstruction comes from Gauge theoretic 4-manifold invariants. Consider instead the following primitive argument showing that some X is not diffeomorphic to some Y; find a manifold M such that M embeds in X but M does not embed in Y. Of course, this argument only gives an advantage over a direct argument using diffeomorphism invariants of X and Y if one has techniques for constructing and obstructing embeddings which are better developed or easier to use than direct diffeomorphism invariants.

I will now introduce a certain type of 4-manifold M for which this is the case.

Definition 1.2 A knot trace $X(K)$ is the 4-manifold obtained by attaching a 0-framed 2-handle to B^4 along K.

The following lemma comes essentially from [FM66], for proof see [HP19].

Lemma 1.3 *For any knot K and closed 4-manifold Y, $X(K) \hookrightarrow Y$ with $\iota_*(H_2(X(K)) = \beta \in H_2(Y)$ if and only if $\overline{K}$ bounds a disk $D^2 \hookrightarrow \mathring{Y}$ with $[D] = \beta \in H_2(\mathring{Y}, \partial)$.*

Throughout, $\mathring{Y}$ is defined to be the complement of an open B^4 in Y. There are many invariants in the literature for obstructing a knot K from bounding such a disk in some $\mathring{Y}$, especially when $\mathring{Y} \cong B^4$. Thus, knot traces provide a compelling source of W for the outline above. In particular, it is well known that Rasmussen's s invariant [Ras10] may be able to obstruct $X(K)$ embedding in S^4 even if $X(K)$ embeds in some homotopy sphere, see [FGMW10]. Since we are particularly interested in this setting, we will be particularly interested in the following type of slice disk:

Definition 1.4 A knot K is *H-slice* in X (or $\mathring{X}$) if K bounds a disk $D^2 \hookrightarrow \mathring{Y}$ with $[D] = 0 \in H_2(\mathring{Y}, \partial)$. A knot K is *slice* if K bounds a disk $D^2 \hookrightarrow B^4$.

In this language, the smooth 4-dimensional Poincare conjecture implies:

Conjecture 1.5 *There does not exist a knot K in S^3 which is H-slice in some homotopy 4-ball Y but such that K is not H-slice in B^4.*

The main theorem of this manuscript is the following:

Theorem 1.6 ([MMP20]) *There exist homeomorphic closed 4-manifolds X and Y and a knot K which is H-slice is Y but not H-slice in X.*

Surprisingly, this provides the first source of exotic closed 4-manifolds which are distinguished by the embedding outline, and provides a proof-of-concept for this approach on the Poincare conjecture.

In the second part of the manuscript, I will discuss joint work with Ciprian Manolescu in which we develop systematic methods for producing knots which are H-slice in some homotopy B^4 but which may not be H-slice in B^4.

2 Using H-sliceness to produce exotica

The goal of this section is to outline a proof of Theorem 1.6. The main technical result with powers Theorem 1.6 is the following:

Theorem 2.1 *[Theorem 1.4 of [MMP20]] Let X be a closed 4-manifold with $b_2^+(X) > 1$ such that there exists a spin^c structure $\mathfrak{s}$ with Bauer-Furuta invariants $BF(X, \mathfrak{s}) \neq 0$. Then there are no spheres $S^2 \hookrightarrow X$ with $S \cdot S \geq 0$ and $[S] \neq 0$.*

This adjunction inequality for the Bauer-Furuta invariants is proved identically to the analogous statement for the Sieberg Witten invariants. We are primarily interested in the following corollary:

Corollary 2.2 *[MMP20] For $(X, \mathfrak{s})$ a closed 4-manifold and spin^c structure such that $BF(X, \mathfrak{s})$ are very nice[1], if some knot $\overline{K}$ bounds $D^2 \hookrightarrow \mathring{X}$ with $D \cdot D \geq 0$ and $[D] \neq 0$ then K is not H-slice in X.*

Sketch of corollary The "very nice" condition is precisely what is needed to conclude that $BF(X\#X, \mathfrak{s}\#\mathfrak{s}) \neq 0$[2]. Now suppose some knot $\overline{K}$ bounds $D^2 \hookrightarrow \mathring{X}$ with $D \cdot D \geq 0$ and $[D] \neq 0$ and suppose for a contradiction that K is H-slice in X with H-slice disk D'. Then $D \cup D' \hookrightarrow X\#X$ is a 2-sphere with $S \cdot S \geq 0$ and $[S] \neq 0$, violating Theorem 2.1. □

The advantage of Corollary 2.2 is that it allows one to prove an obstructive claim with a constructive method; one simply needs to demonstrate the existence of a certain disk (D) for $\overline{K}$ to obstruct the existence of an H-slice disk for K. This is how we prove Theorem 1.6. The following sketch gives an particularly straightforward example; the reader can modify this method to produce other examples as an exercise.

Sketch of Theorem 1.6 It is well known that the left hand trefoil bounds a disk D in $K3$ (thus in $K3\#\overline{\mathbb{C}P^2}$) with $D \cdot D = 0$ and $[D] \neq 0$ (see proof in [MMP20]). Corollary 2.2 then implies that the right hand trefoil is not H-slice in $K3\#\overline{\mathbb{C}P^2}$.

It is also well known that the right hand trefoil is H-slice in $\mathbb{C}P^2$ (thus in $\#_3\mathbb{C}P^2\#_{20}\overline{\mathbb{C}P^2}$). Further, the intersection forms of $K3\#\overline{\mathbb{C}P^2}$ and $\#_3\mathbb{C}P^2\#_{20}\overline{\mathbb{C}P^2}$ are isomorphic, thus the manifolds are homeomorphic by work of Freedman. Since the right hand trefoil is H-slice in one manifold and not the other, the manifolds are not diffeomorphic. □

[1] for precise condition, see [MMP20]

[2] The reason we work with Bauer-Furuta invariants is precisely because they do not necessarily vanish under connected sum. This observation has subsequently been used to greater advantage, see [IMT21]

We remark that this proof relies on the fact that $K3\#\overline{\mathbb{C}P^2}$ has nice, non-vanishing Bauer-Furuta invariants, and it is also the case that $\#_3\mathbb{C}P^2\#_{20}\overline{\mathbb{C}P^2}$ does not. Thus there is a more direct proof, without reference to H-sliceness, that the manifolds are non-diffeomorphic. It is of considerable interest to produce exotic 4-manifolds distinguished by H-sliceness in the absence of another proof.

3 Systematically producing knots which are H-slice in a homotopy ball

We now turn our attention to the problem of systematically producing knots K with are H-slice in some 4-manifold Y which is homeomorphic to B^4 but which may not be slice in B^4. We will be aided by the following folklore lemma.

Lemma 3.1 *If knots K and J have $\partial(X(K)) \cong \partial(X(J))$ and K slice then J is H-slice in some Y homeomorphic to B^4.*

Remark 3.2 For proof of the lemma, we refer the reader to [MP21], but we note here that for explicit knots K and J as in the lemma, the proof of the lemma constructs an explicit Y homeomorphic to B^4.

3.1 Starting with a slice knot

In the most direct attempt to apply Lemma 3.1, one would like to take some slice knot K and construct a J with $\partial(X(K)) \cong \partial(X(J))$. We remind the reader that $\partial(X(K))$ is the familiar 3-manifold obtained by 0-framed Dehn surgery along K, denoted $S^3_0(K)$. We develop previous work of the author [Pic19] to give a fully general construction of pairs of knots which share a 0-surgery. Our construction is called an *RBG link*; for details of the construction, see [MP21]. For this exposition it suffices to know the statement:

Theorem 3.3 *Any RBG link has a pair of associated knots K_B and K_G with $S^3_0(K_B) \cong S^3_0(K_G)$. Conversely, for any knots K and J with homeomorphism $\phi : S^3_0(K) \to S^3_0(J)$ there exists and RBG link with associated knots $K_B \cong K$ and $K_G \cong J$.*

Theorem 3.3 gives a complete method for understanding 0-surgery homeomorphisms when they exist, but it does not help the user necessarily find a J for their favorite (in our setting, slice) knot K. In fact, it is known that there are knots K for with it is not possible to find a distinct J with $S^3_0(K) \cong S^3_0(J)$, for example:

Theorem 3.4 ([Gab83]) *The unknot, both trefoils and the figure eight knot are characterized by their 0-surgeries.*

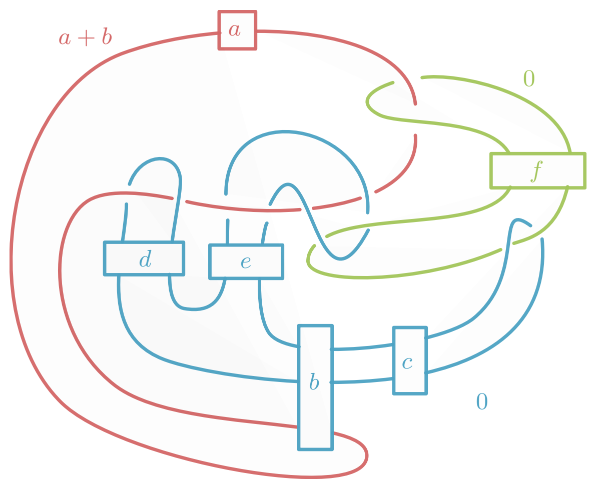

Figure 1: A many parameter family of RBG links

At present, it is technically out of reach to systematically produce a distinct knot J with $S_0^3(K) \cong S_0^3(J)$ for any given slice knot K. However, there are some situations for which such a J can be readily produced, for example:

Proposition 3.5 *[Pic20] If the unknotting number if K is one, there exists an BRG link with $K_B = K$.*

A part of the work in [MP21] consists of, for a handful of slice knots K, the production of infinite families of knots J_i each with $S_0^3(K) \cong S_0^3(J_i)$. Of course, the goal of producing such J_i (which are H-slice in a homotopy B^4 by Lemma 3.1) is to find that $s(J_i) \neq 0$. Unfortunately, for all the J_i we build, either $s(J_i) = 0$ or the knot J_i was too large and s could not be computed.

While these computations overall result in a failed attempt, we point out that these examples do contribute a new source of simple 4-manifolds Y_i homeomorphic to B^4, see Remark 3.2. We fail to show these Y_i are not diffeomorphic to B^4, but we also cannot systematically standardize them[3]; we hope the study of these new simple potential counterexamples will motivate the development of new techniques.

3.2 Starting with a pair

Our attempts at systematizing approach in Subsection 3.1 are hobbled by the technical difficulties inherent in producing a knot J with $S_0^3(J) \cong S_0^3(K)$ for a given slice knot K.

[3]Since our preprint appeared, a graduate student Kai Nakamura has already developed new arguments which standardize some of our examples. His work is in progress.

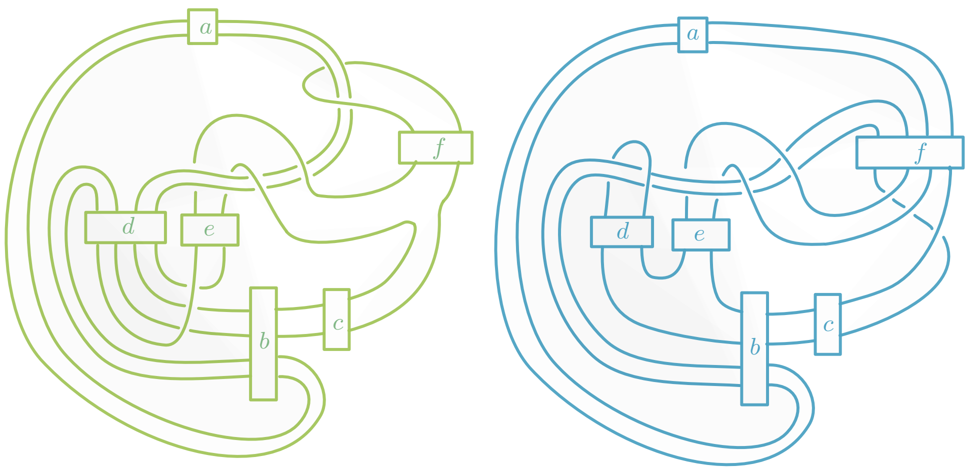

Figure 2: A many parameter family pairs of knots K_B and K_G with the same 0-surgery. We only consider elements with small parameters, for details see [MP21]

Even when it is possible to produce such a J, at present time that J is built by hand, thus it is impractical to build and compute J and $s(J)$ for large families of slice knots K.

The second part of the work in [MP21] attempts to wield Lemma 3.1 from a different, more systematic, perspective. We begin with a many parameter family of *RBG* links, see Figure 1. From this family of links, we get a large (3375 element) family of pairs of knots with $S^3_0(K_B) \cong S^3_0(K_G)$, see Figure 2. We are interested in whether this family contains a pair such that one knot is slice and the other has $s \neq 0$. For ease of exposition, I will refer to a search for a pair with K_B slice and $s(K_G) \neq 0$; of course the search with the roles of B and G reversed must be performed also.

With help of a computer, we searched the family for pairs that may have this property. To begin, we searched for pairs with $s(K_B) = 0$ and $s(K_G) \neq 0$. Finding many such pairs, we are interested in whether any of those K_B are actually slice. We computed all readily computable[4] sliceness invariants for these potentially slice K_B; these other sliceness invariants showed that many of the remaining K_B are not slice, but surprisingly, not all. Thus

Theorem 3.6 ([MP21]) *If any of the 5 topologically slice knots in Figure 3 are slice then the smooth 4-dimensional Poincare conjecture is false.*

[4] After our preprint appeared, Nathan Dunfield and Sherry Gong introduced a new computer program for computing twisted Alexander polynomials. Computations of the twisted Alexander polynomials ruled out 16 of our candidates, our Theorem has been updated to include their calculations.

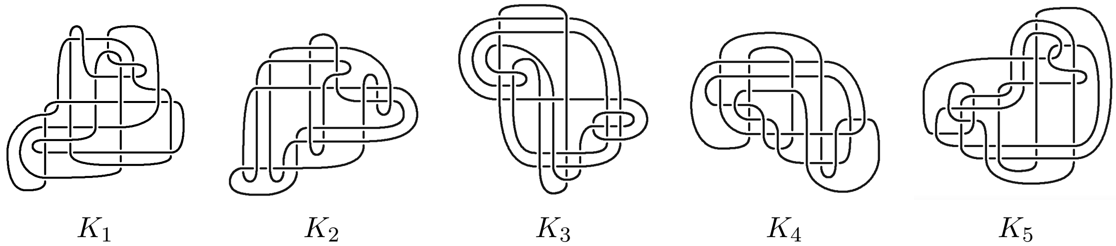

Figure 3: Candidates for slice knots.

Recently, in forthcoming work, Kai Nakamura developed a new argument which shows that the 5 knots from Theorem 3.6 are not slice. However, Kai's arguments do not show that any such examples coming from any such search on any family of RBG links are not slice. So while the particular examples of our paper our now known not to provide counterexamples to the Poincare conjecture, the process may still provide interesting examples. There is ongoing work in the community to continue to wield and systematize our work, as well as the work of Dunfield, Gong, and Nakamura, towards better understanding the examples generated by the techniques in Subsections 3.1 and 3.2.

References

[AP10] Anar Akhmedov and B. Doug Park. Exotic smooth structures on small 4-manifolds with odd signatures. *Invent. Math.*, 181(3):577–603, 2010.

[AR16] Selman Akbulut and Daniel Ruberman. Absolutely exotic compact 4-manifolds. *Comment. Math. Helv.*, 91(1):1–19, 2016.

[Don83] S. K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[FGMW10] Michael Freedman, Robert Gompf, Scott Morrison, and Kevin Walker. Man and machine thinking about the smooth 4-dimensional Poincaré conjecture. *Quantum Topol.*, 1(2):171–208, 2010.

[FM66] Ralph H. Fox and John W. Milnor. Singularities of 2-spheres in 4-space and cobordism of knots. *Osaka Math. J.*, 3:257–267, 1966.

[Fre82] Michael H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[Gab83] David Gabai. Foliations and the topology of 3-manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 8(1):77–80, 1983.

[HP19] Kyle Hayden and Lisa Piccirillo. The trace embedding lemma and spinelessness. preprint, arXiv:1912.13021, 2019.

[IMT21] Nobuo Iida, Anubhav Mukherjee, and Masaki Taniguchi. An adjunction inequality for Bauer Furuta type invariants, with applications to sliceness and 4-manifold topology. preprint, arXiv:2102.02076, 2021.

[MMP20] Ciprian Manolescu, Marco Marengon, and Lisa Piccirillo. Relative genus bounds in indefinite four-manifolds. preprint, arXiv:2012.12270, 2020.

[MP21] Ciprian Manolescu and Lisa Piccirillo. From zero surgeries to candidates for exotic definite four-manifolds, 2021.

[Pic19] Lisa Piccirillo. Shake genus and slice genus. *Geom. Topol.*, 23(5):2665–2684, 2019.

[Pic20] Lisa Piccirillo. The Conway knot is not slice. *Ann. of Math. (2)*, 191(2):581–591, 2020.

[Ras10] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

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